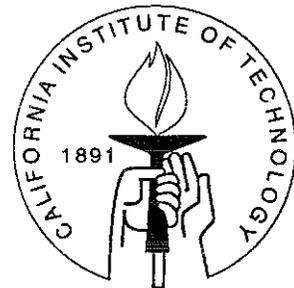


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DISCRETE PRICING AND INSTITUTIONAL DESIGN OF DEALERSHIP MARKETS

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Abstract

This paper models trade in dealership markets when the price grid is in discrete units. Strategic interaction among market makers is complex: Because prices are no longer determined by a zero expected profits condition, priority rules and the timing of offers — do market makers submit price schedules first, or do traders first submit their orders and then market makers set prices — have significant effects on equilibrium outcomes. Discreteness effectively limits competition and permits market makers to offer profitable quotes. In order-driven institutions where traders first submit orders, absolute time priority leads to the “best” price schedule, one which is “better” than that obtained from quote-driven institutions where brokers submit schedules first. This may explain the institutional structure of the NYSE.

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Discrete Pricing and Institutional Design of Dealership Markets

Dan Bernhardt * Eric Hughson †

1 Introduction

This paper focuses on the effects of institutional design of trading markets featuring discrete price grids. We explore the associated ramifications for strategic pricing and demand behavior when agents may be asymmetrically informed. There is little room for institutional design in the canonical insider trading model. Such a model features a set of agents trading claims to an asset in a competitive dealership market. Some, but not all, traders have private information about the asset's value. Uninformed market makers set a *continuous* price schedule which, given their information, yields them zero expected profits given the net order flow (see e.g. Kyle (1985)).

Pricing then is determined solely by this zero expected profit condition. When the pricing function is continuous, market makers cannot earn positive profits of any magnitude — competing market makers would skim off all profits by undercutting by an arbitrarily small amount. Equilibrium then requires that each price exactly equal the expected value of the asset conditional on the order size.

Consequently, there exists a unique schedule which yields market makers zero expected profits order by order.¹ Priority rules which detail how a trade is divided when market makers set the same price do not affect outcomes, nor does the timing of trade. So too, the outcome does not depend on whether the institutional design is a so-called quote-driven open book in which market makers first set price schedules and then traders select their orders, or whether it is order-driven so that traders first submit their orders and then market makers set prices, or whether, as on the NYSE, market makers guarantee some minimum bid-ask-spread up to some maximum trade size, but set price given the order for larger transactions.²

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¹Glosten's (1992) limit order model yields a different schedule. Each limit order earns zero expected profits unconditionally, losing money to large orders and making money from small orders.

²See Madhavan (1992) for a comparison of market and quote-driven institutions when pricing is continuous.

When the price grid is discrete, however, market makers cannot generally set a price which results in zero expected profits conditional on the order size. Hence, equilibrium cannot be determined by a zero profit condition. Consequently, the strategic interplay among agents is dictated by the institutional framework. The past few years have seen a proliferation of different institutional designs as exchanges experiment to find the “optimal” design.

In this paper, we consider two classes of institution, quote-driven and order-driven systems. One can interpret the quote-driven institution as an open limit order book, while the order-driven institution corresponds to market order submission through a specialist who is not required to set a binding quote for large orders. We show that the potential for strictly positive market maker profits means that both market maker profits and the effective pricing schedule (the schedule of “best prices” faced by traders) vary wildly with the institutional rules which dictate both the priority rule and the timing of trade. Consequently, a failure to capture the actual institutional arrangements which dictate the strategic environment can lead to misleading predictions.

Empirical work also strongly suggests that the effects of the discrete price grid on strategic interaction should not be ignored. The price impact of information appears to be small (see e.g. Glosten and Harris (1989)) — perhaps \$.01 to \$.03 per share for a 1000 share order. The typical price grid on the NYSE, \$.125, dwarfs this adverse selection component of the pricing function. The question which the empirical findings beg is why would agents react strategically to such small informational effects on prices, yet ignore the implications of the discrete price grid? Existing work either ignores discreteness, or treats the observed price as a rounded version of some “true” price (Alger (1992), Gottlieb and Kalay (1985), Ball (1988), Harris ((1986), (1989)), etc.), or postulates that trade occurs when the exogenous stochastic “true” price crosses an “eighths boundary” (Cho and Frees (1988)). In contrast, this paper does not gloss over the strategic consequences of discreteness to focus solely on the relatively far smaller informational effects.

The environment we consider is standard save for the restriction that there is a minimum unit size for prices — agents, some of whom are informed, trade claims to a risky security in a dealership market. Like Glosten and Milgrom (1985) we impose almost no restrictions on the stochastic informational processes (i.e. they are not restricted to normally distributed informational events (e.g. Kyle (1985))). Unlike Glosten and Milgrom, we endogenize the order sizes of the informed.

Two market makers compete sequentially in their price quotes. We first consider a quote-driven institution in which first the market makers post price quotes and then traders submit orders. The first market maker posts a price for each transaction level at which he is willing to transact. Then a competing market maker matches or beats whichever quotes he chooses. Given their desired order size, traders then select the best price among the quotes offered. In the event that both market makers offer the best quote, the order is split between them according to some priority rule.

We then consider an order-driven institution in which traders first submit orders and

then the two market makers sequentially set the prices at which they are willing to take the transaction. This institution, it turns out, features the same equilibrium outcomes as when the market makers guarantee a minimum bid-ask spread up to some maximum quantity and for larger orders set higher prices given the order submitted, as on the NYSE.

In both formulations, even though market makers are “competitive” within the discrete institutional environment, the discrete pricing environment does not facilitate competition. Discreteness introduces a strategic advantage to market makers by limiting the incentives to undercut: a market maker must undercut by an entire grid unit. This enables market makers to earn substantially greater profits than would obtain were the market makers merely to “round” prices from the true value to the nearest market price (e.g. to the nearest eighth). Discreteness can thus explain the high price of a seat on the NYSE.

The quote-driven institution features an effective price schedule that is a step function with flat spots. Informed agents react to this price schedule by concentrating their trade at those transaction levels where were they to trade more, they would receive less favorable prices: most levels feature *no* informed trade. This strategic reaction by the informed leads to uncompetitive pricing: beating the first market maker’s schedule on any *single* order size draws heavy informed trade.

The “obvious” comparative statics do not obtain. Neither market maker profits, nor the effective price schedule need be monotone in the probability of informed trade. For instance, when the probability of informed trade falls, to discourage undercutting on a particularly profitable order size, the first market maker may revise his price upward on another, less profitable, trade size. Informed trade would shift, making undercutting on the profitable order size less attractive. Consequently, even though the probability of informed trade falls, the effective price schedule may become less competitive.

In the important case of absolute time priority, the first market maker sets his price schedule, which becomes the effective price schedule, to maximize his profits subject to the constraint that the second market maker earn negative profits undercutting on any *set* of orders. We show, however, that other priority rules may lead to more competitive pricing than absolute time priority, because they may encourage the first market maker to set low prices for some orders, on which he expects to incur losses. Throughout we present examples which highlight these counterintuitive results.

The order-driven institution has a unique equilibrium. This equilibrium involves a mixed strategy on the part of the second market maker who, for most transaction levels, probabilistically undercuts the first market maker’s quote. The endogenous volatility introduced by this mixed pricing strategy plus the bounce from one discrete price grid point to another combine to make asset prices more volatile than the value of the underlying asset.

The probability the second market maker undercuts provides the correct incentives

for insiders to trade at each of these quantities. This means that the expected effective price schedule must be *strictly* increasing in signed trade size except where the bid-ask spread is at a minimum. In turn, the inside trade provides the correct incentives for the second market maker to undercut — it leaves him indifferent between matching the first market maker’s quote and undercutting. These joint restrictions pin down uniquely both the equilibrium expected effective price schedule and the insider’s equilibrium trading strategy. *Equilibrium outcomes are unchanged by Ginsi trading where market makers can split orders, charging different prices on each portion.* Ginsi trading does not smooth the effective amount of discreteness in the price grid.

On those transactions where the second market maker undercuts with positive probability — that is, on most transactions — *the first market maker sets a price which exceeds the expected value of the asset conditional on the order size by exactly $\frac{d}{\alpha}$* , where α is the share of the transaction taken by the first market maker when both market makers offer the same quote. Phrased differently, informed trade is such that the expected value of the asset conditional on the order flow plus $\frac{d}{\alpha}$ equals a feasible price. Since $\alpha \leq 1$, it is not profit-maximizing for the first market maker to set a price which makes it unprofitable for the second market maker to undercut. The first market maker *does not* just “round” his price quote to the nearest feasible price, but rather sets an even less competitive price. This is in sharp contrast to Cho and Frees (1988) who postulate that agents time their trades to take place when the asset value crosses the feasible price at which they transact.

The comparative statics are “intuitive”. As the first market maker’s share α increases, the two market makers offer increasingly attractive price quotes at the expense of the second market maker’s profits. To see this, observe that from the mark-up of $\frac{d}{\alpha}$, conditional on not being undercut, the first market maker’s expected profits per share equal d independent of both his share α and the level of informed trade. Since the mixing second market maker must be indifferent between undercutting and not, his profits must then be exactly $\frac{(1-\alpha)d}{\alpha}$, independent of whether or not he undercuts. Note too, that as the price grid d becomes finer, market maker profits fall, vanishing completely as the price grid becomes arbitrarily fine. This suggests that proposals to the NYSE to reduce the size of the price grid will result in better prices for traders.

It is possible to compare the quote and order-driven institutions from a welfare perspective. When market makers set price schedules first, because of the strategic response of informed trade to prices, less competitive pricing obtains. In the case of absolute time priority ($\alpha = 1$), the effective price schedule set when traders first submit market orders and then prices are set is at least as competitive as that set when market makers submit price schedules first and then traders submit orders. Hence, both informed and uninformed traders strictly prefer to submit their orders first and then have the market makers compete according to absolute priority on price (as is essentially the case on the NYSE).

2 The Model

Risk neutral agents trade claims to a single risky asset. The risky asset's value is given by $\Delta + \delta$, where $E(\delta) = 0$. δ is the current innovation to the commonly known established value of the asset, Δ . The innovation δ is drawn from the continuous density $f(\cdot)$ on bounded support $[\underline{m}, \bar{m}]$, $-\Delta \leq \underline{m} < 0 < \bar{m}$. This distribution is assumed to be common knowledge. Traders submit orders which are in integer multiples of round lots $x > 0$; all prices set are in integer multiples of $d > 0$. A single trader arrives at the market in the period. This agent must trade through one of two uninformed market makers, M_1 and M_2 . The trader is informed with probability γ . An informed trader observes the innovation δ and trades upon this information. An uninformed trader inelastically demands an integer number of round lots of the risky asset, $t \in \{L, L+1, \dots, H\}$, where $L < 0 < H$. The probability the liquidity trader demands tx , $t \in \{L, \dots, H\}$ is $\ell(t) > 0$. The entire transaction is consummated at a single price. It is convenient, but not necessary for the analysis, to assume that Δ , \underline{m} and \bar{m} are integer multiples of the price grid, d . Since the magnitude of the round lot unit x does not affect the analysis or results, we subsume it in our notation. Hence an order of t is an order of tx .

3 Quote-Driven Institutions

Figure 1a illustrates the timing. First, market maker M_1 selects a price schedule which details for each transaction level $t \in \{L, L+1, \dots, H\}$,³ a feasible price $\{p_1(t) : p_1(t) \in \{kd\}_{k=-\infty}^{\infty}\}$, where k is an integer. Having seen M_1 's quotes, market maker M_2 can offer a possibly more attractive feasible price, $p_2(p_1(\cdot), t)$, for any transaction quantity and win the sale⁴. Given these two price schedules, an informed trader selects an order size. Both informed and uninformed agents trade with the market maker offering the better quote. If $p_2(t) = p_1(t)$,⁵ the two market makers split the transaction, with M_1 taking fraction $\alpha > .5$. This division could be probabilistic. We consider sharing rules other than absolute time priority because sometimes transactions are shared by floor traders at the specialist's post. The effective price schedule is given by:

$$\bar{p}(t) = \begin{cases} \min\{p_2(t), p_1(t)\} & t > 0 \\ \max\{p_2(t), p_1(t)\} & t < 0 \end{cases}$$

Equilibrium: An equilibrium is a feasible pricing function for M_1 , $\{p_1^*(t)\}_{t=L}^H$; a pricing function for M_2 conditional on M_1 's price schedule, $\{p_2^*(p_1(\cdot), t)\}_{t=L}^H$; a choice of dealers by informed and liquidity traders; and a set of demands by the informed, $t^*(\delta, \bar{p}(\cdot))$; such that:

(a) M_1 selects his price schedule $\{p_1^*(t)\}_{t=L}^H$ to maximize expected profits:

$$\pi_1(p_1^*(\cdot)) = \sum_{t=L}^H I_1(p_1^*(t), p_2^*(p_1(\cdot), t)) [(1 - \gamma)\ell(t)(p_1^*(t) - \Delta)t +$$

³We focus on these quantities since, in equilibrium, only these orders will be observed, and prices for greater orders must only be "large or small enough".

