DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

DOMINANT AND NASH STRATEGY MECHANISMS FOR THE ASSIGNMENT PROBLEM

Mark Olson

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Abstract

In this paper, I examine the problem of matching or assigning a fixed set of goods or services to a fixed set of agents. I characterize the social choice correspondences that can be implemented in dominant and Nash strategies when transfers are not allowed. This is an extension of the literature that was begun by Gibbard (1973) and Satterthwaite (1975), who independently proved that if a mechanism is nonmanipulable it is dictatorial. For the classes of mechanisms that are described in the paper, the results imply that the only mechanisms that are implementable in dominant and Nash strategies are choice mechanisms that rely only on ordinal rankings. I also describe a subclass of mechanisms that are Pareto optimal. In addition, the results explain the modeling conventions found in the literature — that when nontransfer mechanisms are studied individuals are endowed with ordinal preferences, and when transfer mechanisms are studied individuals are endowed with cardinal preferences.

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1. Introduction

Many allocation problems involve the assignment or matching of a set of individuals to a set of objects. This allocation problem appears in a variety of settings. A school administrator assigns office space to faculty members. A university computer group administers computer resources. A space agency schedules antenna time to spacecraft. All these allocation problems have two properties in common: the task of matching one group to another and the institutional feature that money is not used.

Most of the research on matching problems has concentrated on the two-sided matching problem or marriage problem made famous by Gale and Shapley (1962), (Roth and Sotomayor (1989) provide an extensive review). In two-sided matching problems, members of one group are to form partnerships with a member of another group and members of each group have preferences for members of the opposite group. A common example is the matching of sports teams and players.

In this paper, we examine the problem of matching or assigning a fixed set of goods or services, which we will generically call slots, to a fixed set of agents. We consider the problem from the point of view of a planner who wishes to design a set of rules or procedures (usually called a mechanism), the outcomes of which satisfy certain criteria. This problem is generally known as one-sided matching or the assignment problem. In one-sided matching problems, only one of the groups have preferences for members of the other group. An example of one-sided matching is the matching of office space and faculty members. The research on the one-sided matching problem has tended to propose rules and procedures and is generally found in the operations research literature. Two general classes of mechanisms are available to solve this problem: mechanisms that use transfers and mechanisms that do not use transfers. In this paper, we are concerned with mechanisms that do not use transfers.

We extend the literature that was begun by Gibbard (1973) and Satterthwaite (1975), who independently proved that if a mechanism is nonmanipulable it is dictatorial. A planner is interested in nonmanipulable rules since many rules require knowledge of individual preferences or information. These preferences are privately known and it may not be in the best interest of individuals to truthfully reveal their private information (or preferences) to the planner. That is, agents may be able to manipulate the outcome by misrepresenting their preferences. Manipulation may result in unsatisfactory outcomes; so the planner would like to know what mechanisms are manipulable and to what extent.

The Gibbard-Satterthwaite theorem (hereafter G-S) establishes that a voting scheme (mechanism) must be either manipulable or dictatorial when all possible transitive orderings over the set of alternatives are allowed, and the set of alternatives is finite. Results by Barbera and Peleg (1990), Moreno and Walker (1990), Satterthwaite and Sonnenschein (1981), and Zhou (1990b) establish the Gibbard-Satterthwaite result when the domain of preferences is restricted. Barbera and Peleg show that a G-S result holds when preferences are required to be continuous, and the set of alternatives is a metric space (not necessarily a subset of $\Re^n$). Moreno and Walker add the restriction that some dimensions of the social decision do not affect all the participants. Zhou establishes a G-S-type result in economies with public goods when the set of admissible preferences are continuous and convex. For domain restrictions that typically satisfy economic models Satterthwaite and Sonnenschein (1981) (hereafter SS) provide a result similar to the G-S result. The SS environment requires a condition called *nonbossy* for a serial dictatorship to hold. The results of these papers indicate that there may not exist "satisfactory" mechanisms without the use of monetary transfers or other incentive tax schemes. Common incentive schemes that do not use money are waiting in line, inspection, and punishment.

Our environment differs from the previous environments in two ways: 1) by the restriction on the allocation space, and 2) by the restriction on the domain of preferences. In the problem I pose, allocations that a planner may make are constrained by two requirements: a feasibility constraint (at most one slot is assigned to each agent), and the institutional requirement (incentive taxes such as "money" or waiting in line cannot be used to allocate slots, and lotteries over the alternatives are not allowed). This problem will be referred to as the one-sided matching problem (also known as the assignment problem).[1]

The domain of preferences is restricted by the assumption that individuals are selfish and that their utility does not depend on the slots allocated to others. This restricted domain of preferences does not satisfy the conditions for the proof of the G-S result. For instance, Barbera and Peleg's proof of the Gibbard-Satterthwaite theorem requires that preferences having a single best alternative are not excluded from the domain; in our environment there are no preferences that have a single best alternative and individuals are indifferent over large sets of outcomes. The matching environment differs from the environment constructed by Moreno and Walker, since allocations to

[1] For a more detailed discussion of the assignment problem, see Chapter 8 of Roth and Sotomayor (1989), or Shapley and Shubik (1972).

one individual can affect another individual (*e.g.*, if person 1 is assigned slot A, then 2 cannot be assigned this slot).

In the environment discussed in this paper, there are two ways an agent can behave "strategically." The first is by manipulating the outcome of a social choice function (SCF) and the second is by corrupting the outcome of a SCF. A SCF is manipulable if an agent can improve the outcome for himself by misrepresenting his true preferences, while a social choice function is corruptible if an agent can change the outcome to another agent without changing the outcome for himself. The ability of an agent to manipulate an outcome has been widely discussed and is the main condition that restricts the outcomes of the social choice functions of the papers cited above, while the ability of an agent to corrupt an outcome has received scant attention. Noncorruptible SCFs have been discussed by Ritz (1985) and Satterthwaite and Sonnenschein (1981), who call a noncorruptible mechanism nonbossy.[2]

The mechanisms that have been proposed in the literature to match agents to goods can be classified by the message space and the allocation rule (which may be a procedure or algorithm) used to make the match. There are two basic types of message spaces: ordinal ranking and cardinal ranking. In a mechanism (the message space and allocation rule) that uses an ordinal ranking message space, individuals are asked to submit a preference ranking over slots (*e.g.,* I like 1 better than 2, and 2 better than 3). In a mechanism that uses a cardinal ranking message space, individuals are requested to choose from a subset $C \subset \Re^k_+$ (*e.g.,* 100 points must be divided among the different slots).

I classify the procedures found in the literature into three categories based on their operational characteristics—positional, chit, and choice.[3] Positional mechanisms are mechanisms wherein the message space is an agent's ordinal ranking over slots. Chit mechanisms have a cardinal message space. Choice mechanisms allow the individuals to choose from an available set of slots.

Positional mechanisms are discussed first. In a positional mechanism each agent submits a ranking over slots. The planner (or central coordinator) places a numerical value on each ranking and then determines the outcome by maximizing (or minimizing)

[2] Ritz's definition of corruptible is more general than the SS definition of bossy; Ritz defines corruptible for choice correspondences; SS define bossy for direct mechanisms.

[3] This is not an exhaustive classification of all possible mechanisms, only of those commonly found in the literature.

an objective function defined on the numerical values. Various objective functions are possible. For instance, agents submit rankings[4] $r^i = (r_{i1}, \ldots, r_{ik})$, the most preferred good getting the highest number and so on, whereas the r's are chosen from a set of k specific weights $W = \{w_1, \ldots, w_k\}$. If $W = \{1, \ldots, k\}$, then this is similar to a Borda count. Given a submission of ranks, the assignment is determined by the allocation that maximizes the sum of weights. Gardenfors (1973) shows that assignments generated in this fashion satisfy conditions of neutrality, symmetry, unanimity, monotonicity, and Pareto optimality (see Gardenfors for definitions). In section 9 we show that for this class of mechanisms agents have incentives to misrepresent their ordinal rankings over slots.

A second type of assignment is determined by the allocation that minimizes the worst case (*e.g.,* an assignment is made to make the agent with the lowest ranked slot as high as possible). This procedure was described by Proll (1972) and Wilson (1977). Hylland and Zeckhauser (1979) report on a similar procedure used by Harvard administrators in 1977 to assign students to housing. The administrators first assigned students to their first choice if possible; they then assigned the remaining students to their second choice, and so on (this is called a bottleneck procedure). Hylland and Zeckhauser (1979) report that the Harvard administrators believed they observed students acting strategically. If students believed their first choice was first among many others, they might list their less popular second choice as first.[5]

A second class of mechanisms includes chit mechanisms. A chit mechanism is one wherein the message space allows each person to allocate a certain number of points (or chits) to any of the items which he wishes. The only difference between chit and positional mechanisms is the message space. Operationally, chit mechanisms use "funny money" (sometimes called chits) instead of money (exchangeable currency) as the medium of exchange. A ***chit*** is a medium of exchange whose value is determined solely in the context of the given assignment problem (environment), and has no value for goods or services outside the assignment problem. An example of a chit mechanism is the implicit market mechanism of Hylland and Zeckhauser (1979).

A third class of mechanisms includes choice mechanisms. They are similar to

[4] For some situations, when the number of slots is large, asking agents to submit rankings over all slots is impractical. Wilkonson (1972) suggests a solution where unranked slots are given a ranking one lower than the lowest ranked slot.

[5] See Hylland and Zeckhauser (1979), page 255, note 6.

positional mechanisms, since they only require an individual's rankings over slots. I consider them separately because they can be implemented by procedures that require individuals to choose slots from an available set, so the entire ranking does not need to be obtained. Examples of choice mechanisms are the deferred-acceptance procedure and serial-dictatorship mechanisms. The deferred-acceptance mechanism is based on the Gale-Shapley algorithm used to solve the marriage problem.[6] For the serial dictatorship an ordering of agents is chosen, the first agent chooses her slot, the second agent chooses her slot of those remaining, and so on. These mechanisms will be discussed in more detail in a later section.

Except for the choice mechanisms, none of the literature on matching mechanisms explicitly considers incentive problems.[7] The only result in the matching environment is by Zhou (1990a) who proved that when the number of agents is greater than 2, there exists no mechanism that satisfies symmetry, *ex ante* Pareto optimality, and strategy-proofness. The results presented in this paper characterize the class of social choice rules that can be implemented when nonstrategic behavior (behavior that is both nonmanipulable and noncorruptible) is a condition; the properties of these rules are then discussed.

In this paper we present four basic results: 1) nonstrategic rules must be ordinal; that is, an individual's assignment from the social choice function does not change when his ordinal preferences do not change; 2) nonstrategic social choice functions must be choice mechanisms; 3) the allocation space is rich (in the sense of Dasgupta, Hammond, and Maskin (1979), hereafter DHM), and hence a social choice function is implementable in dominant strategies if and only if it is implementable in Nash strategies; 4) a subclass of nonstrategic social choice functions called serial dictators are Pareto optimal.

For the classes of mechanisms that we described above, the results imply that the only mechanisms that are implementable in dominant and Nash strategies are choice mechanisms that rely only on ordinal rankings. The class of mechanisms we call chits are not implementable since they rely on cardinal information. In addition, the results explain the modeling conventions found in the literature—that when nontransfer mechanisms are studied individuals are endowed with ordinal preferences, and when

[6] See Roth and Sotomayor (1989) and the classic reference Gale and Shapley (1962).

[7] Hylland and Zeckhauser made the assumption that when there are many agents, each agent's contribution is small and hence there is no incentive to be dishonest.

transfer mechanisms are studied individuals are endowed with cardinal preferences. We will discuss these results in the section on applications.

