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ASSET PRICES AND VOLUME
IN A BEAUTY CONTEST

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Abstract

The dynamics of prices and volume are investigated in a market where agents disagree about the fundamental value of the asset. The distribution of beliefs is not taken to be common knowledge. The resulting infinite hierarchy of beliefs is solved by making the assumption that, prior to the first trading round, agents consider themselves to be average. Speculation is shown to generate substantial volatility and volume, bid and transaction price predictability, rich patterns of volume, and an inverse relationship between changes in transaction prices and the number of trading rounds without volume.

Keywords: Disagreement, Speculation, Beliefs Hierarchy, Financial Markets, Volume, Volatility.

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1 Introduction

It has long been acknowledged that speculation may explain prices and volume in financial markets (see Keynes [1936]). Speculators trade because they expect to earn short-term capital gains, rather than on the basis of fundamental information about the long-term value of the asset. Harrison and Kreps [1978] defined speculation as follows:

“An investor may buy the stock now, so as to sell it later for more than what he thinks it is actually worth, thereby reaping capital gains.”

Tirole [1982], however, argued that speculation cannot occur in the standard analytical framework of asset pricing models, namely rational expectations.¹ One way it may occur is if agents disagree. The disagreement has to be authentic. Unlike in Harsanyi’s mutual consistency requirement (Harsanyi [1968]), or in Aumann’s impossibility of agreeing to disagree (Aumann [1976]), disagreement must not be the result of updating from a common prior using different information. For if it were, any trade for reasons other than insurance would be impossible because of the winner’s curse (see Milgrom and Stokey [1982] or Morris [1990]). This is not to say that authentic disagreement is irrational. It merely reflects agents’ fundamental differences in their assessment of uncertainty. As discussed in depth in Kurz [1992], there may still be disagreement even after a prolonged period of comprehensive learning.

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¹See Allen, Morris and Postlewaite [1992] for an extension of Tirole’s analysis.

Despite the fact that disagreement and speculation have long been advanced as potential explanations of prices and volume in financial markets, there have been very few attempts at exploring to what extent they do. The paucity of analyses should not come as a surprise. When agents disagree, the distribution of beliefs in the economy is of interest. If this distribution is *not* common knowledge (would it be plausible to assume otherwise?), the computation of equilibria becomes intractable, because agents posit and update higher-order beliefs.

A speculator buys the asset to resell it. He knows that the resale price will depend on the belief of the buyer about the fundamental value of the asset (the buyer's first-order belief). Hence the speculator must have beliefs over the first-order beliefs of the others (*i.e.*, second-order beliefs). The buyer also is a speculator. Hence, the price at which she buys the asset will, in turn, depend on her own second-order beliefs. Hence the initial holder of the asset must have beliefs about the second-order beliefs of the others, *i.e.*, third-order beliefs. Iterating the argument, a possibly infinite hierarchy of beliefs emerges.

Keynes [1936] coined the term *beauty contest* when spelling out the beliefs hierarchy. In his words:

“... each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not the case of choosing those which, to the best of one's judgement, are really the prettiest, nor even those the average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some I believe, who practise the fourth, fifth and higher degrees.”

Böge and Eisele [1979] and Mertens and Zamir [1985] proved that an equilibrium exists in static games with infinite beliefs hierarchies. Recently, El-Gamal [1992] did so for a dynamic model without strategic interaction as well. Nevertheless, to the best of our knowledge, explicit solutions of such models have not been obtained yet. The existence proofs do not provide an algorithm with which to calculate particular equilibria. The goal of this paper is to characterize an explicit solution, in order to analyze prices and volume in financial markets when agents disagree.

One way to obtain explicit solutions would be to arbitrarily cut the infinite regress of beliefs. Townsend [1983] did this in a related context. Or one could assume, as in El-Gamal [1992], that agents think that beliefs agree from a certain order on. We take an alternative, arguably more natural route, which does not involve an approximation argument. We assume that each agent *a priori* thinks that her beliefs are average. It

works as follows. Assume agent i has first-order belief μ_i . *A priori*, he thinks that the first-order beliefs of the other agents are on average equal to μ_i . Hence, his second order belief is also μ_i . It is common knowledge that all agents behave similarly. So agent i knows that the first *and* second order beliefs of the agent j are equal to μ_j . Now, *a priori*, i expects μ_j to be μ_i . Consequently, the third-order belief of i is also equal to μ_i . Iterating, this argument provides a solution to the infinite hierarchy of beliefs.

In order to avoid problems of strategic behavior (manipulation of beliefs), and to be able to concentrate on the impact of disagreement on prices and volume, we set up our model as follows. Borrowing ideas from the pairwise meetings literature (see Wolinsky [1990]), we assume that, each trading round, the holder of the single unit of the asset meets in the marketplace a person drawn at random from the population. The holder offers to sell this unit through an unspecified mechanism that induces the bidder to reveal truthfully his reservation value. The holder then decides to hand over the asset, or to keep it. Whoever holds the asset at the end moves on to the next trading round. The other agent leaves the market. After a certain number of rounds, trading halts and the asset pays its dividend to the holder. Reflecting the fact that in financial markets the past history of trades can be traced, we assume that every agent, including present and future entrants, updates her beliefs from the initial trading round on, *i.e.*, using the complete history of trades and prices.

There are basically three dimensions in which speculation enriches asset pricing theory. First, it adds a resale option to the privileges of ownership. Agents now value an asset not only for its dividend, but also for the option to resell it to a future buyer who values the asset more highly. Second, it introduces learning about the distribution of beliefs. The learning process generates price movements even when no news about the asset's fundamental value flows to the market. Third, it imports beliefs hierarchies into models of financial markets. The possibility of disagreement at higher levels of beliefs generates trading patterns that cannot be obtained in models with a simple resale option, even if agents have to learn about its value.

In order to distinguish the contribution of each of these facets to explaining the behavior of prices and volume in financial markets, we consider three different cases. In the first (benchmark) case, the distribution of beliefs is common knowledge. In the two other cases, it is uncertain. The agents have different priors about it. They rationally use the information available in the market and apply Bayes' rule to learn about it. In the second case, in order to focus exclusively on learning, we specify the model such that all higher-order beliefs agree after just one observation. In the last case, agents have different posteriors even after observing each other's beliefs. In that version of the model, higher-order beliefs agree only asymptotically.

The resale option has value even when the distribution of beliefs is common knowledge. In fact, in this case, an outside observer, unaware of the absence of fundamental

information flows, could erroneously conclude that prices behave as in a standard rational expectations asset pricing model. For instance, the best offer in the market is a martingale, and volume cannot be used to predict subsequent price changes.

In the second case, where the agents must learn about the distribution of beliefs, this observational equivalence disappears (even asymptotically, see Bossaerts [1992]). In our model, volume can be used to predict prices and, from the point of view of the econometrician, who would be able to determine the true distribution of beliefs in the economy, the best offers are no longer a martingale. Yet, just like in a model where higher-order beliefs agree and are correct, its predictions concerning trading patterns are thin: the holder will always be the agent with the highest first-order belief, and trading dies out as the horizon gets closer.

In contrast, when agents' higher-order beliefs disagree, the holder and the agent with the highest first-order belief need not be the same anymore. This generates richer trading patterns. Because of the diverse patterns of volume it creates, disagreement at higher-order levels makes models of speculation interesting in their own right. This should underscore the pertinence of any assumption that keeps the richness of the hierarchy of beliefs intact yet enhances the model's tractability. Our assumption that investors *a priori* consider themselves to be just average, delivers this.

Unfortunately, the full model with higher-order disagreement cannot be solved analytically. We characterize its properties using simulations on the basis of numerical integration. Because of the simplicity of their trading implications (the holder and agent with highest first-order beliefs are identical), however, models with agreeing higher-order beliefs are analytically tractable and we present some features of such models first.

The paper is organized as follows. In the next section, we present the model. In Sections 3 and 4, we examine the two cases where the distribution of beliefs is common knowledge and where the agents agree after observing each other's first-order belief. In Section 5 we report the results of simulations, in order to assess the impact of higher-order disagreement on prices and volume. Finally, Section 6 offers some concluding comments. The proofs are in the Appendix.

2 The Model

2.1 Assumptions

Consider a market for one risky asset with random payoff X at time $T + 1$. At times $t = 1, \dots, T$, a new agent enters the market and bids to buy the asset from the agent who holds it. The latter has the choice between selling at that price or keeping the asset.

If she keeps it, she may sell the asset later to another agent or keep it until the end, and receive X . There is only one unit of the asset and short sales are not allowed. For simplicity there is no discounting and agents are risk neutral. Also for simplicity the new agents entering the markets are assumed to bid their reservation value. The latter reflects their own perception of the value of the final cash flow *and* the value of the option to resell the asset.

Agents have different beliefs about the final payoff X .² Agent t , who enters the market at time t , expects the asset to pay off μ_t at time $T + 1$. This is hereafter referred to as the agent's first-order belief. First-order beliefs are i.i.d draws from a distribution with expectation θ^* . This means that the average first-order belief in the economy is θ^* :

$$E_{\theta^*}(\mu_t) = \theta^*,$$

where $E_{\theta^*}(\cdot)$ denotes that the expectation is taken given the knowledge of the parameter θ^* . For simplicity, assume θ^* is the only parameter of the distribution of first-order beliefs.