⁴The analysis extends straightforwardly to additional market makers (Bernhardt and Hughson (1991)).

⁵To reduce notation, we often use the notation $p_2(t) \equiv p_2^*(p_1(\cdot), t)$ to refer to the price that M_2 sets for order t , and $p_2^*(t) \equiv p_2(p_1^*(\cdot), t)$ to denote the equilibrium price.

$$\gamma\chi^*(t)(p_1^*(t) - (\Delta + E[\delta|t]))t^*(\delta, \bar{p}(\cdot)),$$

given M_2 's optimal response, $\{p_2^*(p_1(\cdot), t)\}_{t=L}^H$. $E[\delta|t]$ is the expectation of δ conditional on *an insider* trading t . $\chi^*(t)$ is the probability an insider observes an innovation which leads to a trade of t . $I_1(p_1(t), p_2(t))$ is an indicator function detailing M_1 's share of the order:

$$I_1(p_1(t), p_2(t)) = \begin{cases} 1 & \text{if } p_2(t) > p_1(t), t > 0 \quad \text{or } p_2(t) < p_1(t), t < 0 \\ \alpha & \text{if } p_2(t) = p_1(t) \\ 0 & \text{if } p_2(t) < p_1(t), t > 0 \quad \text{or } p_2(t) > p_1(t), t < 0. \end{cases}$$

(b) M_2 selects his price schedule $\{p_2^*(p_1(\cdot), t)\}_{t=L}^H$ to maximize expected profits:

$$\begin{aligned} \pi_2(p_2^*(p_1(\cdot), \cdot)) = \sum_{t=L}^H I_2(p_1(t), p_2^*(p_1(\cdot), t)) [(1 - \gamma)\ell(t)(p_2^*(p_1(\cdot), t) - \Delta)t + \\ \gamma\chi^*(t)(p_2^*(p_1(\cdot), t) - (\Delta + E[\delta|t]))t^*(\delta, \bar{p}(\cdot))], \end{aligned}$$

where $I_2(\cdot)$ is an indicator function detailing M_2 's share of the order, $I_2(\cdot) = 1 - I_1(\cdot)$.

(c) An agent trading t shares maximizes profits by trading with the market maker who offers the best price, and dividing his transaction between the two market makers, fraction α to the M_1 and fraction $(1 - \alpha)$ to M_2 if the market makers set the same price.

(d) The informed choose order size, $t^*(\delta, \bar{p}(\cdot))$, to maximize expected profits:

$$t^*(\delta, \bar{p}(\cdot)) = \max\{\operatorname{argmax}_t(\Delta + \delta - \bar{p}(t))t\}.6 \quad (1)$$

The price of the zero transaction volume is not identified as it does not affect profits, so we assume without loss of generality:

Assumption 1: $p_1^*(0) = p_2^*(0) = \Delta$.

We first provide some intuitive and useful background results. First, if M_2 undercuts M_1 , he undercuts by the smallest amount possible. To undercut by more would both attract more informed trade and needlessly reduce revenues. Second, on the set of trade sizes where M_2 undercuts, he expects a net profit. While M_2 may expect to lose money on some trades, his expected net profit on those order levels where he undercuts must be positive. This result reflects the fact that M_2 can always set the same prices as M_1 and share informed trade with M_1 (at a better price). Third, if M_1 sets a lower price than

⁶The max operator uniquely selects a particular preferred trade quantity in the probability zero event of indifference.

M_2 , then he must expect losses from that trade. Otherwise M_2 would profitably match him, because market maker identity does not affect informed trade. Last, if conditional on handling a given order a market maker expects losses, then the market makers set different prices. This again reflects that M_2 does not want to share any of M_1 's losses and informed trade depends only on the effective price schedule. Throughout the remainder of this paper, we focus on buy orders; $t > 0$. Propositions for sell orders follow analogously. Proofs are in the appendix.

Proposition 1 *In any equilibrium,*

- (a) M_2 never undercuts M_1 by more than d : $p_2(t) \geq p_1(t) - d$, $\forall t$.
- (b) M_2 expects positive profits from the set of transactions on which he undercuts.
- (c) On any transaction where M_1 sets a price lower than M_2 's price, he expects losses.
- (d) If the price set for a particular order level is such that market makers expect negative profits then only one market maker sets that price.

Example 1 will illustrate that either market maker may expect to lose money on a given order.

Proposition 2 *An equilibrium exists in which the effective price schedule is monotone increasing in quantity demanded: $\bar{p}^*(t) \geq \bar{p}^*(t - 1)$. The price schedule is strictly monotone increasing at trade levels where there is informed trade.*

A pure strategy equilibrium follows because agents move sequentially with the informed trader moving last. Each agent has a best response among the finite number of possibly optimal alternatives. Equilibria with non-monotonic effective price schedules may also exist. Equilibrium non-monotonicities can occur only at orders which M_1 does not handle. Such transaction levels feature no informed trade so that M_1 will be undercut by M_2 . Since M_1 is undercut he does not care how high his initial quote is.

The effective price schedule is a step function with flat spots (see Figure 2a). The informed maximize profits by trading as much as they can at a given price. At other transaction levels, market makers recognize they are trading with an uninformed trader, and hence earning positive expected profits. In equilibrium, however, it is not profitable to offer better prices for those trade sizes, for that would draw informed trade.

Corollary 3 *Informed demand, $t^*(\delta, \bar{p}(\cdot))$, is monotone increasing in δ . The innovation space, $[\underline{m}, \bar{m}]$ can therefore be partitioned into*

$$\underline{m} = \delta_{\bullet}^* < \delta_1^* < \dots < \delta_m^* = \bar{m},$$

where if $\delta > 0$, $t^*(\delta, \bar{p}(\cdot)) \geq 0$ and for $\delta \in [\delta_i^*, \delta_{i+1}^*)$, $t^*(\delta, \bar{p}(\cdot)) = t_i^*$, and $t_i^* < t_{i+1}^*$. That is, δ_i^* is the smallest signal resulting in a trade of t_i^* .

Proposition 4 *The bid-ask spread is always positive, even for the smallest orders. If $t > 0$ then $\bar{p}^*(t) > \Delta$. If $t < 0$ then $\bar{p}^*(t) < \Delta$.*

There is a positive bid-ask spread at all transaction levels even if there is arbitrarily little private information or the probability that a given trader is informed is arbitrarily small. The intuition is that if a market maker sets a price without an adverse selection component, $p(t) = \Delta$, he does not gain from the uninformed, and will lose needlessly to the informed who observe small innovations, $|\delta| < d$. It is only profitable for the informed to trade on such information if there is no adverse selection component to price.

Proposition 5 *If there is not absolute time priority, i.e. if $\alpha < 1$, then both market makers expect to earn positive profits in equilibrium.*

The next proposition shows how to construct the equilibrium in the case of absolute time priority: M_1 sets his schedule to maximize his profits subject to the constraint that it is not profitable for M_2 to undercut him on any set of orders so that M_1 handles every transaction, and M_2 earns no profits.

Proposition 6 *Suppose that there is absolute time priority, i.e. $\alpha = 1$. If the price schedule satisfies $\Delta - 3d \leq p_1(t) \leq \Delta + 3d$ for all $t \in \{L, L+1, \dots, H\}$, then M_1 selects his price schedule, $\{p_1^*(t)\}_{t=L}^H$, to maximize profits subject to the constraint that undercutting by M_2 on any set of transaction levels is unprofitable. Further, since M_1 handles every transaction in equilibrium, he has no incentive to update his price schedule after M_2 has moved.*

M_1 's price schedule must make it unprofitable for M_2 to undercut on any set of transactions, rather than just making it unprofitable to undercut on any single transaction. M_2 may find it unprofitable to undercut on one transaction because that order would then draw heavy informed trade. M_2 may do better to beat M_1 's quotes on several orders, expect to lose money on those levels with informed trade, but make it up from uninformed trade at other levels.

For most assets, trading only occurs within the bounds provided in the proposition. We have been unable to prove the more general theorem, but it is difficult to imagine that given absolute time priority it is ever the case that M_1 sets a schedule other than that given by the solution to the program above. It requires that it be profitable for M_1 to set a very uncompetitive price schedule despite the fact that he will be undercut.

Corollary 7 *When there is absolute time priority so that the conditions of proposition 5 hold, M_1 's price schedule is monotone increasing in transaction size, t : $p_1(t) \geq p_1(t-1)$.*

Surprisingly, absolute time priority need not lead to the most competitive price schedule consistent with profit maximizing by the market makers, the one which minimizes the losses of liquidity traders (due to the adverse selection). Example 1 illustrates that while absolute time priority maximizes the incentives of M_2 to undercut, more competitive schedules may obtain when M_1 has a share $\alpha < 1$ and expects to incur a loss conditional on handling some trades. It also demonstrates that the first market maker's price schedule need not be monotone in signed volume if $\alpha \neq 1$. It suggests that there are no comparative statics results with respect to the priority rule. Example 1 shows that the equilibrium can possess the following *counterintuitive* features.

(a) M_1 's expected profits need not increase with increases in his share, α ; M_2 's expected profits need not fall with increases in α .

(b) The effective price schedule $\bar{p}_\alpha(\cdot)$ may become less competitive as α increases; absolute priority need not lead to the most competitive pricing.

(c) M_1 may expect losses from handling some trade sizes.

(d) M_1 's price schedule, $p_1(t)$, may not be monotone increasing in order size, t .

(e) Market maker profits can rise with increases in the probability of informed trade, γ .

(f) The effective price schedule, $\bar{p}_\gamma(\cdot)$, may become more competitive with increases in the probability of informed trade, γ .

Example 1: Equilibrium strategies and profits as a function of M_1 's share, α when market makers set prices first							
M_1 's share α	M_1 's strategy $\{p_1(1), p_1(2)\}$	M_2 's reaction $\{p_2(1), p_2(2)\}$	Equilibrium prices $\{\bar{p}(1), \bar{p}(2)\}$	Insider trade (round lots)	M_1 profits	M_2 profits	Insider profits
$\alpha < .619$	{10,11}	{10,11}	{10,11}	1	$.84\alpha$	$.84(1 - \alpha)$	1
$.619 < \alpha < .8$	{9,10}	{-,10}	{9,10}	1	$.08 + .4\alpha$	$.4(1 - \alpha)$	2
$\alpha > .8$	{10,10}	{10,-}	{10,10}	2	$-.4 + .8\alpha$	$.8(1 - \alpha)$	2

The price grid is in units of $d = 1$ and $\Delta = 0$. The asset shock δ can take on two values $\pm(11 - \epsilon)$, ϵ arbitrarily small with equal probability; liquidity traders buy (sell) either 1 or 2 round lots with probabilities $\ell(\pm 1) = .4$, $\ell(\pm 2) = .1$; and the probability a trader is informed is $\gamma = .8$.

If his share is small enough, M_1 can earn greatest profits by setting high prices of $p_1(1) = 10$, $p_1(2) = 11$. If M_2 matches $p_1(1) = 10$, $p_1(2) = 11$, then M_1 receives $\alpha[(.2)(.4)(10) - (.8)(.5)(11 - 10) + (.2)(.1)(11)(2)] = .84\alpha$ and M_2 receives $.84(1 - \alpha)$.

However, M_2 can earn $.32$ from undercutting M_1 with $p_2(1) = 9$, $p_2(2) = 10$, taking all sales, losing money on transactions of 1 share and making money on transactions of 2 shares: if transaction prices of $\bar{p}(1) = 9$, $\bar{p}(2) = 10$ are set, the insider prefers to buy a single share. Hence if $\alpha > .52/.84 \cong .619$, M_2 has an incentive to undercut on both transaction sizes.