This paper is divided into sections. In Section 2, we describe the formal model. In Section 3, strategic behavior is described. In Section 4, we show the equivalence of various notions of implementability for our environment. In Section 5, necessary conditions for dominant-strategy implementation are presented. In Section 6, the serial dictator is described and shown to characterize the class of nonstrategic rules. In Section 7, Nash implementation is presented. In Section 8, the optimality of implementable rules is presented. In Section 9, we discuss some of the results in the context of the categories of rules presented in the introduction. Finally, in Section 10, we make some concluding remarks.

2. Formal description of the model

The environment consists of n agents and k goods or services to be allocated, which we call slots. Let $N = \{1,\ldots,n\}$ index the set of agents, and let $K = \{1,\ldots,k\}$ index the set of slots. It is assumed that both N and K are finite and nonempty. Let $\mathcal{A}$ be the set of feasible (*i.e.,* at most, one slot may be assigned to each agent), deterministic allocations of K to N, including the zero allocation where no agent receives a slot.

Since agents either receive or do not receive a slot, an allocation in $\mathcal{A}$ can be denoted by a feasible allocation matrix of zeros and ones. That is, $a \in \mathcal{A}$ is an $n \times k$ matrix consisting of at most a single 1 in each row and column, where an element $a_{ij} = 1$ if agent i is assigned slot j, and $a_{ij} = 0$, if he is not. We also define $a^i = (a_{i1},\ldots,a_{ik})$.

We provide the following definition:

DEFINITION 2.0. An allocation $x \in \mathcal{A}$ is ***weakly feasible (WF)*** if $\sum_j x_{ij} \leq 1\ \forall i \in N$, $\sum_i x_{ij} \leq 1\ \forall j \in K$, and $\sum_i \sum_j x_{ij} \leq \min(k,n)$. If $n = k$, this definition reduces to the requirement that $\sum_j x_{ij} \leq 1$, $\forall i \in N$, and $\sum_i x_{ij} \leq 1$, $\forall j \in K$.

Efficiency and monotonicity of preferences will imply that either all slots are allocated or every agent is allocated a slot. The following definition is used:

DEFINITION 2.1. An allocation $x \in \mathcal{A}$ is ***strictly feasible (SF)*** if $\sum_j x_{ij} \leq 1 \ \forall i \in N$, $\sum_i x_{ij} \leq 1 \ \forall j \in K$, and $\sum_i \sum_j x_{ij} = \min(k,n)$. If $n = k$, this definition reduces to the requirement that $\sum_j x_{ij} = 1$, $\forall i \in N$, and $\sum_i x_{ij} = 1 \ \forall j \in K$.

The preferences of each agent depend upon the slot allocated and the agent's type. An agent's type parameterizes the value he places on the goods being allocated. Let $\Theta^i \subset \Re^k$ be a set of possible types for agent i, $\forall i \in N$. Let $\Theta^N = \underset{i \in N}{\times} \Theta^i$. A $\theta \in \Theta^N$ will be called a profile. The number of agents and slots is fixed, so the feasible set is independent of the profile. Each agent i, of type θ^i, evaluates each outcome $x \in \mathcal{A}$ through a valuation function $U(x, \theta^i) = \sum_j x_{ij} \theta^i_j$. The quantity $U(x, \theta^i)$ represents the willingness to pay of agent i of type θ^i for outcome x.

Agents may be indifferent between distinct outcomes since they are selfish; that is, they care only about the slots allocated to them. When lotteries are not allowed (the outcome space is $\mathcal{A}$), and agents are selfish, there is no loss of generality in the linear description of utility since there are a finite number of slots. That is, when agents are selfish and the outcome space is $\mathcal{A}$, then for any utility function $\hat{U}(x)$, there is a θ^i such that $U(x, \theta^i) = \hat{U}(x)$.

Although agents may be indifferent between outcomes, our results require that agents not be indifferent between slots. The following definition is used:

DEFINITION 2.2. A preference domain $\langle U, \Theta^N \rangle$ satisfies ***strict individual preferences (SIP)*** if $\forall i \in N$, $\forall \theta^i \in \Theta^i$, $U(m, \theta^i) \neq U(l, \theta^i)$ $\forall m \neq l \in K$; where $U(j, \theta^i) \equiv$ the utility to type θ^i of slot j. That is, each agent has strict preferences over slots. Given the definition of $U(\cdot)$, SIP holds if and only if $\theta^i_l \neq \theta^i_m$, $\forall m, l \in K$, $m \neq l$, $\forall i \in N$.

For every possible profile, the planner wishes to choose a single allocation from the set of feasible allocations; in addition, the planner requires these assignments to satisfy some criteria. That is, she wishes to implement a ***social choice function***[9] ***(SCF)***, $f: \Theta^N \mapsto \mathcal{A}$, that selects an outcome in $\mathcal{A}$ for every profile in Θ^N. Alternatively, we can describe a ***social choice correspondence (SCC)***, $f: \Theta^N \mapsto \mathcal{P}(\mathcal{A})$, which selects a nonempty subset of $\mathcal{A}$ for every profile in Θ^N, where $\mathcal{P}(\mathcal{A})$ denotes the power set of $\mathcal{A}$. A SCF is a single-valued SCC. In theorem A.1 in Appendix A, we show that in our environment we can restrict attention to SCFs.

Given a SCF, the planner must then choose a procedure or device to obtain these allocations. For example, she may ask agents to place numerical values between 0 and 1 on each slot and then make the assignment that maximizes the sum of the valuations. These procedures contain two parts, a message space and an outcome rule. The combination of message space and outcome rule is a game form, also called a mechanism. The planner chooses a mechanism to "implement" her choice of social choice rule. A SCF is implementable if there exists a game form (message space and outcome rule) such that the equilibria (under some appropriate solution concept) of this game corresponds to the outcomes of the social choice function.

We make two assumptions, as described by Palfrey (1990), about the planner's ability to implement a social choice rule with a mechanism: 1) the commitment assumption: The planner may commit to any feasible outcome rule, and he is committed to his choice; 2) the control assumption: The planner may choose any message space and the agents must communicate exactly one message from this message space and may not communicate with each other. We remark that if two different mechanisms can fully implement a SCF, each agent and the planner are indifferent between them.

There are a number of solution concepts that can be applied; in this paper we will be concerned with two solution concepts: dominant-strategy implementation and Nash implementation (Dasgupta, Hammond and Maskin (1979), and Maskin (1986) discuss these solution concepts in detail). The resulting mechanisms can be significantly different under the two solution concepts (*e.g.,* the divide-the-cake problem). Another solution concept is Bayesian implementation (see the discussion in Palfrey (1990), which also includes references to other solution concepts). Bayesian

[9] The literature often interchanges and confuses the terms social choice function, voting scheme, and mechanism. In this paper the term social choice function is used to describe the type of outcome or allocation that the planner may wish to obtain; a mechanism or voting scheme is a procedure or device to obtain allocations.

mechanisms for the assignment problem are explored in Olson (1991).

Dominant-strategy and Bayesian implementation are solution concepts that are consistent with the assumption of incomplete information, while Nash implementation requires complete information (for the agents but not the planner). Dominant-strategy implementation is more robust than Bayesian implementation since it is prior-independent. Bayesian implementation requires the stronger condition that the information structure is common knowledge, the designer knows the common prior, and Bayesian rationality among all players is common knowledge. Dominant-strategy implementation tends to be more stable than Nash implementation since there are fewer equilibria, but there are fewer instances where dominant-strategy implementation is possible. All the solution concepts have multiplicity problems under certain conditions, but the problem is least difficult under dominant-strategy implementation (see Mookerhjee and Reichelstein (1989), and Ledyard (1986)).

More formally: let (g,S) denote a mechanism (or game form), where $g{:}\, S \mapsto \mathcal{A}$, $S=(S^1,\ldots,S^n)$ and S^i is the strategy space (or message space) for agent $i \in N$. Let $E_g{:}\,\Theta^N \mapsto S$ be an equilibrium correspondence. A ***dominant-strategy equilibrium*** for profile $\theta \in \Theta^N$ of a mechanism (g,S) is an n-tuple of strategies $\tilde{s}=(\tilde{s}^1,\ldots,\tilde{s}^n) \in S$ such that $(\forall i \in N)(\forall s \in S)$ $(U(g(\tilde{s}^i,s^{-i}),\theta^i) \geq U(g(s),\theta^i))$. Let $DE_g(\theta) \subset S$ be the dominant-strategy equilibria for profile θ of mechanism (g,S). A ***Nash equilibrium*** for profile $\theta \in \Theta^N$ of a mechanism (g,S) is an n-tuple of strategies $\tilde{s}=(\tilde{s}^1,\ldots,\tilde{s}^n) \in S$ such that $(\forall i \in N)(\forall s^i \in S^i)$ $(U(g(\tilde{s}),\theta^i) \geq U(g(s^i,\tilde{s}^{-i}),\theta^i))$. Let $NE_g(\theta) \subset S$ be the Nash equilibria for profile θ of mechanism (g,S).

There are a number of different notions of implementation. The strongest is full implementation, which is applied to general game forms. A weaker concept is truthful implementation, which is defined for direct mechanisms. A direct mechanism is a mechanism in which the strategy space S^i for each agent $i \in N$ is the set of possible types Θ^i. In direct mechanisms, agents report their types (not necessarily their true types) to the planner, and then the planner makes an assignment based on these reported types.

Implementation in dominant strategies and implementation in Nash strategies are now defined.

DEFINITION 2.3. A SCF $f\colon \Theta^N \mapsto \mathcal{A}$ is ***implementable in Nash strategies*** if $(\forall\theta \in \Theta^N)(\forall a \in f(\theta))\exists(g,S)[g(NE_g(\theta)) \subset f(\theta)$ and $a \in g(NE_g(\theta))]$.

DEFINITION 2.4. A SCF $f\colon \Theta^N \mapsto \mathcal{A}$ is ***fully implementable in dominant strategies*** if there is a mechanism (g,S) such that $(\forall\theta \in \Theta^N)$ $[g(DE_g(\theta)) = f(\theta)]$.

DEFINITION 2.5. A SCF $f\colon\Theta^N \mapsto \mathcal{A}$ is ***truthfully implementable in dominant strategies*** if there is a direct mechanism $g\colon \Theta^N \mapsto \mathcal{A}$ such that $(\forall\theta \in \Theta^N)$ $[(\forall i \in N)(\forall\hat{\theta}^i \in \Theta^i)(\ U(g(\theta),\theta^i) \geq U(g(\hat{\theta}^i,\theta^{-i}),\theta^i)\ )$ and $g(\theta) \in f(\theta)]$.

The simplest and most direct means of implementing a SCF is to ask agents to report their type, then calculate from this information the assignment using the SCF. This is a particular type of direct mechanism, where the outcome rule is the SCF to be implemented. This notion of implementation was used in the G-S theorem and the other results mentioned in the introduction.

3. Strategic behavior

In this section we simplify the notion of implementation and use the concept of strategy-proof SCFs. A strategy-proof SCF truthfully implements itself in dominant strategies. In the next section I show that this is not a restriction in this environment if we are interested in full implementation. A SCF $f\colon \Theta^N \mapsto \mathcal{A}$ is ***strategy-proof*** if

$$(\forall\theta \in \Theta^N)(\forall i \in N)(\forall\hat{\theta}^i \in \Theta^i)\ [U(f(\theta),\theta^i) \geq U(f(\hat{\theta}^i,\theta^{-i}),\theta^i)].$$

A SCF $f\colon\Theta^N \mapsto \mathcal{A}$ is ***manipulable*** if for some $i \in N$, $(\exists\theta \in \Theta^N)(\exists\hat{\theta}^i \in \Theta^i)$ such that $U(f(\hat{\theta}^i,\theta^{-i}),\theta^i) > U(f(\theta),\theta^i)$. In this case we say i manipulates f at θ with $\hat{\theta}^i$. If a SCF is strategy-proof, then an agent is not able to improve his allocation (be assigned a more preferred slot) by lying about his type to the planner. This restriction reduces the strategic options to the agent and hence the possible allocations. This form of strategic behavior has been studied quite extensively (Muller and Satterthwaite (1986) provide a good review).