Except for the arrival of new agents with different beliefs, there is no information revelation in the market. Our model is *not* an asymmetric-information rational expectations model. The different first-order beliefs are not signals about the fundamental value of the asset. On observing the beliefs of the other agents, agent i does not alter his first-order beliefs. *Agents agree to disagree.*

All agents know that there are differences in beliefs. The fact that the first-order beliefs are i.i.d draws from a given distribution, characterized by a parameter θ , is common knowledge. But the agents do not know the exact value of θ , θ^* . They have priors about it, in the form of a distribution over θ . The priors differ across agents. When they observe additional information, all agents update according to Bayes' law. We assume that the agent who enters the market at time t has observed the sequence of offers up to time t . She uses this information to learn about θ .

When the holder of the asset meets an agent with a higher fundamental valuation (first-order belief), he may choose to keep the asset instead of selling. He will do so knowing that he may be able to sell in the future when meeting new agents. Because of these future trading opportunities, the agent values the asset beyond his first-order belief. This incremental value will be referred to as the *resale option value*. Speculation in the sense of Harrison and Kreps [1978] arises: even if convinced that the asset has a low fundamental value, an agent may want to buy the asset at a high price, anticipating a sale at an even higher price.

²Equivalently, we could interpret the differences between agents as differences in private value. However, in financial markets, differences in beliefs seem a more relevant feature.

In order to determine the resale option value, agents must assess the price at which future agents may be willing to buy the asset. Consider the holder of the asset at time t . He values the option to resell a lot if he expects the next agents to have high first-order beliefs. So, to compute the value of the asset, the agent must have beliefs about the first-order beliefs of the other agents, *i.e.*, second-order beliefs. The agent also anticipates that, in addition to the fundamental value they assign to the asset, subsequent agents will value the resale option as well. Their valuation of the resale option will, in turn, depend on their second-order beliefs. Hence, agent t must have beliefs about the second-order beliefs of the next agents, *i.e.*, third-order beliefs. Iterating the argument we find that fourth, fifth, and even infinite-order beliefs must be considered. This is an infinite regress.

In order to solve the infinite regress, we make the following assumption. Each agent *a priori* believes he has an average belief, *i.e.*, he thinks that, on average, other agents have the same first-order belief as he does:

$$\forall j : E_i(\mu_j) = \mu_i,$$

where $E_i(\cdot)$ denotes the expectation operator, from the point of view of agent i , *i.e.*, using his priors. The fact that each agent *a priori* thinks he is average is common knowledge. This assumption solves the infinite regress problem because it implies that the *a priori* n th-order belief of agent i is his first-order belief. For example, in the case of second-order beliefs:

$$E_i(E_j(X)) = E_i(\mu_j) = \mu_i.$$

In the case of third-order beliefs, the same simplification applies:

$$E_i(E_j(E_k(X))) = E_i(E_j(\mu_k)) = E_i(\mu_j) = \mu_i.$$

The generalization to n th-order beliefs is straightforward. Note that the assumption also implies that, on average, agents correctly estimate the beliefs of others:

$$E_{\theta^*}(E_i(\mu_j)) = E_{\theta^*}(\mu_i) = \theta^*.$$

Consequently, the model has the flavor of a standard rational expectations model, but only *on average*.

2.2 Decisions

At time t , a new agent i , with first-order belief μ_i , enters the market. She bids to buy the asset at her reservation value: r_i^t , where the subscript i denotes the identity of the agent and the superscript t denotes the point in time. The current holder sells the asset if r_i^t exceeds his own reservation value. The decision process of the agents and the

determination of the reservation values can be most clearly described in a simple two-period case. At times $t = 0, 1$ and 2 new agents enter the market, and the asset pays off X at time $T + 1 \equiv 3$. The different trading sequences, corresponding to trades at times 1 and 2 are represented in the tree in Figure 1.

The reservation values are determined by backward induction.

§ At time 2, agent 2 enters the market. Her reservation value for the asset equals her first-order belief about X , μ_2 .³ Agent 2 buys the asset from the current holder if the first-order belief of the latter is lower than μ_2 . If agent 1 had bought the asset from agent 0 at time 1, agent 2 buys the asset at time 2 whenever $\mu_2 > \mu_1$. If agent 0 had preferred to keep the asset at time 1, agent 2 buys the asset at time 2 whenever $\mu_2 > \mu_0$.

§ At time 1, agent 1 knows that he can either sell the asset at time 2, and obtain μ_2 , or keep it and obtain μ_1 . So, his time-1 reservation value, and, hence, his offer, is:

$$r_1^1 = E_1^1(\max[\mu_1, \mu_2]),$$

where $E_i^t(\cdot)$ is the expectation operator from the point of view of agent i at time t (formed using the priors of agent i , updated to time t). Similarly, the time-1 reservation value of agent 0 equals:

$$r_0^1 = E_0^1(\max[\mu_0, \mu_2]).$$

Agent 0 sells the asset whenever $r_1^1 > r_0^1$.

§ At time 0, anticipating choices at later times, agent 0 has reservation value:

$$\begin{aligned} r_0^0 &= E_0^0(\max[r_0^1, r_1^1]) \\ &= E_0^0(\max[E_0^1(\max[\mu_0, \mu_2]), E_1^1(\max[\mu_1, \mu_2])]). \end{aligned}$$

In general, when the time horizon is $T \in N$, the formula for the reservation price of agent i at time t is:

$$r_i^t = E_i^t(\max[r_i^{t+1}, r_{t+1}^{t+1}]).$$

It is somewhat difficult to write explicitly. However it retains the same structure as in the two-period example, *i.e.*, expectations over maxima of expectations over maxima...

$$r_i^t = E_i^t(\max[E_i^{t+1}(\max[\dots]), E_{t+1}^{t+1}(\max[\dots])]).$$

This expression is complex, because (i) expectations are taken over maxima (non-linear functions) of random variables, and because (ii) the expectation operators differ not only

³Remember that agents will not learn about X . Hence the first-order belief of agent 2 about X is unaffected by the offers of the previous agents.

in terms of information sets but also to the extent that the agents have different prior beliefs. Because of the latter, the law of iterated expectations cannot be applied. In the next section we analyze a case where simplifications are possible.

To conclude this section, we analyze the behavior of the best offer in the market at time t (denoted b_t). It is defined as the maximum of the reservation values of the holder of the asset and the new agent who enters the market at time t :

$$b_t = \max[r_t^t, r_{(t-1)^*}^t],$$

where $(t-1)^*$ denotes the index of the agent who holds the asset at the end of the $t-1$ st trading round, and consequently holds the asset at the beginning of the t th round. By definition, b_t is the reservation value of the agent who holds the asset at the end of the t th round. This agent expects to be able to sell the asset at time $t+1$ at the reservation value of the new entrant r_{t+1}^{t+1} or to keep the asset at his own reservation value. The maximum of the two is the best offer in the market at time $t+1$, b_{t+1} . Since the agents are risk-neutral, one can state the following lemma.

Lemma 1 *From the point of view of the agent who holds the asset, the best offer in the market is a martingale:*

$$b_t = E_{t^*}^t(b_{t+1}).$$

The martingale property is simply a rationality condition, very much in the spirit of the early results of Samuelson [1965]. Note however that the martingale result only holds from the perspective of the holder of the asset. Under other beliefs, the best offer may not be a martingale. If θ^* is common knowledge, however, the martingale property also holds under the ‘true’ probability measure. We state this as a corollary.

Corollary 1 *If θ^* is common knowledge,*

$$E_{\theta^*}(b_{t+1}) = b_t.$$

3 Analytical Results

Assume that any two agents agree over the distribution of first-order beliefs in the economy after sharing their information sets. We shall refer to this as *Assumption A*. It means that:

$$\forall i, j < t: E_i^t(\mu_k) = E_j^t(\mu_k).$$

(The next section will provide a parametrization were this is true.)

3.1 Reservation Values

Reservation values are given in the next proposition.

Proposition 1 *Under Assumption A, the reservation value for agent i at time t equals:*

$$r_i^t = E_i^t(\max[\mu_i, \mu_{t+1}, \mu_{t+2}, \dots, \mu_T]),$$

for all $t < T$, $i \leq t$.

(All proofs are in the Appendix.) From Proposition 1, it follows that the speculative component of the value of the asset, *i.e.*, the difference between the reservation and fundamental values, is nonnegative:

$$r_i^t - \mu_i = E_i^t \max[\mu_i, \mu_{t+2}, \dots, \mu_T] - \mu_i \geq 0.$$

In the rational expectations literature, this speculative component is referred to as the ‘bubble’ (see Tirole [1982]).

Proposition 1 also implies that reservation values are increasing in the first-order beliefs. We state this as a corollary.

Corollary 2 *Under Assumption A,*

$$r_i^t > r_j^t \Leftrightarrow \mu_i > \mu_j.$$

The intuition of the corollary is the following. Under assumption A, and after observing each other’s first-order beliefs, agents have the same higher-order beliefs. Hence, they have the same perception of the resale option value and their reservation values differ only because their first-order beliefs do not agree. As a result of Corollary 2, trade occurs only when the newly arrived agent has a higher first-order belief than the current holder of the asset. This implies that, at time t , the current holder of the asset is the agent with the highest first-order belief since time 0.

Reservation prices can be more precisely analyzed by computing explicitly the expectation in Proposition 1. We state the result as a corollary.

Corollary 3 *Under Assumption A, the reservation value of agent i at time t equals:*

$$r_i^t = \mu_i + \int_{\Theta} \left(\int_{\mu_i}^{\infty} P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] > s) ds \right) f_i(\theta) d\theta,$$

where the probability $P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] > s)$ is conditional on θ , and $f_t(\cdot)$ is the density of θ , given the sequence μ_0, \dots, μ_t for any agent $0 \leq i \leq t$.