If $\alpha > .619$ and M_1 sets $p_1(1) = 10$, $p_1(2) = 10$, he expects to earn $-.4 + .8\alpha$. Since the informed trade two lots in this case, M_2 matches only on trades of a single unit and expects to make $.8(1 - \alpha)$. However, if M_1 sets $p_1(1) = 9$, $p_1(2) = 10$, he expects to earn $-.08 + .4\alpha$. M_2 matches only on trades of two round lots and expects to make $.4(1 - \alpha)$. For $\alpha < .8$, M_1 sets the very competitive prices of $p_1(1) = 9$, $p_1(2) = 10$, and for $\alpha > .8$, M_1 sets a slightly less competitive schedule, $p_1(1) = 10$, $p_1(2) = 10$.

(a) Absolute time priority, $\alpha = 1$, does not generate the tightest bid-ask spreads because for less extreme priority rules M_1 sets prices for some orders for which he expects to lose money. The intuition for the humped relationship between α and the effective price schedule is the following: for small α , M_1 need not set a competitive schedule because M_2 gains enough from sharing orders to make undercutting unattractive. For larger α , the profits from sharing are lower so that M_2 would undercut $p_1(1) = 10$, $p_1(2) = 11$ with $p_2(1) = 9$, $p_2(2) = 10$. To avoid being undercut on every transaction, M_1 must set a schedule for which he expects to lose money on orders of either 1 or 2 round lots. For smaller α , he prefers to have the heavy liquidity trade at 1 round lot to himself and incur slightly heavier insider trading losses by setting $p_1(1) = 9$, $p_1(2) = 10$. For sufficiently larger α , M_1 does not mind sharing liquidity trade at 1 unit because he receives the lion's share. He prefers to set a less competitive schedule in which losses to informed traders are less: $p_1(1) = 10$, $p_1(2) = 10$.

(b) Clearly, this implies that there need not be a monotonic relationship between M_1 's share of the order if he sets the same price as M_2 , α , and his profits. If M_1 receives too large a share, the incentive for M_2 to undercut increases enough that to discourage undercutting M_1 must adjust his price schedule downward.

(c) Consequently, no monotonic relationship need exist between the probability a trader is informed, γ , and either M_1 's profits or the effective price schedule. More informed trade reduces the incentive for M_2 to undercut M_1 , so that the share α that M_1 receives must be larger for M_2 to have an incentive to offer a better price schedule.

(d) M_2 's expected profits exceed M_1 's if and only if $.75 > \alpha > .619$.

(e) Were the example altered slightly so that $\ell(1) = .39$, $\ell(2) = .1$, $\ell(3) = .01$, and α were sufficiently high, then M_1 would set prices $p_1(1) = 10$, $p_1(2) = 12$, $p_1(3) = 11$; his price schedule would not be monotone in quantity although the effective price schedule, $\bar{p}(1) = 10$, $\bar{p}(2) = 11$, $\bar{p}(3) = 11$ would be. M_1 does not have to worry about M_2 undercutting $p_1(3) = 11$ because that would draw informed trade, and the gain in uninformed trade is small. \square

Bernhardt and Hughson (1991) construct examples in which the second market maker expects negative profits conditional on taking a particular order. Finally, one can show that as the price grid becomes finer, the effective price schedule can feature greater or smaller bid-ask spreads and that market maker profits can either rise or fall. The reasoning is similar to that for α or γ . As the grid becomes finer, the amount by which M_2 must undercut is reduced. This can encourage undercutting where none occurred before, but undercutting by less may also follow because M_2 undercuts by as little as possible. In turn, M_1 considers these strategic effects when choosing his price schedule. Since the strategic effects are ambiguous, then so too are the effects on profits. However, the following limiting result does obtain:

Proposition 8 *As the price grid becomes arbitrarily fine, then M_1 's expected profits go to zero, order by order.*

Intuitively, as the grid becomes arbitrarily fine, undercutting affects aggregate market maker profits by arbitrarily little so that M_2 can appropriate any market maker profits. M_2 's profits go to zero with the grid size if there is additional competition from other market makers (see Bernhardt and Hughson (1991)).

It is the discreteness in the price grid which leads to positive market maker profits. If the price schedule is continuous, then M_1 cannot earn positive profits of any magnitude — M_2 would skim off all profits by undercutting by an arbitrarily small amount, cream skimming without affecting informed trade. In equilibrium with a continuous price schedule, M_1 sets each price exactly equal to the expected value of the asset conditional on the order size.

4 Order-Driven Institutions

Figure 1b illustrates the timing. First a trader arrives at the market and submits an order t . Neither market maker knows whether this trader is informed. M_1 then sets a feasible price at which he is willing to take the other side of that trade, $p_1(t) \in \{kd\}_{k=-\infty}^{\infty}$, where k is an integer. Having seen M_1 's quote, M_2 can offer a possibly more attractive feasible price $p_2(p_1(t), t)$ and take the trade. The market maker offering the best price wins the transaction, where if $p_2(t) = p_1(t)$, the two market makers split the order with M_1 taking fraction $\alpha \geq .5$.

Equilibrium demands a mixed pricing strategy from M_2 in order to provide the right trading incentives for the informed. An informed agent, whose trading strategy determines the profitability of the market makers' schedules, strictly prefers that M_2 make more attractive offers. In turn, the trading strategy of the informed must leave M_2 indifferent between beating M_1 's quote to obtain the entire transaction and matching M_1 's price and sharing the transaction. The only way for the informed agent's trading strategy to be consistent with market maker pricing and vice versa is for M_2 to adopt a mixed pricing strategy⁷. Because equilibrium outcomes are unaffected by the assumption, to reduce notation we assume that M_1 adopts a pure strategy; we only consider mixed pricing strategies for M_2 . Later we detail how the analysis extends. Let $\omega_{2t}(kd)$ be the probability that M_2 sets price $p_2(t) = kd$.

Equilibrium: An equilibrium is a feasible price for each transaction level t at which M_1 is willing to accept the order, $p_1^*(t) \in \{kd\}_{k=-\infty}^{\infty}$; a vector of probabilities of feasible prices for each transaction level t at which M_2 is willing to take the trade conditional on the price, $p_1(t)$, set by M_1 : $\{\omega_{2t}^*(kd | p_1(t))\}_{k=-\infty}^{\infty}$; and a set of demands by the informed, $t^*(\delta)$, such that:

(a) For each possible order received, $t = \dots - 1, 0, 1, 2, \dots$, the price set by M_1 , $p_1^*(t) \in \{kd\}_{k=-\infty}^{\infty}$, maximizes his expected profits:

⁷Were M_2 to undercut for sure on trade t where equilibrium demanded a mixed strategy, then "more" informed traders would want to trade t . But then it would be strictly less profitable for M_2 to undercut than to match...

$$\begin{aligned} \pi_1(p_1(t)) = & E_{p_2^*(t), \delta} [I_1^*(p_1(t), p_2^*(p_1(t), t))((1 - \gamma)\ell(t)t(p_1(t) - \Delta) \\ & + \gamma\chi^*(t)t^*(\delta)(p_1(t) - (\Delta + \delta)))]t, \end{aligned}$$

where the rational expectations, conditioned on the transaction size, are taken over M_2 's optimal price choice, $p_2^*(t)$, and δ .

(b) The second market maker selects $\{\omega_{2t}^*(kd|p_1(t))\}_{k=-\infty}^{\infty}$ to maximize expected profits:

$$\begin{aligned} \pi_2(\{\omega_{2t}^*(kd|p_1(t))\}_{k=-\infty}^{\infty}, p_1(t)) = & \sum_{k=-\infty}^{\infty} (\omega_{2t}^*(kd|p_1(t))I_2^*(p_1(t), p_2(t) = kd) \\ & \times ((1 - \gamma)\ell(t)(kd - \Delta)t + \gamma\chi^*(t)E[t^*(\delta)(kd - (\Delta + \delta))|t]). \end{aligned}$$

The first term on the right hand side is the expected profit from trading with the uninformed; the second is the expected loss trading with the informed.

(c) The informed choose order size, $t^*(\delta)$, to maximize expected profits:

$$t^*(\delta) = \max\{\operatorname{argmax}_t(\Delta + \delta - E_{p_2(t)}[\bar{p}(t)])t\}.$$

Propositions 7-11 characterize the properties that any equilibrium must have. We first show that as long as M_1 's share, α , is less than one, that each market maker expects strictly positive profits from each order. This implies that the probability that M_1 is undercut is less than one.

Proposition 9 *In equilibrium, unless there is absolute time priority (i.e. $\alpha < 1$),*

- (a) *Each market maker expects strictly positive profits from any price he charges.*
- (b) *Consequently, the probability that M_1 handles the transaction is strictly positive.*
- (c) *And M_1 never undercuts M_2 in equilibrium: $p_2(t) = \bar{p}(t)$.*

We now provide the analog for Proposition 3: there is a strictly positive bid-ask spread at all transaction levels. The intuition is again that if a market maker sets a price of Δ , he does not gain from the uninformed, and with positive probability he will lose needlessly to some informed who receive small signals, $|\delta| < d$. This result contrasts with that of Easley and O'Hara (1987) who find no adverse selection component for small transactions only because they consider a sparse finite support for the signal space so that the informed want to submit large orders despite a positive bid-ask spread.

Proposition 10 *Equilibrium prices feature a strictly positive adverse selection component, even for the smallest transactions. If $t > 0$ then $\bar{p}(t) > \Delta$. If $t < 0$ then $\bar{p}(t) < \Delta$.*

The next result details that the only order sizes, $0 < t \leq H$, with no insider trade have price $\bar{p}(t) = \Delta + d$. In turn, for there to be informed trade at those greater quantities where $p_1(t) > \Delta + d$, the expected effective price schedule must be strictly increasing in t .

Proposition 11 *Define $T_1 > 0$ to be the maximum trade size t such that $p_1(t) = \Delta + d$.*

(a) *For transactions $t = 1, 2, \dots, T_1 - 1$, $p_1(t) = \Delta + d$ and there is no informed trade.*

(b) *For greater transaction quantities, $t = T_1, T_1 + 1, \dots, H$, the probability the trader is informed is strictly positive.*

(c) *The expected effective price schedule is strictly monotone increasing in trade size:*

$$E_{p_2(T_1)}[\bar{p}(T_1)] < E_{p_2(T_1+1)}[\bar{p}(T_1 + 1)] < \dots < E_{p_2(H)}[\bar{p}(H)],$$

where the expectations are taken over M_2 's price.

(d) *If M_2 undercuts M_1 , he loses money conditional on trading with an informed trader.*

These results contrast sharply with those generated by the quote-driven institution: recall that there the informed trade only at a few transaction levels and that because there is no mixing, the effective price schedule is a step function with flat spots. When market makers set schedules first, mixing over price schedules does not affect informed trade because an informed trader can see the schedule realization before submitting an order.

It is clear from Proposition 9 that the same equilibrium outcomes obtain were each market maker initially to set a transaction level $T > 0$ such that provided the order size does not exceed T the market maker guarantees to accept the order at the minimum bid price of $\Delta + d$. For greater quantities, traders submit their orders and then market makers make their (higher) price offers. With this formulation, the guarantee is $T = T_1$. Informed trade is unaffected by the guarantee so the equilibrium outcome must be unchanged. This formulation captures the practice on the NYSE: the specialist for a stock guarantees prices for sufficiently small orders. The guarantee is to take a trader's buy (sell) order at the specified bid (ask) price provided the order size does not exceed the specified bound.

Corollary 12 *Informed demand, $t^*(\delta)$, is monotone increasing in the signal, δ : $\delta > \delta'$ implies that $t^*(\delta) \geq t^*(\delta')$. The innovation space, $[\underline{m}, \bar{m}]$ can be partitioned by*

$$\underline{m} < \delta_L^* < \delta_{L-1}^* < \dots < \delta_{T_1}^* = -d; d = \delta_{T_1}^* < \delta_{T_1+1}^* < \dots < \delta_H^* < \bar{m}.$$

For $\delta \in (-d, d)$ the insider does not trade; for $0 < \delta \in [\delta_t^*, \delta_{t+1}^*)$, $t^*(\delta) = t$; for $0 > \delta \in [\delta_{t-1}^*, \delta_t^*)$, $t^*(\delta) = t$.

The existence of signals $\delta_{T_1}^*$ and $\delta_{T_1}^*$ requires that a round lot transaction has an adverse selection component of only d . If there is “too” little liquidity trade then T_1 is 0: each order size features informed trade and the difference between the minimum ask price and Δ exceeds d . In this case, the minimum bid–ask spread set by M_1 is *at least* $3d$.