A second form of strategic behavior, which has received very little attention[10], is the ability of an agent to change another agent's allocation without changing his own. Mechanisms with this property are labeled ***bossy***, by Satterthwaite and Sonnenschein

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[10] I have found only two references in the literature to this concept, Satterthwaite and Sonnenschein (1981) and Ritz (1985).

(1981), and labeled ***corruptible*** by Ritz (1985). A rule is bossy (or corruptible) if an agent can maintain her allocation at the same time she causes changes in the allocations that other agents receive. Satterthwaite and Sonnenschein do not consider whether "nonbossiness is a reasonable or desirable condition to require of a mechanism." Although we also require a mechanism to be nonbossy, in our environment it is a reasonable requirement.

In the context of this paper we refer to SCFs as being corruptible and to mechanisms as being bossy. We define these concepts formally:[11]

DEFINITION 3.1. The SCF $f\colon\Theta^N \mapsto \mathcal{A}$ is ***noncorruptible (NC)*** if $(\forall\theta \in \Theta^N)$ $(\forall i, j \in N)(\forall\tilde{\theta}^j \in \Theta^j)[f^j(\theta) = f^j(\theta^{-j},\tilde{\theta}^j) \Rightarrow f^i(\theta) = f^i(\theta^{-j},\tilde{\theta}^j)]$. The SCF $f\colon\Theta^N \mapsto \mathcal{A}$ is ***corruptible*** if $(\exists\,\theta \in \Theta^N)(\exists\, i, j \in N)(\exists\,\hat{\theta}^j \in \Theta^j)[f^j(\theta) = f^j(\theta^{-j},\hat{\theta}^j) \Rightarrow f^i(\theta) \neq f^i(\theta^{-j},\hat{\theta}^j)]$.

DEFINITION 3.2. The mechanism (g,S) $g\colon S \mapsto \mathcal{A}$ is ***bossy*** if $(\exists\, s \in S)$ $(\exists\, i, j \in N$ and a $\hat{s}^j \in S^j)[g^j(s) = g^j(s^{-j},\hat{s}^j) \Rightarrow g^i(s^{-j},\hat{s}^j) \neq g^i(s^{-j},\hat{s}^j)]$. The mechanism (g,S) $g\colon S \mapsto \mathcal{A}$ is ***nonbossy*** if it is not bossy. An agent $i \in N$ is said to be bossy if she can change the outcome for some agent $j \in N$.

We combine the two notions of strategic behavior and say that a SCF is ***nonstrategic*** if it is both noncorruptible and strategy-proof. As an example to see the existence of noncorruptible, strategy-proof mechanisms, observe the following example:

EXAMPLE 3.3. Let $n = k = 3$, and let $\Theta^i = \{\text{A,B,C,D,E,F}\}$ $\forall i \in N$. Define the allocation rule:
$f^1(C,\theta^2,\theta^3) = 1,\ f^2(C,\theta^2,\theta^3) = 2,\ f^3(C,\theta^2,\theta^3) = 3;$
$f^1(D,\theta^2,\theta^3) = 1,\ f^2(D,\theta^2,\theta^3) = 3,\ f^3(D,\theta^2,\theta^3) = 2,\ \ \forall\theta^2,\theta^3,$
where $f^i() = j$ denotes that agent i receives slot j. Agent 1 is bossy, since by changing his type from C to D, he changes the allocation to agents 2 and 3 but not to himself. The mechanism is strategy-proof for agent 1 since he always receives slot 1 (we make the assumption that when an agent is indifferent between being truthful and misrepresenting his type, he will be truthful), and for agents 2 and 3, since they cannot affect the outcome of the mechanism. Observe that the conditions of noncorruptibility and strategy-proofness are true for whatever meaning we give to the types θ^i as long as

[11] The definitions of noncorruptible and nonbossy vary slightly from SS and Ritz but are consistent with their usage in the matching environment.

the individuals are selfish. □.

When a mechanism is strategy-proof and nonbossy, then an agent cannot improve his position directly by manipulating the outcome. But when a mechanism is strategy-proof and bossy, an agent may be able to improve his position indirectly by taking a "bribe" from the other agents. In the example above, agent 1 may be able to induce either agent 2 or 3 to pay him to choose in their favor.If a SCF is corruptible, the planner's problem of predicting outcomes becomes more difficult. The planner must model (try to predict) the behavior of agents who may be able to bribe or coerce another agent. If a SCF is noncorruptible, then the task of predicting behavior, and hence the the outcome of a mechanism, is simpler. In the language of implementation theory, the addition of the noncorruptibility restriction reduces the possible equilibria of a mechanism.

4. Implementation

It is well known that if a SCF is implementable, then it is truthfully implementable; that is, there exists a direct mechanism such that truth-telling is an equilibrium. The converse is not always true. However, in the Appendix, we show that in the one-sided matching environment, if a SCF is truthfully implementable and noncorruptible, then it is fully implementable in dominant strategies. This equivalence allows us to restrict the planner's matching problem to direct mechanisms. We also show that full implementation is equivalent to strategy-proofness, when SCFs are noncorruptible, which allows us to restrict the problem even further to self-implementable SCFs.

We formally state the theorem, which is proven in the Appendix.

THEOREM 4.1. If preferences satisfy SIP, and if a SCF is noncorruptible, then strategy-proofness, truthful implementation, and full implementation in dominant strategies are all equivalent.

Proof: See Appendix A. □.

In this section we have shown that there is no loss of generality by focussing on SCFs rather than on SCCs and on strategy-proofness rather than on full implementation in dominant strategies. The ability to reduce the class of feasible mechanisms (or implementable SCCs) represents a significant reduction in the complexity of the problem. This reduction has been achieved by making two assumptions. The first assumption is a restriction on the domain of preferences; we assume that individuals have strict preferences over slots and that they are selfish. The second assumption is a restriction on the strategic behavior that is allowed; we enforce the requirement that SCFs are noncorruptible and strategy-proof.

5. Dominant-strategy implementation (ordinality)

In this section we describe a necessary condition for implementation in dominant strategies. We first investigate a necessary condition in our environment for a SCF to be truthfully implementable in dominant strategies. From DHM we know that a SCF is truthfully implementable in dominant strategies if and only if it satisfies independent person-by-person monotonicity (IPM) (Maskin (1986), and DHM (1979)). First we provide some preliminary definitions.

DEFINITION[12] **5.1.** A SCF $f: \Theta^N \mapsto \mathcal{A}$ satisfies ***independent person-by-person monotonicity (IPM)*** if $\forall \theta \in \Theta^N$, $\forall i \in N$, $\forall \bar{\theta}^i \in \Theta^i$, and $\forall \{a, \mathrm{b}\} \subseteq \mathcal{A}$ such that $a \in f(\theta)$, and $U(a, \bar{\theta}^i) > U(b, \bar{\theta}^i)$, it must be that $b \notin f(\theta^{-i}, \bar{\theta}^i)$.

DEFINITION 5.2. A ***rank function*** is a function $r(\theta^i) = (r_1(\theta^i), \ldots, r_k(\theta^i))$, $r: \Theta^i \mapsto \prod\{r_1, r_2, \ldots, \mathrm{r}_k\}$, such that $r_j(\theta^i) > r_l(\theta^i)$ if and only if $U(j, \theta^i) > U(l, \theta^i)$; where $U(j, \theta^i) \equiv$ the utility to type θ^i of slot j, and $\prod\{A\}$ is the set of all permutations of the set A.

DEFINITION 5.3. A SCF is ***individually ordinal*** if $\forall \theta \in \Theta^N$, $\forall i \in N$, $\forall \bar{\theta}^i \in \Theta^i$, such that $r(\theta^i) = r(\bar{\theta}^i)$, then $f^i(\theta) = f^i(\theta^{-i}, \bar{\theta}^i)$, where f^i is the allocation to agent i. A SCF is ***ordinal*** if $\forall \theta \in \Theta^N$, $\forall i \in N$, $\forall \bar{\theta}^i \in \Theta^i$, such that $r(\theta^i) = r(\bar{\theta}^i)$, then $f(\theta) = f(\theta^{-i}, \bar{\theta}^i)$.

A SCF is individually ordinal if an agent's assignment does not change when his ordinal preferences do not change. An SCF is ordinal if the group assignment does not change when an agent's ordinal preferences do not change. If a SCF is ordinal, then it is individually ordinal, but if a SCF is individually ordinal, it is not necessarily ordinal. For an individually ordinal SCF to be ordinal it must also be noncorruptible. To prove

[12] This is not the definition of IPM that appears in DHM (1979), but it is the definition given in Laffont and Maskin (1982), and Maskin (1986).

our result we show that if a SCF satisfies IPM, then it is individually ordinal. We state the theorem below and prove it in Appendix B.

THEOREM 5.4. If preferences satisfy SIP, if $\mathcal{A}$ is the allocation space, and if a SCF can be truthfully implemented in dominant strategies, then the SCF is individually ordinal. In addition, if the SCF is noncorruptible then it is ordinal.
Proof. See Appendix B. □.

We have shown that an ordinal condition is necessary for IPM when preferences satisfy SIP, and when the allocation space is $\mathcal{A}$, but is it sufficient? The answer is no. We give an example of a SCF that is ordinal but cannot be truthfully implemented in dominant strategies.

EXAMPLE 5.5. Let $n = k = 3$, and $r\colon \Theta^i \mapsto \prod\{r_1, r_2, r_3\}$, be the rank function, where $r_1 = 1.0$, $r_2 = 0.5$, and $r_3 = 0$. Let $f(\theta) = \underset{x \in \mathcal{A}}{\operatorname{argmax}} \sum_i \sum_j r_j(\theta^i) \cdot x_{ij}$; ties are resolved by giving the lower indexed agent his preferred slot, and x is strictly feasible.[13] Let $U(x,\theta^i) = \sum_j x_{ij}\theta^i_j$, and $\theta^i = (\theta^i_1, \theta^i_2, \theta^i_3) \in \Re^3$.

Since f depends on θ only through the rank function r, f is ordinal. We will show that truth is not a dominant strategy for some θ. For our example we will define 2 types: type A, $\theta(\mathrm{A}) = (1.0, 0.5, 0)$ and type B $\theta(\mathrm{B}) = (0, 1.0, .5)$.

Two allocations of the SCF f are $f(\mathrm{A},\mathrm{A},\mathrm{B}) = (1,3,2)$, and $f(\mathrm{A},\mathrm{B},\mathrm{B}) = (1,2,3)$, where $f(i,j,k)$ is the allocation if agent 1 is type i, agent 2 is type j, and agent 3 is type k, $f(\cdot) = (l,m,n)$ is the allocation of l to agent 1, m to agent 2, and n to agent 3.

When $\theta^{-2} = (1,3)$, a report of type A, by agent 2, gives him slot 3. A report of type B gives agent 2 slot 2. So when agent 2 is a type A, he is better off reporting a type B, which gives him his 2^{nd} ranked slot 2, instead of his 3^{rd} ranked slot 3. □.

In summary, we have proven that if a SCF is truthfully implementable in dominant strategies, then it is individually ordinal, and if in addition, it is noncorruptible, then it is ordinal.

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[13] This is a an example of a positional mechanism.

6. Serial dictator

In the previous sections we showed that if a SCF is implementable in dominant strategies then it is ordinal and that there is no loss of generality in requiring strategy-proofness. Hence, we will restrict ourselves to finding strategy-proof ordinal mechanisms.

