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# A GAME-THEORETIC INTERPRETATION OF SUN TZU'S THE ART OF WAR

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## Abstract

Over twenty five hundred years ago the Chinese scholar Sun Tzu, in **The Art of War**, attempted to codify the general strategic character of conflict and, in the process, offer practical advice about how to win military conflicts. His advice is credited with having greatly influenced both Japanese military and business practices, as well as Mao Tse-Tung's approach to conflict and revolution. The question, however, is whether or to what extent Sun Tzu anticipated the implications of the contemporary theory of conflict -- game theory. The thesis of this essay is that he can be credited with having anticipated the concepts of dominant, minmax, and mixed strategies, but that he failed to intuit the full implications of the notion of equilibrium strategies. Thus, while he offers a partial resolution of "he-thinks-that-I-think" regresses, his advice remains vulnerable to a more complete strategic analysis.

**A Game-Theoretic Interpretation of Sun Tzu's The Art of War**[1]

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The formulation of general strategic principles -- whether applied to war, parlor games such as Go, or politics -- has long fascinated scholars. For some, such as the Chinese strategist Sun Tzu, this fascination is motivated by the necessity for formulating immediate practical advice, while for others the motivation is merely by intellectual curiosity. Regardless of motivation, however, the study of strategic principles is of interest because it grapples with fundamental facts of human existence -- first, people's fates are interdependent; second, this interdependence is characterized generally by conflicting goals; and, finally as a consequence of the first two facts, war is not accidental but is the purposeful extension of a state's policies and must be studied in a rational way.

Sun Tzu's **The Art of War**, written more than two thousand years ago, is, of course, our first written record of the attempt to understand strategy and conflict in a coherent and general way. Its age, however, is less important to us than the fact that it was written at a time of prolonged conflict within an emerging China -- in an era in which the leaders of competing kingdoms possessed considerable experience in the conduct of diplomacy, strategic maneuver, and in the art of war. As such, then, we should presume that **The Art of War** codifies the insights of an era skilled at strategy and tactics. The better we understand it, the better we understand not only the era in which Sun Tzu wrote, but also the essentials of conflict today.

But just as we might suppose that Sun Tzu offers insight into the past and the present, we cannot suppose that our understanding of strategy has not progressed in twenty five centuries. New modes of analysis, including the development of decision theory and the application of mathematics have all entered the domain of strategic analysis to generalize and refine our thinking. Thus, for scholars interested in understanding Sun Tzu's particular contribution, we must address the issue of how to best interpret and analyze his writings in the context of these advances, since it is in this way that we maximize **The Art of War**'s contemporary relevance. This is the issue with which this paper deals.

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Rather than begin our discussion with Sun Tzu himself, we begin with the theory of strategic behavior developed during the latter part of this century called game theory. Briefly, game theory, which we can view as either a branch of mathematics or of political science and economics (Ordeshook 1986), seeks to isolate general, abstract principals of decision making when the outcomes of people's choices depend on what others decide and when everyone is aware of their mutual interdependence. Thus, while we might interpret much of **The Art of War** as modeling and analyzing a particular type of game -- interactions of pure conflict -- game theory is concerned with the general issue of interdependent decision making, including the possibility that people may choose to cooperate so as to achieve mutually beneficial ends. Hence, game theory's applications include not only strategic military planning, but also an analysis of the decisions confronting the heads of business firms as they compete for profits or market share, of political candidates who must compete in order to win elections, of members of committees who compete to form alternative coalitions, and of nation-states who compete to secure advantageous positions in alliances.

Game theory and **The Art of War**, then, each bring something different to our understanding of strategy. Game theory offers generality and mathematical precision and it allows us to ascertain the logical coherence of our ideas about strategic interaction; Sun Tzu provides a specific application of general principals, and demonstrates the art of rendering logical and abstract reasoning practical. The plan of this essay, then, is to review the essential components of game theory in such a way that we can explore the extent to which Sun Tzu's writings are consistent with or illustrate parts of that theory.

