GEOPHYSICS, VOL. 57, NO. 7 (JULY 1992); P. 927-932, 4 FIGS.

Singular value decomposition of the velocity-reflector depth tradeoff, Part 1: Introduction using a two-parameter model

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ABSTRACT

A singular value decomposition (SVD) of a two-parameter model serves to introduce several characteristics of a raypath inversion of the standard reflection seismology recording geometry. Two important families of eigenvectors consist of constructive interference and destructive interference of velocity and reflector depth. The eigenvalue that corresponds to the velocity-reflector depth destructive interference is very sensitive to the maximum ray angle in the data. For a cable length equal to twice the reflector depth, the theoretical linear resolution is quite high. The relative weighting between velocity and reflector depth is not critical so long as the weight is near 1.0.

INTRODUCTION

The accurate resolution of reflector depth is an important objective of seismic data processing. Relative depths help geoscientists map structural shapes and determine optimum well locations. Absolute depths aid engineers in the drilling of wells.

Determining reflector depths from seismic data requires that the reflection time be converted to depth using the seismic wave velocity of the subsurface. In the absence of sufficient velocity information from nearby well measurements or geologic data, seismic data are used to determine the velocities.

A common practice for determining velocities from seismic data is common mid-point (CMP) stacking semblance analysis (Taner and Kohler, 1969). This approach is valid when the velocity field is laterally invariant over the width of the CMP gather.

Velocity analysis methods that address laterally variable velocity fields are currently under development (Bishop et al., 1985; Tarantola, 1986; Bording et al., 1987; Mora, 1987; Kennett et al., 1988; Stork and Clayton, 1991; Williamson, 1986; Sword, 1987; Fowler, 1988; Van Trier, 1990; Biondi, 1990; Etgen, 1990; van der Made, 1988; Sherwood et al., 1986; Julien et al., 1988). These procedures generally involve unraveling the signature from the velocity variations in data collected from different view angles through the field of interest. The unraveling is an inversion process that is sometimes called tomography.

A singular value decomposition (SVD) of the seismic reflection experiment can give insight into the resolution characteristics of the linear aspects of velocity analysis methods and the corresponding reflector resolution. SVDs with this objective are performed by Bube and Resnik (1984), Bishop et al. (1985), and in Part 2 of this paper. A similar approach was taken in Wiggins, Larner, and Wisecup (1976) to analyze the resolution of surface consistent reflection statics. Bickel (1990) and Toldi (1985) analyze the ambiguity between reflector depth and velocity for stacking velocities in the presence of velocity variations.

In this paper, I analyze the SVD of only a two-parameter model. This model, shown in Figure 1, contains a constant velocity and a flat reflector. It is illuminated by one CMP gather. The analysis of this model serves to introduce the SVD method, demonstrate two types of eigenvectors that represent important families of the more general multiparameter SVD, and describe relative weighting of parameters with different dimensionality.

We are especially interested in the ambiguity between velocity and reflector depth variations. With only vertical raypaths, velocity can be adjusted to compensate for a change of reflector depth. As a result, there is no net effect on the data and a complete ambiguity between velocity and reflector depth exists. However, with additional rays at different azimuths, this ambiguity becomes resolvable.

The results show, for a maximum offset equal to twice the reflector depth (maximum ray angle ≈ 45 degrees), the velocity reflector depth ambiguity corresponds to an eigenvalue of 0.10. This value indicates that the theoretical

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reflector depth resolution is approximately one tenth the resolution of the data times the square root of the data redundancy.

SETTING UP THE MATRIX FOR SVD

For the two-parameter model shown in Figure 1, consisting of a slowness variation Δs and reflector depth variation Δd the traveltime deviation of the kth ray Δt_k is expressed as:

$$\Delta t_k = \Delta s \cdot 2 \cdot \frac{d}{\cos(\theta_k)} + \Delta d \cdot 2 \cdot \cos(\theta_k) \cdot s + O(\Delta s \cdot \Delta d, \Delta s^2, \Delta d^2, \cdots), \tag{1a}$$

where

d = the depth of the reflector, and
θ_k = the angle from vertical of the kth ray.