Satterthwaite and Sonnenschein (1981) established a Gibbard-Satterthwaite-type theorem in a classical economic environment (alternatives are a compact and convex subset of $\Re^l$). Their result relies on a differentiable allocation mechanism and a number of other technical conditions. The environment constructed for the assignment problem (without lotteries) lacks convexity, and we do not require a mechanism to be differentiable. Hence, the results of this paper do not fall into the class of environments established by Satterthwaite and Sonnenschein, although we obtain a similar result.

Satterthwaite and Sonnenschein (1981, p. 588) describe a serial dictatorship[14] as follows:

> Serial dictatorship means that the mechanism consists of one or more hierarchies of agents where the highest ranking agent in each hierarchy selects his allocation from a feasible set that is exogenously given, the second highest ranking agent selects his allocation from a feasible set that depends on the first agent's choice, the third highest ranking agent selects his allocation from a feasible set that depends on the first and second agents' choices, etc. Consequently, an agent who is high on a hierarchy is a dictator to those agents lower on that hierarchy in the sense that he can affect what is available to them to choose among and they can not affect him reciprocally. He is not, however, necessarily a dictator in the stronger senses of being able to choose any technologically feasible outcome for himself and being able to impose particular outcomes on the other agents.

The serial dictator is a member of the class of ***sequential-choice mechanisms***. A sequential choice mechanism is a mechanism where there is an ordering (or hierarchy) of agents, which may depend on the profile. Given this ordering, each agent in turn is allowed to choose their slot from an option set, which is a nonempty set of slots presented to them. These option sets have the feature that each agent's decision affects only the option sets of those agents that are lower on the hierarchy. Associated with each sequential-choice mechanism is an ordering $I(\theta) = \{i_1, \ldots, i_n\} \in \mathbb{P}(N)$, which may vary with θ, and where $\mathbb{P}(N)$ is the set of permutations of agents. In the serial mechanism the option set to agent i_1 is K, the option set to i_2 is a subset of K, and so

[14] An early reference to a serial dictatorship is found in Luce and Raiffa (1957, p. 344), who observe that a serial dictatorship "is consistent with all of Arrow's conditions except nondictatorship."

on. That is, each agent has her turn to choose slots from a set of slots, whose elements are affected by those agents who are placed before her in the ordering.

The following definitions are used formally to define a sequential-choice mechanism and the serial dictator:[15]

DEFINITION 6.1. For a given direct mechanism $x:\Theta^N \mapsto \mathcal{A}$, agent i ***affects*** agent j at $\theta \in \Theta^N$ if $(\exists \bar{\theta}^i \in \Theta^i) \ni x^j(\theta) \neq x^j(\bar{\theta}^i, \theta^{-i})$. We write this as $i\,A(\theta)j$.

DEFINITION 6.2. For a given direct mechanism $x:\Theta^N \mapsto \mathcal{A}$, $A(\theta)$ is ***acyclic*** at $\theta \in \Theta^N$ if $\forall (i_1, i_2, \ldots, i_n) \in \mathbb{P}(N)$ $\quad i_1 A(\theta) i_2, i_2 A(\theta) i_3, \ldots, i_{n-1} A(\theta) i_n \Rightarrow i_1 A(\theta) i_n$. $I(\theta) = \{i_1, \ldots, i_n\}$ is the ordering that is induced by the acyclic mechanism.

DEFINITION 6.3. A direct mechanism $x:\Theta^N \mapsto \mathcal{A}$ is a ***sequential choice mechanism*** if $\forall \theta \in \Theta^N$, $A(\theta)$ is acyclic.

DEFINITION 6.4. $o(i,\theta) = \{a \in \mathcal{A} \mid \exists \bar{\theta}^i \ni a = f(\theta^{-i}, \bar{\theta}^i)\}$; $\quad O(i,\theta) = \{l \in K \mid \exists \bar{\theta}^i \ni l = f^i(\theta^{-i}, \bar{\theta}^i)\}$.

Given $\theta^{-i} \in \Theta^{N-i}$, $o(i,\theta)$ is the set of agent i's options at profile θ that she can receive by deviating her messages, and $O(i,\theta)$ is the set of slots that agent i can receive at profile θ by deviation her messages. Note: Both $O(i,\theta)$ and $o(i,\theta)$ do not depend on θ^i and clearly $(\forall \theta \in \Theta^N)[f(\theta) \in o(i,\theta)]$ and $O(i,\theta) = o^i(i,\theta)$.

DEFINITION 6.5. A direct mechanism $x:\Theta^N \mapsto \mathcal{A}$ is a ***serial dictator*** if $\forall \theta \in \Theta^N$, $A(\theta)$ is acyclic and $O(i_n, \theta) \subset O(i_{n-1}, \theta) \subset O(i_{n-2}, \theta) \subset \cdots \subset O(i_1, \theta) = K$.

This definition of a serial dictator is broader than the description in the introduction. That description describes a ***simple serial dictator*** wherein an ordering $I = \{i_1, i_2, \ldots, i_n\}$ of agents is fixed and does not depend on the profile. Given this fixed ordering, agents choose their slots in turn: i_1 goes first, then i_2 chooses her slot from those remaining, and so on. The definition provided here (and also in SS) allows the ordering of agents to vary with $\theta \in \Theta^N$. The main theorem and an important lemma are stated below:

[15] These definitions follow Satterthwaite and Sonnenschein (1981).

THEOREM 6.6. If preferences satisfy SIP, $f\colon \Theta^N \mapsto \mathcal{A}$ is strategy-proof and noncorruptible; then for all $\theta \in \Theta^N$, $A(\theta)$ is acyclic.
Proof. See Appendix C. □.

LEMMA 6.6d (Asymmetry). If preferences satisfy SIP and $f\colon \Theta^N \mapsto \mathcal{A}$ is strategy-proof and noncorruptible; then $(\forall \theta \in \Theta^N)(\forall i, j \in N)\ iA(\theta)j \Rightarrow$ either:

1) $o(i,\theta) = o(j,\theta)$, or 2) $\sim jA(\theta)i$.

Proof. See Appendix C. □.

In the previous lemma we showed that an agent i affects another agent j then agent j cannot affect agent i except in one instance. In that instance agent i and agent j both have the same option set and can affect each others outcome. This case describes the nature of the ordering that is induced by a sequential choice mechanism. When both agents can affect each other at a profile the relative ordering of agent i and j does not affect the nonstrategic outcome at that profile.

7. Nash Implementation

We now investigate to see whether the use of Nash equilibria can expand the class of implementable SCFs. We do this by applying the result by DHM which states that if the environment is rich, then a SCF implementable in Nash strategies is truthfully implementable in dominant strategies. We show that our domain of preferences is rich[16] when the allocation space is $\mathcal{A}$.

DEFINITION 7.1. A class $\langle U, \Theta^i \rangle$ of utility functions is ***rich*** if $\forall$ pairs $\{\theta^i, \hat{\theta}^i\} \subset \Theta^i$, and $\forall \{a, b\} \subset \mathcal{A}$, such that
i) $U(a,\theta^i) \geq U(b,\theta^i) \Rightarrow U(a,\hat{\theta}^i) \geq U(b,\hat{\theta}^i)$, and ii) $U(a,\theta^i) > U(b,\theta^i) \Rightarrow U(a,\hat{\theta}^i) > U(b,\hat{\theta}^i)$;
there exists a $\bar{\theta}^i \in \Theta^i$, such that $\forall\ c \in \mathcal{A}$,
a) $U(a,\theta^i) \geq U(c,\theta^i) \Rightarrow U(a,\bar{\theta}^i) \geq U(c,\bar{\theta}^i)$, and b) $U(b,\hat{\theta}^i) \geq U(c,\hat{\theta}^i) \Rightarrow U(b,\bar{\theta}^i) \geq U(c,\bar{\theta}^i)$.

PROPOSITION 7.2. If preferences satisfy SIP then the domain of preferences is rich.
Proof. See Appendix D. □.

[16] Rich is also known as monotonically closed. DHM (1979) and Laffont and Maskin (1982) provide discussions of this concept.

When our domain of preferences is rich, we can apply the following result from DHM (theorem 7.2.3). If the domain of preferences $\langle U, \Theta^i \rangle$ is rich $\forall i \in N$, then if a SCF is implementable in Nash equilibrium, it is truthfully implementable in dominant strategies. This result follows since for rich domains and single-valued choice functions, monotonicity implies independent, weak monotonicity (IWM), which implies independent person-by-person monotonicity (IPM). If we also add the requirement that the SCF is noncorruptible, then it is fully implementable in dominant strategies. In addition, by applying DHM (theorem 7.1.1), if a SCF is truthfully implemented in dominant strategies, then it is truthfully implemented in Nash strategies. It is not necessarily true that a SCF that is fully implemented in dominant strategies is fully implemented in Nash strategies.

Therefore, the Nash solution concept does not allow us to implement more SCFs than the dominant-strategy solution concept. Furthermore, even if we use the Nash solution concept, we can fully implement only those SCFs that are ordinal. This does not imply that if we use a cardinal SCF, there are no Nash strategies, but that there are additional equilibria that do not result in the implementation of the SCF.

8. *Optimality of Implementable Rules*

In this section we discuss the optimality properties of implementable rules. A minimal requirement for optimality of a SCF is Pareto optimally (PO). If an allocation is not PO, then either there is an agent who can be made strictly better off by taking the surplus, or there are at least two agents who can be made strictly better off by trading.

DEFINITION 8.1. A SCF $f: \Theta^N \mapsto \mathcal{A}$ is ***Strong Pareto Optimal*** **(SPO)** if $\forall \theta \in \Theta^N$, and all $a \in f(\theta)$, there does not exist a $b \in \mathcal{A}$, such that $\forall i \in N$, $U(b, \theta^i) \geq U(a, \theta^i)$ and for some $j \in N$, $U(b, \theta^j) > U(a, \theta^j)$.

In general, sequential-choice mechanisms are not Pareto optimal. However, the serial dictator can be readily seen to be strong Pareto optimal, since every agent is matched with her most preferred slot in the available set. It is readily seen that if a SCF is SPO then it must be strictly feasible. This allows us to restrict attention to the set of strictly feasible (SF) allocations if we want to restrict ourselves to PO allocations.

A strictly feasible allocation may not be PO, but if it is not strictly feasible, then it is not PO.

In previous sections we restricted agents' preferences over slots to be strict. This is different from the restriction of strict preferences used in DHM and in much of the dominant-strategy literature. In most implementation papers preferences are strict over outcomes. In our case, agents are selfish and are indifferent between allocations that give them the same slot but give other agents different slots. Because of this indifference, many of the results of DHM and others are not applicable.

DHM show that when the preference domain is rich and consists of strict preferences, if a SCF satisfies citizen sovereignty (CS) and IPM, then the SCF is weak Pareto optimal and we obtain similar results, but for our environment we also require that the SCF be noncorruptible. We include a definition of CS and state the proposition which is proven in Appendix E.

DEFINITION 8.2. A SCF $f:\Theta^N \mapsto \mathcal{A}$ satisfies ***citizen sovereignty (CS)*** if $\forall a \in \mathcal{A}^*$, $\exists\, \theta \in \Theta^N$, such that $a \in f(\theta)$, where $\mathcal{A}^* \equiv \{a \in \mathcal{A} \mid a \text{ is strictly feasible}\}$. That is, the mapping f is onto the set of strictly feasible allocations.

PROPOSITION 8.3. If preferences satisfy SIP, then if a SCF is noncorruptible, satisfies IPM, and CS, it is strong Pareto optimal (SPO).[17]

Proof. See Appendix E. □.

The following is an example wherein a mechanism satisfies strategy-proofness and noncorruptibility but not CS.

EXAMPLE 8.4. Let $N = \{1,2,3\}$, $K = \{1,2,3\}$. Let f be the SCF that is implemented by the following mechanism:

Let $O^1(\cdot) = \{2,3\}$: the option set available to agent 1.