An insider seeing signal $\delta_t^* > 0$ is just indifferent between trading t round lots and $t + 1$. The value of the extra round lot just balances the cost of the price increase on the first t lots:

$$(\Delta + \delta_t^* - E[\bar{p}(t+1)]) = (t+1)(E[\bar{p}(t+1)] - E[\bar{p}(t)]).$$

Our next result is that if M_2 undercuts, he undercuts by the minimum necessary to take the entire transaction. Second, if he undercuts with positive probability, M_2 must be indifferent between undercutting and not.

Proposition 13 *If for some transaction level t , M_2 sets more than one price with positive probability in equilibrium, then*

- (a) M_2 must expect the same profits from each price.
- (b) M_2 either matches M_1 's quote or undercuts by d : $p_2(t) \in \{p_1(t), p_1(t) - d\}$.
- (c) If M_2 undercuts M_1 with positive probability, i.e. if $\omega_{2t}^*(p_1(t) - d | p_1(t)) > 0$, then

$$\alpha(p_1(t) - \Delta - \frac{\gamma\chi^*(t)}{(1-\gamma)\ell(t) + \gamma\chi^*(t)} E[\delta|t]) = d. \quad (2)$$

Condition (c) that M_2 must be indifferent between undercutting and not provides strong restrictions on the equilibrium. The left hand side of (2) is M_1 's profit conditional on not being undercut. This must exactly equal the price grid size, d . Equation (2) underlies the following key result.

Proposition 14 *For $t > T_1$, if $p_1(t) = p_1(t+1) \dots = p_1(t+j)$, then the greater is the trade size, the greater are M_1 's expected profits:*

$$\pi_1(t) = t\omega_2^*(t)d < \pi_1(t+1) = (t+1)\omega_2^*(t+1)d < \dots < \pi_1(t+j-1) = (t+j-1)\omega_2^*(t+j-1)d,$$

where $\omega_2^*(t)$ is redefined as the equilibrium probability that M_2 undercuts M_1 on order t . On trade size $t + j$, an upper bound on M_1 's profits is the grid size, d . The likelihood of insider trade falls with the trade size in such a way as to keep the expected innovation from the perspective of the market makers (who do not know whether the trader is informed) constant:

$$\begin{aligned} \frac{\gamma x^*(t)}{\gamma x^*(t) + (1-\gamma)\ell(t)} E[\delta|t] &= \frac{\gamma x^*(t+1)}{\gamma x^*(t+1) + (1-\gamma)\ell(t+1)} E[\delta|t+1] = \dots \\ &= \frac{\gamma x^*(t+j-1)}{\gamma x^*(t+j-1) + (1-\gamma)\ell(t+j-1)} E[\delta|t+j-1]. \end{aligned}$$

The intuition is that M_2 must undercut with decreasing probability as the order size increases to maintain the equilibrium incentives for some informed traders to transact at each trade level. Hence as order size increases, M_1 receives a greater portion of the same (per share) pie, so that his expected profits per share are increasing in the order size. In turn, the informed's trading strategy must leave M_2 just indifferent between undercutting and not. Consequently, the conditional probability of informed trade must fall as the order size increases. In sharp contrast, with the quote-driven institution, recall that the insider trades as much as possible at a given price and will not trade other quantities.

M_1 does not set a very competitive price schedule (see Figure 2b). If the probability of being undercut is positive⁸, then (see equation 2) his profit-maximizing quote exceeds the expected value of the asset conditional on the order size by exactly $\frac{d}{\alpha} \geq d$:

$$p_1(t) = \Delta + E[\delta|t] + \frac{d}{\alpha}.$$

Since $\alpha \leq 1$, M_1 *does not* just “round” his price quote up to the nearest feasible price above the asset's expected value, but rather sets an even less competitive price. Further, for $\alpha \neq 1$, M_2 earns positive profits when he undercuts. Note that one implication is that the informed trader's strategy is such that $E[\delta|t] + \frac{d}{\alpha}$ is a *feasible price*. If for two different (positive) transaction levels M_2 undercuts M_1 with positive probability, then the difference in the expected innovation is exactly equal to the difference in the prices that M_1 sets, kd .

Propositions 9 and 11 combine to have strong implications for market maker profits:

1. For small order sizes $T_{-1} < t < T_1$, M_1 's profits per share exceed those of M_2 , i.e., $\alpha d > (1 - \alpha)d$.
2. For any larger order where M_2 does not undercut M_1 with positive probability, M_1 's expected profits per share exceed M_2 's, but are bounded from above by d .
3. For most large transactions $t > T_1$, M_2 undercuts M_1 with positive probability. On these orders M_1 's expected profits per share, conditional on not being undercut exactly equal the grid size d . Hence, M_1 's unconditional expected profits per share are $(1 - \omega_2^*(t))d$. M_2 must be indifferent between undercutting and not. Hence, M_2 's expected per share profits are $\frac{(1-\alpha)}{\alpha}d$, independent of the quantity traded as well as, whether or not he undercuts.

⁸In practice on the NYSE, for most orders with prices $p_1(t) > \Delta + d$, the transaction price does not increase by d with each additional round lot, suggesting that the probability of undercutting is generally positive.

Either market maker may expect greater profits from a given transaction. M_1 expects greater profits than M_2 if and only if M_2 is not too likely to undercut, i.e. if $\omega_2^*(t) < \frac{2\alpha-1}{\alpha}$. The probability of undercutting, $\omega_2^*(t)$, is pinned down by the equilibrium condition for the insider which determines his order choices (which must, in turn, leave M_2 indifferent between beating and matching M_1 's price).

In the case of absolute time priority, M_2 's expected profits are zero. It cannot be profitable for M_2 to undercut M_1 or he would always do so. But then M_1 could have obtained those profits by setting the lower price as well. Hence with absolute time priority, additional market makers have no effect on outcomes: the same equilibrium expected price schedule obtains. The order of play of the market makers is also irrelevant: the same prices obtain when the market makers set prices simultaneously, but one market maker is given absolute priority as when market makers select schedules sequentially and there is absolute time priority.

The next proposition details that there is only one expected effective price schedule consistent with optimization by both the informed trader and the market makers. Intuitively were there multiple equilibrium price schedules then there would exist some greatest transaction level t where the expected effective prices differ. But then, for the schedule with the higher price, informed trade would be less. But then there would be a greater incentive for market makers to set lower prices, a contradiction.

Proposition 15 *The expected effective price schedule, $\{E_{p_2(t)}[\bar{p}(t)]\}_{t=L}^H$, is unique.*

Corollary 16 *The equilibrium trading strategy by the informed, $\{\delta_j^*\}_{j=L}^H$, is unique.*

Example 2: Equilibrium strategies and profits									
Case A: Traders submit orders first						Case B: Market makers set prices first			
Trade size t	p_1	Prob. $p_2 < p_1$	$E[\bar{p}]$	Signals leading to trade size t	Pr(informed) given t	Equil. price: $\bar{p}(t) = p_1(t)$	Signals leading to trade size t	Pr(informed) given t	Expected value given t
0	0	0.000	0.000	$0.000 < \delta < 1.000$	1.000	0	$0 < \delta < 1$	1.000	0.500
1	1	0.000	1.000	Never	0.000	1	None	0.000	0.000
2	1	0.000	1.000	$1.000 < \delta < 1.832$	0.333	1	$1 < \delta < 3$	0.545	0.981
3	2	0.722	1.278	$1.832 < \delta < 3.206$	0.452	2	None	0.000	0.000
4	2	0.241	1.759	$3.206 < \delta < 4.000$	0.323	2	$3 < \delta < 4$	0.375	1.138

The price grid is in units of $d = 1$ and there is absolute time priority ($\alpha = 1$).

$\Delta = 0$, and δ is uniformly distributed on $[-4, 4]$.

The probability of an insider is $\gamma = .325$, and the uninformed are equally likely to buy or sell 1, 2, 3, 4 units.

Case A: Insider expected profits are $2 * \frac{1.832-1}{4}(\frac{1}{2}(1.832 + 1) - 1) + 3 * \frac{3.206-1.832}{4}(\frac{1}{2}(3.206 + 1.832) - 1.278) + 4 * \frac{4-3.206}{4}(\frac{1}{2}(4 + 3.206) - 1.759) = 2.916$. His unconditional expected profits are $.325 * 2.916 = .948$.

Of that amount, only 1.288 comes at the expense of M_1 . M_1 's gains from uninformed are $\frac{1}{4}(1(1 - 0) + 2(1 - 0) + .278 * 3(2 - 0) + .759 * 4(2 - 0)) = 2.685$.

M_1 's expected profits are therefore $.675 * 2.685 - .325 * 1.288 = 1.393$, which exceed those of the informed trader. The uninformed also lose an additional $\frac{1}{4}(.722 * 3 + .241 * 4) = .783$ to M_2 , yielding 3.47 in losses. Since $\alpha = 1$, M_2 expects zero profits conditional on taking a transaction.

Case B: Since $\alpha = 1$, M_1 's strategy is the equilibrium pricing function. If M_2 attempts to undercut to $p_2(3) = 1$ at transaction level 3, all informed traders who receive signals $\delta > 1$ trade 3 units, and M_2 makes losses because the expected value of the asset given $t = 3$ is 1.477.

Insider expected profits are reduced to 2.5. Since $\alpha = 1$, M_1 's gains from the uninformed equal total uninformed losses, $4.25 > 3.47$. M_1 's expected profits increase to $.675 * 4.25 - .325 * 2.5 = 2.056$ from 1.393.

M_1 cannot set a less competitive schedule. If M_1 sets $p_1 = \{1, 2, 2, 2\}$, M_2 can undercut at $t = 2$ and $t = 3$ with $p_2 = \{1, 1, 1, 2\}$, lose at $t = 3$, but profit overall. When γ is instead .375, equilibrium prices are the same, but M_1 would now expect losses at $t = 2$ (the expected value is 1.09), even though there is absolute time priority.

The construction of the equilibrium is as follows: $\delta^*(4) = 3.206$ leaves M_2 indifferent between undercutting $p_1(4) = \Delta + 2$ and not:

$$\begin{aligned} (1 - \alpha)((1 - \gamma)\ell(4)(p_1(4) - \Delta) + \gamma\chi^*(4)(p_1(4) - \Delta - E[\delta|4])) \\ = (1 - \gamma)\ell(4)(p_1(4) - 1 - \Delta) + \gamma\chi^*(4)(p_1(4) - 1 - \Delta - E[\delta|4]), \end{aligned}$$

where $\chi^*(4) = \int_{\delta^*(4)}^{\bar{m}} dF(\delta)$, and $E[\delta|4] = E[\delta|\delta^*(4) < \delta \leq \bar{m}]$. Similarly, $\delta^*(3)$ leaves M_2 indifferent to undercutting M_1 on trades of 3 round lots:

$$\begin{aligned} (1 - \alpha)((1 - \gamma)\ell(3)(p_1(3) - \Delta) + \gamma\chi^*(3)(p_1(3) - \Delta - E[\delta|3])) \\ = (1 - \gamma)\ell(3)(p_1(3) - d - \Delta) + \gamma\chi^*(3)(p_1(3) - d - \Delta - E[\delta|3]), \end{aligned}$$

where $\chi^*(3) = \int_{\delta^*(3)}^{\delta^*(4)} dF(\delta)$, and $E[\delta|3] = E[\delta|\delta^*(3) < \delta \leq \delta^*(4)]$; $\delta^*(2) = 1$.

In turn, the probability that M_2 undercuts M_1 on trades of 3 or 4 round lots $\omega(3), \omega(4)$ pin down $\delta^*(3), \delta^*(4)$ by leaving the insider indifferent who sees such asset innovations indifferent between the quantities which he trades:

$$\begin{aligned} 2(\delta^*(3) - p_1(2)) &= 3(\delta^*(3) - (p_1(3) - \omega(3)d)); \\ 3(\delta^*(4) - p_1(3) - \omega(3)d) &= 4(\delta^*(4) - (p_1(4) - \omega(4)d)). \end{aligned}$$

The expected profits of the first “competitive” market maker *actually exceed* those of the informed trader. Indeed, since $p_1(3) = 2 > 1.832 = \delta^*(3)$, M_1 expects a profit from trading with some *informed* traders: such informed traders hope to make positive profits when M_2 undercuts M_1 . In contrast, whenever M_2 undercuts M_1 , conditional on trading with an informed trader, M_2 *always* expects negative profits. Note lastly that M_1 's profits fall when he raises his quote from $\Delta + 1$ to $\Delta + 2$, but increase when his quote stays constant.