Corollary 3 shows explicitly that the reservation value is the sum of the fundamental and option or speculative values.

Corollary 3 can be used to characterize the process of the reservation values further.

Corollary 4 *Under Assumption A, r_i^t is a submartingale under agent i 's beliefs, i.e.,*

$$r_i^t \geq E_i^t(r_i^{t+1}).$$

This result reflects the decrease in the speculative value of the asset as the horizon gets closer. The decline is due to the loss of resale opportunities and the decrease in uncertainty about θ as the number of draws of first-order beliefs increases. Corollary 4 does not contradict Lemma 1 (which stated that the best offer is a martingale from the point of view of the holder). In the case of Lemma 1, the decrease in the option value is offset by the fact that the new entrant might have higher beliefs than the current holder, so that the latter would sell and earn capital gains.

When θ^* is common knowledge, the expression for the reservation value can be simplified further.

Corollary 5 *If θ^* is known, the reservation value of agent i at time t is:*

$$r_i^t = \mu_i + \int_{\mu_i}^{\infty} (P_{\theta^*}(\max[\mu_{t+1}, \dots, \mu_T] > s) ds.$$

In this case, the impact of μ_i on the reservation value can be easily analyzed. An increase in μ_i implies an increase in the fundamental value of the asset for the agent. But it also implies a decrease in the speculative value of the asset, i.e., in:

$$\int_{\mu_i}^{\infty} P_{\theta^*}(\max[\mu_{t+1}, \dots, \mu_T] > s) ds.$$

If μ_i is large, the agent does not expect to find another agent with a higher first-order belief, hence the agent does not expect to sell. When θ^* is unknown, the impact of μ_i on the speculative component is more ambiguous. An increase in μ_i may imply an increase in the belief that somebody with a higher first-order belief will be met, but the higher-order beliefs of the latter have to be taken into account as well.

As the number of draws of first-order beliefs increases, higher-order beliefs become more precise. In the limit, θ^* is perfectly known and the reservation values are as given in the Corollary 5. Note that even in this case, the value of the asset is above its fundamental value. There continues to be speculation when θ^* is known.

3.2 The Best Offer in the Market

Using Proposition 1 and Corollary 2, the following obtains.

Proposition 2 *Under Assumption A, the best offer in the market at time t (b_t) equals:*

$$b_t = E_i^t(\max[\mu_0, \dots, \mu_T]),$$

where i is any integer between 0 and t .

Proposition 2 reflects the rational expectation by the market that the asset will be owned by the agent with the maximum first-order belief. Since the agents are risk neutral and learning, their reservation value for the asset is equal to their conditional expectation of this maximum.

From Proposition 2, the best offer in the market at time t can be obtained analytically. We state this as a corollary.

Corollary 6 *Under Assumption A, the best offer in the market at time t equals:*

$$b_t = \mu_t^* + \int_{\Theta} \left(\int_{\mu_t^*}^{\infty} P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] > s) ds \right) f_t(\theta) d\theta,$$

where

§ the probability $P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] > s)$ is conditional on θ ,

§ $f_t(\cdot)$ is the density of θ , given the sequence μ_0, \dots, μ_t for any agent $0 \leq i \leq t$,

§ $\mu_t^* \equiv \max[\mu_0, \dots, \mu_t]$.

4 A Parametric Example

We now propose a distribution that fits Assumption A. This enables us to analyze more explicitly (i) the learning process, (ii) the speculative component of prices and (iii) the econometric implications of our analysis for the evolution of prices and volume.

Assume that the first-order beliefs, μ_i , are i.i.d. draws from an exponential distribution with parameter θ . This implies that:

$$E_{\theta^*}(\mu_i) = \theta^*.$$

Assume further that the priors of agent i about θ are inverse gamma. The inverse gamma distribution depends on two parameters, which we denote α_i and β_i . We will use the shorthand notation $IG(\alpha_i, \beta_i)$ for the distribution. Its mean equals $((\alpha_i - 1)\beta_i)^{-1}$ and its variance (the inverse of the precision) $((\alpha_i - 1)^2(\alpha_i - 2)\beta_i^2)^{-1}$. We set: $(\alpha_i, \beta_i) = (2, 1/\mu_i)$. In that case, agents' higher-order beliefs have means:

$$E_i(\theta) = \frac{1}{(\alpha_i - 1)\beta_i} = \mu_i,$$

and, hence,

$$E_i(\mu_j) = E_i(E_\theta(\mu_j)) = E_i(\theta) = \mu_i.$$

That is, *a priori*, agents think they have average first order beliefs, as required.

The sequence of offers r_0^0, \dots, r_t^t can be inverted to obtain the sequence of first-order beliefs: $\mu_\bullet, \dots, \mu_t$. The agents use this information to update their higher-order beliefs. In the exponential-inverse gamma case, the posterior based on a sample of $\mu_\bullet, \dots, \mu_t$ is $IG(2 + t, (\sum_{i=0}^t \mu_i)^{-1})$. Notice that the posteriors always agree. In particular, for all $i \leq t$,

$$E_i^t(\theta) = \frac{\sum_{i=0}^t \mu_i}{t + 1},$$

i.e., the posterior expectation of θ is simply the empirical average of the first-order beliefs. As the number of draws increases, the expectation of θ converges to θ^* .

When $(\alpha_i, \beta_i) = (2, 1/\mu_i)$, the results of the previous section can be made more explicit. For example, the reservation value of agent i at time t can be easily computed from Proposition 1.

Corollary 7 *If first-order beliefs are exponential and higher-order beliefs inverse gamma with parameters $(2, 1/\mu_i)$, agent i 's time- t reservation value equals:*

$$r_i^t = \mu_i + \sum_{j=1}^{T-t} C_{T-t}^j \frac{(-1)^{j+1}}{j} \frac{\bar{\mu}_t + j\mu_i}{t+1} \left[1 + j\frac{\mu_i}{\bar{\mu}_t}\right]^{-(t+2)},$$

where $\bar{\mu}_t \equiv \sum_{j=0}^t \mu_j$.

When θ^* is common knowledge, the expression can be simplified further.

Corollary 8 *If first-order beliefs are exponential with parameter θ^* and this is common knowledge, the reservation value of agent i at time t is*

$$r_i^t = \mu_i + \sum_{j=1}^{T-t} C_{T-t}^j (-1)^{j+1} \frac{\theta^*}{j} e^{-j\frac{\mu_i}{\theta^*}}.$$

The formulae for the best offers in the market are similar, except that μ_i^* ($= \max[\mu_0, \dots, \mu_t]$) must be substituted for μ_i .

5 Simulation Results

Until now, we have been looking at cases where agents' higher-order beliefs are correct or where they agree after one observation (Assumption A). Because it was easy to determine who would hold the asset at any moment (Corollary 2), analytical solutions could be obtained. But the aim of the paper is to understand the consequences of full disagreement (when beliefs agree only asymptotically) for prices and volume in financial markets. Because the identity of the holder becomes ambiguous, the model is intractable and prices and volume can only be characterized through simulations based on numerical calculations of reservation values. Using the same numerical approximation, we also simulated the models where agents' higher-order beliefs were correct and where higher-order beliefs agreed after one observation, in order to induce the same numerical error and make the results comparable across models. The availability of analytical solutions, moreover, allowed us to verify that the numerical error was small.⁴

5.1 Description of the Simulations

We assume that the first-order beliefs are i.i.d. exponential with mean θ^* . When agents' higher-order beliefs agree after one observation, we use the same specification as in the previous section. In the third version of the model, where agents' higher-order beliefs agree only asymptotically, we set the prior of agent i to be inverted gamma with parameters $\alpha_i = \mu_i + 1$ and $\beta_i = 1/\mu_i^2$. With this specification, our assumption that agent i *a priori* perceives other agents' beliefs to be hers on average, continues to be satisfied. Also, reservation values are still invertible in the first-order belief, so that the holder can correctly infer the bidder's beliefs hierarchy (we have only been able to verify this invertibility numerically).

Nevertheless, agents who assign a fundamental value below one to the asset ($\mu_i \leq 1$) have zero-precision higher-order beliefs. Since the level of precision of the higher-order beliefs and the value of the resale option are inversely related, there is a trade-off between fundamental value and resale value, to the extent that some agents with low first-order beliefs may initially assign a higher total value to holding the asset than agents with higher first-order beliefs within a certain range.

⁴The numerical error was very small in the range of the distribution of beliefs with most of the (probability) mass, but substantial close to the edges imposed in the numerical approximation; in particular, agents with high first-order beliefs underestimated the resale option, because the distribution of beliefs is necessarily truncated in the numerical approximation.

It may seem that this reversion in reservation values destroys invertibility. For the holder to correctly infer the beliefs of the bidder, however, reservation values should be invertible when evaluated using the history of first-order beliefs *including those of the holder, but excluding those of the bidder*. In contrast, when the holder contemplates the choice of selling the asset to the bidder, she does not consider her own first-order beliefs as part of the history from which she learns the distribution of beliefs, and, hence, determines her reservation value. Yet, she does include the first-order beliefs of the bidder. This difference in information sets makes invertibility compatible with higher valuation by speculators with lower first-order beliefs.

Our specification of the higher-order beliefs is admittedly subtle, but simplifies the calculations substantially. It should be obvious that we could have generalized, at the cost of losing the invertibility of the reservation values in the first-order beliefs. Updating would still be possible, but much more complex. Volume and volatility patterns, though, would potentially be richer.

The simulations were carried out in several steps.