Let $O^2(\cdot) = \{2,3\}$: the option set available to agent 2.

Let $O^3(\cdot) = \{1\}$: the option set available to agent 3.

and $I(\theta) = \{1,2,3\}\ \forall\theta$, the order that agents choose slots.

[17] If we remove the requirement that preferences satisfy SIP then the SCF is weak Pareto optimal; where a SCC $f:\Theta^N \mapsto \mathcal{A}$ is ***Weak Pareto Optimal (WPO)*** if $\forall\theta \in \Theta^N$, and all $a \in f(\theta)$, there does not exist a $b \in \mathcal{A}$, such that $\forall i \in N,\ U(b,\theta^i) > U(a,\theta^i)$.

That is, agent 1 selects first from the set $\{2,3\}$, agent 2 selects second from the remainder of set $\{2,3\}$, and agent **3** receives slot 1. This mechanism satisfies SP, since I doesn't depend on θ, and is noncorruptible (it is also nonconstant), but it does not satisfy CS.
Let $\theta^1 = 1>2>\mathbf{3}$, $\theta^2 = 1>2>\mathbf{3}$, and $\theta^3 = \mathbf{3}>2>1$; then $a = f(\theta)$. But if $b = (1,2,\mathbf{3})$, then $U(b,\theta^i) > U(a,\theta^i)\ \forall i$. So f is not Pareto optimal (strong or weak). □.

The results of this section indicate that if a SCF is strategy-proof and noncorruptible and satisfies citizen sovereignty, then it is Pareto optimal.

9. Application of results to mechanisms

Given the results in the previous sections, what can we say about the types of mechanisms that can be implemented? In the introduction we described three categories of mechanisms that have been proposed in the literature to "solve" the one-sided matching problem. These were positional, chit, and choice mechanisms. In the previous sections we have proven some results that allow us to draw some conclusions about the ability to predict behavior in these mechanisms.

We can readily apply the results of the previous sections to determine the implementability in dominant and Nash-strategy equilibrium of chit and choice mechanisms. Since the outcome of chit mechanisms, as described in the introduction, can be affected by changing cardinal information, we cannot make dominant-strategy-equilibrium predictions, or Nash-strategy-equilibrium predictions. Since a necessary condition for implementation in dominant strategies is that the outcome can only be affected by a change in ordinal information, chit mechanisms are not dominant strategy mechanisms. Choice mechanisms have been shown to be implementable in dominant strategies and if the mechanism is a serial dictator, it is also Pareto optimal. We now discuss the last class of mechanisms—positional mechanisms.

We begin by formally describing a positional mechanism: a mechanism $x: S \mapsto \mathcal{A}$ is ***positional*** if there are strictly monotonic weighting functions $w^i: S^i \mapsto \{w_1, \ldots, w_n\}$, $w_i \in \Re$, $\forall i \in N$, and $x(s)$ maximizes the function $\sum_{ij} x_{ij} w^i(s^i)$, where ties are broken arbitrarily. If a SCF is implemented by positional mechanism, the SCF can be easily seen to be symmetric, and Pareto optimal, but it cannot be implemented in dominant strategies as the following proposition shows.

PROPOSITION 9.1. If a SCF can be implemented by a positional mechanism, then it is not implementable in dominant strategies.

Proof. Without loss of generality let $N = K$. Suppose that the SCF $f: \Theta^N \mapsto \mathcal{A}$ is implementable and let $\theta \in \Theta^N$, such that $\theta^i = \theta^j, \forall i, j \in N$, and $\theta^i_1 > \theta^i_2 > \cdots > \theta^i_n$; θ is an allowable profile in the matching environment. Since f is positional, any strictly feasible allocation $a \in \mathcal{A}$ will maximize $x(a, \theta) = \sum_{i,j} a_{ij} w(\theta^i_j)$, as long as $w(\cdot)$ is strictly monotonic. Without loss of generality let $a = f(\theta) = \{1, \ldots, n\}$; that is, agent i is assigned slot i, $a^i = i$. So $U(a, \theta^1) = \theta^1_1$, $U(a, \theta^2) = \theta^2_2$, and so on.

Let $\hat{\theta}^3$ be such that $\hat{\theta}^3_2 > \hat{\theta}^3_l$, $\forall l \neq 2$. Then $x(a, (\hat{\theta}^3, \theta^{-3}))$ is maximized by $a^3 = 2$, and $a^2 = 3$; this implies that $2A(\theta)3$.

Let $\hat{\theta}^2$ be such that $\hat{\theta}^2_3 > \hat{\theta}^2_l$, $\forall l \neq 3$. Then $x(a, (\hat{\theta}^2, \theta^{-2}))$ is maximized by $a^2 = 2$, and $a^3 = 3$; this implies that $3A(\theta)2$.

But since f is implementable, there is an affects relation $A(\theta)$ induced by f, and the asymmetry property of the affects relation $A(\theta)$ implies that $2A(\theta)3$ and $\sim 3A(\theta)2$, so f cannot be implementable.

□.

The only class of procedures with a dominant strategy prediction is the class of choice mechanisms. An example of one of these mechanisms is the serial dictator, which is Pareto optimal but not symmetric, unless there is a random selection of order.

10. Conclusion

There have been a number of procedures proposed in the literature to "solve" the one-sided matching problem. Almost all of these procedures assume an agent's behavior is nonstrategic. If we assume that an agent's behavior reflects his own best interest, strategic behavior is likely. The importance of understanding an agent's strategic behavior reflects the importance of the planner's ability to determine the outcome of an allocation mechanism. The more that can be said about an agent's strategic behavior, the more precise can be the planner's prediction of the outcome of an allocation mechanism.

Determining an agent's strategic behavior can be quite complex, but in the one-sided matching problem there does exist a class of SCCs whose outcomes are easily predicted. This is the class of noncorruptible and strategy-proof SCFs. In this paper we were able to show that the class of strategy-proof and noncorruptible SCFs does not exclude any SCCs that can be implemented in dominant strategies. But most importantly, we were able to characterize the class of implementable SCFs, that is, those SCFs that are strategy-proof and noncorruptible. We found that SCFs must rely only on ordinal information to be implementable and the Nash solution concept does not enlarge the class of implementable SCFs. We also found that the only implementable SCFs were sequential choice mechanisms, and that a particular member of this class, the serial dictator, was also Pareto optimal.

11. Appendix A

In appendix A, we show that in our environment we can restrict attention to SCFs and that we can restrict attention to strategy-proofness. We show in theorem A.1 that in the one-sided matching environment, which we describe, a nonstrategic SCC is single-valued and hence there is no loss in restricting the matching problem to SCFs.

THEOREM A.1. If preferences satisfy SIP, then if a SCC is noncorruptible and fully implementable in dominant strategies, it is single-valued.

Proof. Suppose $f: \Theta^N \mapsto \mathcal{A}$ is not single-valued; then for $\theta \in \Theta^N$, let $a, b \in f(\theta)$, $a \neq b$. Let (g, S) fully implement f in dominant strategies, $\exists\, \tilde{s}, s \in S(\theta)$, such that $g(\tilde{s}) = b$, and $g(s) = a$. $\tilde{s}^1$ and s^1 are both dominant strategies for θ^1, so $U(g(\tilde{s}^1, s^{-1}), \theta^1) = U(g(s^1, s^{-1}), \theta^1)$. SIP implies that $g^1(\tilde{s}^1, s^{-1}) = g^1(s^1, s^{-1}) = a^1$; non-corruptibility implies that $g^i(\tilde{s}^1, s^{-1}) = g^i(s^1, s^{-1}) = a^i$, $\forall i \in N$; hence $g(\tilde{s}^1, s^{-1}) = g(s^1, s^{-1}) = a$. Similarly, $g(\tilde{s}^1, \tilde{s}^2, s^{-1,2}) = g(\tilde{s}^1, s^{-1}) = a$. Continuing iteratively, $g(\tilde{s}) = a$, which is a contradiction, so f is single-valued. □.

We prove the equivalence of strategy-proofness, truthful implementation, and full implementation, by first proving three lemmas and then combining them into the theorem.

LEMMA A.2. If preferences satisfy SIP, then if a SCF is noncorruptible and truthfully implementable in dominant strategies, it is fully implementable in dominant strategies.

Proof. Let $f: \Theta^N \mapsto \mathcal{A}$ be truthfully implemented in dominant strategies by $g^*: \Theta^N \mapsto \mathcal{A}$. Define $f^*: \Theta^N \mapsto \mathcal{A}$ so that $\forall \theta \in \Theta^N$, $f^*(\theta) = g^*(E_{g^*}(\theta))$, where $E_{g^*}(\theta)$ is the set of dominant-strategy equilibria for preference profile θ in game form g^*. Now by construction, g^* fully implements f^*. Therefore, $f^*(\theta)$ must be a singleton $\forall \theta \in \Theta^N$. But $g^*(\theta) \in f(\theta)$; therefore, $f^*(\theta) \subseteq f(\theta), \forall \theta \in \Theta^N$. Since f is single-valued, $f^*(\theta) = f(\theta), \forall \theta \in \Theta^N$, so g^* fully implements f. □.

We apply two well-known results. 1) If a SCF is strategy-proof, then it truthfully implements itself in dominant strategies. 2) If a SCF is fully implementable in dominant strategies, then it is truthfully implementable in dominant strategies.

LEMMA A.3. If preferences satisfy SIP, and if a SCF is noncorruptible and strategy-proof then, it is fully implementable in dominant strategies.
Proof. Let $f\colon\Theta^N \mapsto \mathcal{A}$ be strategy-proof and noncorruptible. Since f is strategy-proof, it truthfully implements itself in dominant strategies. Since f is also noncorruptible, then by the previous lemma it is fully implementable in dominant strategies. □.

LEMMA A.4. If preferences satisfy SIP, and if a SCF is truthfully implementable in dominant strategies and noncorruptible, then it is strategy-proof.
Proof. Let $f\colon\Theta^N \mapsto \mathcal{A}$ be truthfully implementable in dominant strategies and noncorruptible. Since f is truthfully implementable, $\exists\, g\colon\Theta^N \mapsto \mathcal{A}$ such that $\forall\theta \in \Theta^N$, $\forall i \in N$, $\forall\hat{\theta}^i \in \Theta^i$, $U(g(\theta),\theta^i) > U(g(\hat{\theta}^i,\theta^{-i}),\theta^i)$, $g(\theta) \in f(\theta)$. Since f is truthfully implementable and noncorruptible, it is fully implementable and hence it is single-valued, so $g(\theta) = f(\theta)$, $\forall\theta \in \Theta^N$. Therefore, $U(f(\theta),\theta^i) \geq U(f(\hat{\theta}^i,\theta^{-i}),\theta^i)$, so f is strategy-proof. □.

THEOREM 4.1. If preferences satisfy SIP, and if a SCF is noncorruptible, then strategy-proofness, truthful implementation, and full implementation in dominant strategies are all equivalent.
Proof: Apply previous propositions. □.

12. Appendix B

Our first result relies on the following lemma.

LEMMA B.1. If preferences satisfy SIP, and if a SCF $f\colon\Theta^N \mapsto \mathcal{A}$ satisfies IPM, $a \in f(\theta)$, $c \in f(\overline{\theta}^i,\theta^{-i})$, and $a^i \neq c^i$, then $U(a,\overline{\theta}^i) < U(c,\overline{\theta}^i)$ and $U(c,\theta^i) < U(a,\theta^i)$. (remark: if $a^i = c^i$, then equality holds).
Proof. Suppose our hypothesis holds; then
1) If $a \in f(\theta)$ and $U(a,\overline{\theta}^i) > U(c,\overline{\theta}^i)$, then IPM implies $c \notin f(\overline{\theta}^i,\theta^{-i})$.
2) If $c \in f(\overline{\theta}^i,\theta^{-i})$ and $U(c,\theta^i) > U(a,\theta^i)$, then IPM implies $a \notin f(\theta^i,\theta^{-i})$.
Both 1) and 2) contradict the hypothesis". This implies:

$$U(a,\overline{\theta}^i) \leq U(c,\overline{\theta}^i) \text{ and } U(c,\theta^i) \leq U(a,\theta^i). \tag{B.1}$$

But by individual strict preferences and $c^i \neq a^i$, it must be that $U(a,\overline{\theta}^i) < U(c,\overline{\theta}^i)$ and $U(c,\theta^i) < U(a,\theta^i)$. □.

