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Hamiltonian Electric/Magnetic Duality and Lorentz Invariance

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Abstract

In (3+1) Hamiltonian form, the conditions for the electric/magnetic invariance of generic self-interacting gauge vector actions and the definition of the duality generator are obvious. Instead, (3+1) actions are not intrinsically Lorentz invariant. Imposing the Dirac–Schwinger stress tensor commutator requirement to enforce the latter yields a differential constraint on the Hamiltonian which translates into the usual Lagrangian form of the duality invariance condition obeyed by Maxwell and Born-Infeld theories. We also discuss covariance properties of some analogous scalar models.

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The conditions for duality invariance of $D = 4$ vector gauge theories [1] and more generally of n-form models in $4n$ dimensions [2] are well known. Although the duality transformation generators, Ω, are necessarily defined canonically (rather than covariantly), as is the verification of invariance, namely commutation of Ω with the Hamiltonian, the invariance criterion is usually stated covariantly [3, 4] as a constraint on the Lagrangian. In the present note, we start canonically with an *a priori* purely 3–invariant formulation: Here Ω and the invariance requirement will be easy to find. Instead, the hard part will be to impose Lorentz invariance, thereby recovering the covariant criterion. Our ingredients are simple: (a) the known relation [5] between Lagrangian and (3+1) descriptions for any vector field action depending on $F_{\mu\nu}$ (but, not, for simplicity on derivatives of $F_{\mu\nu}$), (b) the classic Dirac–Schwinger local stress tensor commutator criterion [6] for Lorentz invariance of systems of spin ≤ 1.

Any gauge invariant second-order action, *i.e.*, one in which $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ are dependent variables,

$$I[A_\mu] = \int d^4x L(\alpha, \beta) \quad , \quad \alpha \equiv \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \quad , \quad \beta \equiv \frac{1}{4} F_{\mu\nu} \; {}^*F^{\mu\nu} \quad , \quad {}^*F^{\mu\nu} \equiv \frac{1}{2} \, \epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau} \quad , \tag{1}$$

has the equivalent first order form [2] [5],

$$\tilde{I}[F^{\mu\nu}, A_\sigma] = \int d^4x \tilde{L} = \int d^4x \left((L_\alpha F^{\mu\nu} + \frac{1}{2} L_\beta \; {}^*F^{\mu\nu})(\partial_\mu A_\nu - \partial_\nu A_\mu) + L - 2(\alpha L_\alpha + \beta L_\beta) \right) \tag{2}$$

where $F_{\mu\nu}$ and A_σ are to be varied independently. Here $\tilde{L}$ depends on A_μ *only* in the first term; the (α, β) only involve $F_{\mu\nu}$; subscripts on L mean differentiation with respect to α or β. The $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ relation emerges as the field equation from varying $F_{\mu\nu}$. Our conventions $\epsilon^{0123} = +1$, $\eta = (-, +, +, +)$ imply $\alpha = \mathbf{B}^2 - \mathbf{E}^2$, $\beta = -\mathbf{B} \cdot \mathbf{E}$ with $E^i \equiv F^{0i}$, $B^i \equiv \frac{1}{2} \, \epsilon^{ijk} F_{jk}$.

[2]The special cases for which $L - 2(\alpha L_\alpha + \beta L_\beta) = 0$, such as $L = \sqrt{a\alpha + b\beta} f(\frac{\alpha}{\beta})$, where f is arbitrary, has no Maxwell limit, but it can also be accommodated by use of Lagrange multipliers.

The Gauss constraint from varying A_0 implies that the "true" electric field variable conjugate to $\mathbf{A}$, $-\mathbf{D} \equiv 2L_\alpha \mathbf{E} + L_\beta \mathbf{B}$ is transverse, so that we get the gauge invariant canonical action

$$\tilde{I} = \int d^4x \left(-\mathbf{D}^T \cdot \dot{\mathbf{A}}^T - \tilde{H} \right) \quad , \tag{3}$$

$$\tilde{H}[\mathbf{D}^T, \mathbf{A}^T] = -2\mathbf{B}^2 L_\alpha - L + 2\alpha L_\alpha + \beta L_\beta \quad . \tag{4}$$

Defining a second potential $\mathbf{Z}$ for the transverse $\mathbf{D}$ field by $\mathbf{D} \equiv \nabla \times \mathbf{Z}$ puts $(\mathbf{D}, \mathbf{B})$ on a symmetric footing, as is even clearer if one defines [2] the doublet $\mathbf{A}^a \equiv (\mathbf{A}, \mathbf{Z})$, $\mathbf{B}^a \equiv (\mathbf{B}, \mathbf{D})$

$$\tilde{I} = \int d^4x \left(\frac{1}{2} \epsilon_{ab} \, \mathbf{B}^a \cdot \dot{\mathbf{A}}^b - \tilde{H}(\mathbf{B}^a) \right) \quad . \tag{5}$$