We analyze only the linear problem and thus ignore the higher order terms of Δs and Δd. Equation (1a) is rewritten

$$\Delta t_k = \cos(\theta_k)^{-1} \cdot (2 \cdot d\Delta s) + \cos(\theta_k) \cdot w \cdot \left(2 \cdot \frac{s\Delta d}{w}\right), \tag{1b}$$

where w is a weighting factor for relating the reflector component to the velocity component. The terms in brackets are now in units of two-way traveltime, which will be the parameters used in the SVD analysis. Converting the eigenvectors back to their physical meaning will require taking into account the physical dimensions of the model. In matrix notation, this equation is

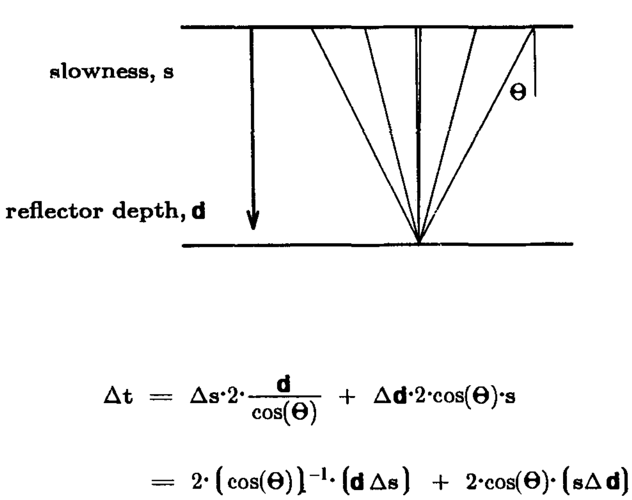


FIG. 1. Two-parameter model used for singular value decomposition (SVD). The model consists of a flat reflector at a depth d and a constant slowness s. The model is illuminated by a common midpoint (CMP) gather with maximum ray angle, Θ. The traveltime variations of rays are proportional to the cosine of the angle for reflector depth perturbations but proportional to the inverse of the cosine of the angle for slowness perturbations.

$$\Delta \mathbf{t} = \underset{\sim}{\mathbf{L}} \Delta \mathbf{s} \tag{1c}$$

where

$$\Delta \mathbf{s} = \begin{bmatrix} 2 \cdot d\Delta s \\ 2 \cdot \dfrac{s\Delta d}{w} \end{bmatrix},$$

$$\Delta \mathbf{t} = \begin{bmatrix} \Delta t_0 \\ \Delta t_1 \\ \cdot \end{bmatrix},$$

$$\underset{\sim}{\mathbf{L}}_{k0} = \cos(\theta_k)^{-1}, \text{ and}$$

$$\underset{\sim}{\mathbf{L}}_{k1} = w \cdot \cos(\theta_k).$$

The equation demonstrates that the resolution between velocity variations and reflector variations results from the path length of a ray through the velocity media being proportional to the inverse of the cosine of its angle from vertical, while the effect of the reflector depth is proportional to the cosine of this angle.

Data collected in this model are a ray set ranging in equal offset increments from vertical to some maximum angle θ_{max}, which approximates one CMP gather of a reflection survey.

The forward problem is set up according to Stork and Clayton (1991) by applying the weights $\underset{\sim}{\mathbf{D}}$ and $\underset{\sim}{\mathbf{S}}$ to equation (1c):

$$\underset{\sim}{\mathbf{D}}^{1/2}\Delta \mathbf{t} = (\underset{\sim}{\mathbf{D}}^{1/2}\underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{S}}^{1/2})(\underset{\sim}{\mathbf{S}}^{-1/2}\Delta \mathbf{s}), \tag{1d}$$

where

$$\underset{\sim}{\mathbf{S}}_{00}^{-1} = \sum_{k=1}^{n} \cos(\theta_k)^{-1} \text{ for the slowness cell; and}$$

$$\underset{\sim}{\mathbf{S}}_{11}^{-1} = \sum_{k=1}^{n} w \cdot \cos(\theta_k) \text{ for the reflector cell;}$$

$$\underset{\sim}{\mathbf{D}}_{kk} = (w \cdot \cos(\theta_k) + \cos(\theta_k)^{-1})^{-1}.$$