In the previous lemma if preferences do not satisfy SIP then $U(a,\overline{\theta}^i) \leq U(c,\overline{\theta}^i)$ and $U(c,\theta^i) \leq U(a,\theta^i)$.

The next two propositions establish that an ordinal condition is necessary for implementation in dominant strategies when the allocation space is $\mathcal{A}$.

PROPOSITION B.2. If preferences satisfy SIP, $\mathcal{A}$ is the allocation space, and if a SCF satisfies IPM, then it is individually ordinal.

Proof. Let $f: \Theta^N \mapsto \mathcal{A}$ be a SCF. Suppose that the hypothesis holds but that f is not individually ordinal. Then for some $i \in N$, $\theta^i, \overline{\theta}^i \in \Theta^i$, where $r(\theta^i) = r(\overline{\theta}^i)$, there exists $a, b \in \mathcal{A}$ such that $a \in f(\theta)$, and $b \in f(\overline{\theta}^i, \theta^{-i})$. If $a^i \neq b^i$, then by lemma B.1, $U(a,\overline{\theta}^i) < U(b,\overline{\theta}^i)$ and $U(b,\theta^i) < U(a,\theta^i)$. But this is a contradiction since θ^i, $\overline{\theta}^i$ have the same ordinal preferences over slots and a, b allocate a single slot to agent i. It must be that either a is preferred to b for both θ^i, $\overline{\theta}^i$, or b is preferred to a for both θ^i, $\overline{\theta}^i$. $\square$.

THEOREM 5.4. If preferences satisfy SIP, if $\mathcal{A}$ is the allocation space, and if a SCF can be truthfully implemented in dominant strategies, then the SCF is individually ordinal. In addition, if the SCF is noncorruptible then it is ordinal.

Proof. By DHM (theorem. 4.3.1), a SCF can be truthfully implemented in dominant strategies if and only if it is IPM. By applying the previous proposition, the first result follows. For the second result, since the SCF is noncorruptible and individually ordinal, it is ordinal. $\square$.

13. Appendix C

We provide some further definitions:[18]

DEFINITION 13.1. For $\theta^i \in \Theta^i$, $x \in \mathcal{A}$, a $\hat{\theta}^i \in \Theta^i$ is a ***reshuffling of θ^i around x*** if $(\forall y \in \mathcal{A})[U(x,\theta^i) \geq U(y,\theta^i) \Leftrightarrow U(x,\hat{\theta}^i) \geq U(y,\hat{\theta}^i)]$. $r(\theta^i,x)$ denotes the set of all reshufflings of θ^i around x, and a $\hat{\theta}^i \in r(\theta^i,x)$ is a reshuffle of θ^i around x.

A reshuffling of θ^i around x is another preference ordering such that x preserves the same ordinal rankings relative to all other alternatives. Observe: if SIP holds, $x,y \in \mathcal{A}$, and $\hat{\theta}^i \in r(\theta^i,x)$; then $[U(y,\theta^i) = U(x,\theta^i) \Leftrightarrow U(y,\hat{\theta}^i) = U(x,\hat{\theta}^i)]$. Some special preference relations will be used. Let ${}^x\theta^i$ be the preferences obtained from θ^i when x is ranked first, all other ordinal preferences remaining the same; and let ${}_x\theta^i$ be the preferences obtained from θ^i when x is ranked last, all other ordinal preferences remaining the same. Let ${}^x\Theta^i$ be the set of preferences that rank x first.

For a direct mechanism $f{:}\,\Theta^N \mapsto \mathcal{A}$, define $\mathcal{A}_f =$ range of f. If $Y \subseteq \mathcal{A}$, then Y^i is the set of slots $Y^i \subseteq K$ that are obtained by i in Y.

DEFINITION 13.2. For $Y \subseteq \mathcal{A}, \theta^i \in \Theta^i$, let

$$C(\theta^i, Y) = \{a \in Y \mid U(a,\theta^i) \geq U(b,\theta^i) \forall b \in Y\}$$

be the choice of agent θ^i in the set Y of allocations, and let

$$C^i(\theta^i, Y) = \{a^i \in Y^i \mid U(a,\theta^i) \geq U(b,\theta^i) \forall b \in Y\}$$

be the set of the best slots obtained by i in the set of allocations Y. If SIP holds, then $C^i(\theta^i, Y)$ is a singleton.

The main theorem is stated below:

THEOREM 6.6. If preferences satisfy SIP, $f{:}\Theta^N \mapsto \mathcal{A}$ is strategy-proof and noncorruptible; then for all $\theta \in \Theta^N$, $A(\theta)$ is acyclic.

The result of Theorem 1. is that for each $\theta \in \Theta^N$, $A(\theta)$ is acyclic; this permits the hierarchies of serial dictators to vary as $\theta \in \Theta^N$. This is a similar result to SS. Before I prove the theorem I prove some lemmas that will be used in the proof.

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[18] A number of concepts from Barbera (1983) are used; Barbera proved the Gibbard-Satterthwaite theorem by a pivotal-voter technique. A similar technique is used in this paper. A mechanism is ***pivotal*** at θ if $\exists \hat{\theta}^i \ni x(\theta^{-i}, \hat{\theta}^i) \neq x(\theta)$. Clearly, if i affects j at θ, then i is pivotal at θ.

LEMMA C.0. For $f: \Theta^N \mapsto \mathcal{A}$, if f is noncorruptible and $(\forall \theta \in \Theta^N)(\forall i \in N)$ $(\forall \hat{\theta}^i \in \Theta^i)[f(\theta) \neq f(\theta^{-i},\hat{\theta}^i) \Rightarrow f^i(\theta) \neq f^i(\theta^{-i},\hat{\theta}^i)]$.

Proof. Suppose $f: \Theta^N \mapsto \mathcal{A}$ satisfies the hypothesis of the lemma. Then $f(\theta) \neq f(\theta^{-i},\hat{\theta}^i)$ implies either a) $f^i(\theta) \neq f^i(\theta^{-i},\hat{\theta}^i)$ or b) $f^j(\theta) \neq f^j(\theta^{-i},\hat{\theta}^i)$ or both. If a) is true then the conclusion is true. If b) is true then noncorruptibility implies $f^i(\theta) \neq f^i(\theta^{-i},\hat{\theta}^i)$ and the conclusion is true. □.

LEMMA 6.6a. If preferences satisfy SIP and $f: \Theta^N \mapsto \mathcal{A}$ is strategy-proof and noncorruptible, then $(\forall \theta \in \Theta^N)(\forall i \in N)[f(\theta) = C(o(i,\theta),\theta^i)]$.

Proof. Suppose not. Let $z = f(\theta)$ and $x \in C(o(i,\theta),\theta^i)$, $x \neq z$. By definition of $o(\cdot)$, $\exists \hat{\theta}^i \ni f(\hat{\theta}^i,\theta^{-i}) = x$ and $z = f(\theta) \in o(i,\theta)$, so $f(\hat{\theta}^i,\theta^{-i}) \neq f(\theta)$. Since f is noncorruptible, by lemma 0, it must be that $x^i \neq z^i$. Since $x,z \in o(i,\theta)$ and $x \in C(o(i,\theta),\theta^i)$, SIP $\Rightarrow U(x,\theta^i) > U(z,\theta^i) \Rightarrow U(f(\theta^{-i},\hat{\theta}^i),\theta^i) > U(f(\theta),\theta^i)$. Hence f is manipulable at θ by i, a contradiction. □.

The above lemma says that for every profile, the outcome must be the best option at that profile for each one of the agents. Without the noncorruptible condition the result of the lemma is $f(\theta) \in C(o(i,\theta),\theta^i)$, and if in addition SIP holds, then $f^i(\theta) = C^i(o(i,\theta),\theta^i)$.

LEMMA 6.6b. If preferences satisfy SIP, $f: \Theta^N \mapsto \mathcal{A}$ is strategy-proof and noncorruptible, then $(\forall \theta \in \Theta^N)(\forall i \in N)[\hat{\theta}^i \in r(\theta^i,f(\theta)) \Rightarrow f(\theta^{-i},\hat{\theta}^i) = f(\theta)]$.

Proof. Suppose not. Then for some $\theta \in \Theta^N$, $i \in N$, $\hat{\theta}^i \in r(\theta^i,f(\theta))$, $f(\theta^{-i},\hat{\theta}^i) \neq f(\theta)$. Since f is noncorruptible, by lemma 0., it must be that $f^i(\theta^{-i},\hat{\theta}^i) \neq f^i(\theta)$. By SIP, either

a) $U(f(\theta^{-i},\hat{\theta}^i),\theta^i) > U(f(\theta),\theta^i)$, or

b) $U(f(\theta),\theta^i) > U(f(\theta^{-i},\hat{\theta}^i),\theta^i)$.

By definition of $r()$ $\hat{\theta}^i \in r(\theta^i,f(\theta)) \Rightarrow [U(f(\theta),\theta^i) > U(y,\theta^i) \Leftrightarrow U(f(\theta),\hat{\theta}^i) > U(y,\hat{\theta}^i)$, $\forall y \in \mathcal{A}$, $y^i \neq f^i(\theta)]$.

If a) is true, then a*) $U(f(\theta^{-i},\hat{\theta}^i),\hat{\theta}^i) > U(f(\theta),\hat{\theta}^i)$.

If b) is true, then b*) $U(f(\theta),\hat{\theta}^i) > U(f(\theta^{-i},\hat{\theta}^i),\hat{\theta}^i)$.

But a) and b*) contradict strategy-proofness, hence $f(\theta^{-i},\hat{\theta}^i) = f(\theta)$. □.

The above lemma states that no agent can change the outcome at a profile by changing his preferences to a reshuffle around this outcome. If f is corruptible then the conclusion of the above lemma is $f^i(\theta^{-i},\hat{\theta}^i) = f^i(\theta)$.

LEMMA 6.6c. If preferences satisfy SIP, $f{:}\Theta^N \mapsto \mathcal{A}$ is strategy-proof and noncorruptible; then $(\forall \in \Theta^N)(\forall i,j \in N)[\hat{\boldsymbol{\theta}}^i \in r(\theta^i, f(\theta)) \Rightarrow o(j,\theta) = o(j,(\theta^{-i},\hat{\theta}^i))]$.
Proof. Let $\hat{\theta} = (\theta^{-i},\hat{\theta}^i)$. We will show that $o(j,\hat{\theta}) \subseteq o(j,\theta)$. We can show that $o(j,\theta) \subseteq o(j,\hat{\theta})$ by a similar argument, since $f(\theta) = f(\hat{\theta})$ by lemma (1b), and $\theta^i \in r(\hat{\theta}^i, f(\hat{\theta}))$ by the definition of $r(\,\cdot\,)$.

