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Dutch Book Arguments and Subjective Probability

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1 Introduction

A Dutch book is a gamble which is sure to lose and a Dutch book argument is an argument of the following general form.

> A rational individual must behave in accordance with precept X, for if they do not, an outsider will be able to gain a financial advantage from her.

Dutch book arguments are often at the core of normative arguments for behaving in accordance with expected utility theory. It is our thesis that such Dutch book arguments while logically and mathematically correct, cannot be used to deduce the properties of subjective probability. We illustrate our point by concentrating on a Dutch book argument due to deFinetti which is generally construed as requiring rational individuals to have subjective probabilities regarding uncertain events. That is, it is argued that it is irrational to have subjective beliefs about the likelihood of events that are not consistent with *some* probability measure.

DeFinetti's argument runs like this. (See [2].)

> Let us suppose that an individual is obliged to evaluate the rate p at which he would be ready to exchange the possession of an arbitrary sum S (positive or negative) dependent on the occurrence of a given event E, for the possession of the sum pS; we will say by definition that this number p is the measure of the degree of probability attributed by the individual considered to the event E, or, more simply, that p is the probability of E (according to the individual considered; this specification can be implicit if there is no ambiguity.)[1]

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[1] [2, p. 62.]

He then goes on to define probability assessments as being *coherent* if it is not possible to bet with the subject in such a way as to be assured of gaining, and argues that it is

> precisely this condition of coherence which constitutes the sole principle from which one can deduce the whole calculus of probability: this calculus then appears as a set of rules to which the subjective evaluation of probability of various events by the same individual ought to conform if there is not to be a fundamental contradiction among them.[2]

While deFinetti refers to the exchange rate p as the individual's probability of the event E, we will use the more noncommittal term *price*. That is, the individual, henceforth called the Bookie, agrees to take bets on or against E at the same rate p.

Given a collection $E_1, \cdots, E_n$ of mutually exclusive and exhaustive events, if their prices $p_1, \cdots, p_n$ are nonnegative and sum to unity, they define a probability measure on the events. What deFinetti proved is that either the p's are nonnegative and sum to unity (and hence determine a probability) or the p_i's are *incoherent*, meaning it is possible to make a combination of bets which guarantee that the bettor makes a profit. These possibilities are mutually exclusive. While deFinetti in his comments indicates that he does not mean the term "incoherent" in a pejorative sense, the implication is clear cut. This is a Dutch book argument in one of its starkest forms.

We are not interested in what deFinetti said per se, rather we are interested in the validity of this mode of argument, and deFinetti's is one of the most unambiguous statements of this approach.

There are two fundamental questions to examine in regard to Dutch book arguments of this type. One is whether we can interpret a bookie's prices as *her* probabilities, the other is whether a rational bookie will set prices consistent with *any* probabilities. Another way to ask the question is, *to what extent can we deduce the calculus of probability form the postulate of rational behavior?*

We begin by describing the logic of a typical Dutch book argument in favor of additive prices.

Let A and B be two mutually exclusive events, and assume that the decision maker's subjective probabilities for the events A, B, and $A \cup B$ are p, q, and r respectively, where $p + q \neq r$, say $p + q < r$. Offer him now the following lotteries. First, pay him p dollars for a bet on A. If A occurs he will pay one dollar (thus his net income is $p - 1$ dollars), and if A does not occur, he will keep the p dollars. The expected gain from this lottery, according to the subject's beliefs, is zero. Of course, a risk averse decision maker will refuse to participate in such a game. Nevertheless, if we replace dollars by cents (or tenth on cents, etc.), and we offer to pay him $p + \varepsilon$, $\varepsilon < r - p - q$, rather than p, then for a

[2] [2, p. 63.]

sufficiently small gamble the decision maker will overcome his risk aversion and agree to the above mentioned terms.

Similarly, he will accept an offer of $q+\varepsilon$ on B, which requires him to pay one dollar in case that B happens. Finally, he is also willing to pay $r-\varepsilon$ dollars for a gamble on the event $A \cup B$, provided he wins one dollar in case the event "A or B" happens. If either A or B happens, he wins one dollar, but must also pay a dollar. He received $p+q+2\varepsilon$ and paid $r-\varepsilon$ for these lotteries, hence his net gain is $p+q-r+\varepsilon < 0$. Note that this procedure involves no uncertainty, and the decision maker will thus lose the above difference with certainty. Such a mechanism is a Dutch book.

Several assumptions are implicit in the above discussion. The assumption that the gambles are small was already introduced by deFinetti himself, to overcome the issue of diminishing utility of money. (But see [2, fn. a., p. 62.].) Nevertheless, the assumption that for a small risk the decision maker comes as close as we wish to risk neutrality does not hold true even within expected utility theory, where it is equivalent to the assumption that the utility function is differentiable [11, fn. 6]. The risk neutrality assumption certainly does not always hold in nonexpected utility models. (See [15] for a general discussion of risk aversion with respect to small risks). However, it is still true that if the decision maker is willing to accept and to buy bets at nonadditive rates, then he will be vulnerable to a Dutch book. Indeed, Samuelson's comment may prove that the decision maker does not announce his true beliefs whenever his utility function is nondifferentiable, but nonadditive rates still expose him to the danger of a Dutch book.

In this paper we point out an overlooked element in the traditional analysis. Namely that the bookie and any potential bettors are playing a game against each other, and the bookie's prices and the bettors' bets are strategic choices. It is thus necessary to analyze the equilibrium of this game in order to draw any conclusion about the relationship between the bookie's prices and her beliefs. Since the bookie moves first and sets her prices, the bettor cannot influence the prices. That is, the appropriate equilibrium concept is subgame perfect equilibrium. Each set of prices for bets constitutes an information set for the bettors, so a strategy for a bettor is a function mapping prices into bets. Subgame perfection requires that these bets maximize the bettor's payoff, however he chooses to evaluate it. Given the bettors' strategy functions, the bookie chooses prices to maximize her payoff given the bets that will actually be placed.