The square root of a diagonal matrix is defined as the matrix of the square root of its elements. We rewrite equation (1d)

$$\mathbf{d} = \underset{\sim}{\mathbf{A}} \cdot \mathbf{m}, \tag{1e}$$

where

$\mathbf{d} = \underset{\sim}{\mathbf{D}}^{1/2}\Delta \mathbf{t}$
$\mathbf{A} = \underset{\sim}{\mathbf{D}}^{1/2}\underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{S}}^{1/2}$, and
$\mathbf{m} = \underset{\sim}{\mathbf{S}}^{-1/2}\Delta \mathbf{s}$.

The weights $\underset{\sim}{\mathbf{D}}$ and $\underset{\sim}{\mathbf{S}}$ set the maximum eigenvalue of the $\underset{\sim}{\mathbf{A}}$ matrix to 1.0 and help correct for heterogeneous ray coverage (Stork and Clayton, 1991).

We seek the SVD (Lanczos, 1961) of the matrix $\underset{\sim}{\mathbf{A}}$.

SVD OF MATRIX $\underset{\sim}{A}$

We first compute the model eigenvectors by:

$$\underset{\sim}{A}^T\underset{\sim}{A} = \underset{\sim}{S}^{1/2}\underset{\sim}{L}^T\underset{\sim}{D}\underset{\sim}{L}\underset{\sim}{S}^{1/2} = \underset{\sim}{V}^T\underset{\sim}{\Lambda}^2\underset{\sim}{V} \quad (2a)$$

where

$\underset{\sim}{V}$ = the matrix of model space eigenvectors, and
$\underset{\sim}{\Lambda}$ = the diagonal matrix of the eigenvalues.

$\underset{\sim}{A}^T\underset{\sim}{A}$ is computed in the Appendix.

Using the results of the Appendix, when θ_{max} = 45 degrees:

$\underset{\sim}{S}_{00}^{-1} = 1.15 \cdot n$ for the slowness cell, and
$\underset{\sim}{S}_{11}^{-1} = 0.88 \cdot n \cdot w$ for the reflector cell,

where

n = the number of rays in the CMP gather used to illuminate the model.

We chose $w = 1.15/0.88 = 1.31$ so that $\underset{\sim}{S}_{ii}$ is the same for both the reflector cell and the slowness cell. This produces:

$$\underset{\sim}{A}^T\underset{\sim}{A} = \begin{bmatrix} 0.5057 & 0.4943 \\ 0.4943 & 0.5057 \end{bmatrix}.$$

The eigenvalues and eigenvectors of this matrix are:

First eigenvector: $(\sqrt{0.5}, \sqrt{0.5})$

first eigenvalue = 1.000,

Second eigenvector: $(\sqrt{0.5}, -\sqrt{0.5})$

second eigenvalue = 0.01147,

These eigenvalues are the square of the eigenvalues of $\underset{\sim}{A}$. The final eigenvalues (square roots) are (1.000, 0.107).

The first eigenvector corresponds to the constructive interference of traveltime from slowness and reflector depth. This eigenvector adjusts the velocity and reflector depth by equal amounts to match the two-way traveltime of the data. The second eigenvector corresponds to the destructive interference between slowness and reflector, or the velocity-reflector depth ambiguity. It has the smaller eigenvalue.

Data space eigenvectors corresponding to these eigenvalues are computed using

$$\underset{\sim}{D}^{1/2}\underset{\sim}{L}^T\underset{\sim}{S}\underset{\sim}{L}\underset{\sim}{D}^{1/2} = \underset{\sim}{A}\underset{\sim}{A}^T = \underset{\sim}{U}^T\underset{\sim}{\Lambda}\underset{\sim}{U}, \quad (2a)$$