Observe that when market makers set prices first and then traders submit their orders that the effective price schedule is less competitive than when the timing is reversed: traders face “worse” prices. In Section 5 we show that this is a general result on institutional design. \square

We now turn to examining the effects of changing the exogenous parameters of the model on the price schedules set. The first finding is that the smaller is M_2 's share $1 - \alpha$, the more attractive it is for M_2 to undercut M_1 . Consequently, both M_1 and M_2 set tighter bid-ask spreads, the smaller is α . This contrasts sharply with the ambiguous effects of α on price schedules when market makers set prices before traders submit orders.

Proposition 17 *The greater is α , the smaller is the bid-ask spread set by each market maker, and hence the smaller is the expected effective market bid-ask spread. All traders, both informed and uninformed, strictly prefer institutions with greater time priority. Let $\alpha' > \alpha''$. Then*

$$p_{1\alpha'}(t) \leq p_{1\alpha''}(t); \quad E[p_{2\alpha'}(t)] \leq E[p_{2\alpha''}(t)], \quad t > 0.$$

If M_2 ever undercuts M_1 's price on some order size t then the greater is α , the strictly "better" is the equilibrium expected effective price for that trade level: $E_{p_2(t)}[\bar{p}_{\alpha'}(t)] < E_{p_2(t)}[\bar{p}_{\alpha''}(t)]$. Traders always prefer that M_1 receive greater priority because it increases M_2 's incentive to undercut and offer a better price. An immediate corollary is that

Corollary 18 *Aggregate market maker profits are decreasing in M_1 's share α .*

The reduction in market maker profits is at M_2 's expense (propositions 9, 11).

For similar reasons less informed trade and finer price grids lead to "better" schedules and hence are preferred by both informed and uninformed traders. Again, this contrasts with the ambiguous predictions obtained for quote-driven institutions.

Proposition 19 *The greater the probability of uninformed trade, $1 - \gamma$, the smaller is the bid-ask spread set by each market maker, and hence the smaller is the expected effective market bid-ask spread.*

Proposition 20 *Let the price grid be reduced from $d = k$ to $d = \frac{k}{2}$, $k > 0$. Then the smaller the grid size, the tighter is the bid-ask spread set by each market maker, and hence the tighter is the expected effective market bid-ask spread.*

Corollary 21 *As the price grid size d becomes finer, expected total market maker profits on any order t fall, vanishing completely as the price grid becomes arbitrarily fine.*

Comparative statics on the division of market maker profits are slightly more ambiguous. For orders where the bid-ask spread is at a minimum, it is clear that increases in uninformed trade raise market maker profits, finer price grids lead to lower profits and increases in M_1 's share α raise M_1 's profits at M_2 's expense.

However, on orders where M_2 undercuts with positive probability, M_1 's expected profits per share are just the probability that he is not undercut times d : $\omega_2^*(t)$ alone determines M_1 's expected profits per share, $(1 - \omega_2^*(t))d$. Since $\omega_2^*(t)$ pins down informed trading behavior, it is clear that sharp results on the relationship between M_1 's profits and α , γ or d do not obtain. For instance, if increasing α leads M_1 to set bid-ask spreads, it will increase his profits if and only if the probability that he is undercut is reduced. Clean results do not obtain even in the limit as α goes to one.

When M_2 undercuts with positive probability, his profits equal $\frac{(1-\alpha)}{\alpha}d$, which are falling in α , increasing in the price grid size, d , and independent of the level of inside trade, γ . On transactions where he does not undercut, his profits are strictly less than $\frac{(1-\alpha)}{\alpha}d$.

4.1 Ginsi Trading

The analysis is robust to so-called ‘‘Ginsi trading’’. The unique Ginsi trading equilibrium features the same insider strategy and expected effective price schedule as that detailed here. In the Ginsi equilibrium, rather than M_2 mixing over prices in equilibrium, both market makers split the orders, offering better prices on the fraction of the order which corresponds to M_2 ’s mixing probability.

To see this is an equilibrium, observe that the risk neutral insider’s trading decisions are unaffected by whether the division is probabilistic due to mixing by M_2 or deterministic due to Ginsi trading. If the expected effective price schedules with and without Ginsi trading are the same, then insider trading decisions will be unchanged. Hence we must show that if insider trading decisions are not altered by Ginsi trading then the market makers will set the same expected effective price schedules. M_2 receives the same profits from matching M_1 on the higher priced portion of the transaction as he does from undercutting and taking that portion himself, so he is indifferent between undercutting further and not. M_2 , however, maximizes profits by matching M_1 on any portion where M_1 sets the lower price in order to receive a share of the profits. Since M_2 will always match or undercut M_1 , M_1 wants to minimize the number of round lots on which the lower price is set: if M_2 will undercut on at least the fraction of the order corresponding to the mixing probability, then M_1 will offer the lower price only on that fraction of the order. Hence the equilibrium Ginsi price schedule corresponds to the equilibrium expected effective price schedule without Ginsi trading. Following the arguments of proposition 12, one can show that this equilibrium is unique.

Thus, allowing Ginsi trading does not overturn any result. The discrete price grid introduces non-trivial strategic interaction among market makers and allows them to earn substantial profits. Even though market makers are ‘‘competitive’’ within the discrete institutional environment and can split orders finely, the discrete environment does not facilitate competition.

The outcome would be different if traders could commit to only transacting with a single market maker. This has the effect of making the price grid arbitrarily fine, since M_2 can now win the entire transaction by undercutting M_1 on a single round lot of the total transaction, thereby winning the entire transaction. Effectively, the strategic effect of the discrete price grid is overcome. In practice this commitment is not feasible, or at least not observed: market orders are often split among several limit orders and perhaps the specialist.

4.2 Robustness

In the case of absolute time priority, equilibrium outcomes are unchanged if market makers have multiple (infinitely many is equivalent to one) opportunities to undercut each other. For less strict priority rules, the effective equilibrium price schedules become less competitive when M_1 can submit improved quotes which match M_2 's. Rather than receiving the entire transaction by undercutting M_1 's initial quote, M_2 only receives share α from being the first to set a better quote because M_1 will match a profitable quote when he has the chance to move again. This reduces the incentives of M_2 to undercut, and in turn, reduces the incentives of M_1 to initially set better prices.

Little changes if M_1 adopts a mixed pricing strategy. Profit maximization requires that M_1 place positive probability on at most two prices which differ by d . Let $p(t)$ be the higher price, and $\omega'_1(t)$ be the probability that M_1 sets a price d lower. M_2 matches $p(t) - d$ for sure, and undercuts $p(t)$ with probability $\omega'_2(t) = \omega_2(t) - \omega'_1(t)$ (where $\omega_2(t)$ is the probability M_2 undercuts were M_1 always to set price $p_1(t)$), so that the aggregate probability that price $p(t) - d$ is set is unchanged and consistent with the equilibrium beliefs of the informed traders, which determine $\delta^*(t-1), \delta^*(t)$. For M_1 to be indifferent between setting $p(t)$ or $p(t) - d$, $\omega'_2(t)$ must satisfy

$$\omega'_2(t)(p(t) - \Delta) - \frac{\gamma\chi^*(t)}{(1-\gamma)\ell(t) + \gamma\chi^*(t)}E[\delta|t] = d.$$

Hence while the equilibrium is not unique, both the equilibrium expected effective price schedule and insider trading strategies must be.

5 Welfare

We conclude by comparing the welfare properties of the quote and order driven institutions. In the case of absolute time priority, the price schedule set by M_1 in the order-driven institution where traders first submit orders is more competitive than the *effective* price schedule in the quote-driven institution where market makers first submit price schedules, in the sense that the bid-ask spreads are uniformly tighter. Hence, the expected losses of uninformed agents are smaller when traders submit their orders first. This may explain why the NYSE features absolute priority, and traders submit their trades before market makers make their quotes, at least for larger orders.

Proposition 22 *Suppose there is absolute time priority. Then the effective price for each buy order in the quote-driven institution is at least as great as the price set by M_1 in the order-driven institution. Hence, on those transactions where M_2 undercuts with positive probability, the expected price in the quote-driven institution is strictly lower than that which obtains in the order-driven institution.*

Intuitively, when market makers set prices first, the informed respond strategically to the price schedule. Mixing over price schedules cannot discipline informed trade because the informed see the schedule realization before selecting their order, and will concentrate their trade where the price realization was “low”. This strategic informed response leads

market makers to set less competitive price schedules. Hence, as Example 2 illustrated, traders earn greater profits by committing — submitting orders first and then receiving price quotes.

6 Conclusion

This paper studies the impact of a discrete price grid on strategic pricing and inside trade in dealership markets. The discrete environment means that prices are no longer determined by a zero expected profits condition for market makers. Discreteness effectively limits competition and leads market makers to set uncompetitive and profitable price schedules. We show that priority rules and the timing of offers — do market makers submit price schedules first, or do traders first submit their orders and then market makers set prices — have significant effects on equilibrium outcomes. With quote-driven institutions where market makers submit price schedules first, “intuitive” comparative statics results need not obtain: the effective price schedule need not be monotonic in the probability of informed trade, the price grid size, or the priority rule.

In contrast, order-driven institutions have sharp “intuitive” properties: the effective price schedule “improves” as the probability of inside trade falls, the price grid becomes finer, or as the first market maker receives a greater portion of the trade. Conditional on not being undercut on those transactions, the first market maker expects the substantial profit *per share* of exactly $\frac{d}{\alpha}$, which exceeds the grid size, d . This is important because it suggests that the commonly used approach for empirical work in market microstructure of rounding to the nearest eighth is not appropriate. The second market maker expects a fraction of this profit whether or not he undercuts, where the fraction is decreasing in the first market maker’s share of the transaction. These results are robust to Ginsi trading. Given that there are significant fixed costs associated with market making, the paper suggests that there is an optimal price grid size, one which minimizes expected trader losses subject to a minimum profit requirement for market makers. Since market maker profits and trader losses are monotonically increasing in d , this grid size is unique.

Finally, we find that the most competitive effective price schedule is set in the order-driven institution where there is absolute time priority, as is the case on the NYSE. Traders get better price quotes in this environment than they do when they get to see price quotes first before submitting their orders.

Because we impose little structure on the distributions of both information and liquidity trade, the analysis can be embedded in more general environments. The extension is immediate to an economy where insiders observe the innovation with noise, and to any preferences for the insider which are monotone increasing in wealth. Because market makers care only about the expected value of the asset given the order size, perturbing preferences has an effect similar to perturbing the distribution of innovations (which is unrestricted). Sunshine trading in which traders commit to executing the transaction several hours later can be incorporated by assuming that market makers recognize that such traders are less likely to be informed so that “their γ ” is smaller. Introducing competing exchanges with the same price grid corresponds exactly to introducing additional

market makers, where the priority rule mandates a probabilistic division of the order if market makers on different exchanges set the same price.

The analysis extends to dynamic environments where information is longer lived, where a trader may implement a dynamic trading strategy in which he splits his trade, provided that, as with Ginsi trading, the trades are handled independently. Intuitively, the solution to a traders' splitting problem just corresponds to a different distribution of trade (which is unrestricted), and which the market makers take into account.

The theory has implications for the magnitude of dynamic quote revisions. For instance, after a large price movement, bid-ask spreads are likely to be wider because the expected value of the asset is generally a feasible price. In contrast, after smaller movements, the expected asset value is generally unchanged, so that quotes are not revised. This means that it is possible to distinguish empirically the predictions of the order-driven model from those of inventory models.

Appendix: Proofs

Proposition 1. (a) Suppose $p_2(t) < p_1(t) - d$. By increasing $p_2(t)$ to $p_1(t) - d$, trade with liquidity traders is unchanged, but revenues are increased, and expected losses with informed traders are no greater (strictly less if they continue to trade t). Hence $p_2(t) < p_1(t) - d$ cannot be profit maximizing.