If, as is typically the case in game theoretic models, the bookie is cognizant of the payoffs of the bettors, (or at least of how Nature chooses the bettors), the relevant consideration is how these bettors will respond. That is, the prices set by the bookie reflect not only her own beliefs, but also the bettors' beliefs. In fact, as Corollary 5 below shows, the bookie's prices may reflect *only* the beliefs of the bettors.

This is obviously true of professional bookies. Their prices are designed to equate the supply and demand of bets, and so reflect the bookies' beliefs about the bettor's beliefs, not about the events. Professional bookies also set nonadditive prices because they take a percentage off the top and will not let the bettors take either side of a bet at a given price. Dutch book arguments only address themselves to the case where the bookie must take both sides of a bet at the same price. In the expected value case with one bettor, such bookie's rates are at least additive. But this result fails more generally.

In this paper we show, by means of an example, that it may indeed happen that the bookie's optimal strategy is to announce nonadditive prices. For this, we have to assume that the bettors are nonexpected utility maximizers. The bookie herself evaluates uncertain prospects by their expected utility. (This utility function can be chosen to be differentiable, so that all of deFinetti's requirements are satisfied.) In this setting we construct a game with a subgame perfect equilibrium with nonadditive prices. It is true that at these prices, it is possible for a bettor to place a combination of bets that will guarantee that he wins, but for the bettors in the game that is not their best course of action. Consequently the bookie does not lose for sure, and in fact, has a higher expected utility from these bets than from not participating at all. In later sections we examine Dutch books arguments in favor of the rule of conditional probability, i.e., $P(A \cap B) = P(A)P(B|A)$, and Bayes' rule. We find that the validity of these arguments also depend on the assumption that the bookie does not behave strategically.

2 The game

There are disjoint events A and B which exhaust the set of possible states. That is, A, B, $A \cup B$, and $\emptyset$ are the only events under consideration. The bookie posts prices a and b for bets on A and B respectively. If a bettor buys $\$x$ of the bet on A his net proceeds are $\$(\frac{x}{a} - x)$ if A occurs and $-\$x$ if B occurs. The bookie must both buy and sell bets at the posted prices. Table 1 summarizes the bettor's monetary payoffs from various bets. Observe that selling x is the same as buying $-x$, so without loss of generality we shall consider the bettors' purchases, which may be negative.

Bettor's Action	Net Payoff on A	Net Payoff on B
Buy x on A and buy y on B	$x/a - x - y$	$y/b - x - y$
Buy x on A and sell y on B	$y - x + x/a$	$y - x - y/b$
Sell x on A and buy y on B	$x - y - x/a$	$x - y + y/b$
Sell x on A and sell y on B	$x + y - x/a$	$x + y - y/b$

Table 1: Bettor's Monetary Payoffs from Combinations of Bets

We impose the following budget constraint on the bettors. Each bettor has only one dollar and is not permitted to buy on credit nor is he allowed to sell a bet (buy a negative quantity) unless he proves that he possesses sufficient funds to pay off in the event he loses. These funds cannot include possible gains from from some other bets.[3] Formally,

[3]Even in deFinetti's analysis, it is impossible to let the bettor use this potential revenue for further betting (thus creating a money pump), because the bookie will refuse to let him bet with her money.

we can write his budget constraint as

$$x^+ + y^+ + x^- \frac{a}{1-a} + y^- \frac{b}{1-b} \leq 1.$$

where, as usual, x^+ denotes $\max\{x, 0\}$ and x^- denotes $\max\{-x, 0\}$.

The following lemma guarantees that without loss of generality the bookie need only consider prices a and b satisfying $a + b \leq 1$. It applies to any bettor whose preferences are monotone with respect to first order stochastic dominance. Its proof is omitted.

Lemma 1 *Assume the bettor's preferences over lotteries are monotone. If $a + b < 1$, then the bettor's optimal bets satisfy $x \geq 0$ and $y \geq 0$. If $a + b > 1$, then the bettor's optimal bets satisfy $x \leq 0$ and $y \leq 0$.*

For the remainder of our results, the bettors are assumed to evaluate lotteries using an *anticipated utility functional.* That is, there is a utility function u and a rank-dependent probability weighting function $g : [0, 1] \to [0, 1]$, strictly increasing and satisfying $g(0) = 0$ and $g(1) = 1$, which determine the *value* of a lottery. (See [1, 9, 13, 18].) The formula for the value of a lottery with (subjective) distribution function F is

$$V(F) = \int u(w)\, d(g \circ F)(w).$$

For a random variable taking on only two values, $v < w$, with probabilities q and $1 - q$, the formula for the value reduces to

$$V(v, q; w, 1-q) = u(v)g(q) + u(w)\left(1 - g(q)\right). \tag{1}$$

We also assume that the probability weighting function g is concave.

We can now evaluate the bettor's choices. Let q denote the bettor's subjective probability of A. Given the prices a and b, the bettor's values are given in Table 2. The next

	Bettor's Value
$\frac{x}{a} < \frac{y}{b}$	$g(q)u\left(\frac{x}{a} - x - y\right) + (1 - g(q))u\left(\frac{y}{b} - x - y\right)$
$\frac{x}{a} > \frac{y}{b}$	$(1 - g(1-q))\,u\left(\frac{x}{a} - x - y\right) + g(1-q)u\left(\frac{y}{b} - x - y\right)$
$\frac{x}{a} = \frac{y}{b}$	$u\left(\frac{x}{a} - x - y\right) = u\left(\frac{y}{b} - x - y\right)$

Table 2: Bettor's Values from Purchases of Bets x on A and y on B

lemma depends on the bettor's evaluating lotteries using an anticipated utility functional and greatly simplifies the analysis of the bettor's best response behavior. It states that if $a + b \leq 1$, then the bettor will either *plunge* by betting everything on A or on B or else *hedge* by betting so as to receive the same payoff in either event. (Remember that if $a + b < 1$, then this payoff will be strictly positive.) The reason for this is that if $a + b \leq 1$ and the bettor chooses x and y to satisfy $\frac{x}{a} < \frac{y}{b}$, then he becomes better off by increasing y up to $y = 1$. Similarly if he chooses to set $\frac{x}{a} > \frac{y}{b}$, he should set $x = 1$, and if he chooses to set $\frac{x}{a} = \frac{y}{b}$, then he should set $x = \frac{a}{a+b}$ and $y = \frac{b}{a+b}$. Again the proof is straightforward.