Because it is unreasonable to suppose, however, that someone writing more than twenty centuries ago could have anticipated all the nuances of strategic interaction that formal mathematical reasoning reveals to us today, we also want to learn which aspects of Sun Tzu's writings fail to take account fully of what we know generally about strategic choice. We begin Section 1 of this essay with a discussion of "pre-game" decision making -- of situations in which there is only one decision maker. Section 2 takes us to the core of game theory and describes how interdependent choice situations differ fundamentally from simple decision problems. In Section 3 we explore more carefully Sun Tzu's writings as they apply to a particular class of games in which decision makers implement their actions sequentially as in parlor games like Chess and Go. Section 4 looks at games that better model the strategic situation that confronts decision makers in battle -- simultaneous move games and games of imperfect information, and introduces the concept of a Nash equilibrium as a general solution for such games. Section 5 considers those situations that do not have simple Nash equilibria, and which cause us to introduce the idea of a mixed strategy. Here we argue that mixed strategies -- strategies that leave something to chance -- are not mere mathematical curiosities, but are in fact a central feature of Sun Tzu's strategic principles.

## 1. Individual Decision Making

In order to appreciate the lessons and perspectives of game theory as well as of Sun Tzu's insights, it is necessary to first consider a situation in which there is a single decision maker whose actions we are trying to understand, and who must choose one action from some set of alternative actions. The usual decision-theoretic representation of such situations requires a specification of the following components:

1. A list of the decision-maker's alternative actions, where this list is exhaustive in the sense that at least one action in the list must be chosen by the decision maker in question and where the actions in the list are exclusive in the sense that one and only one action can be taken.
2. A list of possible outcomes (consequences of actions taken) that is also exhaustive and exclusive.
3. A specification of the relationship between actions and outcomes -- what outcome prevails after a specific act is chosen.

Appropriately, Chapter 1 of Sun Tzu's text begins with a specification of those elements of a strategic situation that mediate between outcomes and actions-- domestic politics (moral influence), weather, terrain, and doctrine (organizational efficiency of the state and armies) [the remaining factor, command (quality of leadership) is a decision variable pertaining to actions and strategies].

An additional important component of any decision-making environment is a specification of the goals (preferences over outcomes) of the participants. Of course, the goals of the king and an Army's leaders are obvious -- to win. Indeed, winning is a necessity if the state is to survive:[2]

> *War is a matter of vital importance to the state; the province of life or death; the road to survival or ruin.* (I,1).
> *Victory is the main object of war.* (II, 3)

Refining goals further, Sun Tzu is concerned also that too myopic a view of victory can weaken a state. Thus,

> *What is essential in war is victory, not prolonged operations.* (II,21)

More specifically, the first few chapters of **The Art of War** contain a number of passages pertaining to protracted conflicts, which we can interpret to mean that one ought to avoid conflicts that deplete one's resources.

> *For to win one hundred victories in one hundred battles is not the acme of skill. To subdue the enemy without fighting is the acme of skill.* (III,3)

Thus, one's goal should not merely be that of winning a battle, but to win at minimal cost so as not to deplete one's resources for future conflicts. In addition, one should use tactics that maximize the gains from victory,

> *Generally in war the best policy is to take the state intact; to ruin it is inferior to this.* (III,1)

Sun Tzu does not ignore the goals of those who must actually implement plans in warfare -- soldiers:

> *The reason troops slay the enemy is because they are enraged... [and] they take booty from the enemy because they desire wealth... Therefore, when in*

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[2] Throughout this essay our quotations of Sun Tzu's The **Art of War** are taken from Samuel B. Griffith's (1963) translation. The notation following each quote refers to Chapter and paragraph as indexed by Griffith.