We have just seen how to recast a manifestly Lorentz invariant action (1) or its alternate form (2) to Hamiltonian form (5), whose (highly non-manifest) invariance is guaranteed by the precise form of the Hamiltonian $\tilde{H}$ in (4). Suppose instead that we begin with the (3+1) Hamiltonian form (5) without any such *a priori* Lorentz invariance properties. Instead, it describes a pair of 3–vectors $\mathbf{B}^a$ in a 3–invariant fashion, but in general does **not** correspond to a Minkowski invariant model. [Even in the simplest, Maxwell case, any explicit reconstruction would require turning the transverse $\mathbf{A}^T$ into a 4–vector, recognizing that $(\mathbf{B}, \mathbf{D})$ are parts of a 6–tensor, etc.] Fortunately, there is a direct Lorentz invariance criterion, for spin ≤ 1 systems, that requires knowledge only of the energy and momentum densities. While these quantities are not uniquely defined from their spatial integrals, it suffices to find an appropriate gauge invariant set. Clearly T^0_0 can be taken to be the Hamiltonian density $\tilde{H}(\mathbf{B}^a)$. As Dirac has taught us, the momentum of any system is dynamics-independent: $\mathbf{P} = \int d^3x \sum_i \pi^i(-\nabla\phi_i)$. In our case, we may therefore take the usual gauge invariant choice $T^{0i} = (\mathbf{D} \times \mathbf{B})^i = \frac{1}{2}(\epsilon_{ba}\mathbf{B}^a \times \mathbf{B}^b)^i$ whose integral is also $\mathbf{P}$; in the absence of gravitation, there is no unique choice for the densities. The Dirac–Schwinger [6] Lorentz invariance condition,

$$[T^{00}(\mathbf{r}),T^{00}(\mathbf{r}')]=\Big(T^{0i}(\mathbf{r})+T^{0i}(\mathbf{r}')\Big)\,\partial_i\delta^3(\mathbf{r}-\mathbf{r}')\ \ , \tag{6}$$

is to be computed through the canonical Poisson bracket (or commutation) relation $[B_i^a(\mathbf{r}),B_j^b(\mathbf{r}')]=\epsilon^{ba}\epsilon_{ijk}\partial^k\delta^3(\mathbf{r}-\mathbf{r}')$, with both sides transverse.

[It should be noted that (6) (or its half-integrated form) is an "on-shell" condition. Thus (6) can verify Lorentz covariance of Hamiltonian forms even if these do not have a simple "off-shell" covariant equivalent. An illustration is provided by the $D=2$ self-dual scalar field [8], $I=\int d^2x(\pi\dot{\phi}-\pi\phi')$ where $T^{00}=T^{01}=\pi\phi'$ and (6) is manifestly obeyed. However, there is no underlying $L((\partial_\mu\phi)^2)$ form.]

We are now in a position to first impose the duality (trivial) and then the Lorentz (nontrivial) invariance on our system (5). The generator Ω of $\mathbf{B}^a$ rotations, $[\Omega,\mathbf{B}_a]=\epsilon_{ab}\mathbf{B}^b$ is obvious,

$$\Omega=-\frac{1}{2}\int d^3x\,\mathbf{A}^a\cdot\mathbf{B}^b\delta_{ab}\ \ . \tag{7}$$

Equally obvious is the vanishing of its commutator with the $\frac{1}{2}\int\epsilon_{ab}\mathbf{B}^a\cdot\dot{\mathbf{A}}^b$ kinetic term. Finally, as advertised, the invariance of the Hamiltonian density is a triviality: $\tilde{H}$ can only depend on the two manifestly duality invariant combinations (u,v) of the three independent space scalars [3]

$$\tilde{H}=\tilde{H}(u,v)\ \ ,\quad (t,u,v)\equiv\left(\mathbf{B}^2\ ,(\mathbf{B}^a\cdot\mathbf{B}^a)\ ,\frac{1}{4}(\epsilon_{ab}\mathbf{B}^a\times\mathbf{B}^b)^2\right)\ \ . \tag{8}$$

The Lorentz invariant L depends on two 4–scalars (α,β), but of course neither necessarily implies the other. Indeed, the hard part is now to implement Lorentz invariance by (6). Note in

[3]This can be obtained more ploddingly from

$$[\tilde{H},\Omega]=\epsilon_{ab}\mathbf{B}^a\cdot\frac{\partial\tilde{H}}{\partial\mathbf{B}^b}=0\,,$$

and solving the ensuing differential equation for $\tilde{H}$.

this connection that the momentum density, being kinematical, is (like $\tilde{H}$) duality invariant, but is also independent of any assumed dynamics.

We find after some calculation that (6) constrains any $\tilde{H}(t,u,v)$ (dual or not) to obey

$$(\tilde{H}_u)^2 + u\tilde{H}_u\tilde{H}_v + v(\tilde{H}_v)^2 + \tilde{H}_t\tilde{H}_u + t\tilde{H}_t\tilde{H}_v = \frac{1}{4} \tag{9}$$

which reduces in our case, $\tilde{H}(u,v)$ to

$$(\tilde{H}_u)^2 + u\tilde{H}_u\tilde{H}_v + v(\tilde{H}_v)^2 = \frac{1}{4} \ . \tag{10}$$

This result already follows from the weaker, "half-integrated", $[T^{00}(\mathbf{r}), \tilde{H}] = \partial_i T^{0i}(\mathbf{r})$ version of (6) which is of course the Hamiltonian statement of the conservation requirement $\partial_\mu T^{0\mu} = 0$. (9) was also proposed in the present context, but from different considerations, some time ago in [7].