Lemma 2 *If u is convex and g is concave, and if $a + b \leq 1$, then the bettor's optimal response is to Plunge on A, Plunge on B, or Hedge. The payoffs are given in Table 3.*

Action	Bettor's value
Plunge on A	$u(-1)g(1-q) + u\left(\frac{1-a}{a}\right)(1 - g(1-q))$
Plunge on B	$u(-1)g(q) + u\left(\frac{1-b}{b}\right)(1 - g(q))$
Hedge	$u\left(\frac{1-a-b}{a+b}\right)$

Table 3:

His optimal strategy depends on which of these three options yields the highest utility. We start with the case of linear utility, $u(x) = x$.

Define the parameters

$$\alpha = \frac{g(1-q)}{1 - g(1-q)}$$

and

$$\beta = \frac{1 - g(q)}{g(q)}.$$

Since g is concave, $\alpha \geq \beta$.

Proposition 3 *If the bettor's utility function u is linear and his probability transformation function g is concave, then the his optimal strategies are:*

- *Plunge on A whenever $\frac{b}{a} \geq \alpha$;*
- *Plunge on B whenever $\frac{b}{a} \leq \beta$;*

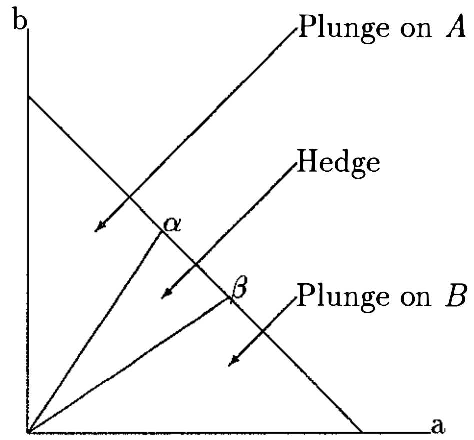

Figure 1: Bettor's optimal response (Linear utility)

- *Hedge whenever* $\beta \leq \frac{b}{a} \leq \alpha$.

These strategies are depicted in Figure 1.

Note that when g is linear, that is, when the bettor is risk neutral, $\alpha = \beta$ and he will buy either on A or on B, but not on both, unless $\frac{b}{a} = \alpha = \beta$, in which case he is indifferent between all three strategies.

It follows from Proposition 3 that if the bettor's utility function is linear, but his probability transformation function is concave, then his optimal strategy depends only on the ratio $\frac{b}{a}$. Since the bettor is buying bets, the bookie always prefers, for a given strategy of the bettor, to set the prices a and b as high as possible. Thus if $a + b < 1$ it pays to raise both proportionately. We thus get the following result.

Theorem 4 *If all bettors maximize an anticipated utility functional with a linear utility function and concave probability transformation function, and the bookie's preferences are monotone, then the bookie's equilibrium strategy requires* $a + b = 1$.

Corollary 5 *If* g *is linear, i.e., the bettor is an expected value maximizer, then the equilibrium strategy of the bookie is to set* $a = q$ *and* $b = 1 - q$.

That is the bookie's prices are the bettor's subjective probabilities, not her own!

Note that the theorem remains true even when the bettors are risk loving (i.e., when the function g is convex and $\alpha \leq \beta$).

Theorem 4 holds whenever there is just one bettor whose preferences exhibit *diversification*. Suppose that there are $s_1, \ldots, s_m$ states of the world, and let $X_1, \ldots, X_n$ by n random variables. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in R^n_+$ such that $\sum \alpha_i = 1$, let $X(\alpha)$ be the random variable yielding $\sum_{i=1}^n \alpha_i X_i(s_j)$ if the state of the world is s_j. The decision maker's preferences exhibit diversification if $X_1 \sim \cdots \sim X_n$ implies that $X(\alpha) \succ X_i$ for every α and i.

Diversification implies, but is not implied, by risk aversion. However, risk aversion and quasiconcavity of the preference relation imply diversification. (See [3]).

Theorem 6 *Suppose that the bookie's preferences are monotonic. If there is just one bettor and his preference relation exhibits diversification, then the bookie's equilibrium strategy satisfies $a + b = 1$.*

Proof: It is always optimal for the bettor to spend the whole dollar. Note that when the rates on A and B are a and b, respectively, then betting $\frac{ax}{a+b}$ on A and $\frac{bx}{a+b}$ on B adds $x\frac{1-a-b}{a+b}$ to the bettors income and so is desired by first order stochastic dominance.

Suppose that when the rates are a and b with $a + b < 1$, the bettor plunges on A. For every $b' > b$, the bettor will plunge on a, as long as a does not change. Indeed, suppose that for (a, b') with $b' > b$ the bettor buys x on A and $1 - x$ on B. He then faces the lottery $X = (\frac{x}{a} - 1, A; \frac{1-x}{b'} - 1, B)$, which he finds to be preferred to the lottery $Y = (\frac{1}{a} - 1, A; -1, B)$. By first order stochastic dominance, the lottery $Z = (\frac{x}{a} - 1, A; \frac{1-x}{b} - 1, B)$ is better than X, and by transitivity it is preferred to Y, the lottery he chose. In other words, the bookie might as well set $a+b = 1$. Similar arguments hold when the bettor plunges on B.