1. The number of periods T was set equal to 10.
2. Three possible values of θ^* were used, namely, 0.6, 0.8 and 1.0.
3. Reservation values were calculated using numerical integration with a mesh equal to 0.1.
4. We generated 10^3 sample paths of $T + 1$ draws of first-order beliefs, discretized over intervals of size 0.1 (*i.e.*, matching the mesh in the numerical calculations).
5. At each point in time, and for each sample path, we computed the reservation value of the holder of the asset, and that of the agent who entered the market, thus determining whether there was a trade and at what price.
6. For each trading round, we computed sample averages and standard deviations over the 10^3 sample paths of the following variables: the beliefs of the holder, the trading volume, the best offer in the market, and the transaction price⁵.

5.2 The Results

5.2.1 Trading Volume

Figures 2 and 3 depict the changes in the first-order beliefs of the holder and the trading volume, respectively, as a function of the trading round. Trading volume gradually dies

⁵Transaction prices are defined in the same way as on the ISSM tape. Let p_t^* denote the transaction price at time t . If there is a trade, $p_t^* = b_t$. Otherwise, $p_t^* = p_{t-1}^*$.

out as time passes. When θ^* is common knowledge or when higher-order beliefs agree after one observation, the holder always has the highest first-order beliefs (of all the agents that have entered the market; Corollary 2 states this for the case when higher-order beliefs agree after one observation). Consequently, the first-order belief of the holder cannot decrease over time, but increases become less likely as time passes (see Figure 2). Increases translate into trades, and, hence, volume gradually decreases (Figure 3).

In the third case, this phenomenon is mitigated. As conjectured above, some agents with low fundamental valuation, but, imprecise higher-order beliefs (*i.e.*, high resale option valuation) pass on trades they would never let go in the first two cases. They even buy from agents with higher fundamental valuation, in the hope of selling later. When the horizon nears and if they have not been able to sell earlier, they unload the asset in the market (remember that they do not attach a high fundamental value to the asset). This is most clearly reflected in Figure 2, where the average change in the first-order beliefs of the holder is much lower in early trading rounds than in the first two cases, but much higher later on. The activity of the agents with low first-order beliefs maintains volume, albeit without offsetting its secular decline. This is depicted in Figure 3.

5.2.2 The Best Offer in the Market

As can be seen in Figure 4, when beliefs agree only asymptotically, the average change in the best offer is positive at the beginning, then it gradually declines until it becomes negative. Eventually it reverts slowly back to 0. What we observe is:

$$E_{\theta^*}(b_t) \neq E_{\theta^*}(b_{t+1}).$$

One might conclude that agents' valuations are biased, that they should realize this (after all, they can perform the simulations that we did), and revise their bids accordingly. However, this line of reasoning is not correct. Agents must not look at the behavior of the best offer in the same way as we do in the simulations. At time t , agents do not know θ^* yet. Agents may want to estimate $E_{\theta^*}(b_{t+1})$ using their own information. That is, at time t , agent i could compute: $E_i^t(E_{\theta}(b_{t+1}))$. But performing this computation does not make sense for them. Indeed, when taking the expectation $E_{\theta}(b_{t+1})$, agents would throw away valuable information about the first t draws of μ_t . Instead, what agents compute is: $E_i^t[E_{\theta}(b_{t+1}|\mu_0, \dots, \mu_t)]$. Hence, to determine if the offer at time t is over- or under-valued, agent i compares $E_i^t[E_{\theta}(b_t|\mu_0, \dots, \mu_t)]$ with $E_i^t[E_{\theta}(b_{t+1}|\mu_0, \dots, \mu_t)]$. The first term equals b_t and the second one $E_i^t(b_{t+1})$. Rationality (and risk neutrality) implies that the two terms should be equal (see Lemma 1).

5.2.3 Transaction Price Changes

Figure 5 displays the average change in the transaction price as a function of the trading round. In light of our discussion about the best offer in the market, the predictability of transaction prices should not come as a surprise. We should, however, draw the reader's attention to the behavior of the transaction price when higher-order beliefs are correct: on average, its change is (significantly) positive and constant. This pattern is reminiscent of the behavior of risk premia in simple rational expectations models. As a matter of fact, an econometrician who ignores the absence of information flow about the asset's payoff and who focuses on the behavior of the transaction price would (incorrectly) infer that he is in a standard rational expectations economy. This impression will be reinforced when he contemplates Figure 7 (to be discussed later), where it is documented that volume cannot be used to predict transaction prices. If he were to focus on bid prices, he would find substantial evidence of a martingale (Figure 4), consistent with standard noisy rational expectations models of the microstructure of a financial market.

5.2.4 Volatility

In our model, there is no release of fundamental information about the likely payoff on the asset. In spite of this, there is volume and volatility. The disagreement among traders creates a resale option, whose value changes over time. Figure 6 illustrates this. Notice the dramatic increase in volatility when there is uncertainty about the distribution of beliefs. However, as learning about the distribution of beliefs progresses, volatility declines.

5.2.5 The Relation Between Volume and Volatility

Figure 7 documents that, when there is uncertainty about the distribution of first-order beliefs, trading volume (or rather the *absence* of it) can be used to predict future price changes. The intuition behind this predictability is simple: high volume (hectic trading) is good news about the distribution of first-order beliefs; the holder is continuously surprised by the heavy buying activity and revises her reservation value (a lower boundary to the transaction price in the subsequent trading round, if there is a trade) upward. Conversely, when buyers stay out of the market, the holder necessarily revises down her beliefs about the resale potential of the asset. When there is a transaction, it will be at a substantially discounted price.

5.3 Simulation Errors

Because analytical expressions for the reservation values are unavailable when agents' higher-order beliefs agree only asymptotically, numerical integration using a finite mesh is used, and the sample paths of the first-order beliefs have to be correspondingly discretized. While altering the size of the mesh did not affect the results qualitatively, one specific pattern did emerge. At this moment we are not sure whether this is evidence of a more general property about smoothness of price changes, or a mere *artefact* of the specific distributional assumptions.

As we increased the mesh from 0.1 to 0.2, we noticed a more pronounced volume and volatility rebound in later trading rounds for the case where higher-order beliefs converge only asymptotically. This effect was clearest when $\theta^* = 0.6$. Figure 8 provides the charts that document this. The late increase in volume and volatility can be attributed to the lower initial change in first-order beliefs of the holder (actually, negative) reported in the top panel of Figure 8. In other words, the increase in the mesh lowers the average first-order belief of the holder in early trading rounds, leading to a more dramatic sell-off in later periods as investors gradually learn about the true distribution of beliefs and run out of resale opportunities.

One could wonder whether it is a general property that if the distribution of beliefs is continuous, price changes are smooth and volume more spread out, whereas, if beliefs are discrete, price decreases (as opposed to increases) are sudden and volume uneven. Our experience with altering the mesh of the discretization in the simulations indicates that this conjecture may be correct.

6 Conclusion

Perhaps the empirically most relevant aspect of models of financial markets with speculating agents is that they predict volume and volatility in the absence of information about the fundamental value of the asset. As Roll has stressed on several occasions (see, e.g., Roll [1984], [1989]), this is the single most puzzling feature of financial markets that cannot be explained in terms of standard asset pricing models. This paper demonstrated that speculation generates substantial volatility and volume without 'fundamental information.' Price and trade patterns are potentially richest when there is disagreement at higher-order levels.

Our model still presents some weaknesses. To single out the most disturbing one: it is assumed that the holder can somehow obtain truthful revelation of the bidder's beliefs without specifying the mechanism that will enable this. There is no strategic interaction. Our intuition is that the availability of alternatives for the holder would force bidders not

to manipulate. It may be that further research along the lines of Green [1991] will prove this intuition to be correct. Nevertheless, given the presence of large players in actual financial markets, the potential to manipulate beliefs is interesting in its own right and deserves to be looked at more closely.

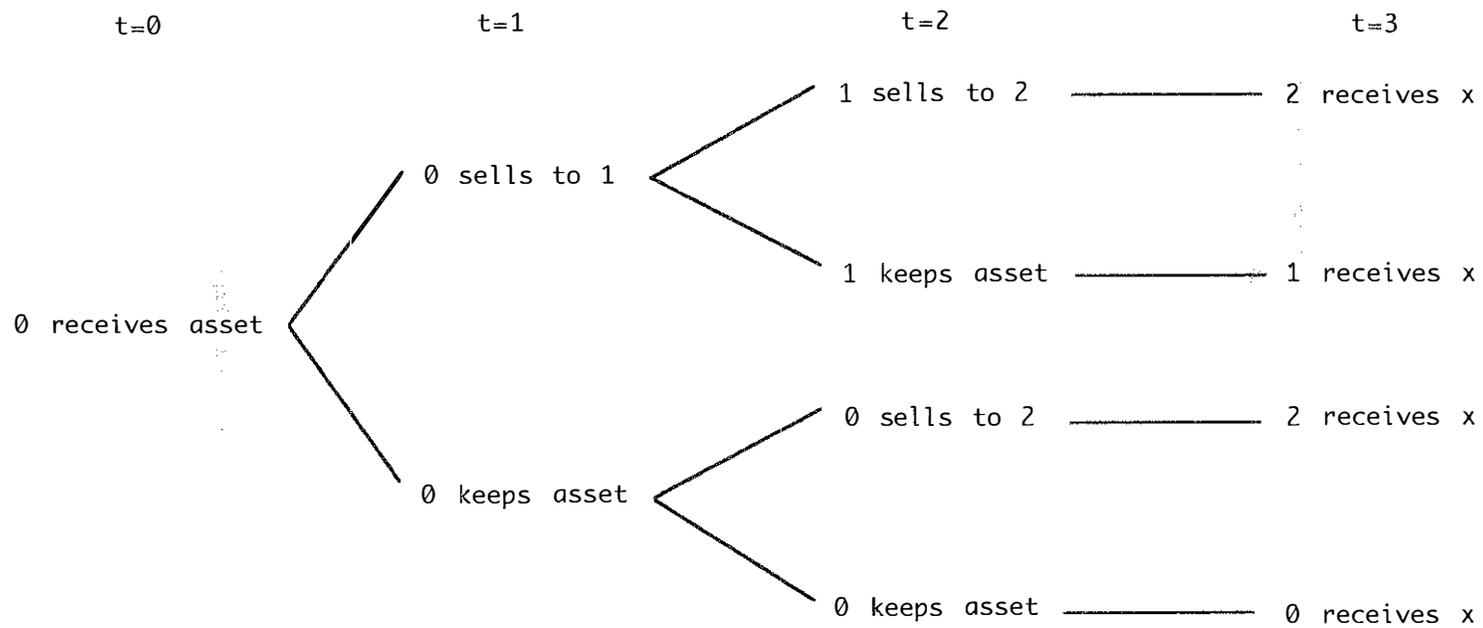


Figure 1: Event Tree With Two Trading Rounds ($T=2$)