*chariot fighting more than ten chariots are captured, reward those who take the first.* (II, 16-18)

In the simplest decision making environment, decision making under certainty, we assume that each action leads to a well defined, specific outcome. In this instance, we need only know the decision maker's preference order over outcomes, since we can suppose that the action leading to the most preferred outcome will be chosen. However, a more general environment, decision making under risk, allows for the possibility that we or the decision maker in question are uncertain as to what outcome prevails after a specific action is taken.

In simple decision theory we assume that the consequences of action are contingent on the "choices" of nature, where by nature we mean an entity that operates without purpose -- an entity that cannot be said to operate in the pursuit of some goal. Because nature has no purpose -- because it is neither purposefully malevolent nor beneficial -- we suppose that it takes its form outside the influence of our actions. Thus, we merely suppose that we can associate probabilities with each of nature's alternative choices.

One example of this situation is a farmer's decision about what crops to plant. The seeds a farmer might sow are the farmer's alternatives; yields are outcomes; preferences are dictated by the relative profitability of outcomes; and nature's choices might correspond to weather patterns for the year. In this instance, we would not normally view nature as a either a benevolent or malevolent creature with goals; instead, we would merely associate probabilities with alternative weather patterns (based, presumably, on the historical record).

With this conceptualization, the consequence of a specific action is a lottery over outcomes, and one question that concerns decision theorists is how people evaluate these lotteries -- how preferences over specific outcomes determines preferences over lotteries over these outcomes. Generally, game theorists assume that we can assign a value to each outcome -- a numerical representation of preference -- such that we can compute the expected value of each action and predict that a decision maker will choose the action that yields the greatest expected value.

To illustrate so that we can later add some strategic complexity, consider Figure 1, which assumes that a decision maker, person 1, must choose between A and B, that nature has two alternatives which it chooses with probabilities p and 1-p respectively, and that the values assigned by the decision maker to specific outcomes are as shown.

In this instance, then, the expected value of alternative A, E(A), equals $4p + 2(1-p) = 2 + 2p$; and the expected value of B, E(B), equals $p + 3(1-p) = 3 - 2p$. Thus the decision maker chooses A if $E(A) > E(B)$, or equivalently, if $2 + 2p > 3 - 2p$ which requires that $p > 1/4$. Hence, if $p = 1/4$, then the decision maker is indifferent between A and B; and if $p < 1/4$, then the decision maker prefers B to A.

| | | Nature | |
|---|---|---|---|
| | | p | 1-p |
| person 1 | A | 4 | 2 |
| | B | 1 | 3 |

Figure 1

On occasion, Sun Tzu offers advice that corresponds to this simple structure. For example,

> *There are five methods of attacking with fire ... There are suitable times and appropriate days on which to raise fires.* (XII,1,4)

There are, however, only a few instances in **The Art of War** in which Sun Tzu focuses on specific decision making problems with nature as the adversary. Indeed, even in the context of the use of fire, Sun Tzu subsequently makes his advice contingent on an enemy's responses,

> *When fire breaks out in the enemy's camp immediately co-ordinate your action from without. But if his troops remain calm bide your time and do not attack.* (XII,7)

Thus, to explore matters further we must turn to game theory and the analysis of interdependent choice.

## 2. The Special Relevance of Game Theory

We do not offer the preceding representation of simple decision making because we believe that it is especially relevant to the analysis of conflict and war, but rather because it serves as a point of departure in securing an understanding of strategy and conflict. Indeed, Sun Tzu offers an early admonition that reveals that he considered such a representation as a mere preliminary:

*All warfare is based on deception.* (I,17)

We cannot deceive nature, because by definition nature is an unthinking anonymous entity. Thus, all warfare entails the strategic interaction of two or more people, and the next issue we must address is the distinction between decision making against nature and decision making against another person.