Suppose now that when the rates are a and b where $a + b < 1$, the bettor mixes, and buys x on A and $1 - x$ on B. By the same argument as before it follows that when the rates are $(a, 1 - a)$ he will not spend more than $1 - x$ on B, and when the rates are $(1 - b, b)$, he will not spend more than x on A. We want to show that for some $a' + b' = 1$, $a' \in [a, 1 - b]$, the bettor will mix $(x, 1 - x)$, thus making the bookie better off. Suppose that such a point does not exist. Let a' be the supremum of the numbers a'' (with corresponding points on the main diagonal) for which the bettor mixes $(y, 1 - y)$, where $y \geq y^* > x$. Also, as long $a'' \geq a'$, the bettor hedges $(z, 1 - z)$ where $x \leq z^* < x$. Suppose, without loss of generality that $y^* = \inf\{y\}$ and $z^* = \sup\{z\}$. Then at $(a', 1-a')$ the bettor is indifferent between hedging y^* on A and $1-y^*$ on B. By diversification, he will prefer hedging with x^* on A and $a - x^*$ on B. ∎

Nonlinear utility

Next we will show that this analysis may fail if there is more than one bettor and the bettors' utility functions are convex, but not linear. When facing two such bettors, the bookie's optimal strategy may be to set nonadditive rates, even when her own subjective probabilities are additive.

We start with the analysis of a bettor's problem. Specifically, consider utilities for the bettors of the following form.

$$u(x) = \begin{cases} e^{kx} & x \geq 0 \\ x+1 & x \leq 0 \end{cases}$$

For $k \geq 1$ this is a (weakly) convex increasing function. Set

$$s = -ln[1 - g(1-q)]$$

and

$$t = -ln[1 - g(q)].$$

The parameters s and t depend only on the bettor's belief q and his preferences through g. By appropriately choosing q we can make s and t arbitrarily large.

The equation of the set (a, b) pairs making the bettor indifferent between plunging on A and plunging on B is

$$b = \frac{a}{1 - a^{\frac{s-t}{k}}}. \tag{2}$$

This curve is labeled "A vs. B" in Figure 2. It is convex if $s > t$ and concave if $s < t$.

The equation of the (a, b) pairs for which plunging on A is indifferent to hedging is given by

$$b = \frac{a^{2\frac{s}{k}}}{1 - a^{\frac{s}{k}}}. \tag{3}$$

This curve is convex and intersects the line $a + b = 1$ at the point $a = \dfrac{k}{s+k}$.

Finally, the locus of (a, b) pairs which the bettor indifferent between plunging on B and hedging is

$$a = \frac{b^{2\frac{t}{k}}}{1 - b^{\frac{t}{k}}}. \tag{4}$$

This curve intersects the $a + b = 1$ line at $a = \dfrac{t}{k+t}$.

Also note that the transitivity of indifference guarantees that if two these curves intersect, then all three of them intersect at the same point.

Figure 2 depicts these loci for the case $s = 1.8$, $t = 1.5$, and $k = 1$. In the first drawing, the curve labeled "A vs. H" is the locus of (a, b) pairs making plunging on A

and hedging indifferent. the other curves are similarly labeled. The regions are labeled with the bettor's preferences. That is, in the region marked "AHB," the bettor prefers plunging on A to hedging to plunging on B. The second drawing indicates the bettor's best responses.

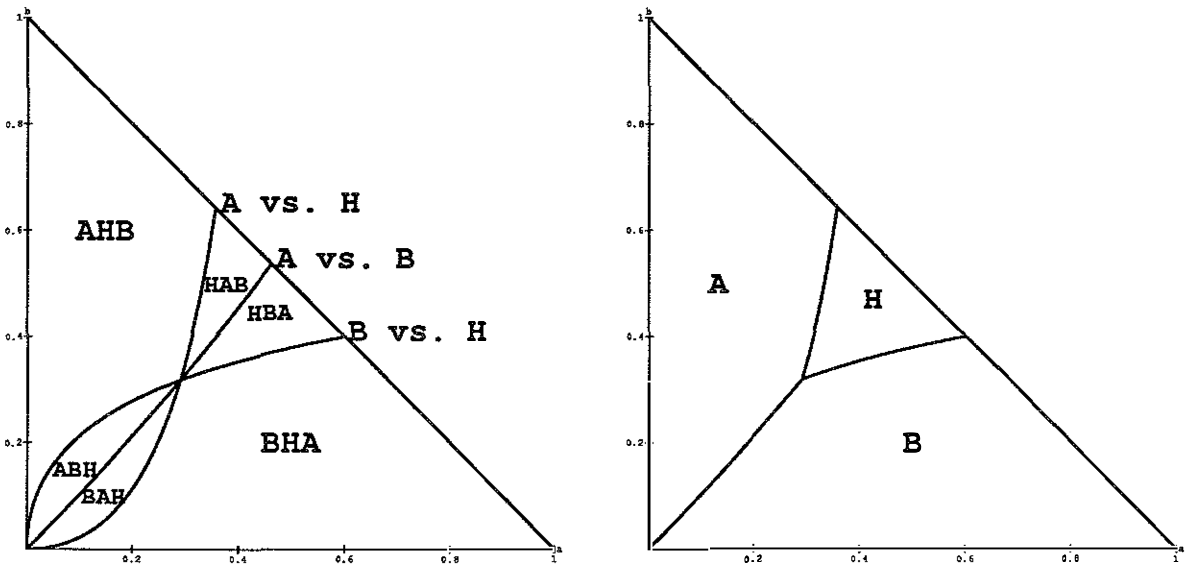

Figure 2: Bettor's choices

We will assume that *when the bettor is indifferent between plunging and hedging, the bettor will plunge.* (We are, by the usual rules of game theory, entitled to do this.) Even in this case, the logic that when the bettor is plunging, the bookie wants to raise the price of the bettor's bet and when he's hedging, the bookie wants to raise both prices, drives the equilibrium prices to satisfy $a + b = 1$.