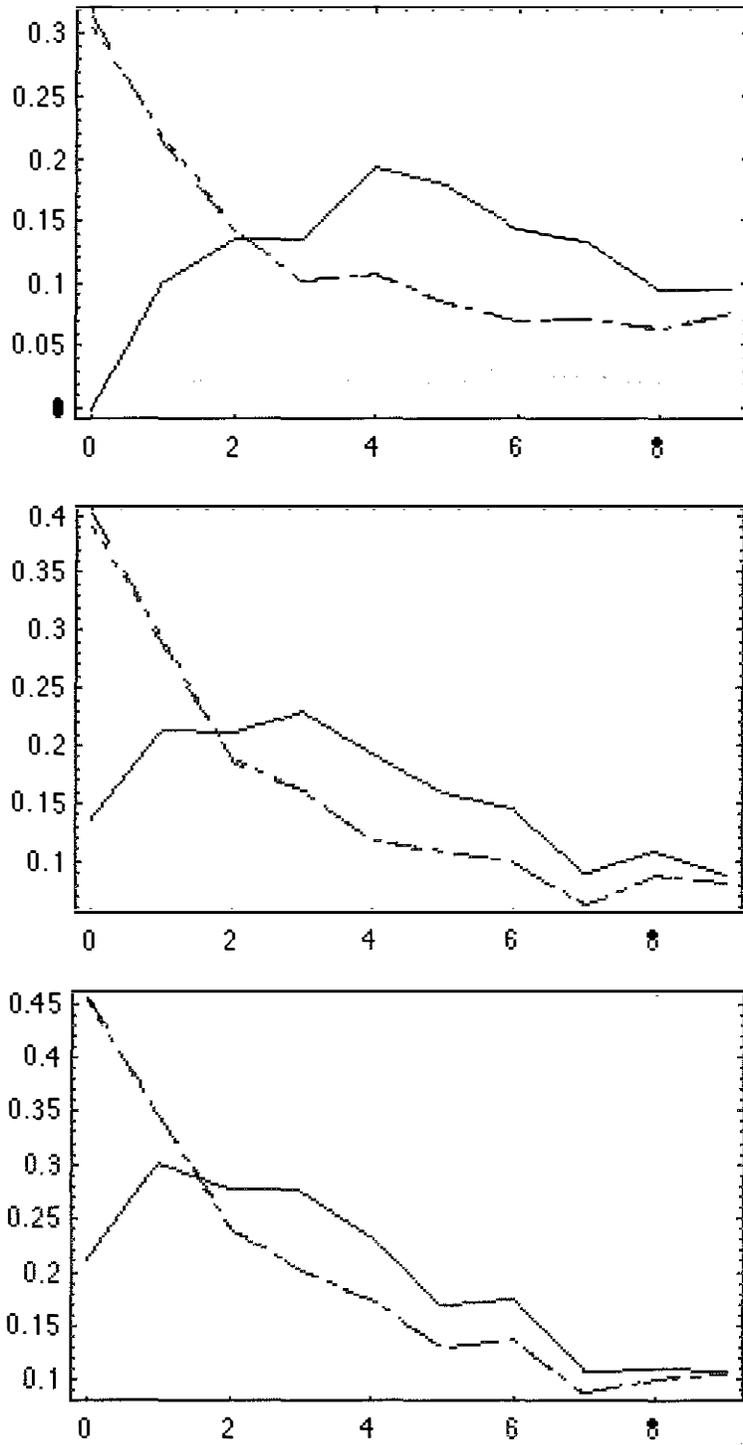


Figure 2: Average Change in First-Order Beliefs of Holder as a Function of the Trading Round, for $\theta^* = 0.6$ (Top), $\theta^* = 0.8$ (Middle) and $\theta^* = 1.0$ (Bottom)

Note: Full line depicts results when higher-order beliefs agree only asymptotically; long dashes depict results when higher-order beliefs agree after one observation; short dashes depict results when higher-order beliefs agree and are correct. The averages are based on 10^3 simulations.

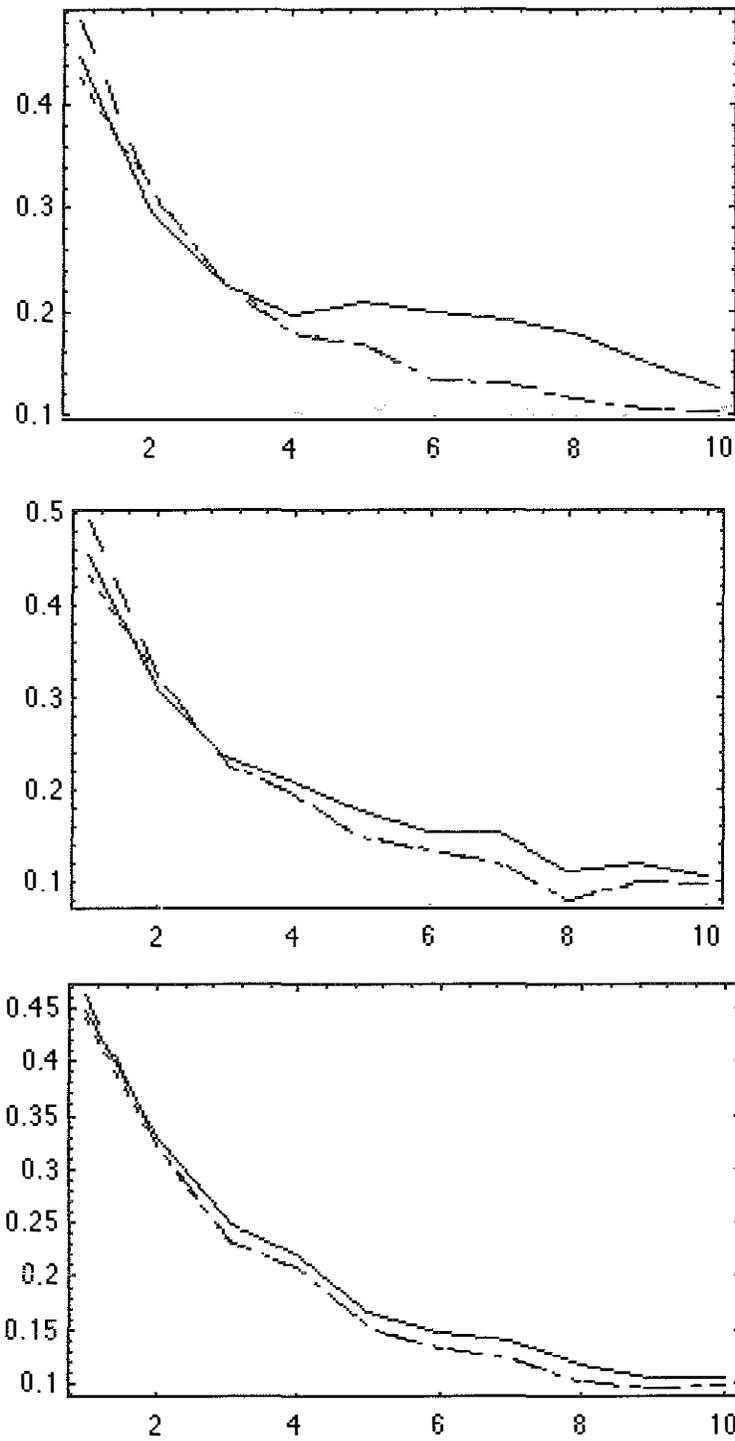


Figure 3: Average Volume as a Function of the Trading Round, for $\theta^* = 0.6$ (Top), $\theta^* = 0.8$ (Middle) and $\theta^* = 1.0$ (Bottom)

Note: Full line depicts results when higher-order beliefs agree only asymptotically; long dashes depict results when higher-order beliefs agree after one observation; short dashes depict results when higher-order beliefs agree and are correct. The averages are based on 10^3 simulations.