where

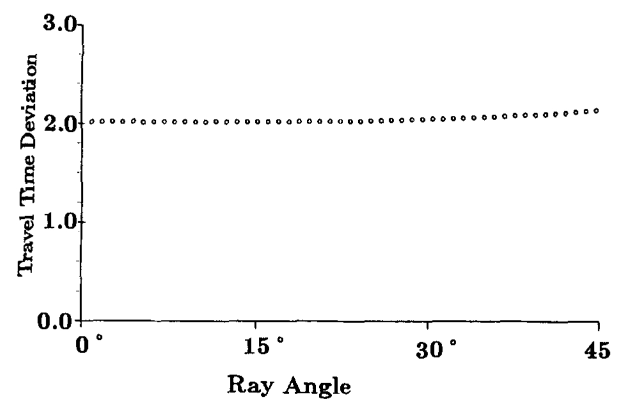
$\underset{\sim}{U}$ = the matrix of the data space eigenvectors.

Figure 2 shows the data space eigenvectors corresponding to these model space eigenvectors. These eigenvectors were derived numerically with the algorithm of Golub and Reinsch (1970) using one CMP gather with 50 offsets. The first data space eigenvector is essentially the average of the traveltimes in the data set corresponding to the constructive interference of slowness and reflector depth. The second data space eigenvector is the variation with offset of the data that is used to resolve the second eigenvector. The second eigenvector relates the hyperbolic moveout of the data to the correct reflector depth and velocity.

The magnitude of the eigenvalue of the second eigenvector is a measure of the resolution of the reflector depth. It is plotted as a function of the maximum ray coverage in the two-parameter model in Figure 3. As could be expected, the eigenvalue is a strong function of the maximum angle available, ranging from 0.1 at 45 degrees to 0.04 at 30 degrees. When accurate resolution of reflector depth is important, getting as much angular coverage as possible is vital.

RELATIVE WEIGHTING OF VELOCITY AND REFLECTOR DEPTH

The factor w determines the weighting between the velocity and the reflector. Earlier, a value of 1.31 was used for w to equally weight velocity and reflector. Had a different value for w been chosen, say $w = 0.1$, the $\underset{\sim}{A}$ matrix and its eigenvalues and eigenvectors would change to:



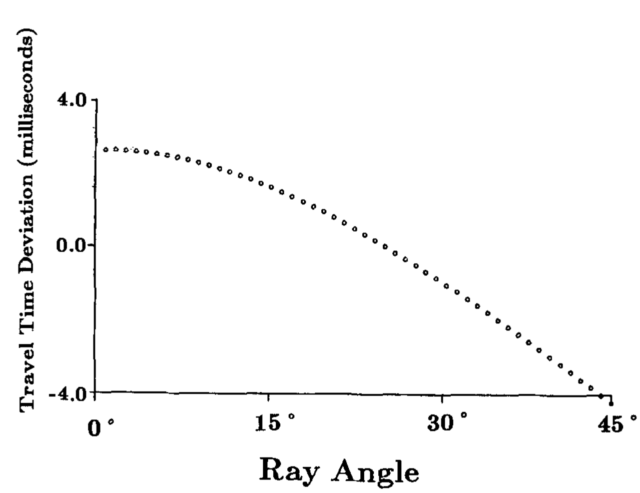


FIG. 2. The two data space eigenvectors of the two-parameter model with nonzero eigenvalues. The upper eigenvector has an eigenvalue of 1.0 and corresponds to the model space eigenvector (1.0, 1.0). The lower eigenvector has an eigenvalue of 0.1 (for Θ_{max} = 45 degrees) and corresponds to the model space eigenvector (1.0, −1.0).

$$\underset{\sim}{S}^{1/2}\underset{\sim}{L}^{T}\underset{\sim}{D}\underset{\sim}{L}\underset{\sim}{S}^{1/2} = n \cdot \begin{bmatrix} 0.9284 & 0.2566 \\ .2566 & 0.0741 \end{bmatrix}$$

The eigenvectors and eigenvalues of this matrix are:

First eigenvector: (0.963, 0.269) eigenvalue = 1.000,

Second eigenvector: (0.269, −0.963)

eigenvalue = 0.00297.

The square roots of these eigenvalues are (1.000, 0.054).