(b) Suppose the premise were not true, i.e.

$$\sum_{t|p_1(t) > p_2(t)} [(1 - \gamma)\ell(t)(p_2(t) - \Delta)t + \gamma\chi^*(t)(p_2(t) - (\Delta + E[\delta|t]))t^*(\delta, \bar{p}(\cdot))] < 0.$$

But then, by increasing $p_2(t)$ on all such orders by d to $p_1(t)$, expected losses are reduced by more than:

$$\alpha \left[\sum_{t|p_1(t) > p_2(t)} [(1 - \gamma)\ell(t)(p_2(t) - \Delta)t + \gamma\chi^*(t)(p_2(t) - (\Delta + E[\delta|t]))t^*(\delta, \bar{p}(\cdot))] \right] > 0.$$

(c) Suppose $p_1(t) < p_2(t)$, but

$$(1 - \gamma)\ell(t)(p_1(t) - \Delta) + \gamma\chi^*(t)(p_1(t) - (\Delta + E[\delta|t])) > 0.$$

But if M_1 expects to make money on an order t , then M_2 can earn positive profits by matching the price. Informed trade at all levels is unaffected, as $\bar{p}(t)$ is unchanged. Hence if M_1 sets a lower price than M_2 's, it must be unprofitable for M_2 to match it.

(d) Suppose the premise were not true, i.e.

$$(1 - \gamma)\ell(t)(\bar{p}(t) - \Delta)t + \gamma\chi^*(t)(\bar{p}(t) - (\Delta + E[\delta|t])) < 0 \quad \text{but } p_1(t) = p_2(t).$$

But if $p_1(t) = \bar{p}(t)$, informed trade is unaffected if M_2 sets price $p_2(t) > \bar{p}(t)$. Hence M_2 's expected profits are increased by avoiding the loss incurred by potentially trading t . \square

Lemma 23 : *Equilibrium price schedules can be bounded: $\Delta + \underline{m} - d \leq p_1^*(t) \leq \Delta + \bar{m} + d$; $\Delta + \underline{m} \leq p_2^*(t) \leq \Delta + \bar{m}$.*

Proof. Consider an order $0 < t \leq H$, for which there is a positive probability that the trader is a liquidity trader. Suppose M_1 sets a price $p_1^*(t) \geq \Delta + \bar{m} + d$. Then M_2 can set a price $p_2^*(p_1(\cdot), t) = p_1^*(t) - d$, and win the sale. Further, for such orders M_2 earns greater profits by doing so (and leaves M_1 with none), as $p_1^*(t) - d > (1 - \alpha)p_1^*(t) > 0$ (recall M_2 's share is at most .5). This order is necessarily a liquidity trader's for such an order is unprofitable. Since M_2 's prices at other trade levels are unaffected by the choice of $p_1^*(t) \geq \Delta + \bar{m} + d$, M_1 's profits are unaffected by this choice. An analogous argument for $L \leq t < 0$, demonstrates that a lower bound on prices is $\Delta + \underline{m} - d$. Hence the set of

potentially optimal prices for M_1 can be restricted to the set $\Delta + \underline{m} - d \leq p_1^*(t) \leq \Delta + \overline{m} + d$, and hence M_2 , who may undercut, to prices $\Delta + \underline{m} \leq p_2^*(p_1(\cdot), t) \leq \Delta + \overline{m}$. \square

Proposition 2. For any innovation δ , and price schedules consistent with lemma 1, an informed agent's optimal order is contained in the set $\{L, L + 1, \dots, H\}$, a finite set. Hence there is a best element. For any given price schedule of M_1 , for each transaction level $\{L, \dots, H\}$, the set of possibly optimal prices for M_2 is contained in $\{\Delta + \underline{m}, \Delta + \underline{m} + d, \dots, \Delta + \overline{m}\}$. Since the set $\{L, \dots, H\}$ is finite, as is $\{\Delta + \underline{m} - d, \Delta + \underline{m} + d, \dots, \Delta + \overline{m} + d\}$, for any given price schedule of M_1 there is a best response for M_2 . Analogously, there must be a best price schedule for M_1 . Hence an equilibrium exists.

As the proof for $t < 0$ is analogous consider only buy orders. Suppose M_1 's price schedule is monotone in t : $p_1(t - 1) \leq p_1(t)$. If $p_2(t) > p_1(t) = \bar{p}(t)$, arbitrarily assign it the value $p_1(t) + d$. From Proposition 1, if $p_2(t) < p_1(t)$ is profit-maximizing then $p_2(t) = p_1(t) - d$. Suppose $p_2(t) < p_2(t - 1)$. Then $p_2(t - 1) \geq p_1(t - 1)$ and $\bar{p}(t - 1) = p_1(t - 1)$. It cannot be the case that $p_1(t) > p_1(t - 1)$, else $p_2(t) = p_1(t) - d$, so that $p_2(t - 1) = p_1(t)$, and $\bar{p}(t - 1) = \bar{p}(t)$. But then informed traders will not trade at $t - 1$, so that M_2 wants to at least match $p_1(t - 1)$. This implies that $p_1(t) = p_1(t - 1)$, which, in turn, must equal $p_2(t - 1)$ since there will be no informed trade there. But if $p_2(t) < p_2(t - 1)$ then informed traders will still not trade at $t - 1$, even if M_2 undercuts. Hence undercutting $p_2(t - 1) = p_1(t - 1) - d$ yields M_2 greater expected profits. But then his schedule is monotone. But then the effective price schedule is monotone. For if $\bar{p}(t) < \bar{p}(t - 1)$, if $\bar{p}(t - 1) = p_1(t - 1)$, then M_2 increases his profits at $\bar{p}(t - 1)$ by undercutting $\bar{p}(t - 1)$ by d . And if $\bar{p}(t - 1) = p_1(t - 1) - d$, $p_1(t) \geq p_1(t - 1)$ (by assumption), so that $p_2(t) \geq p_1(t - 1) - d$. Monotonicity of $\bar{p}(\cdot)$ in t follows.

Suppose now that M_1 's schedule is not monotone, i.e. that $p_1(t - 1) > p_1(t)$. Then M_2 's profit-maximizing response is to set $p_2(t - 1) = p_1(t - 1) - d$. Let M_1 instead quote a price $p_1'(t - 1) = p_1(t) + d$. Then the profit-maximizing response by M_2 is to set $p_2'(t - 1) = p_1(t)$, as there is no informed trade at $t - 1$ and the profits from the rest of the schedule remain unchanged. Hence $\bar{p}(t - 1) \leq \bar{p}(t)$ as desired. The proof now follows as above.

An informed trader selects a transaction level, $t^*(\delta, \bar{p}(\cdot))$, which maximizes (1). Suppose that $\delta > 0$. Then for any given price, \bar{p} , such that $\Delta + \delta - \bar{p} > 0$, the informed trader maximizes profits by buying as much as he can at that price. \square

Corollary 1. Profit maximizing implies that this partition solves

$$(\Delta + \delta_i^* - \bar{p}(t_i^*))t_i^* = (\Delta + \delta_i^* - \bar{p}(t_{i+1}^*))t_{i+1}^*.$$

Rewriting yields:

$$(t_{i+1}^* - t_i^*)(\Delta + \delta_i^* - \bar{p}(t_{i+1}^*)) = t_i^*(\bar{p}(t_{i+1}^*) - \bar{p}(t_i^*)). \square$$

Proposition 3. Consider $t > 0$. Without loss of generality consider the largest such t such that M_2 sets $p_2^*(t) \leq \Delta$. Then his expected profits from such transactions are

negative. For consider an innovation $\delta < d$. It is profitable for the insider to purchase t , but not to purchase anything at a price greater than Δ . Expected profits from trading with the uninformed are zero. If M_1 also sets that price, then informed trade is unaffected if M_2 raises his price of an order of t to $p'_2(t)$ (leaving other prices unchanged), and M_2 gains from not earning negative profits on t . If $p_1^*(t) > \Delta$, then M_2 gains on liquidity trade by setting $p'_2(t) = \Delta + d$, and does not lose to the informed even if he handles them at some other transaction level (it must be less profitable for the informed, else they would have traded there originally). Hence M_2 gains, receiving greater profits from setting $p'_2(t) = \Delta + d$ for all t where he might set $p_2^*(t) = \Delta$.

Now suppose M_1 sets a price $p_1^*(T) \leq \Delta$. He receives informed demand at T but not at lesser quantities. For τ , $0 < \tau < T$, M_2 optimally sets the same price he would were $p'_1(T) = \Delta + d$, as he would with the price $p_1^*(T) \leq \Delta$: Informed trading is unchanged (they do not trade) for such transactions. The only way for M_1 to earn profits on τ , $0 < \tau < T$, is if $p_1^*(\tau) = \Delta + d$, as otherwise M_2 undercuts by d and win all such sales: it can only be possibly optimal to set a price $p_1(t) < \Delta + d$ for at most one t . Further, setting $p'_1(T)$ increases earnings from liquidity traders on T , and the total cost of informed trade shifted to transactions $T+1, \dots, H$ to both M_1 and M_2 is bounded from above by the cost of those informed to M_1 when they to trade at price $p_1^*(T)$ for T , else the informed could have increased their profits in the face of the original schedule by trading greater quantities, a contradiction. Since M_2 's expected profits on those orders $T+1, \dots, H$ do not increase when M_1 sets $p'_1(T)$, it must be that the increase in expected losses to M_1 at these greater quantities is bounded from above by the reduction in expected losses from trading with the informed at T . Hence, it could not have been optimal for M_1 to set a price $p_1(T) \leq \Delta$. \square

Proposition 4. If $\alpha < 1$ and $\pi_1 > 0$ then $\pi_2 > 0$ because M_2 can offer the same schedule as M_1 . To see that M_1 expects positive profits, consider the schedule

$$p_1^*(t) = \begin{cases} \Delta + \bar{m} + d & t > 1 \\ p_1 & t = 1 \\ \Delta - \underline{m} - d & t \leq 0 \end{cases}$$

where p_1 is chosen so that given the effective price schedule at other trade levels M_2 would expect to earn non-positive profits trading one round lot at $p_1 - d$, but trade at p_1 is strictly profitable. M_1 will not be undercut on 1 unit and any other schedule must generate at least these profits for him. M_2 's profits are at least what he would obtain from matching M_1 's schedule and may be greater. \square

Proposition 5. We must show that $\{p_1^*(t)\}_{t=L}^H$ solves

$$\max_{p_1^*(\cdot)} \sum_{t=L}^H [(1 - \gamma)\ell(t)(p_1^*(t) - \Delta)t + \gamma\chi^*(t)(p_1^*(t) - (\Delta + E[\delta|t]))t^*(\delta, p_1^*(\cdot))]$$

s.t.

$$\begin{aligned} \sum_{\{T_1, \dots, T_j\}} (1 - \gamma) \ell(T_i) [(p_1^*(T_i) + d - \Delta) T_i + \gamma \chi^*(T_i) (p_1^*(T_i) + d - (\Delta + E[\delta | T_i])) T_i^*(\bar{p}(\cdot), \delta)] < 0, \\ \text{for any collection of } \{T_1, \dots, T_j\}, T_i < 0 \\ \sum_{\{T_1, \dots, T_j\}} [(1 - \gamma) \ell(T_i) (p_1^*(T_i) - d - \Delta) T_i + \gamma \chi^*(T_i) (p_1^*(T_i) - d - (\Delta + E[\delta | T_i])) T_i^*(\bar{p}(\cdot), \delta)] < 0, \\ \text{for any collection of } \{T_1, \dots, T_j\}, T_i > 0. \end{aligned}$$

As the proof is analogous consider only buy orders. Suppose that the so-called profit-maximizing schedule by M_1 leads M_2 to undercut him on a set of orders $\{T_1, \dots, T_N\}$ by d (offering better prices for other orders is unprofitable). Further, if M_1 were instead to have set the schedule $\{\bar{p}(\cdot)\}$ by setting M_2 's prices on those orders T_1, \dots, T_N then it must be optimal for M_2 to undercut further as otherwise M_1 's profits would be increased by setting $\{\bar{p}(\cdot)\}$ rather than the so-called optimal schedule (he obtains M_2 's non-negative profits and informed trade is unaffected), a contradiction of profit maximization by M_1 . Denote the non-empty subset of these transactions on which M_2 undercuts by an additional d by $\{\tau_1, \dots, \tau_j\}$. Thus, the price schedule features some order levels where M_2 undercuts M_1 's so-called optimal schedule by $2d$. But, by Proposition 3 this requires that the so-called optimal price set by M_1 be at least $\Delta + 3d$. Let M_1 again set a schedule which matches those prices. Now if M_2 undercuts on no transactions other than τ_1, \dots, τ_j , then M_1 's so-called optimal schedule could not have been optimal for the same reason as above. So suppose that this undercutting shifts informed trade such that it is also optimal to undercut on an additional set of orders $\{\Upsilon_1, \dots, \Upsilon_m\}$. But it must be that $p_1(\Upsilon_1), \dots, p_1(\Upsilon_m)$ each exceed $\min\{p_1(\tau_1), \dots, p_1(\tau_j)\}$, else when undercutting on $\{\Upsilon_1, \dots, \Upsilon_m\}$ even more informed trade is lured back to orders $\Upsilon_1, \dots, \Upsilon_m$ than before (that is, no informed traders who would trade quantities $t < \min\{\tau_1, \dots, \tau_j\}$ given the original price schedule would prefer to trade a greater quantity $\Upsilon_1, \dots, \Upsilon_m$ given the revised schedule. Put differently, it is just as profitable for M_2 to undercut on these transactions with the M_1 's revised schedule as it was to undercut the original (which was assumed to be unprofitable). Hence transaction levels $\Upsilon_1, \dots, \Upsilon_m$ must feature prices greater than $\Delta + 3d$, a contradiction of the proposition's statement. \square