To see that this distinction is important, suppose we replace nature in Figure 1 with a second decision maker, person 2, who must also choose between two alternatives, say C and D. Thus, the consequences of person 1's decisions depend on what 2 does, and vise versa. Suppose moreover that this person evaluates the outcomes differently than 1 -- specifically, suppose as is most likely the case in military conflict situations that 2 has precisely the opposite preferences of 1. Figure 2 portrays this new situation, where the first number in each cell corresponds as before to 1's preferences, and where the second number corresponds to 2's preferences.

| | | Person 2 | |
|---|---|---|---|
| | | C | D |
| person 1 | A | 4, 1 | 2, 3 |
| | B | 1, 4 | 3, 2 |

Figure 2

There are now two ways in which we can characterize the scenario that confronts persons 1 and 2: (1) <u>perfect information</u> -- one person must choose first and the other chooses second, after learning the choice of the first person; and (2) <u>imperfect information</u> -- both persons choose either simultaneously or ignorance of what each other will do. With perfect information, case (1), the situation can be analyzed straightforwardly. If, for example, person 1 chooses first, then he knows that if he chooses A, person 2 will respond with D, whereas if he chooses B, person 2 will respond with C. Person 1, then, ought to choose A because this alternative maximizes his minimum gain.

Matters achieve considerably greater complexity with imperfect information, case (2), and it is here that we see the profound difference between Figures 1 and 2. Specifically, we can now imagine the following thought process for person 1 as he contemplates his options:

> I think that I should choose A, because it offers me my best choice and provides a better guarantee than B (2 versus 1). But then ... if person 2 reasons as I do, he will infer that I will chose A, in which case he will choose D, in which case I should choose B (since I prefer a payoff of 3 to a payoff of 2). But then ... if person 2 reasons as I do further, he will infer my decision to switch to B in response to his choice of D, in which case he will conclude that C is his better choice; in which case I ought to choose A. But then again, if he anticipates my reasoning, he will conclude that I will choose A, in which case he will respond with D ... and so on.

Of course, circular reasoning will also characterize person 2's thinking. So suppose that person 1, exasperated and perplexed, concludes that person 2 will choose between C and D with equal probability. In this instance, the expected value from A equals $4/2 + 2/2 = 3$, whereas the expected value from B equals $3/2 + 1/2 = 2$. Person 1, then, might decide to choose A. But again, if person 1 believes that 2 will anticipate 1's tentative speculation that 2 will choose randomly, 1 should also anticipate that 2 can infer 1's choice of A, in which case 2 will not choose randomly at all and will instead choose D, because it is a best response to A. But once again, if 1 anticipates this reasoning himself ... and so on and so forth.

Such circular reasoning does not arise in simple decision making scenarios, since by assumption nature does not reason. Thus, the situation changes drastically once we admit the possibility that "nature" is not an anonymous entity, but is a person capable of the same strategic thought as our initial decision maker. And this change requires the development of new tools for thinking about rational individual action.

Implicit in our argument about circular reasoning, however, is the presumption that not only is each decision maker aware of this situation, but each is aware that the other is aware, and so on. Game theorists refer to this assumption as the assumption of <u>common knowledge</u>. In the context of Sun Tzu's analysis, such common knowledge will arise if both sides to a conflict have read **The Art of War** (or have an equivalent source of advice), and if both sides are aware that the other side has access to this volume. In assessing the game theoretic credentials of **The Art of War**, then, one question we must ask is whether and to what extent Sun Tzu's advice accommodates alternative solutions to the dilemma of circular reasoning that common knowledge admits.

In order to make this evaluation, we must first discuss the general components of a game. Briefly, these components are much those of the decision problem we outlined previously, except that now we must make allowance for the fact that there are two or more decision makers. We also must make allowances for the possibility of complex strategic interaction in which people interact over long intervals of time. To illustrate these ideas as we present them, we encourage the reader to keep a parlor game such as Chess in mind. A description of any game, then, necessarily includes the following:

1. A list of relevant decision makers or players. In Chess, there are but two players -- white and black -- whereas in card games there can be many more players.
2. A description of the strategic situation in terms of who moves at what time, and in what order. In Chess, white moves first, then black, etc. Chess also allows long sequences of moves. In other games, however, players may have to move simultaneously, and they may only allow short sequences, including only a single choice for each player.
3. A specification of the alternative actions that each person can take at every opportunity that person has to act. In Chess, white's first move allows ten choices -- each of eight pawns, plus either knight. This list of alternatives will expand or decrease as the game proceeds, as other pieces are freed or eliminated;
4. A specification of what each person knows about the previous choices of other players. In Chess, a person knows all the earlier moves of the opponent. In other games, such as Poker, these moves may be hidden from view; and in other circumstances, players may have to make some choices simultaneously;
5. A description of each player's goals -- an evaluation of all of the final outcomes that the game allows. In Chess as in war these outcomes include victory and defeat, with the presumption that all players prefer victory.

The final idea we must introduce is that of a strategy -- a plan of action for how to play the game. This plan -- one for each player -- specifies the alternative action a player should choose for each and every contingency that that player can encounter as the game unfolds. Strategies, then, consist of contingent actions, and take the form: "If my opponent does ...., then I will respond with ...; but if my opponent chooses ..., then respond with ..." In Chess a strategy for White consists of an opening move, a response to each of Black's potential responses to the opening move, and so on.

It is important to appreciate the centrality of this concept of a strategy to game theory. If we have modeled the situation well, we can anticipate every possible contingency that a player can confront, including those contingencies established by nature. Thus, a strategy should allow a player to adjust his choices in accordance with things learned as the game proceeds. And because a strategy thereby allows for shifting tactics depending on all possible contingencies, we can think of the outcome of a game as being fully determined after all players choose their strategies (up to the uncertainties created by nature), but before those strategies are actually implemented. Once strategies are chosen events unfold according to plans, as the full character of each player's strategy is revealed over time.

This concept of a strategy might seem odd to those who are unfamiliar with strategically complex situations, but it should not be unfamiliar to a reasonably skilled Chess or Go player. Unsurprisingly, then, Sun Tzu seems to have fully appreciated this idea. In addition to his early emphasis of correctly assessing situation, Sun Tzu argues strongly for the necessity for formulating one's plans based on rational calculation beforehand:

> *With many calculations, one can win; with few one cannot. How much less chance of victory has one who makes none at all!* (I, 28)

and as to consequences,

> *Thus a victorious army wins its victories before seeking battle; an army destined to defeat fights in the hope of winning.* (IV,14)
> *Therefore a skilled commander seeks victory from the situation and does not demand it of his subordinates. He selects his men and they exploit the situation.* (V,21,22)

Put differently, after describing a situation's strategic structure, including the strategies available to all antagonists, the problem is to identify suitable strategies for players, including the strategies that opponents are likely to choose.

Sun Tzu offers some specific advice about the general viability of alternative strategies ("disrupt his alliances; attack cities only when there is no alternative; when five times his strength, attack him; if double his strength, divide him; if equally matched you may engage him; and if in all respects unequal, be capable of eluding him for a small force is but booty for one more powerful."). Nevertheless, his general intent is clear -- to analyze the diversity of interdependent choice situations in warfare and to deduce efficient strategies -- plans of action that lead to victory, broadly defined. Consistent with the idea that it is not particular moves but the overall strategy of an opponent that must be countered, he argues correctly that,

> *what is of supreme importance in war is to attack the enemy's strategy.* (III,4)
>
> *Thus, those skilled in war subdue the enemy's army without battle. They capture his cities without assaulting them and overthrow his state without protracted operations.* (III,10)

### 3. Solving Sequential Games

The issue, then, is how to identify good strategies so that we can resolve the circular reasoning that arises from games like the one we offer in Figure 2. At this point we must return to the distinction offered earlier between two types of situations: (1) situations in which the players make their choices in sequence -- one after the other -- as in parlor games like Chess, Go, and Tic-Tac-Toe; and (2) situations in which players must formulate strategies and make choices simultaneously, or, equivalently, in which choices must be made in ignorance of the opponent's actions. In this section we consider the first type of situation, games of perfect information.