Clearly the *Ansätze* $\tilde{H} = \tilde{H}(u)$ and $\tilde{H} = \tilde{H}(u+v)$ yield the Maxwell and Born-Infeld solutions $\tilde{H}(u) = \frac{1}{2}u$, and $\tilde{H} = \sqrt{1+u+v} - 1$, respectively. [The overall $\tilde{H}$ normalization is forced by the kinetic terms.] To summarize, any $\tilde{H}$ depending only on u,v and obeying (10) defines a duality and Lorentz invariant model.

Going back to the full Lagrangian formulation, using our inputs (1–4) will yield the duality constraint in terms of the original second order $L(\alpha,\beta)$ of (1). The calculations are a bit tedious and we merely sketch the steps. Express u, v in terms of $(t \equiv \mathbf{B}^2, \alpha, \beta)$:

$$\begin{aligned} v &= 4L_\alpha^2(t^2 - \alpha t - \beta^2) \ , \\ u &= t(1 + 4L_\alpha^2 + L_\beta^2) - 4\alpha L_\alpha^2 - 4\beta L_\alpha L_\beta \ . \end{aligned} \tag{11}$$

Next write $(d\alpha, d\beta)$ in terms of (dt, du, dv) by solving for these differentials using (11). Using (4), namely $\tilde{H}(t,\alpha,\beta) = -2L_\alpha t - L + 2\alpha L_\alpha + \beta L_\beta$, rewrite $d\tilde{H}(t,\alpha,\beta)$ as $d\tilde{H}(t,u,v) = \tilde{H}_t dt + \tilde{H}_u du +$

$\tilde{H}_v dv$. But in this basis, $\tilde{H}_t = 0$, while our calculation yields

$$\tilde{H}_t = \frac{\beta(1 - 4L_\alpha^2 + L_\beta^2) + 2\alpha L_\alpha L_\beta}{2(2\beta L_\alpha - tL_\beta)} \quad . \tag{12}$$

Hence we have reproduced the covariant conditions of [3, 4]:

$$L_\alpha^2 - \frac{\alpha}{2\beta} L_\alpha L_\beta - \frac{1}{4} L_\beta^2 = \frac{1}{4} \; . \tag{13}$$

[As a consistency check, we note that (13) also implies $(\tilde{H}_u, \tilde{H}_v) = \left(\frac{\beta}{2}\, , \frac{-L_\beta}{4L_\alpha}\right)(tL_\beta - 2\beta L_\alpha)^{-1}$; substituting these into (10) shows that it is satisfied provided (13) holds.] In this form, the Maxwell and Born-Infeld solutions are $L = -\frac{1}{2}\alpha$ and $L = 1 - \sqrt{1 + \alpha - \beta^2}$, respectively.

An amusing parallel to our procedure arises for a massless scalar field whose (3+1) variables are (π, ϕ), with canonical Lagrangian $L = \pi\dot{\phi} - H(\pi^2, (\nabla\phi)^2)$. The Dirac covariance requirement is the familiar [4] equation

$$H_x H_y = \frac{1}{4} \quad , \quad (x, y) \equiv (\pi^2, (\nabla\phi)^2) \quad . \tag{14}$$

Particular solutions include $H = \frac{1}{2}(x + y)$, the free field ("Maxwell"), and $H = \sqrt{(1 + x)(1 + y)} - 1$, ("Born-Infeld"). Covariantly, $L = -\frac{1}{2}(\partial_\mu\phi)^2$ and $L = 1 - \sqrt{-det[\eta_{\mu\nu} + (\partial_\mu\phi)(\partial_\nu\phi)]} = 1 - \sqrt{1 + (\partial_\mu\phi)^2}$, respectively, as follows from the Legendre transform equivalent of (2): From a general $L(z)$, $z \equiv \frac{1}{2}(\partial_\mu\phi)^2$,

$$I[\pi^\mu, \phi] = \int d^4x \tilde{L} = \int d^4x \left(L' \pi^\mu(\partial_\mu\phi) + L - 2zL'\right) \; , \tag{15}$$

where π^μ and ϕ are to be varied independently, L is to be regarded as a function of z only, $'$ denotes differentiation with respect to z, and the $\pi_\mu = \partial_\mu\phi$ relation follows from varying π_μ.

Although we have explicitly worked with one–form potentials in $D = 4$, the same procedure can be applied to obtain the duality criteria for actions with $(2n+1)$–form potentials in $D = 4(n+1)$ spaces, using a spacetime decomposition with electric/magnetic $(2n + 1)$ forms. Unfortunately, the number of spacetime as well as spatial invariants grow so rapidly with dimension that the explicit steps become untractable beyond $D = 4$.

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