The eigenvectors are now unbalanced: the first one has a greater velocity component. This imbalance will bias the inversion toward placing traveltime deviations into slowness variations, a process observed in Stork (1988). The eigenvalue has also decreased as a result of unbalanced weighting. This effect is analyzed in Figure 4, where the magnitude of the smaller eigenvalue is plotted as a function of the factor w. The plot demonstrates that velocity-reflector depth ambiguity has the largest eigenvalue when the weight is about 1.3 for a vertical ray, which will produce an equal net weight for the reflector and velocity. The optimal weight is slightly greater than 1.0 to compensate for the rays off vertical, which are affected more by velocity variations and less by reflector depth variations.

The eigenvalue is not very sensitive to the relative weighting. As long as the weighting is close to equal, this parameter should not be a serious concern.

However, some proposed inversion methods for reflection tomography that do not take care with the weighting have effective weighting of less than 0.1.

VARIANCE ANALYSIS OF THE TWO-PARAMETER MODEL

Model resolution can be related to data resolution using the eigenvalue of the velocity-reflector depth ambiguity.

The covariance matrices of model and data are related by (Aki and Richards, 1980):

$$\langle \underset{\sim}{S}^{-1/2}\Delta s \Delta s^{T}\underset{\sim}{S}^{-1/2}\rangle = \sigma_{\Delta t}^{2}\underset{\sim}{D}\underset{\sim}{V}\underset{\sim}{\Lambda}^{-2}\underset{\sim}{V}^{T},$$

where

Δs = the model vector consisting of a slowness parameter and reflector depth parameter, and

$\sigma_{\Delta t}$ = the variance of the data, Δt.

Using the results from the two-parameter model analysis for θ_{max} = 45 degrees and w = 1.31, gives us

$$\langle \Delta s \Delta s^{T}\rangle = \frac{1}{2.31} \cdot \sigma_{\Delta t}^{2}\underset{\sim}{V}\underset{\sim}{\Lambda}^{-2}\underset{\sim}{V}^{T} \cdot \frac{1}{1.15 \cdot n},$$

where

$$\underset{\sim}{V}\underset{\sim}{\Lambda}^{-2}\underset{\sim}{V}^{T} = \frac{1}{2} \cdot \begin{bmatrix} \lambda_0^{-2} + \lambda_1^{-2}, & \lambda_0^{-2} - \lambda_1^{-2} \\ \lambda_0^{-2} - \lambda_1^{-2}, & \lambda_0^{-2} + \lambda_1^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 44.2 & -43.2 \\ -43.2 & 44.2 \end{bmatrix}.$$

By substituting and rearranging

$$\langle \Delta s \Delta s^{T}\rangle = \sigma_{\Delta t}^{2} \cdot \begin{bmatrix} 16.6 & -16.3 \\ -16.3 & 16.6 \end{bmatrix} \cdot \frac{1}{n}.$$

It is clear that the variance is dominated by the smaller eigenvalue.

The lower right entry of the matrix is the covariance of the reflector depth in two-way traveltime. Its resolution ($\sqrt{\text{variance}} \approx$ resolution) is approximated by:

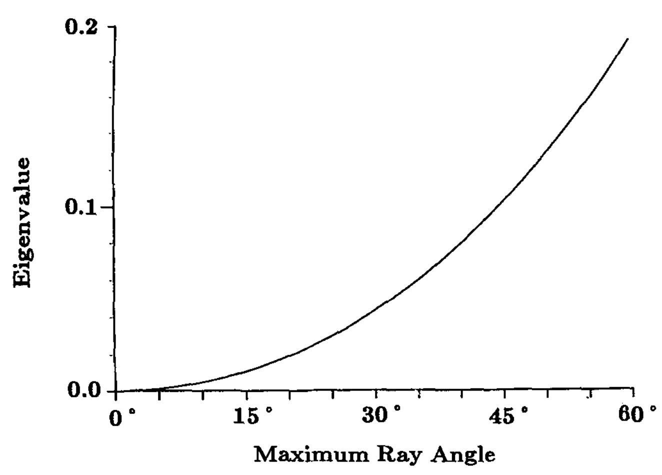


FIG. 3. The dependence of the smaller eigenvalue on the maximum ray angle of the CMP gather. The smaller eigenvalue is a description of how well ambiguous velocity and reflector depths can be resolved. The eigenvalue is a strong function of the maximum ray angle.