Consider a game like Chess in which one player moves first, then the other, then the first again, then the second again, and so on. Suppose we can map out all actions in sequence so that all possibilities are covered. Some sequences will end quickly, as when one person makes a series of skilled moves against an ineffective opponent. However, it is a well known result of game theory that, regardless of the complexity of the situation, in principal the game's outcome will be determinate and we will be able to specify an unambiguously best strategy (plan of action for playing the game) for each person.

This result is general. For example, recall that if the game in Figure 2 is played sequentially with person 1 choosing first, then, as we have previously argued, the eventual outcome is (2,3). Restated in terms of strategies, person 1 confronts only a single contingency -- the necessity for choosing first. Thus, he has only two strategies, A and B. Person 2, on the other hand, confronts two contingencies -- person 1's two choices -- and thereby he has four strategies:

s1: Choose C regardless of what person 1 chooses;

s2: Choose C if 1 chooses A, but choose D is 1 chooses B.

s3: Choose D if 1 chooses A, but choose C if 1 chooses B.

s4: Choose D regardless of what person 1 chooses.

Looking at Figure 2 directly, however, we can see that if 1 chooses A, then 2 will choose D, whereas if 1 chooses B, then 2 will choose C. Clearly, then, 1 should choose A and 2 should respond with D. Thus, with sequential decisions, the outcome (2,3) will prevail. In other words, we can straightforwardly deduce that person 1 will choose the strategy "A" and that person 2 will choose the strategy "Choose D if 1 chooses A, but choose C if 1 chooses B."

Game theory tells us, then, that in principle, games such as Chess can be "solved." Practical considerations, of course, render it impossible to map out all moves in such games, which keeps us playing them as tests of skill and experience. On the other hand, a simpler game such as Tic-Tac-Toe, because it too can be solved, is interesting only to children who have not yet comprehended its strategic structure. The skill and experience of playing chess well, though, is much like the skill at playing any complex game -- using our experience and skills to simplify a complex strategic structure so that we can learn strategies that ought to be avoided, and so that we can identify when our opponent is using a good or a poor strategy. Skill and experience also help us reduce a game's complexity so that, although its simplified form may not match that of Tic-Tac-Toe, its general principles can be understood and optimal contingent plans formulated.

And it is evident that the formulation of optimal contingent plans characterizes Sun Tzu's intent. For example,

> *The doctrine of war is to follow the enemy situation in order to decide on battle.* (XI,60)

That one should formulate plans contingent on an enemy's dispositions is, of course, self-evident. More interestingly, it is important to keep in mind the implications of the fact that games such as Chess can, in principal, be as thoroughly analyzed by one person as they can by that person's adversary. And barring any advantage to the player who moves first, if the game is symmetric in all other respects -- if no player has a natural superiority in terms of the size and quality of one's army and in terms of the quality of leadership -- then it is a well known theoretical result of game theory that the eventual outcome must be a draw. Sun Tzu obviously intuited this important theorem:

> *Invincibility depends on one's self; the enemy's vulnerability on him.* (IV,2). *It follows that those skilled in war can make themselves invincible but cannot cause an enemy to be certainly vulnerable.* (IV,3) *Therefore it is said that one may know how to win, but cannot necessarily do so.* (IV,4)

And, as in Chess or any game which allows sequential moves,

> *the skilled commander takes up a position in which he cannot be defeated and misses no opportunity to master his enemy.* (IV.13)

Of course, so important is this fact that Sun Tzu could assert with respect to the parameters of conflict that might generate an asymmetric conflict,

> *There is no general who has not heard of these five matters* [moral influence, weather, terrain, command, doctrine]. *Those who master them win; those who do not are defeated* (I,9)

We should consider one final issue with respect to games in which the players move sequentially -- namely, whether there is any advantage to moving first or second. Of course, whether and what type of advantages accrue to the first mover in a game in which the players move in sequence depends on the specific structure of the situation under consideration. For example, Sun Pin, writing one hundred years after

Sun Tzu, recounts the now famous story of the horse race in which the opponent possessed three horses generally superior to T'ien Chi, the commander and chief of Ch'i. Sun Pin's advice was simple: pair the worst horse against the opponent's best, the best against the opponent's second best, and the second best against the opponent's worst. In this way, two out of three races can be won. Clearly, moving second by having the option of determining the pairings of the horses confers the advantage on T'ien Chi.