Proposition 6. We must show that for any $\nu > 0$, there exists a $\zeta > 0$ such that for $d < \zeta$, M_1 's expected profits from handling order t are less than ν . For a given transaction level $t > 0$, and effective price schedule, $\{\bar{p}(\cdot)\}$, observe that if schedule $\{\bar{p}'(\cdot)\} = \bar{p}(L), \dots, \bar{p}(t-1), \bar{p}'(t) = \bar{p}(t) - d, \bar{p}(t+1), \dots, \bar{p}(H)$ were set, that for all $\nu > 0$, there exists a $\zeta > 0$ such that for $d < \zeta$, $E\pi(\bar{p}'(t)) > E\pi(\bar{p}(t)) - \nu$, and for $\tau \neq t$, $E\pi(\bar{p}(\tau)) \geq E\pi(\bar{p}(t))$. That is, arbitrarily small price cuts have arbitrarily small effects on informed trade and hence on profits given the continuous density $f(\cdot)$ for δ .

Hence, if $E\pi_1(t) > \nu$, $0 < \alpha \leq 1$, there exists a ζ , $d < \zeta$ such that $(1 - \alpha)E\pi_2(p_2(t) = p_1(t)) < E\pi_2(p_2(t) = p_1(t) - d)$ (being slightly loose with notation), and for any given set of price quotes by the market makers, profits of M_2 at all other transaction levels are at least as great when he undercuts M_1 on transaction t . Hence M_1 's profits must go to zero. \square

Lemma 24 *In equilibrium, for $t \in \{L, L + 1, \dots, H\}$ an upper bound for prices $p_i^*(t)$ is $\Delta + \bar{m}$; and a lower bound for prices is given by: $\Delta + \underline{m}$: $\Delta + \underline{m} \leq p_i^*(t) \leq \Delta + \bar{m}, i = 1, 2$.*

Proof. Analogous to that of Lemma 1.

Proposition 7. Both market makers can ensure themselves non-negative expected profits by setting price $\Delta + \bar{m}$. M_2 can earn positive profits by matching M_1 's price and sharing the transaction unless the M_1 's price leaves them with zero expected profits:

$$(1 - \gamma)\ell(t)(p_1(t) - \Delta)t + \gamma\chi^*(\delta|t)E[(p_1(t) - (\Delta + \delta))t^*(\delta)|t] = 0.$$

But then M_1 can set price $p'_1(t) = p_1(t) + d$, and earn positive expected profits since M_2 can earn positive profits if and only if he matches $p'_1(t)$ and does not undercut. It cannot be optimal for M_1 to set a price which is undercut for sure because that leaves him with zero profits: by setting a price which generates positive profits if matched and is such that were M_2 to undercut he would not expect positive profits, M_1 can ensure himself positive expected profits. Since M_1 always expects positive profits, M_2 can ensure himself positive profits by matching M_1 's price; M_1 never undercuts M_2 . \square

Proposition 8. Immediate. For $t > 0$, a price $p(t) < \Delta$ is unprofitable. But a price of Δ is also strictly unprofitable. If price Δ is set with positive probability for transaction size t , then $E_{p_2(t)}[\bar{p}(t)] < \Delta + d$ since M_2 never undercuts M_1 by more than d . Define T to be the trade size which maximizes $T(\Delta + d - E_{p_2(T)}[\bar{p}(T)])$. Profits from liquidity trade are zero, and an insider observing an innovation $d - \epsilon < \delta < d$, ϵ small, will trade T . Since such innovations occur with positive probability, the market maker must expect negative profits, a contradiction of proposition 7. \square

Proposition 9. If the probability of insider trade for some trade size $t > 0$ is zero and there is liquidity trade, then since $\alpha \geq .5$, M_2 will undercut M_1 with probability 1 if $p_1(t) > \Delta + d$, leaving M_1 with zero profits. But then M_1 cannot be maximizing profits since M_1 can earn strictly positive profits by setting $p_1(t) = \Delta + d$. An informed trader selects an order t^* which maximizes $(\Delta + \delta - \bar{p}(t))t$. An informed trader trading at a price $\Delta + d$ maximizes profits by trading as much as possible at that price. Following the arguments above, there will be some inside traders who will see innovations δ , $d < \delta < d + \epsilon$, ϵ small, who maximize profits by trading at that price. If the expected effective price schedule were not strictly monotone increasing in trade size for $t > T_1$, then $E[\bar{p}(t)] \geq E[\bar{p}(t + 1)]$ some t . But then there is no informed trade in equilibrium at t , so that M_2 should always undercut M_1 on trades of size t , a contradiction of proposition 7. Since for order sizes greater than T_1 , there is an informed trader who expects positive profits equal to the market maker's expected losses, conditional on trading with an informed trader at the most attractive price for that trade size, the market maker expects to lose money. \square

Corollary 3. For $\delta \in [-d, d]$, the insider does not trade. For other levels, the insider can earn positive profits by trading. Profit maximizing implies that this partition solves

$$(\Delta + \delta_i^* - E[\bar{p}(t)])t = (\Delta + \delta_i^* - E[\bar{p}(t + 1)])(t + 1).$$

Rewriting yields:

$$(\Delta + \delta_i^* - E[\bar{p}(t + 1)]) = (t + 1)(E[\bar{p}(t + 1)] - E[\bar{p}(t)]). \quad \square$$

Proposition 10. Follows from lemma 2 and profit-maximization. If the expected profits to setting one price or the other were greater, then profit maximization requires that the more profitable price be set. Hence (2) must hold. \square

Proposition 11. From Proposition 10, $d = \alpha(p_1(t) - \Delta - \frac{\gamma\chi^*(\bar{t})}{\gamma\chi^*(\bar{t})+(1-\gamma)\ell(\bar{t})} E[\delta|\bar{t}])$, for $\bar{t} = t, t+1, \dots, t+j-1$. $\frac{\gamma\chi^*(\bar{t})}{\gamma\chi^*(\bar{t})+(1-\gamma)\ell(\bar{t})} E[\delta|\bar{t}]$ is the expected innovation given transaction t from a market maker's perspective: neither market maker knows the identity, informed or uninformed, of the trader with whom he transacts. This expected innovation does not vary across order sizes for which M_1 offers the same price. Hence $\frac{\pi_1(p_1(t))}{\pi_1(p_1(\bar{t}))} = \frac{t(1-\omega_2^*(t))}{\bar{t}(1-\omega_2^*(\bar{t}))}$, $\bar{t} = t, t+1, \dots, t+j-1$. \square

Proposition 12. Suppose the equilibrium expected effective price schedule is not unique. That is, suppose there are at least two equilibrium expected effective price schedules, $\{E_{p_2'(t)}[\bar{p}'(t)]\}_{t=L}^H \neq \{E_{p_2''(t)}[\bar{p}''(t)]\}_{t=L}^H$. Consider the greatest transaction $\tau > 0$, such that there is a difference between the two schedules, where without loss of generality, $E[\bar{p}'(\tau)] < E[\bar{p}''(\tau)]$. Then $\delta_{\tau+1}^{*'} \geq \delta_{\tau+1}^{*''}$, where the relationship is strict unless $\tau = H$. If $E[\bar{p}'(\tau-1)] \geq E[\bar{p}''(\tau-1)]$, then $\delta_{\tau}^{*'} \leq \delta_{\tau}^{*''}$ so there is more informed trade at τ in equilibrium ' than in the '' equilibrium, and losses to informed trade are greater in the ' equilibrium. The level of liquidity trade at τ is the same in both conjectured equilibria. It is then straightforward to show that this cannot be consistent with profit maximizing by both market makers in both equilibria: if informed trade is greater then expected prices must be greater. Hence it must be the case that $E[\bar{p}'(\tau-1)] < E[\bar{p}''(\tau-1)]$, and that $\delta_{\tau-1}^{*'} > \delta_{\tau-1}^{*''}$. Continuing this argument at $\tau-1$, it must be that $E[\bar{p}'(\tau-2)] < E[\bar{p}''(\tau-2)]$, and $\delta_{\tau-2}^{*'} > \delta_{\tau-2}^{*''}$, and so on: $E[\bar{p}'(\tau-j)] < E[\bar{p}''(\tau-j)]$, and $\delta_{\tau-j}^{*'} > \delta_{\tau-j}^{*''}$, $j = 1, 2, \dots, \tau$. But consider the least transaction level $T > 0$ such that there is informed trade in the ' equilibrium. It must be that $E[\bar{p}'(T)] < E[\bar{p}''(T)]$, and $\delta_T^{*'} < \delta_T^{*''}$, a contradiction. \square

Proposition 13. Suppose not. Then there is a least $\tau > 0$ such that $E_{p_2(\tau)}[\bar{p}_{\bullet}'(\tau)] > E_{p_2(\tau)}[\bar{p}_{\bullet}''(\tau)]$. Hence, $\delta_{\tau}^{*'} > \delta_{\tau}^{*''}$. Suppose that $\delta_{\tau+1}^{*'} \leq \delta_{\tau+1}^{*''}$, so there is more informed trade at τ in the ' economy than the '' economy. Consider any $p_1(\tau)$. It is more attractive to undercut $p_1(\tau)$ given α' than α'' even for fixed levels of informed trade since the profits to offering a better price are independent of α and the profits to matching $p_1(\tau)$ fall with α , a contradiction. Further if $p_{1\bullet}'(\tau) > p_{1\bullet}''(\tau)$ then equilibrium in both economies necessarily requires M_2 to always undercut $p_{1\bullet}'(\tau)$, a contradiction of profit maximizing by M_1 . Hence, it must be that $\delta_{\tau+1}^{*'} > \delta_{\tau+1}^{*''}$, so that $E_{p_2(\tau+1)}[\bar{p}_{\bullet}'(\tau+1)] > E_{p_2(\tau+1)}[\bar{p}_{\bullet}''(\tau+1)]$. Continuing this argument as in proposition 12, it must be that $\delta_H^{*'} > \delta_H^{*''}$ and $E_{p_2(H)}[\bar{p}_{\bullet}'(H)] > E_{p_2(H)}[\bar{p}_{\bullet}''(H)]$. But then there is less informed trade at H in the ' economy than the '' economy, so it is more attractive to offer better prices at H in the ' economy, a contradiction. \square

Proposition 15. As in Proposition 13. As d falls, the minimum bid-ask spread shrinks. Further, because M_2 can undercut M_1 by a smaller quantity and obtain the entire order, undercutting is more profitable so market makers offer better prices. \square

Proposition 16. Index the price schedules when market makers set price schedules

first by M , and price schedules set when traders submit orders first by T . By construction when traders submit orders first, M_2 's price schedule earns him zero expected profits. Further, were the timing reversed so that market makers set price schedules first, then M_2 would expect strictly negative profits from pricing below $p_1^T(\cdot)$ on any set of transactions.⁹ Indeed for any schedule $p_1^M(\cdot) > p_1^T(\cdot)$, the losses M_2 expects from pricing below $p_1^T(\cdot)$ on any set of trades are even greater than those which would obtain were M_1 to set schedule $p_1^T(\cdot)$ (it draws even more informed trade). Let $\{\tau_1, \dots, \tau_j\}$ be the hypothesized set of transactions such that $p_1^M(\tau) > p_2^M(\tau) \geq p_1^T(\tau)$. Let $\{\rho_1, \dots, \rho_k\}$ be the hypothesized set of transactions such that $p_1^M(\rho) = p_1^T(\rho) > p_2^M(\rho)$, i.e. those transactions where M_2 undercuts M_1 when market makers move first and M_1 sets the same price as when traders move first. But, following the above argument, M_2 would earn greater expected profits by undercutting only on transactions τ_1, \dots, τ_j , since undercutting on ρ_1, \dots, ρ_k given schedule $p_1^T(\cdot)$ is unprofitable.