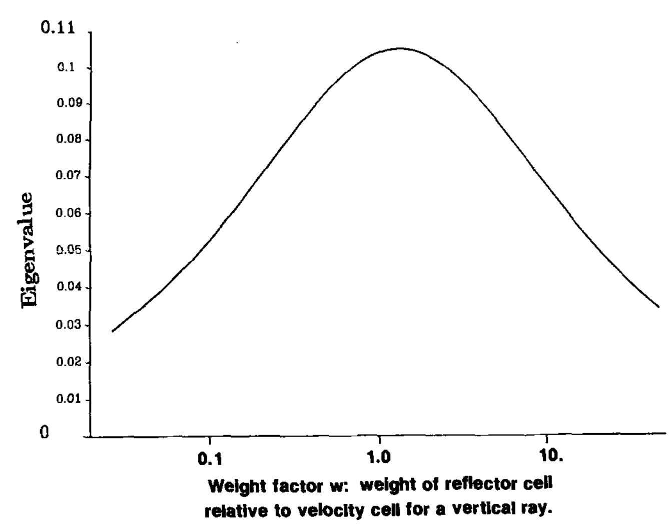


FIG. 4. The dependence of the smaller eigenvalue on the weight of the reflector parameter relative to the velocity parameter for a vertical ray. This is factor w used in the text. The smaller eigenvalue is not a strong function of the reflector weight so long that the weight is near 1.0.

$$\Delta s_{\text{reflector}} \approx \frac{4.0}{\sqrt{n}} \cdot \sigma_{\Delta t},$$

or

$$\Delta s_{\text{reflector}} \approx \frac{\sqrt{\lambda_0^{-2} + \lambda_1^{-2}}}{\sqrt{5.3} \cdot \sqrt{n}} \cdot \sigma_{\Delta t} \approx \frac{\lambda_{\min}^{-1}}{2.3 \cdot \sqrt{n}} \cdot \sigma_{\Delta t}.$$

A maximum ray angle of 45 degrees, a data resolution of 4 ms, and 100 independent data points should resolve a reflector depth to 3 ms, which is about 10 ft (3m) in 7000 ft/s (213 m/s) media. Resolution for other maximum ray angles will be proportional to the inverse of the eigenvalue for the slowness-reflector depth ambiguity. The predicted linear resolution of this variance analysis of the velocity-reflector depth ambiguity is theoretically quite high.

CONCLUSION

This analysis demonstrates that potential exists in the raw data for highly accurate resolution of reflector depth. However, other aspects, such as laterally variant velocities, anisotropy, three dimensionality, inelasticity, edge effects, and nonlinearities are probably more serious factors than data accuracy for velocity analysis.

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APPENDIX

NUMERICAL COMPUTATION OF EQUATION (2a)

The summations

$$\underset{\sim}{S}_{00}^{-1} = \sum_{k=1}^{n} \cos(\theta_k)^{-1},$$

$$\underset{\sim}{S}_{11}^{-1} = \sum_{k=1}^{n} w \cdot \cos(\theta_k),$$

and

$$(\underset{\sim}{L}^T \underset{\sim}{D} \underset{\sim}{L})_{ij} = \sum_{k=0}^{n} \ell_{ki} \cdot \underset{\sim}{D}_{kk} \cdot \ell_{kj}$$

are converted to integrals using the relationship

$$\sum_{k=1}^{n} g(\theta_k) = \int_{h=0}^{h_{\max}} dh \cdot g(\theta(h)) \cdot \frac{n}{h_{\max}},$$

where h is the offset corresponding to a ray with angle θ and n is the number of rays in a CMP gather.

A change of variables converts offset coordinates to angle coordinates:

$$\theta(h) = \tan^{-1}\left(\frac{h}{d}\right)$$

$$\frac{d\theta}{\cos^2(\theta)} = \frac{dh}{d},$$

where d is the depth to the reflector. Since we simulate a cable length equal to the reflector depth $h_{max} = d$.