This, however, is not necessarily the case when the bookie plays against *more than one* bettor.

Suppose that there are *two* bettors, I and II. Their optimal strategies are indicated in Figure 3. There are five points of special interest, labeled P, Q, R, S, and T in Figure 3. Point Q has the largest a for which both bettors will plunge on A, and point T has the largest b for which both bettors will plunge on B. At point R, Bettor I hedges while II plunges on A. At S, I plunges on B, while II hedges. The segment joining R and S has both bettors hedging. Finally at point P, Bettor I is plunging on B and Bettor II is plunging on A. It is easy to see that the bookie's expected utility will be maximized at one of P, Q, T, or on the segment $\overline{RS}$. Letting p denote the bookie's subjective probability of A, her expected utilities are given in Table 4. It is possible to choose values for p, s_I, t_I, k_I, s_{II}, t_{II}, and k_{II}, and a concave increasing utility u for the bookie so that point P has the highest expected utility. For instance, choose $k_I = 2.857$, $k_{II} = 1$, $s_I = 28.57$, $t_I = 1$, $s_{II} = 3$, $t_{II} = 12$ (these are the values used to plot Figure 3)

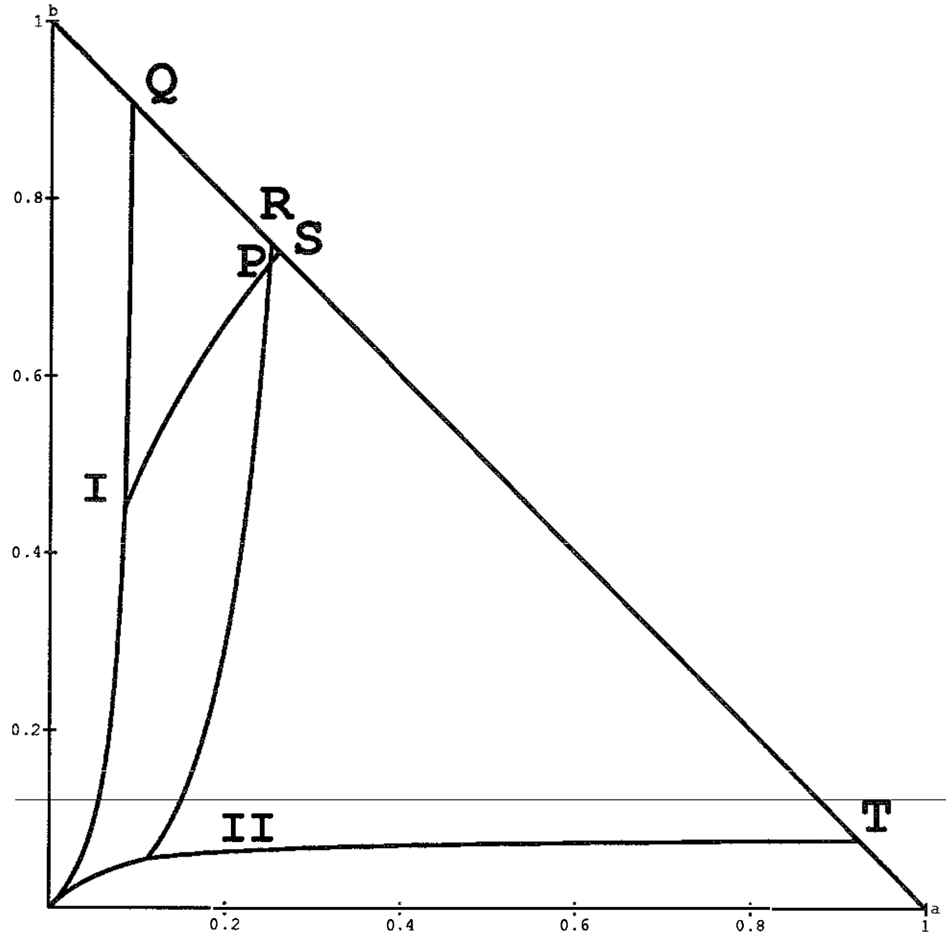

Figure 3: Two bettors

P	$pu\left(2-\frac{1}{a}\right)+(1-p)u\left(2-\frac{1}{b}\right)$
Q	$pu\left(2-p\frac{2}{a}\right)+(1-p)u\,(2)$
R	$pu\left(1-\frac{1}{a}\right)+(1-p)u\,(1)$
S	$pu\,(1)+(1-p)u\left(1-\frac{1}{b}\right)$
T	$pu\,(2)+(1-p)u\left(2-\frac{2}{b}\right)$
$\overline{RS}$	$u(0)$

Table 4: Bookie's candidate strategies.

and $p = .2$. For the bookie's utility choose

$$u(x) = \begin{cases} x & x \geq -2 \\ 4x+6 & x \leq -2. \end{cases}$$

Then to three decimal places, the bookie's expected utilities are given in Table 5. In the

Point	(a,b)	Expected utility
P	(.248,.727)	0.080
Q	(.091,.909)	-13.200
R	(.250,.750)	-0.400
S	(.259,.741)	-0.215
T	(.923,.077)	-71.600
$\overline{RS}$		0.000

Table 5: Bookie's candidate strategies: Numerical example.

equilibrium of this game the bookie does not post additive prices.

To be fair, there is another equilibrium, where the bettors hedge when indifferent, in which the bookie posts additive prices and her expected value is zero. The point is that there is at least one equilibrium (in fact the bookie's favorite) in which she sets nonadditive prices.