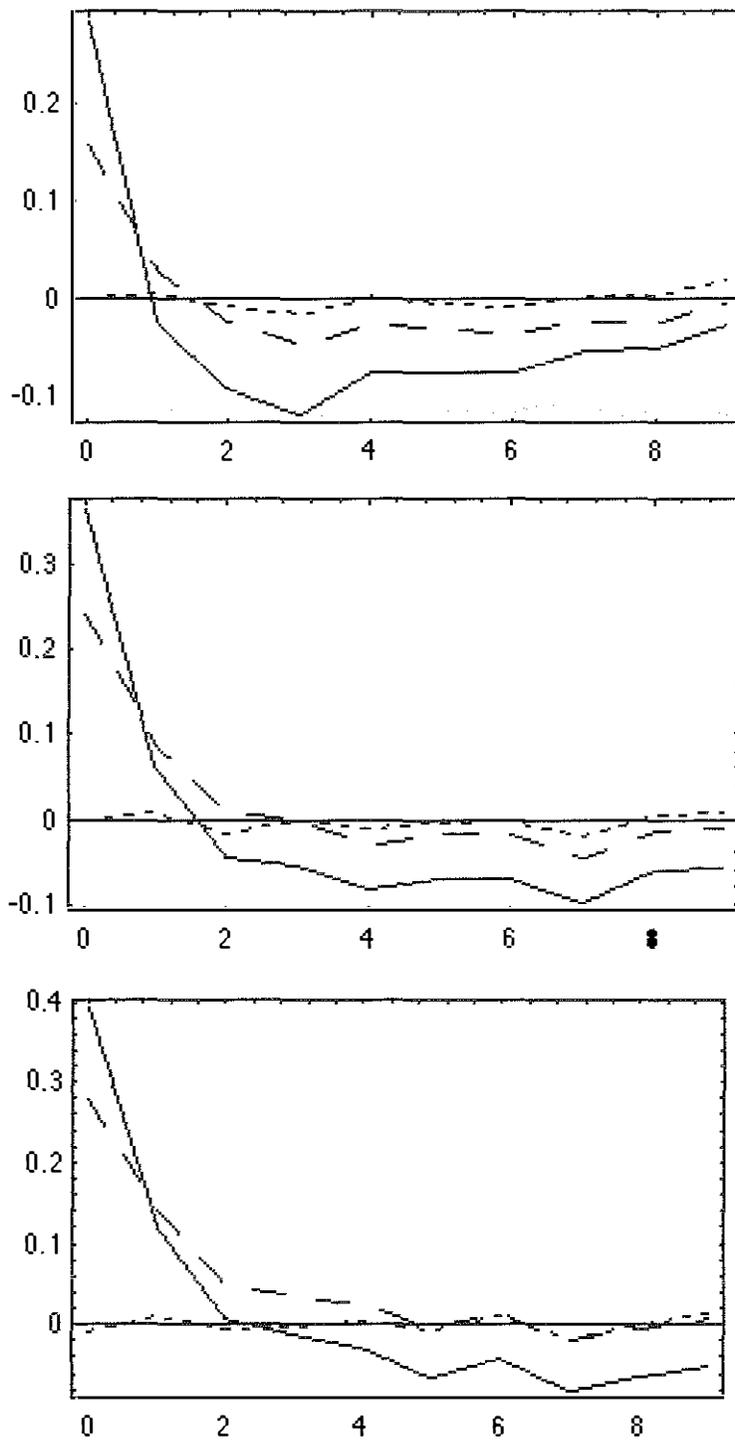


Figure 4: Average Change in Best Offer as a Function of the Trading Round, for $\theta^* = 0.6$ (Top), $\theta^* = 0.8$ (Middle) and $\theta^* = 1.0$ (Bottom)

Note: Full line depicts results when higher-order beliefs agree only asymptotically; long dashes depict results when higher-order beliefs agree after one observation; short dashes depict results when higher-order beliefs agree and are correct. The averages are based on 10^3 simulations.

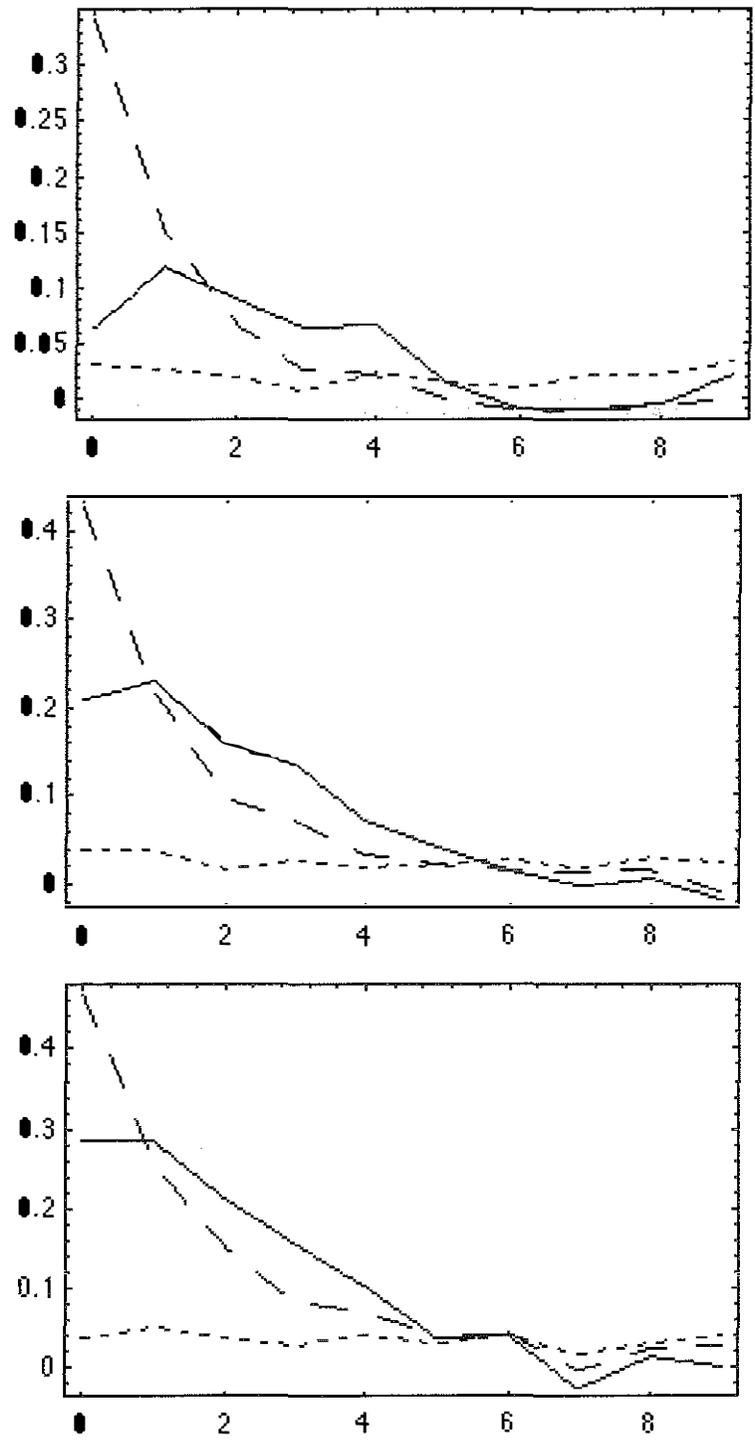


Figure 5: Average Change in Transaction Price as a Function of the Trading Round, for $\theta^* = 0.6$ (Top), $\theta^* = 0.8$ (Middle) and $\theta^* = 1.0$ (Bottom)

Note: Full line depicts results when higher-order beliefs agree only asymptotically; long dashes depict results when higher-order beliefs agree after one observation; short dashes depict results when higher-order beliefs agree and are correct. The averages are based on 10^3 simulations.