In order to interpret this example properly, however, we must keep in mind the distinction between actions and strategies. Strategies are plans -- rules for selecting actions as contingencies arise -- whereas actions are merely pieces of the plan. Sun Pin's example is especially simple because action (the assignment of horses) is equivalent to a strategy. In Chess, on the other hand, as well as in the complex maneuvers of war, the equivalence between actions and strategy is lost.

With respect now to the issue of advantages, for the class of games we are considering here -- games of pure conflict -- the advantage belongs to the player who reveals his <u>strategy</u> last. But this does not necessarily mean that the advantage belongs to the player who makes a second <u>choice</u> in a sequential game. In Chess, for example as in Tic-Tac-Toe, the advantage belongs to whoever moves first, which is different from revealing one's strategy. Now consider Sun Tzu's view of advantages. In one instance, he argues that rather than engage the enemy as the first move, certain terrain dictates forcing the enemy to move first:

> *Ground equally disadvantageous for both the enemy and ourselves to enter is indecisive. The nature of this ground is such that although the enemy holds out a bait I do not go forth but entice him by marching off. When I have drawn out half his force, I can strike him advantageously.* (X,4)

But at another point Sun Tzu appears to argue for the advantage of the first move:

> *Generally, he who occupies the field of battle first and awaits his enemy at ease; he who comes later to the scene and rushes into the fight is weary. And therefore those skilled in war bring the enemy to the field of battle and are not brought there by him.* (VI,1,2)
> *Ground which both we and the enemy can traverse with equal ease is called*

*accessible. In such ground, he who fights first takes high sunny positions convenient to his supply routes can fight advantageously.* (X,2)

This apparent confusion, however, is resolved if we keep in mind the difference between strategy and action. In the first cited passage, the positions of the players have already been determined -- the circumstance of the game are fixed -- and from that point, it is best to reveal one's strategy second. In contrast, in the second cited passage, the field of battle has yet to be determined, and Sun Tzu is in fact arguing that it is better to be the one who dictates which game is to be played or, equivalently, which player is to be assigned which position in the game. In this second passage, then, Sun Tzu is referring to the first move in the game. Thus, there is no confusion, and we can conclude that Sun Tzu does in fact appreciate the advantages of choosing one's strategy second.

### 4. **Games With Imperfect Information**

Pre-war preparations -- evaluating one's domestic power, recruiting skilled commanders, training troops, and choosing whether or not to engage in war -- proceed sequentially so that one's character, as well as that of an enemy, is revealed as events unfold. Tactics are chosen by different rules. The success of battle tactics depends on contingencies, including the opponent's preparations and tactics, that often become apparent only after it is too late to condition on them, and decisions must be made with imperfect information. In some instances, this imperfect information arises because choices are made simultaneously by all antagonists, whereas at other times choices are not simultaneous but information is imperfect because choices are hidden from view. Regardless of its source, the game theorist's approach to imperfect information is to identify classes of games for which this fact matters little. Referring to Figure 3, notice that each player has a <u>dominant</u> choice -- A for person 1 and D for person 2 -- where by dominant we mean an action that is better than all others, regardless of what other persons choose. In Figure 3, A is better than B for person 1 regardless of what 2 chooses, whereas D is better than C for 2, regardless of what 1 chooses. When decision makers have dominant choices the analysis of ultimate decisions and outcomes avoids the circular reasoning that we applied to Figure 2.

|  |  | Person 2 |  |
| --- | --- | --- | --- |
|  |  | C | D |
| person 1 | A | 4, 1 | 3, 2 |
|  | B | 2, 3 | 1, 4 |