When the bookie sets the rates a on A and b on B, she also sets the rate $a+b$ on $A \cup B$. Indeed, a bettor may now bet $\frac{a}{a+b}$ on A and $\frac{b}{a+b}$ on B, thus winning $\frac{1}{a+b}$ in case either A or B happen, which is, of course, the same as a bet of one dollar on $A \cup B$ when the rate on this event is $a+b$. It follows that in the above example the bookie could have announced the rate $a+b$ on $A \cup B$, rather than the rate 1, but this would not have changed the equilibrium strategies. Since $A \cup B$ is the sure event, this by itself would probably have been considered by deFinetti as leading to an immediate Dutch book. Nevertheless, the major point of the section is established, as we show the existence of an equilibrium with nonadditive rates.

3 Conditional probability

We now examine arguments regarding computation of conditional probabilities. Consider the following two state process. At time 0, Nature chooses event A or A^c. If A^c occurs, the game is over. If A occurs, then at time $t = 1$ Nature chooses B or B^c. For example, A may be the development of a new product, and B may be its success in the market. (See [10] for an experiment based on this interpretation.) Suppose a decision maker has subjective probabilities p that A occurs at time 0, q that B occurs at time 1 given that A has already occurred, and r that both A and B will occur. If $r \neq pq$, say $r > pq$, then the following Dutch book is possible.

At time 0, a bettor will bet \$1 on A at the price p and sell the bookie a \$1 bet on $A \cap B$ at the price r. If A occurs, then the bettor wins $\$\frac{1}{p}$, which at time 1 he bets on B at the rate q. If B occurs, then the bettor nets $\$\frac{1}{pq} - \frac{1}{r} > 0$. If B^c occurs, then the bettor breaks even, because he lost the proceeds from his \$1 bet on A, but wins \$1 from the bet he sold on $A \cap B$. Similarly, if A^c occurs, the bettor breaks even. Thus the bettor can never lose, but can win.[4]

It turns out that this Dutch book too depends on the assumption that since there is a potential sure profit, someone will take it, regardless of any other available options. As is the case with the Dutch book against nonadditive probabilities, here too it is possible to find examples where the bookie's optimal strategy is to announce rates violating the rule of conditional probability.

Assume one bookie and two bettors, all of whom maximize expected value. Let p_i, q_i, r_i, $i = 0, 1, 2$, be the subjective probabilities the bookie (agent 0) and the two bettors (agents 1 and 2) assign to the events A, B, and $A \cap B$, respectively. Assume further that $p_i q_i = r_i = r$, $i = 0, 1, 2$. In other words, all three agents agree on the probability of the event $A \cap B$, and all three of them satisfy the rule of conditional probability.

If the bookie offers the rates a on A and b on B, then she also offers the rate of ab on $A \cap B$, as a bettor may put a dollar on A, and buy, in case he wins, a bet of $\frac{1}{a}$ dollars on B. Since all agents are risk neutral, and they all agree on the probability of the event $A \cap B$, it follows that the bookie's expected gain from bets on $A \cap B$ is never positive, but if the bookie fixes the rate r on $A \cap B$, the expected value of any bets on this event is zero. If the bookie announces the rate $c = r$ on $A \cap B$ and finds it optimal to set the rates a on A and b on B such that $ab \neq r$, then there is an optimal strategy for the bookie at which $ab \neq c$, as switching from ab to r cannot make her worse off.

We start with an analysis of a bettor's optimal strategy. To simplify notation, we omit the index i. As before, his initial budget is one dollar. If $a < p$, he wants to buy one dollar on A, since such a bet has a positive expected return. This is the case regardless of what he wants to do with respect to the event B. Of course, if A does not happen he will not be able to play on B, but since he believes that the expected value of his budget, after the uncertainty concerning the event A is resolved, is more than one, he will first play on A. Upon winning, he will buy $1/a$ on B if $b < q$, and sell $\frac{1}{a}(1-b)$ on B if $b > q$.[5]

Things are different when $a > p$. In this case the bettor wants to sell on A, but then he will not be able to buy or sell at a later date on B. If A happens then he looses, and

[4] For another discussion of Dutch books and conditional probabilities, see [16].

[5] If he has $\frac{1}{a}$ dollars, the maximal x he may sell on B satisfies $x - \frac{x}{b} = \frac{1}{a}$, hence $x = \frac{1}{a}(1-b)$.

if A does not happen, he wins but B can no longer happen. Of course, if he plays on B, he will buy on B if $b < q$ and sell if $b > q$. His expected returns are $\frac{q-b}{b}$ and $\frac{b-q}{1-b}$, respectively. On the other hand, if he sells $\frac{a}{1-a}$ on A, his expected return is $\frac{a-p}{1-a}$. Obviously, $\frac{a-p}{1-a} \geq \frac{q-b}{b}$ if and only if $b \geq q\frac{1-a}{1-b}$, and $\frac{a-p}{1-a} \geq \frac{b-q}{1-b}$ if and only if $b \geq 1-(1-q)\frac{1-a}{1-p}$.

If the bettor buys x dollars on A, then the bookie's expected gain is $x - x\frac{p_0}{a}$. If he sells x on A, then her expected gain is $x\frac{p_0}{a-x}$. Similar analysis holds for bets on B. If the bettor buys x on A and in case he wins he buys $\frac{x}{a}$ on B, the bookie's utility is $x - xp_0\frac{q_0}{a}b$. If, after A happens, the bettor sells $x\frac{b}{a}(1-b)$ on B, the bookie's utility is $x[1-p_0\frac{1-q_0}{a}(1-b)]$. This discussion is summarized in Table 6.