The resulting integral for the reflector cell is

$$\underset{\sim}{S}_{11}^{-1} = \int_{h=0}^{h_{max}} dh \cdot w \cdot \cos(\theta(h)) \cdot \frac{n}{h_{max}}$$

$$= w \cdot \int_{\theta=0}^{\theta_{max}} d\theta \cdot \frac{\frac{d}{h_{max}}}{\cos(\theta)} \cdot \frac{n}{\tan(\theta_{max})}$$

$$= \frac{n}{\tan(\theta_{max})} w \cdot \ln\left|\frac{1}{\cos(\theta_{max})} + \tan(\theta_{max})\right|.$$

The integral for the slowness cell is

$$\underset{\sim}{S}_{00}^{-1} = \int_{h=0}^{h_{max}} \frac{dh}{\cos(\theta(h))} \cdot \frac{n}{h_{max}} = \int_{\theta=0}^{\theta_{max}} \frac{d\theta}{\cos^3(\theta)}$$

$$\cdot \frac{n}{\tan(\theta_{max})} = \frac{n}{\tan(\theta_{max})} \cdot \left(\frac{\tan(\theta_{max})}{2 \cdot \cos(\theta_{max})}\right.$$

$$\left. + \frac{1}{2} \cdot \ln\left|\frac{1}{\cos(\theta_{max})} + \tan(\theta_{max})\right|\right).$$

$$(\underset{\sim}{L}^T\underset{\sim}{L})_{00} = \int_{h=0}^{h_{max}} \frac{\cos(\theta(h))^{-2}\, dh}{w \cdot \cos(\theta(h)) + \cos(\theta(h))^{-1}} \cdot \frac{n}{h_{max}}$$

$$= \int_{\theta=0}^{\theta_{max}} \frac{\cos(\theta)^{-3}\, d\theta}{w \cdot \cos(\theta)^2 + 1} \cdot \frac{n}{\tan(\theta_{max})}.$$

$$(\underset{\sim}{L}^T\underset{\sim}{L})_{01} = w \cdot \int_{h=0}^{h_{max}} \frac{dh}{w \cdot \cos(\theta(h)) + \cos(\theta(h))^{-1}} \cdot \frac{n}{h_{max}}$$

$$= w \cdot \int_{\theta=0}^{\theta_{max}} \frac{\cos(\theta)^{-1}\, d\theta}{w \cdot \cos(\theta)^2 + 1} \cdot \frac{n}{\tan(\theta_{max})}.$$

$$(\underset{\sim}{L}^T\underset{\sim}{L})_{11} = \cdot w^2 \cdot \int_{h=0}^{h_{max}} \frac{\cos(\theta(h))^2\, dh}{w \cdot \cos(\theta(h)) + \cos(\theta(h))^{-1}} \cdot \frac{n}{h_{max}}$$

$$= \cdot w^2 \cdot \int_{\theta=0}^{\theta_{max}} \frac{\cos(\theta)\, d\theta}{w \cdot \cos(\theta)^2 + 1} \frac{n}{\tan(\theta_{max})}.$$

These integrals can be determined numerically.

For $\theta_{max} = 45$ degrees:

$\underset{\sim}{S}_{ii}^{-1} = 0.88 \cdot n \cdot w$ for the reflector cell

$\underset{\sim}{S}_{ii}^{-1} = 1.15 \cdot n$ for the slowness cell

and for $w = 1.31$:

$$\underset{\sim}{L}^T\underset{\sim}{D}\underset{\sim}{L} = n \cdot \begin{bmatrix} 0.5816 & 0.5684 \\ 0.5684 & 0.5816 \end{bmatrix}.$$

Combining the results produces our $\underset{\sim}{A}^T\underset{\sim}{A}$ matrix:

$$\underset{\sim}{S}^{1/2}\underset{\sim}{L}^T\underset{\sim}{D}\underset{\sim}{L}\underset{\sim}{S}^{1/2} = n \cdot \begin{bmatrix} 0.5057 & 0.4943 \\ 0.4943 & 0.5057 \end{bmatrix}.$$