Suppose $o(j,\hat{\theta}) \not\subseteq o(j,\theta)$ and let $y \in o(j,\hat{\theta})$, $y \notin o(j,\theta)$, and let $x = f(\theta)$.
By lemma (1b), $f(\theta) = f(\hat{\theta}) = x$.
$x = f(\theta) \in o(j,\theta)$ by definition of $o(\,\cdot\,)$, so $x \neq y$.
By lemma (1a), $f(\theta) = C(o(j,\theta),\theta^j)$, and $x^j = f^j(\theta) = C^j(o(j,\theta),\theta^j)$.
Let ${}^y\theta^j$ denote the preferences obtained from θ^j by lifting y to first place, all other rankings remaining the same. Since $y \notin o(j,\theta)$, $x^j = f^j(\theta) = C^j(o(j,\theta), {}^y\theta^j)$.
By lemma (1a), $f^j(\theta^{-j},{}^y\theta^j) = C^j(o(j,\theta^{-j},{}^y\theta^j),{}^y\theta^j)$.
Also $C^j(o(j,\theta),{}^y\theta^j) = C^j(o(j,\theta^{-j},{}^y\theta^j),{}^y\theta^j)$, since $o(j,\cdot)$ does not depend on θ^j. So $f^j(\theta) = f^j(\theta^{-j},{}^y\theta^j) = x$. But $y \in o(j,\hat{\theta}) \Rightarrow y = f(\hat{\theta},{}^y\theta^j) = C(o(j,\hat{\theta}),{}^y\theta^j)$ by lemma (1a), and y is best for ${}^y\theta^j$. Expanding the arguments of $f(\,\cdot\,) \Rightarrow y = f(\theta^{-ij},{}^y\theta^j,\hat{\theta}^i)$ and $x = f(\theta^{-ij},{}^y\theta^j,\theta^i)$. Since $\hat{\theta}^i$ and θ^i maintain the same relative ordinal preference between x and y, i can either manipulate $(\theta^{-ij},{}^y\theta^j,\hat{\theta}^i)$ or $(\theta^{-ij},{}^y\theta^j,\theta^i)$, which is a contradiction to the strategy-proofness of f. $\square$.

This lemma states that an agent cannot change the option set of any other agent by a reshuffling of his preferences.

Example with simple serial dictator:

Let $n = k = 5$ and the agent ordering is $\{1,2,3,4,5\}$ for each profile. Let the profile be such that the best slot for agent 1 is slot 1, the best slot for agent 2 is slot 2, and the ordinal preferences for agent 3 are $(1,2,3,4,5)$. The ordering assigns slot 1 to agent 1, slot 2 to agent 2, and slot 3 to agent 3; the option set for agent 4 is $\{4,5\}$. Suppose agent 3's preferences are reshuffled around the outcome $(2,1,3,5,4)$, then every agent's assignment remains the same. Let agent 3's preferences be changed to

(2,4,3,1,5), which is not a reshuffle around the outcome, then agent 3 is assigned slot 4 and the option set for agent 4 is $\{3,5\}$.

In the following lemmas if E is a relation, then $\sim E$ denotes not E; the cardinality of set A, is denoted by $|A|$.

LEMMA 6.6d (Asymmetry). If preferences satisfy SIP and $f\colon\Theta^N \mapsto \mathcal{A}$ is strategy-proof and noncorruptible then $(\forall\theta \in \Theta^N)(\forall i,j \in N)\ iA(\theta)j \Rightarrow$ either:

1) $o(i,\theta) = o(j,\theta)$, or 2) $\sim jA(\theta)i$.

Proof. Suppose the hypothesis of the lemma holds, then $iA(\theta)j$ requires that $\exists\,\hat\theta^i \ni f^j(\theta^{-i},\hat\theta^i) \neq f^j(\theta)$. Suppose that $jA(\theta)i$ then $\exists\,\hat\theta^j \ni f^i(\theta^{-j},\hat\theta^j) \neq f^i(\theta)$. Let $y = f(\theta^{-i},\hat\theta^i)$, $z = f(\theta^{-j},\hat\theta^j)$ and $x = f(\theta)$. We will look at 2 mutually exclusive and exhaustive cases.

Case 1: $\forall\,\hat\theta^i,\hat\theta^j \ni f^j(\theta^{-i},\hat\theta^i) \neq f^j(\theta)$ and $f^i(\theta^{-j},\hat\theta^j) \neq f^i(\theta)$, $f(\theta^{-i},\hat\theta^i) = f(\theta^{-j},\hat\theta^j)$.

The definition of $o(\cdot)$ implies that $\{x,y\} = o(i,\theta)$ and $\{x,z\} = o(j,\theta)$. But $x = z$, so $\{x,y\} = o(i,\theta)$ and $\{x,y\} = o(j,\theta)$. Therefore $o(i,\theta) = o(j,\theta)$ and $|o(j,\theta)| = 2$.

Case 2: $\exists\,\hat\theta^i,\hat\theta^j \ni f^j(\theta^{-i},\hat\theta^i) \neq f^j(\theta)$ and $f^i(\theta^{-j},\hat\theta^j) \neq f^i(\theta)$, $f(\theta^{-i},\hat\theta^i) \neq f(\theta^{-j},\hat\theta^j)$.

The construction of y, z, and x implies that $y \neq z$, $y^j \neq x^j$ and $z^i \neq x^i$. By lemma (1a), $x,y \in o(i,\theta)$ and $x,z \in o(j,\theta)$. $z^i \neq x^i \Rightarrow z \notin f(\theta)$ and if $U(z,\theta^i) > U(x,\theta^i)$ then $z \notin o(i,\theta^{-i})$.

Let $\acute\theta^i = {}_z\theta^i$ if $U(x,\theta^i) > U(z,\theta^i)$: put z last
$\acute\theta^i = {}^z\theta^i$ if $U(z,\theta^i) > U(x,\theta^i)$: put z first
Let $\acute\theta^j = {}_y\theta^j$ if $U(x,\theta^j) > U(y,\theta^j)$: put y last
$\acute\theta^j = {}^y\theta^j$ if $U(y,\theta^j) > U(x,\theta^j)$: put y first
For $k \neq i,j$, $\acute\theta^k = \theta^k$, so $\acute\theta = (\theta^{-ij},\acute\theta^i,\acute\theta^j)$.
$\acute\theta^i$ and $\acute\theta^j$ are reshuffles of θ^i and θ^j about x.

Let $\ddot\theta^i = {}^y\acute\theta^i$ if $U(x,\theta^i) > U(z,\theta^i)$: put y first, z last
$\ddot\theta^i = {}^z({}^y\theta^i)$ if $U(z,\theta^i) > U(x,\theta^i)$: put z first, y second
Let $\ddot\theta^j = {}^z\acute\theta^j$ if $U(x,\theta^j) > U(y,\theta^j)$: z first, y last
$\ddot\theta^j = {}^y({}^z\theta^j)$ if $U(y,\theta^j) > U(x,\theta^j)$: y first, z second

Note that SIP and $y^j \neq x^j$, $z^i \neq x^i$ eliminate the equality in the above definitions.

We now show that $y = C(o(i,\acute{\theta}),\ddot{\theta}^i)$ and $y = f(\acute{\theta}^{-i},\ddot{\theta}^i)$.

Since $\acute{\theta}^j$ is a reshuffle of θ^j around $x = f(\theta)$, then by lemma (1c), $o(i,\acute{\theta}) = o(i,\theta)$ and $y \in o(i,\acute{\theta})$. By construction $\ddot{\theta}^i$ ranks y first or second. If $\ddot{\theta}^i$ ranks y first then $y = C(o(i,\acute{\theta}),\ddot{\theta}^i)$. If $\ddot{\theta}^i$ ranks y second, $\ddot{\theta}^i$ ranks z first, but $z \notin o(i,\theta)$, so $z \notin o(i,\acute{\theta})$; and hence, $y = C(o(i,\acute{\theta}),\ddot{\theta}^i)$. By lemma (1a), $y = f(\acute{\theta}^{-i},\ddot{\theta}^i)$.

By a similar argument, we can show $z = C(o(j,\acute{\theta}),\ddot{\theta}^j)$, and $z = f(\acute{\theta}^{-j},\ddot{\theta}^j)$.

By construction, $\ddot{\theta}^j \in r(y,\acute{\theta}^j)$, and $\ddot{\theta}^i \in r(z,\acute{\theta}^i)$, so by lemma (1b), $y = f(\theta^{-ij},\ddot{\theta}^j,\ddot{\theta}^i)$, and $z = f(\theta^{-ij},\ddot{\theta}^i,\ddot{\theta}^j)$, which is a contradiction since $y \neq z$ and f is single-valued. □.

LEMMA 6.6e (Transitivity). If preferences satisfy SIP, $f{:}\,\Theta^N \mapsto \mathcal{A}$ is strategy-proof and noncorruptible; then $(\forall\theta \in \Theta^N)(\forall i,j \in N)[iA(\theta)j \ \&\ jA(\theta)k \Rightarrow \sim kA(\theta)i]$.

Proof. Suppose $\exists\, i,j,k \in N \ni iA(\theta)j$, $jA(\theta)k$, and $kA(\theta)i$. Without loss of generality let $i = 1$, $j = 2$, and $k = 3$. $1A(\theta)2 \Rightarrow \exists\,\hat{\theta}^1 \ni f^2(\theta^{-1},\hat{\theta}^1) \neq f^2(\theta)$, $2A(\theta)3 \Rightarrow \exists\,\hat{\theta}^2 \ni f^3(\theta^{-2},\hat{\theta}^2) \neq f^3(\theta)$, and $3A(\theta)1 \Rightarrow \exists\,\hat{\theta}^3 \ni f^1(\theta^{-3},\hat{\theta}^3) \neq f^1(\theta)$.

Let $y = f(\theta^{-1},\hat{\theta}^1)$, $z = f(\theta^{-2},\hat{\theta}^2)$, $w = f(\theta^{-3},\hat{\theta}^3)$, and $x = f(\theta)$; then $y^2 \neq x^2$, $z^3 \neq x^3$, and $w^1 \neq x^1$. Noncorruptibility implies $y^1 \neq x^1, z^2 \neq x^2, w^3 \neq x^3$. Asymmetry implies $y^3 \neq x^3$, $z^1 \neq x^1$, and $w^2 \neq x^2$.

By definition of $o(i,\cdot)$ $x,y \in o(1,\theta)$, $x,z \in (2,\theta)$, and $w,z \in o(3,\theta)$. $z \neq x \Rightarrow z \notin f(\theta)$, $y \neq x \Rightarrow y \notin f(\theta)$, $w \neq x \Rightarrow w \notin f(\theta)$.

Define ${}^{x:y}\theta^1$ to by the preferences defined by ${}^x({}^y\theta^1)$ if $U(x,\theta^1) > U(y,\theta^1)$, and by ${}^y({}^x\theta^1)$ if $U(y,\theta^1) > U(x,\theta^1)$. Define ${}_{x:y}\theta^1$ to by the preferences defined by ${}_x({}_y\theta^1)$ if $U(x,\theta^1) > U(y,\theta^1)$ and by ${}_y({}_x\theta^1)$ if $U(y,\theta^1) > U(x,\theta^1)$.

Let $\acute{\theta}^1 = {}_{w:z}\theta^1$ if $U(x,\theta^1) > U(w,\theta^1)$: put $(w{:}z)$ last
$\acute{\theta}^1 = {}_z({}^w\theta^1)$ if $U(w,\theta^1) > U(x,\theta^1)$: put w first, z last
Let $\acute{\theta}^2 = {}_{y:w}\theta^2$ if $U(x,\theta^2) > U(y,\theta^2)$: put $(y{:}w)$ last
$\acute{\theta}^2 = {}_w({}^y\theta^2)$ if $U(y,\theta^2) > U(x,\theta^2)$: put y first, w last
Let $\acute{\theta}^3 = {}_{z:y}\theta^3$ if $U(x,\theta^3) > U(z,\theta^3)$: put $(z{:}y)$ last
$\acute{\theta}^3 = {}_y({}^z\theta^3)$ if $U(z,\theta^3) > U(x,\theta^3)$: put z first, y last

For $k \neq 1,2,3$ $\acute{\theta}^k = \theta^k$, so $\acute{\theta} = (\theta^{-123},\acute{\theta}^1,\acute{\theta}^2,\acute{\theta}^3)$. $\acute{\theta}^1,\acute{\theta}^2$, and $\acute{\theta}^3$ are reshuffles of

$\theta^1, \theta^2, \theta^3$ about $x = f(\theta)$.