Note that since she is an expected value maximizer, the bookie's utility from a set of gambles is the sum of the utilities she receives from the gambles themselves. Let $p_0 = q_0 = \frac{1}{6}$, $p_1 = \frac{1}{9}$, $q_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$, and $q_2 = \frac{1}{18}$. A numerical analysis of the bookie's options implies that her optimal strategy is to set $a = 0.362$, $b = 0.179$. Clearly, $ab - \frac{1}{36} = 0.037$.

4 The Bayesian Bookie

The next "Dutch book" argument is weaker than the previous examples as it only shows that the bookie's expected return is negative, but it does not imply that she can never win. It concerns the incentives to use Bayes' rule.

The following game is played. There are two urns indexed by $\{1,2\}$. Urn 1 has 10 white balls and 20 black balls, and Urn 2 has 20 white balls and 10 black balls. The Master of Ceremonies (MC) secretly chooses an urn and draws a ball from it randomly. He shows the ball to the Bookie and the Bettor. The Bookie then announces odds on which urn the MC used. The Bettor now bets on the urns. The MC then reveals which urn he used and the bet is settled as follows. When the Bookie posts odds of λ against Urn 2 and the Bettor bets s on Urn 2 and t against Urn 2, the Bettor receives $\lambda s - t$ if Urn 2 was the MC's choice and $-s + \frac{t}{\lambda}$ if Urn 1 was the MC's choice.

A Bayesian bookie would use a prior probability $q = (q_1, q_2)$ over the MC's choice of urns and compute the following conditional probabilities.

$$\text{Prob(Urn 1|White)} = \frac{\frac{1}{3}q_1}{\frac{1}{3}q_1 + \frac{2}{3}q_2}$$

Case	Bettor's Optimal Strategy	Bookie's Utility
$a \leq p$, $b \leq q$	Buy 1 on A. If A happens, then buy $\frac{1}{a}$ on B.	$1 - p_0 \frac{q_0}{a} b$
$a \leq p$, $b \geq q$	Buy 1 on A. If A happens, then sell $\frac{b}{a}(1-b)$ on B.	$1 - p_0 \frac{1-q_0}{a}(1-b)$
$a \geq p$, $b \geq q\frac{1-a}{1-p}$	Buy 1 on B.	$1 - \frac{q_0}{b}$
$a \geq p$, $b \leq q\frac{1-a}{1-p}$	Buy 1 on B.	$1 - \frac{q_0}{b}$
$a \geq p$, $b \geq 1-(1-a)\frac{1-q}{1-p}$	Sell $\frac{b}{1-b}$ on B.	$\frac{q_0 - b}{1-b}$
$a \geq p$, $q\frac{(1-a)}{(1-p)} \leq b$, $b \leq (1-q)\frac{1-a}{1-p}$	Sell $\frac{a}{1-a}$ on A.	$\frac{p_0 - a}{1-a}$

Table 6: Conditional probability example.

$$\begin{aligned}
\text{Prob}(\text{Urn } 2|\text{White}) &= \frac{\frac{2}{3}q_2}{\frac{1}{3}q_1 + \frac{2}{3}q_2} \\
\text{Prob}(\text{Urn } 1|\text{Black}) &= \frac{\frac{2}{3}q_1}{\frac{2}{3}q_1 + \frac{1}{3}q_2} \\
\text{Prob}(\text{Urn } 2|\text{Black}) &= \frac{\frac{1}{3}q_2}{\frac{2}{3}q_1 + \frac{1}{3}q_2}
\end{aligned}$$

This would lead him to compute the posterior odds λ_W against Urn 2 when a white ball is drawn as

$$\lambda_W = \frac{\text{Prob}(\text{Urn } 1|\text{White})}{\text{Prob}(\text{Urn } 2|\text{White})} = \frac{\frac{1}{3}q_1}{\frac{2}{3}q_2} = \frac{1}{2}\frac{q_1}{q_2};$$

and the posterior odds λ_B against Urn 2 when a black ball is drawn as

$$\lambda_B = \frac{\text{Prob}(\text{Urn } 1|\text{Black})}{\text{Prob}(\text{Urn } 2|\text{Black})} = \frac{\frac{2}{3}q_1}{\frac{1}{3}q_2} = 2\frac{q_1}{q_2}.$$

Thus regardless of what his prior is, a Bayesian bookie will choose a plan of action (λ_B, λ_W) satisfying

$$\frac{\lambda_W}{\lambda_B} = \frac{1}{4}.$$

Freedman and Purves [5] show that unless a Bookie sets Bayesian posterior odds, he will be vulnerable to very weak sort of Dutch book. Suppose the Bookie is not a Bayesian and chooses a plan of action (λ_B, λ_W) with $\frac{\lambda_W}{\lambda_B} \neq \frac{1}{4}$. For example, suppose he sets $\lambda_B = 3$ and $\lambda_W = 1$. Then consider the following plan of action for the Bettor:

- If a white ball is drawn, bet 750 on Urn 2 and 250 against Urn 2.
- If a black ball is drawn, bet 25 on Urn 2 and 975 against Urn 2.

How would our Bettor expect to fare ex ante?

- If the MC chooses Urn 1, then with probability $\frac{1}{3}$ he draws a white ball. In this case the Bookie posts odds $\lambda_W = 1$ against Urn 2 (even odds), and the Bettor bets 750 on Urn 2 and 250 against, so the Bettor loses 500. With probability $\frac{2}{3}$ a black ball is drawn, the Bookie posts odds $\lambda_B = 3$ against Urn 2, the Bettor bets 25 on Urn 2 and 975 on Urn 1, so he wins $-25 + \frac{975}{3} = 300$. The Bettor's expected winnings are thus $\frac{100}{3} > 0$.
- If the MC chooses Urn 2, then with probability $\frac{2}{3}$ he draws a white ball, the Bookie posts $\lambda_B = 1$ against Urn 2, the Bettor bets 750 on Urn 2 and 250 against, and so wins 500. With probability $\frac{1}{3}$, the MC draws a black ball, the Bookie posts odds $\lambda_B = 3$ against Urn 2, the Bettor bets 25 on Urn 2 and 975 against, and so loses $-(3 \times 25 - 975) = 900$. The Bettor's expected winnings are again $\frac{100}{3} > 0$.