Finally, M_1 never sets a price $p_1^M(t) < p_1^T(t)$. For suppose otherwise. He expects to lose money on any transaction with price below $p_1^T(t)$, and such pricing makes it more attractive for M_2 to undercut on other transactions. Rather than price below $p_1^T(t)$, M_1 does better to raise his prices at least to $p_1^T(t)$. Let z be the decrease in losses on those transactions. Setting these higher prices leads M_2 to undercut less since there is more informed trade at other trade levels. Since those transactions where M_2 no longer undercuts were profitable given the original schedule, a strict upper bound on the increase in losses for those order levels t where M_1 left his price unchanged (i.e. those t such that $p_1^M(t) \geq p_1^T(t)$) is z . Hence it cannot be optimal for M_1 to set a price $p_1^M(t) < p_1^T(t)$. \square

⁹When $\alpha = 1$, the effective price schedule, profits and insider trade are unique, but the schedule set by M_1 need not be if he receives exactly zero profits from setting some price and taking the order, or setting a price α higher and getting undercut for sure. To fix arguments we suppose that M_1 sets his schedule so that he is not undercut with probability one - this is just a convenience.

Figure 1a.
Market Makers Set Price Schedules; Then Traders Submit Orders

Time Line

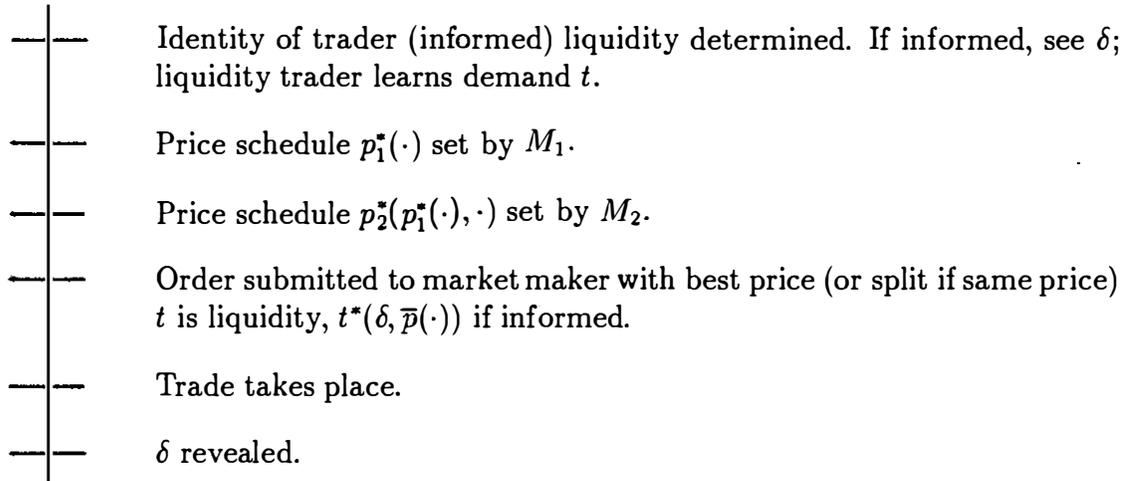


Figure 1b.
Traders Submit Orders; Then Market Makers Set Prices

Time Line

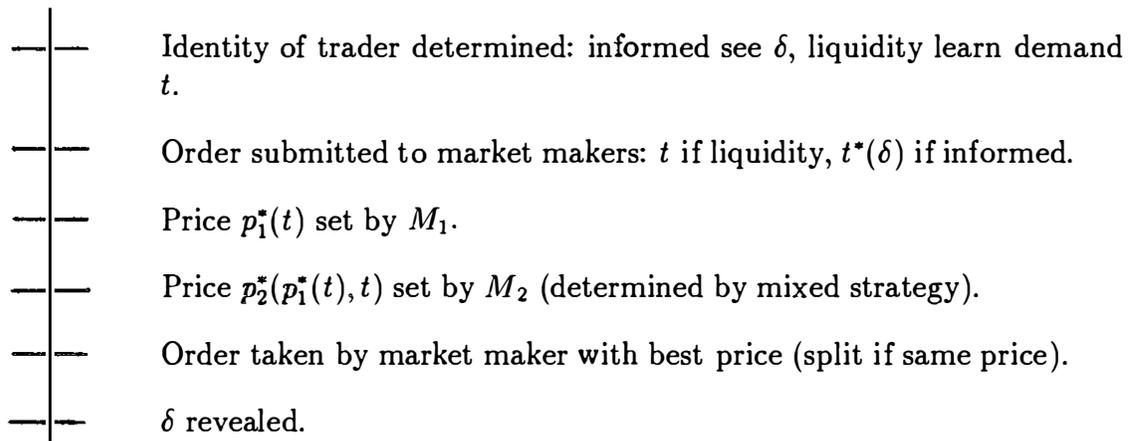


Figure 2a

Price Schedule ($t > 0$), when market makers set prices first

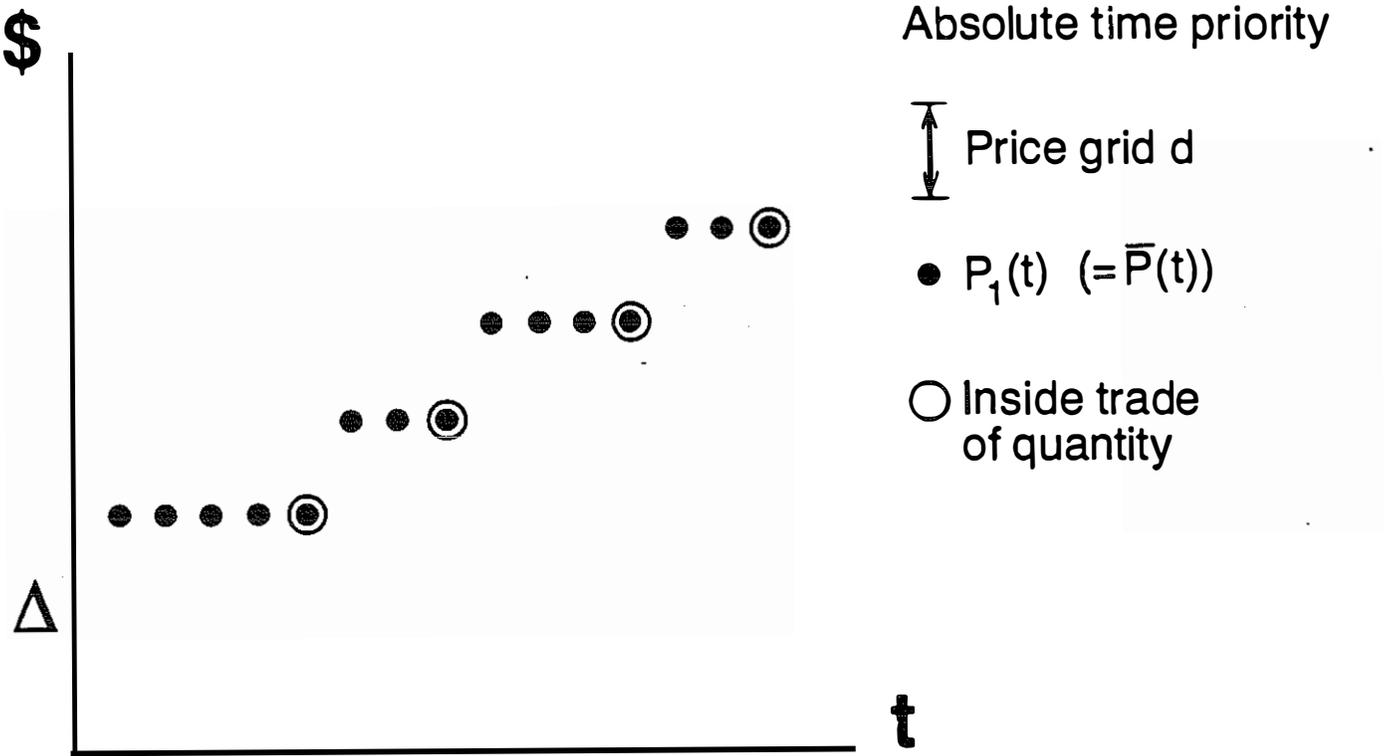
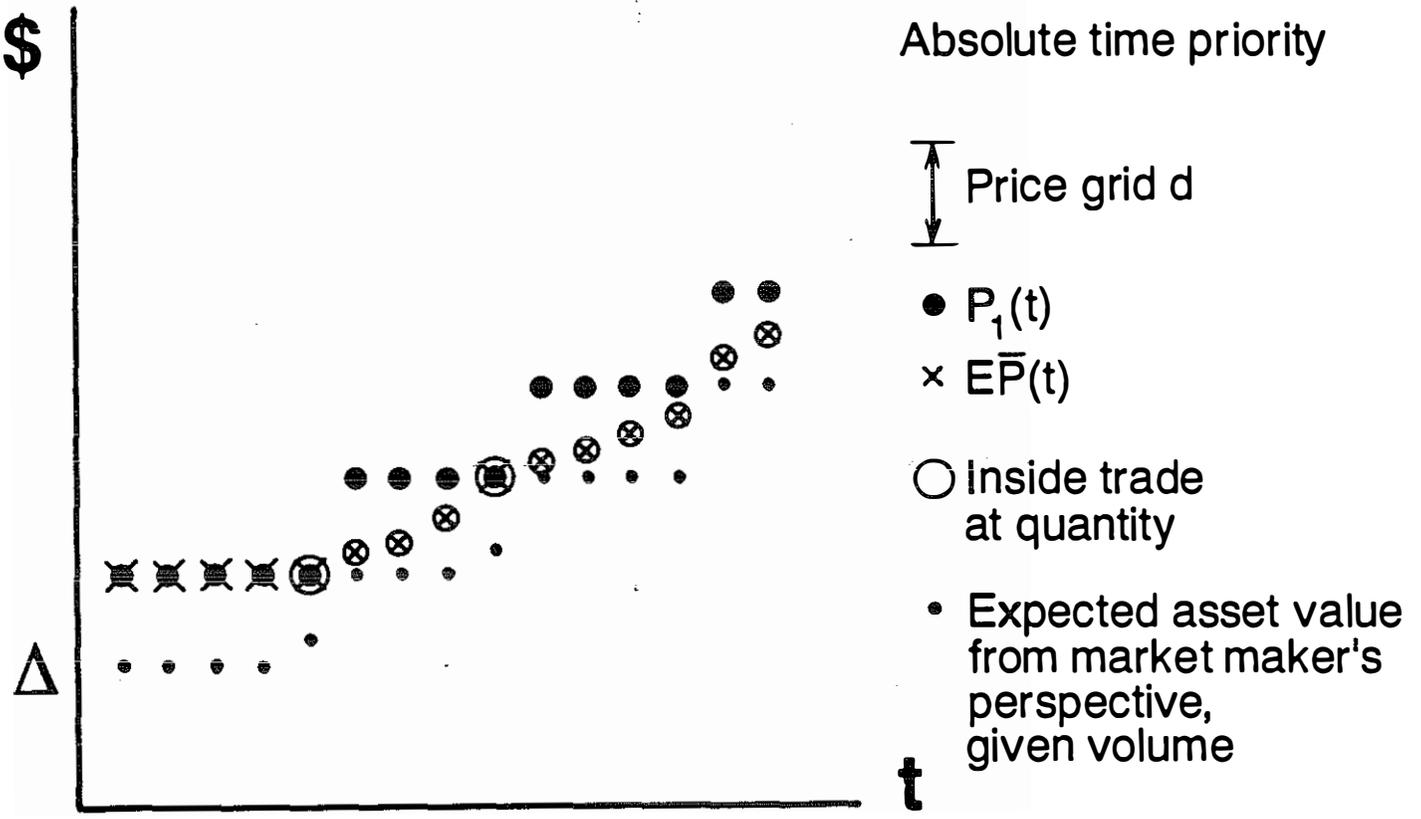


Figure 2b

Price Schedule ($t > 0$), when orders are submitted first



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