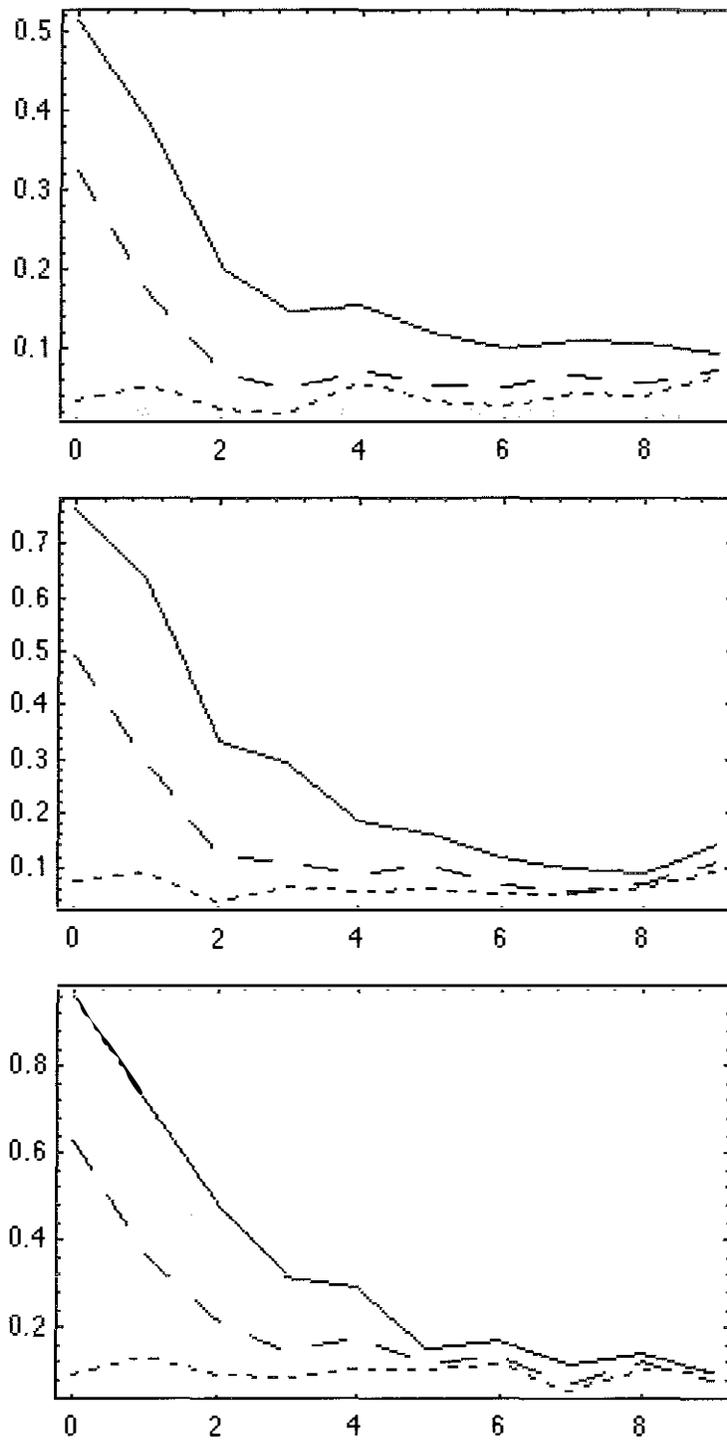


Figure 6: Standard Deviation of Transaction Price Change as a Function of the Trading Round, for $\theta^* = 0.6$ (Top), $\theta^* = 0.8$ (Middle) and $\theta^* = 1.0$ (Bottom)

Note: Full line depicts results when higher-order beliefs agree only asymptotically; long dashes depict results when higher-order beliefs agree after one observation; short dashes depict results when higher-order beliefs agree and are correct. The averages are based on 10^3 simulations.

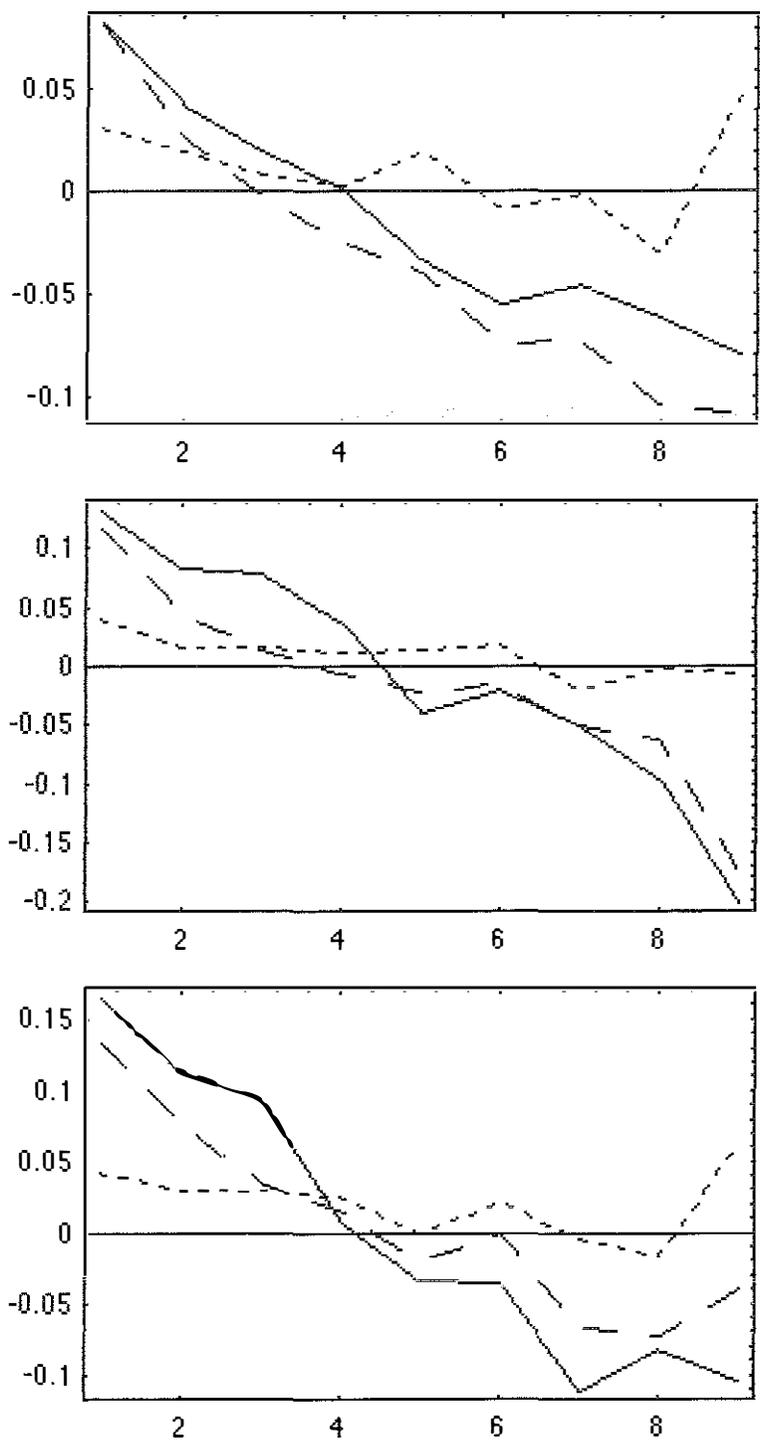


Figure 7: Average Transaction Price Change as a Function of Number of Trading Rounds without Volume, for $\theta^* = 0.6$ (Top), $\theta^* = 0.8$ (Middle) and $\theta^* = 1.0$ (Bottom)

Note: Full line depicts results when higher-order beliefs agree only asymptotically; long dashes depict results when higher-order beliefs agree after one observation; short dashes depict results when higher-order beliefs agree and are correct. The averages are based on 10^3 simulations.

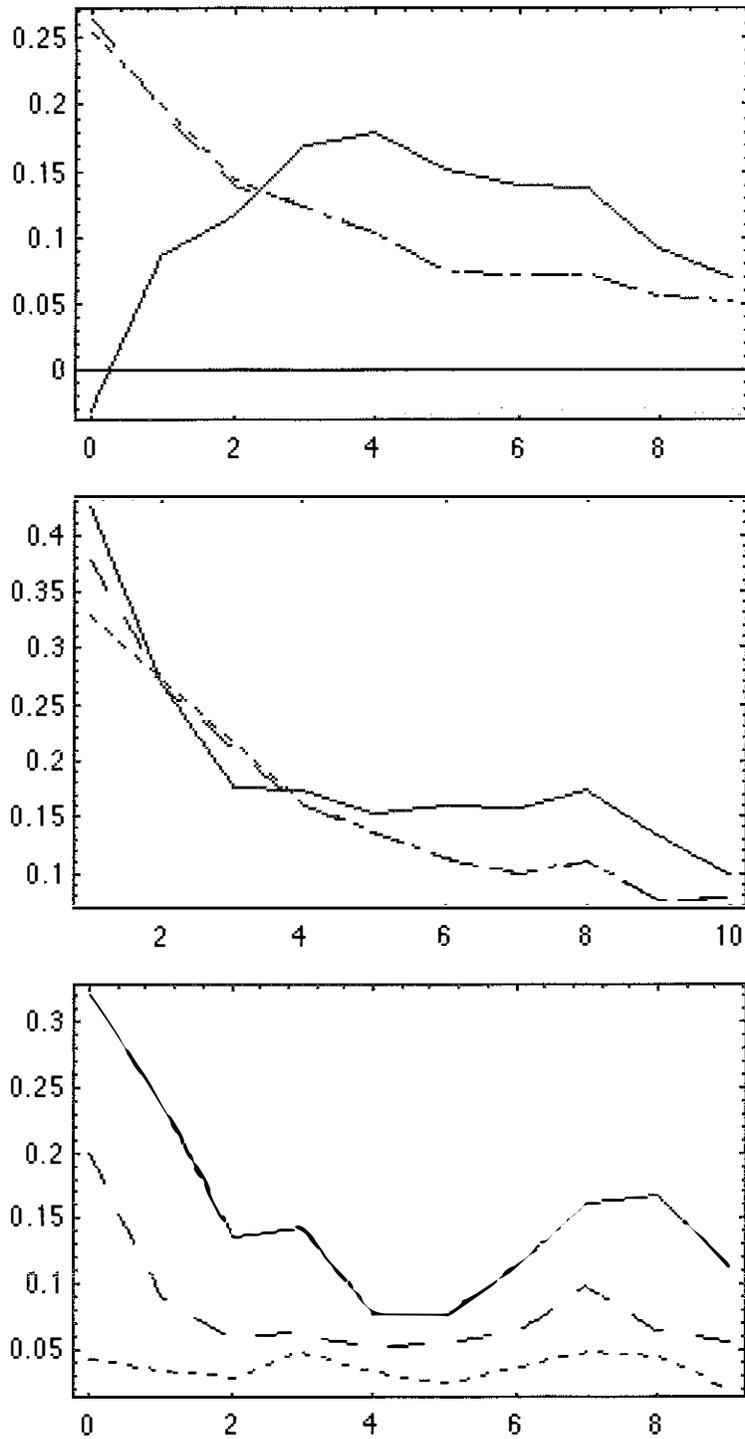


Figure 8: Average Change in First-Order Beliefs of Holder (Top), Average Volume (Middle) and Standard Deviation of Transaction Price Change (Bottom) as a Function of the Trading Round, for $\theta^* = 0.6$, based on a cruder numerical approximation.