With these odds, regardless of which urn is actually chosen, the Bettor expects to make money ex ante. Note that he does not make money in all cases. Freedman and Purves [5] prove that unless the Bookie uses Bayesian posterior odds for some prior, then such a betting strategy always exists.

This is of course a much weaker Dutch Book than those of the other two possible violations of the basic rules of probability theory, as it does not imply that the bookie must lose money.[6] Nevertheless, even this conclusion fails once the bookie plays strategically. It turns out that even if there is only one possible bettor the bookie may find it optimal to set rates that do not agree with Bayes' rule, provided the bettor is not an expected utility maximizer.

Suppose that the bookie is an expected value maximizer and the bettor maximizes an anticipated utility functional with the concave probability transformation function

$$g(p) = \frac{e - e^{1-p}}{e - 1}.$$

We have already seen that in that case the bookie's optimal strategy is to set additive rates (Theorem 4). Let the bookie's initial beliefs be given by $q = (0.25, 0.75)$ and the bettors beliefs are $q' = (0.5, 0.5)$. After observing a white ball their beliefs change to $\left(\frac{1}{7}, \frac{6}{7}\right)$ and $\left(\frac{1}{3}, \frac{2}{3}\right)$, respectively, while after observing a black ball they will change to $\left(\frac{2}{5}, \frac{3}{5}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}\right)$, respectively.

Let the bookie announce the rates $(p_W, 1-p_W)$ and $(p_B, 1-p_B)$ after observing a white or a black ball, respectively. In the notation of Proposition 3 we obtain $\alpha_W = 3.343p_W$, $\beta_W = 1.230p_W$, $\alpha_B = 0.813p_B$, and $\beta_B = 0.299p_B$. Let the bettor's budget constraint be one dollar. It follows from Proposition 3 that the bettor's optimal strategies are:

- If the MC draws a white ball, bet a dollar on Urn 1 if $p_W \leq 0.230$ and bet one dollar on Urn 2 if $p_W \geq 0.448$.
- If the MC draws a black ball, bet a dollar on Urn 1 if $p_B \leq 0.552$ and bet one dollar on Urn 2 if $p_B \geq 0.770$.

Whatever the bettor decides to do, the bookie would like to set the rate on the event upon which the bettor plays to be as high as possible. Therefore, she will set p_W to be either 0.230 or 0.552, and p_B will be either 0.552 or 0.230. Moreover, the rates p_W and

[6] It is impossible to expose the bookie to a Dutch book even if the bettor is allowed to bet on the urns before and after the MC draws the ball. Suppose that he first bets one dollar on Urn 1. (There is of course no reason to bet on both urns as such a bet yields zero with probability one). If the MC chooses Urn 2 and draws a white ball the rates are set by the bookie to be $\frac{1}{3}$ and $\frac{2}{3}$. The bettor must now bet at least two dollars on Urn 2 to end with a non-negative outcome. However, if the MC chooses Urn 1 and draws a white ball this strategy forces the bettor to lose a dollar.

p_B are set independently. Since the bookie is risk neutral, it is easy to verify that she will set the rates $p_W = 0.230$ and $p_B = 0.552$. It is easy to verify that

$$\frac{\lambda_W}{\lambda_B} = \frac{\frac{p_W}{(1-p_W)}}{\frac{p_B}{(1-p_B)}} = 0.242 \neq 0.25.$$

In this case, in equilibrium the bookie will announce rates disagreeing with Bayes' rule.

5 Conclusions

Several recent nonexpected utility models are based on the assumption that decision makers do not obey some of the basic rules of probability theory. Fishburn [4], Schmeidler [12], and Gilboa [6] presented models where people use nonadditive probabilities. Segal [14] analyses a model where decision makers do not multiply the probabilities in a multi-stage lottery. There is evidence that experimental subjects have a tendency to violate at least this last rule (see, for example, [10, 8] and the references in [14]). We do not claim that the only reason for these violations is that people behave strategically. Nor do we want to suggest that the correct interpretation of the above mentioned models is game theoretic. However, we believe that these models and empirical evidence cannot be rejected as irrelevant on the grounds that violations of probability theory exposes the decision maker to a Dutch book.

All these models analyze the behavior of a single agent. Dutch books must involve at least two agents, therefore the correct framework is game theoretic, and one must assume that agents behave strategically. In particular, the subject who appears to be falling for a Dutch book mat be strategic. Green [7] presents a Dutch book argument in favor of quasi-convex preferences. His argument assumes that the person offering choices to the subject is much more sophisticated than the subject. Our approach is more symmmetric in that the subject bookie is at least as sophisticated as the bettors.

Yaari [17] argued against giving too much normative weight to Dutch books, as they may lead to the conclusion that only expected value maximization is rational. While we do not reject the value of Dutch book arguments, we do reject their usual interpretation.

It seems to be too strong to conclude from this paper that strategic behavior can rationalize all apparent Dutch books. A well known defense of the transitivity axiom is based on the following argument. Suppose that a consumer (strictly) prefers A to B, B to C, but C to A, and assume that he holds a ticket for A. Offer to trade his ticket on A and a small sum of money for a ticket on C, then from C to B and back to A, just to find himself with his initial option less a positive sum of money. One might try to use the approach of this paper to argue that announcing this nontransitivity may be rational provided that the other agent's preferred option is not A, otherwise he will not let the nontransitive agent to get back his original option. However, this objection holds only if the other agent cannot induce the consumer to move once more from A to C. But in that case there is no reason to assume that he will be able to convince him to complete even one cycle.

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