Let $\ddot{\theta}^1 = {}^y\acute{\theta}^1$ if $U(x,\theta^1) > U(w,\theta^1)$: put y first, $(w{:}z)$ last
$\ddot{\theta}^1 = {}_z({}^w({}^y\theta^1))$ if $U(w,\theta^1) > U(x,\theta^1)$: put w first, y second, z last
Let $\ddot{\theta}^2 = {}^z\acute{\theta}^2$ if $U(x,\theta^2) > U(y,\theta^2)$: z first, $(y{:}w)$ last
$\ddot{\theta}^2 = {}_w({}^y({}^z\theta^2))$ if $U(y,\theta^2) > U(x,\theta^2)$: y first, z second, w last
Let $\ddot{\theta}^3 = {}^w\acute{\theta}^3$ if $U(x,\theta^3) > U(z,\theta^3)$: put w first, $(z{:}y)$ last
$\ddot{\theta}^3 = {}_y({}^z({}^w\theta^3))$ if $U(z,\theta^3) > U(x,\theta^3)$: put z first, w second, y last

Since $\acute{\theta}^k$ are reshuffles for $k = 1,2,3$ by lemma (1c), $o(1,\acute{\theta}) = o(1,\theta)$, $o(2,\acute{\theta}) = o(2,\theta)$ and $o(3,\acute{\theta}) = o(3,\theta)$. Since $o(1,\acute{\theta}) = o(1,\theta)$, $C(o(1,\acute{\theta}),\ddot{\theta}^1) = C(o(1,\theta),\ddot{\theta}^1)$. By construction, $y \in o(1,\theta)$, and $\ddot{\theta}^1$ ranks y first or second. If $\ddot{\theta}^1$ ranks y first then $y = C(o(1,\acute{\theta}),\ddot{\theta}^1)$. If $\ddot{\theta}^1$ ranks y second, then $U(w,\theta^1) > U(x,\theta^1)$, and $x \in f(\theta)$, implies that $w \notin f(\theta)$, and $w \notin o(1,\theta)$. Hence, $y = C(o(1,\acute{\theta}),\ddot{\theta}^1)$. Similarly $w = C(o(3,\acute{\theta}),\ddot{\theta}^3)$ and $z = C(o(2,\acute{\theta}),\ddot{\theta}^2)$.

By lemma (1a), $y = f(\acute{\theta}^{-1},\ddot{\theta}^1)$, $z = f(\acute{\theta}^{-2},\ddot{\theta}^2)$ and $w = f(\acute{\theta}^{-3},\ddot{\theta}^3)$. But $\ddot{\theta}^1 \in r(w,\acute{\theta}^1)$, $\ddot{\theta}^1 \in r(z,\acute{\theta}^1)$, $\ddot{\theta}^2 \in r(w,\acute{\theta}^2)$ $\ddot{\theta}^2 \in r(y,\acute{\theta}^2)$, and $\ddot{\theta}^3 \in r(z,\acute{\theta}^3)$, $\ddot{\theta}^3 \in r(y,\acute{\theta}^3)$; hence, by lemma (1b), $y = f(\acute{\theta}^{-123},\ddot{\theta}^1,\ddot{\theta}^2,\ddot{\theta}^3)$, $z = f(\acute{\theta}^{-123},\ddot{\theta}^1,\ddot{\theta}^2,\ddot{\theta}^3)$ and $w = f(\acute{\theta}^{-123},\ddot{\theta}^1,\ddot{\theta}^2,\ddot{\theta}^3)$, which is a contradiction, since $y, z,$ and w are distinct.

If the $y, z,$ and w are not distinct, then the conditions of lemma (1d) are observed and the lemma still follows. $\square$.

Given the previous lemmas, the proof of the theorem is straightforward.

Proof of Theorem 6.6. Without loss of generality, suppose that $1A(\theta)2$, $2A(\theta)3, \ldots, n-1A(\theta)n$. By the transitivity of $A(\cdot)$, $1A(\theta)n$. By asymmetry, $1A(\theta)n \Rightarrow \sim nA(\theta)1$. Hence, $\sim nA(\theta)1$, and $A(\theta)$ is acyclic. $\square$.

14. Appendix D

PROPOSITION 7.2. If preferences satisfy SIP then the domain of preferences is rich.
Proof. Without loss of generality let $\theta_1 > \theta_2 > \cdots > \theta_k$. The definition of rich and of $U(\cdot)$ implies:
i) $\sum_j(a_j - b_j)\theta_j \geq 0 \Rightarrow \sum_j(a_j - b_j)\hat{\theta}_j \geq 0$, and
ii) $\sum_j(a_j - b_j)\theta_j > 0 \Rightarrow \sum_j(a_j - b_j)\hat{\theta}_j > 0$.
i) and ii) are true for a_l and b_m such that $l \leq$ m and $\hat{\theta}_l > \hat{\theta}_m$.
Choose $\overline{\theta}$ such that $\overline{\theta}_l > \overline{\theta}_m$ and $\overline{\theta}_m > \overline{\theta}_j$, for$j \neq l$ or m.
Then for preferences to be rich, any $c_p \in \mathcal{A}$, where c_p assigns the p^{th} slot, must satisfy:
a) $a_l\theta_l - c_p\theta_p \geq 0 \Rightarrow a_l\overline{\theta}_l - c_p\overline{\theta}_p \geq 0$, and
b) $b_m\hat{\theta}_m - c_p\hat{\theta}_p \geq 0 \Rightarrow b_m\overline{\theta}_m - c_p\overline{\theta}_p \geq 0$.
Both the right-hand side and left-hand side of a) are true for $l \leq p$, and both the right-hand side and left-hand side of b) are true for $m \leq p$. Therefore, a) and b) hold for any $c \in \mathcal{A}$, so $\mathcal{A}$ is rich. □.

15. Appendix E

In this appendix we provide the conditions sufficient for a Pareto optimal outcome. We do this by first providing conditions for a SCF to be S-IWM*, which is similar to DHMs notion of independent weak monotonicity. Our definition S-IWM* does not involve coalitions and is used only to obtain intermediate results. We first provide these definitions and then prove the proposition.

DEFINITION[19] E.1. A SCF $f:\Theta^N \mapsto \mathcal{A}$ satisfies ***independent weak monotonicity (IWM)*** if $\forall\theta \in \Theta^N$, $\forall C \subseteq N$, $\forall\,\overline{\theta}^C \in \prod_{i \in C}\Theta^i$, and $\forall\{a,b\} \subseteq \mathcal{A}$ such that $a \in f(\theta)$, and $(\forall i \in C)[U(a,\theta^i) \geq U(b,\theta^i) \Rightarrow U(a,\overline{\theta}^i) > U(b,\overline{\theta}^i)]$; it must be that $b \notin f(\theta^{-C},\overline{\theta}^C)$.

[19] This is the definition of IWM that appears in DHM. Neither Laffont and Maskin (1982) or Maskin (1986) have definitions of IWM.

DEFINITION E.2. A SCF $f: \Theta^N \mapsto \mathcal{A}$ satisfies ***S-IWM**** if $\forall \{\theta, \hat{\theta}\} \subset \Theta^N$, and $\forall \{a, b\} \subset \mathcal{A}$ such that:
1) $a \in f(\theta)$, and 2) $(\forall i \in N)[U(a, \theta^i) \geq U(b, \theta^i) \Rightarrow U(a, \hat{\theta}^i) \geq U(b, \hat{\theta}^i)]$; then $b \notin f(\hat{\theta})$.

We first provide a lemma that is a similar to lemma B.1 in Appendix B.

LEMMA E.3. If preferences satisfy SIP, and if a SCF $f: \Theta^N \mapsto \mathcal{A}$ satisfies IPM, and is noncorruptible, $a \in f(\theta)$, $c \in f(\overline{\theta}^i, \theta^{-i})$, and $a \neq c$, then $U(a, \overline{\theta}^i) < U(c, \overline{\theta}^i)$ and $U(c, \theta^i) < U(a, \theta^i)$.
Proof. Suppose our hypothesis holds; then noncorruptibility and $a \neq c$ imply that $a^i \neq c^i$; lemma B.1 is then applied to obtain the result. $\square$.

As in lemma B.1 if preferences do not satisfy SIP in the previous lemma then $U(a, \overline{\theta}^i) \leq U(c, \overline{\theta}^i)$ and $U(c, \theta^i) \leq U(a, \theta^i)$. The difference between lemma B.1 and the previous lemma is that in lemma B.1, $a^i \neq c^i$ and noncorruptibility is not a condition. In the previous lemma $a \neq c$, and noncorruptibility implies $a^i \neq c^i$.

LEMMA E.4. If preferences satisfy SIP, then if a SCF is noncorruptible and satisfies IPM, it is S-IWM*.
Proof. Suppose that $f: \Theta^N \mapsto \mathcal{A}$ satisfies the hypothesis of the lemma; then for $\theta, \hat{\theta} \in \Theta^N$, $a \in \mathcal{A}$, such that $a \in f(\theta)$, and $(\forall i \in N)(\forall b \in \mathcal{A}, b \neq a)$ $[U(a, \theta^i) \geq U(b, \theta^i) \Rightarrow U(a, \hat{\theta}^i) \geq U(b, \hat{\theta}^i)]$, but $b \in f(\hat{\theta})$; we will show that this is a contradiction.

Suppose that $c \in f(\hat{\theta}^1, \theta^{-1})$, for some $c \in \mathcal{A}$. If $c \neq a$, then by noncorruptibility, SIP, IPM and lemma **E**.3, $U(a, \hat{\theta}^1) < U(c, \hat{\theta}^1)$ and $U(c, \theta^1) < U(a, \theta^1)$. But $[U(a, \theta^1) \geq U(c, \theta^1) \Rightarrow U(a, \hat{\theta}^1) \geq U(c, \hat{\theta}^1)]$, so it must be that $c = a$, and $a \in f(\hat{\theta}^1, \theta^{-1})$.

Similarly, $a \in f(\hat{\theta}^1, \hat{\theta}^2, \theta^{-1,2})$, continuing iteratively, $a \in f(\hat{\theta})$. But f is a SCF, so it is single-valued; hence $b \notin f(\hat{\theta})$. $\square$.

The main proposition is stated and proven below.

PROPOSITION 8.3. If preferences satisfy SIP, then if a SCF is noncorruptible, satisfies IPM, and CS, it is strong Pareto optimal (SPO).

Proof. Suppose that $f: \Theta^N \mapsto \mathcal{A}$ satisfies the hypothesis but that f is not SPO. Then there exist a $\bar{\theta} \in \Theta^N$ and a pair $\{a, b\} \subseteq \mathcal{A}$, such that $\forall i \in N$, $U(a, \bar{\theta}^i) \geq U(b, \bar{\theta}^i)$, and $b \in f(\bar{\theta})$. Since preferences satisfy SIP, either $a = b$ or there exists a $j \in K$ such that $U(a, \bar{\theta}^j) > U(b, \bar{\theta}^j)$. By CS, there exists a $\theta \in \Theta^N$ such that $a \in f(\theta)$. By the previous lemma, f satisfies S-IWM*. Since $(\forall i \in N)[U(a, \theta^i) \geq U(b, \theta^i) \Rightarrow U(a, \bar{\theta}^i) \geq U(b, \bar{\theta}^i)]$ and $a \in f(\theta)$, S-IWM* implies that $b \notin f(\bar{\theta})$, a contradiction. □.

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