Note: Full line depicts results when higher-order beliefs agree only asymptotically; long dashes depict results when higher-order beliefs agree after one observation; short dashes depict results when higher-order beliefs agree and are correct. The averages are based on 10^3 simulations.

Appendix

Proof of Proposition 1:

The proof is by induction.

At time $T - 1$, the reservation value of agent i is $E_i^{T-1}(\max[\mu_i, \mu_T])$. Hence, at that time the property holds. Assume that at $t + 1$ the property holds, *i.e.*, the reservation value of agent i is:

$$r_i^{t+1} = E_i^{t+1}(\max[\mu_i, \mu_{t+2}, \dots, \mu_T]).$$

We now show that the property holds at time t . At time $t + 1$, the agent can keep the asset, and obtain his reservation value. Alternatively, he can sell the asset to agent $t + 1$ at the reservation value of the latter, which equals:

$$r_{t+1}^{t+1} = E_{t+1}^{t+1}(\max[\mu_{t+1}, \mu_{t+2}, \dots, \mu_T]).$$

From Assumption A,

$$r_{t+1}^{t+1} = E_i^{t+1}(\max[\mu_{t+1}, \mu_{t+2}, \dots, \mu_T]).$$

If $\mu_i < \mu_{t+1}$, agent i sells the asset and obtains:

$$r_i^{t+1} = E_i^{t+1}(\max[\mu_{t+1}, \dots, \mu_T]) = E_i^{t+1}(\max[\mu_i, \mu_{t+1}, \dots, \mu_T]).$$

If instead $\mu_i > \mu_{t+1}$, agent i keeps the asset and obtains:

$$r_i^{t+1} = E_i^{t+1}(\max[\mu_i, \mu_{t+2}, \dots, \mu_T]) = E_i^{t+1}(\max[\mu_i, \mu_{t+1}, \mu_{t+2}, \dots, \mu_T]).$$

So, in both cases he obtains:

$$E_i^{t+1}(\max[\mu_i, \mu_{t+1}, \dots, \mu_T]).$$

At time t , agent i anticipates this, so his reservation value for the asset equals:

$$r_i^t = E_i^t[E_i^{t+1}(\max[\mu_i, \mu_{t+1}, \dots, \mu_T])].$$

Since both expectation operators are taken from the perspective of agent i , we can apply the law of iterated expectations. Hence the reservation value of the asset for agent i at time t is:

$$r_i^t = E_i^t(\max[\mu_i, \mu_{t+1}, \dots, \mu_T]).$$

□

Proof of Corollary 2:

$$r_i^t > r_j^t \Leftrightarrow E_i^t(\max[\mu_i, \mu_{t+2}, \dots, \mu_T]) > E_j^t(\max[\mu_j, \mu_{t+2}, \dots, \mu_T]).$$

However, under Assumption A, $E_i^t(\cdot) = E_j^t(\cdot)$. Hence,

$$r_i^t > r_j^t \Leftrightarrow E_i^t(\max[\mu_i, \mu_{t+2}, \dots, \mu_T]) > E_i^t(\max[\mu_j, \mu_{t+2}, \dots, \mu_T]).$$

So,

$$r_i^t > r_j^t \Leftrightarrow \mu_i > \mu_j.$$

□

Proof of Corollary 3:

From Proposition 1,

$$r_i^t = E_i^t(\max[\mu_i, \mu_{t+1}, \dots, \mu_T]).$$

This can be rewritten:

$$r_i^t = \mu_i + \int_{\ominus} \int_{\mu_i}^{\infty} (s - \mu_i) dP_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s) f_t(\theta) d\theta.$$

$\int_{\mu_i}^{\infty} (s - \mu_i) dP_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s)$ can be rewritten:

$$\int_{\mu_i}^{\infty} s dP_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s) - \mu_i \int_{\mu_i}^{\infty} dP_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s).$$

Now:

$$\int_{\mu_i}^{\infty} dP_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s) = 1 - P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < \mu_i),$$

and

$$\int_{\mu_i}^{\infty} s dP_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s) = - \int_{\mu_i}^{\infty} s d(1 - P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s)).$$

Integrating by parts, one obtains:

$$[-s(1 - P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s))]_{\mu_i}^{\infty} + \int_{\mu_i}^{\infty} (1 - P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s)) ds.$$

Simplifying,

$$r_i^t = \mu_i + \int_{\ominus} \int_{\mu_i}^{\infty} (1 - P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] < s)) ds f_t(\theta) d\theta.$$

□

Proof of Corollary 4:

From Corollary 3:

$$r_i^t = \mu_i + E_i^t \Phi(\mu_i, \theta, t),$$

where

§ $\Phi(\mu_i, \theta, t) \equiv \int_{\mu_i}^{\infty} P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] > s) ds$.

(Note that Φ is non-increasing in its third argument.)

§ E_i^t is taken over θ .

Consequently:

$$r_i^t \geq E_i^t(r_i^{t+1}) \leftrightarrow E_i^t \Phi(\mu_i, \theta, t) \geq E_i^t(E_i^{t+1}(\Phi(\mu_i, \theta, t))).$$

This holds because

$$\Phi(\mu_i, \theta, t) \geq \Phi(\mu_i, \theta, t + 1).$$

□

Proof of Corollary 5:

Follows immediately from Corollary 3.

□

Proof of Proposition 2:

Let i be the agent who holds the asset after the t th round of trade. Hence,

$$\mu_i = \max[\mu_0, \dots, \mu_t].$$

Combining this with Proposition 1, the proposition obtains.

□

Proof of Corollary 6:

Combine Proposition 2 with Corollary 3.

□

Proof of Corollary 7:

From Corollary 3, the reservation value of agent i at time t is:

$$r_i^t = \mu_i + \int_{\Theta} \left(\int_{\mu_i}^{\infty} P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] > s) ds \right) f_t(\theta) d\theta.$$

To compute this more explicitly, we first analyze $\int_{\mu_i}^{\infty} P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] > s) ds$, then we integrate over θ .

Analyze $\int_{\mu_i}^{\infty} P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] > s) ds$ first. The probability that one exponential variable with parameter θ is smaller than s equals:

$$1 - e^{-s/\theta}.$$

The probability that $T - t$ exponential variables are smaller than s equals:

$$[1 - e^{-s/\theta}]^{T-t} = \sum_{j=0}^{T-t} C_{T-t}^j (-1)^j e^{-js/\theta},$$

or

$$1 + \sum_{j=1}^{T-t} C_{T-t}^j (-1)^j e^{-js/\theta}.$$

So, the probability that $T - t$ exponential variables are larger than s equals:

$$P_{\theta}(\max[\mu_{t+1}, \dots, \mu_T] > s) = \sum_{j=1}^{T-t} C_{T-t}^j (-1)^{j+1} e^{-js/\theta}.$$

We now integrate this over $s \in [\mu_i, \infty]$.

$$\int_{\mu_i}^{\infty} \sum_{j=1}^{T-t} C_{T-t}^j (-1)^{j+1} e^{-js/\theta} ds = \sum_{j=1}^{T-t} C_{T-t}^j (-1)^{j+1} \int_{\mu_i}^{\infty} e^{-js/\theta} ds.$$

This can be rewritten:

$$\sum_{j=1}^{T-t} C_{T-t}^j (-1)^{j+1} (\theta/j) e^{-j\mu_i/\theta}.$$

Integrate over θ :

$$r_i^t = \mu_i + \int_{\Theta} \sum_{j=1}^{T-t} C_{T-t}^j (-1)^{j+1} (\theta/j) e^{-j\mu_i/\theta} f_t(\theta) d\theta,$$

or

$$r_i^t = \mu_i + \sum_{j=1}^{T-t} C_{T-t}^j (-1)^{j+1} / j \int_{\Theta} \theta e^{-j\mu_i/\theta} f_t(\theta) d\theta.$$

The latter integral can be written more explicitly:

$$\int_{\Theta} \theta e^{-j\mu_i/\theta} \left(\frac{1}{\theta} \frac{e^{-1/(j\beta_t\theta)}}{(\alpha_t - 1)!} \left(\frac{1}{\beta_t\theta} \right)^{\alpha_t} \right) d\theta,$$

where

$$\beta_t = \left(\sum_{s=0}^t \mu_s \right)^{-1} \equiv (\bar{\mu}_t)^{-1}$$

and

$$\alpha_t = t + 2$$

are the parameters of the inverse gamma distribution (the posterior beliefs about θ of the agents in the market at time t). The integral can be rewritten:

$$\int_{\Theta} \theta \left(\frac{1}{\theta} \frac{e^{-\frac{j\mu_i + \bar{\mu}_t}{\theta}}}{(t+1)!} \left(\frac{\bar{\mu}_t}{\theta} \right)^{t+2} \right) d\theta,$$

or

$$\left(\frac{\bar{\mu}_t}{j\mu_i + \bar{\mu}_t} \right)^{t+2} \int_{\Theta} \theta \left(\frac{1}{\theta} \frac{e^{-\frac{j\mu_i + \bar{\mu}_t}{\theta}}}{(t+1)!} \left(\frac{j\mu_i + \bar{\mu}_t}{\theta} \right)^{t+2} \right) d\theta.$$

The integral is the expectation of an inverse gamma variable with parameters $(t+2, (j\mu_i + \bar{\mu}_t)^{-1})$, *i.e.*,

$$\frac{j\mu_i + \bar{\mu}_t}{t+1}.$$

Substituting this in the formula for the reservation value, the proposition obtains.

□

Proof of Corollary 8: Combine Proposition 2 with elements from the proof of Corollary 7.

□

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