Figure 3

We should emphasize that the possibility of a dominant choice is not altogether unrelated to the issue of simultaneous versus sequential choice. Indeed, a game with sequential choices such as Chess also occasions dominant strategies, and thus we can view the concept of dominance as a generalization of previous analysis. To see what we mean by this, recall that if the game in Figure 2 is played sequentially with person 1 choosing first and person 2 choosing second, person 1 has two strategies, A and B, whereas person 2 has four strategies which we denoted s1, s2, s3, and s4. Consider now the game portrayed in Figure 4, which describes the outcome that prevails for each of the eight possible joint choices of strategies by persons 1 and 2. Notice in particular that although person 1 does not have a dominant strategy, s3 is dominant for person 2 -- it is never worse and is sometimes better than s1 and s4, and it is uniformly better than s2. Thus, person 2 should choose s3. But, since person 1 is also assumed to be cognizant of this game, person 1 should be able to infer that 2 will use s3, in which case person 1 should choose A. This reasoning, of course, leads to precisely the same outcome we deduced earlier -- person 1 chooses A, 2 chooses D in response, and the outcome (2,3) prevails.

|  |  | Person 2 |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | s1 | s2 | s3 | s4 |
| person 1 | A | 4,1 | 4,1 | 2,3 | 2,3 |
|  | B | 1,4 | 3,2 | 1,4 | 3,2 |

Figure 4

Sequential choice, then, is sufficient to generate dominant strategies (but not necessary), and although as we argue later, Sun Tzu seems to have understood the consequences of moving from a game of simultaneous choice to one with sequential choices, much of his strategic analysis consists of identifying dominant and dominated strategies. First, with respect to dominated strategies -- strategies that ought to be avoided in any context:

> *You should not encamp in low lying ground ... You should not linger in desolate ground ... There are some roads not to follow; some troops not to strike; some cities not to assault; and some ground which should not be contested.* (VIII,1-7)
>
> *do not ascend to attack ... When an advancing enemy crosses water do not meet him at the water's edge ... Do not take positions downstream.* (IX, 2-6)

And with respect to dominant strategies,

> *In enclosed ground resourcefulness is required. In death ground, fight.* (VIII,5,6).
>
> *Encamp on high ground facing the sunny side; Fight downhill ... After crossing a river you must move some distance away from it. It is advantageous to allow half his force to cross and then strike ... Cross salt marshes speedily.* (IX, 1-8)

We have introduced this notion of a dominant strategy not only because it helps us interpret parts of **The Art of War**, but also because the existence of such strategies helps us avoid circular reasoning. Specifically, notice the special feature of the cell characterized by the joint choice of (A,D) in Figure 3 -- once "at" such a cell, neither player has any incentive unilaterally to change his decision. Indeed, it is this characteristic of (A,D) that terminates circular reasoning. The existence of dominant strategies, however, is not essential to offering a solution to a game. Consider Figure 5 and the cell corresponding to the joint choice of (B,E). Notice first that neither player has a dominant choice -- for example, A is best for 1 if 2 chooses D, B is best if 2 chooses E, and C is best if 2 chooses F. Nevertheless, once at (B,E) neither player has a unilateral incentive to defect to some other choice. Thus, (B,E) terminates circular reasoning as well -- in other words, the existence of dominant choices are sufficient to terminate such reasoning, but they are not necessary.

| | | Person 2 | | |
|---|---|---|---|---|
| | | D | E | F |
| | A | 8,1 | 3,6 | 1,8 |
| Person 1 | B | 7,2 | 5,4 | 6,3 |
| | C | 2,7 | 4,5 | 9,0 |

Figure 5

Because it terminates circular reasoning, cells such as (B,E) -- called Nash equilibrium after the theorist who proved a number of important results about them for different classes of games -- is profoundly important as an idea about ultimate choices. Consequently, game theorists have devoted considerable efforts at analyzing the properties of equilibria, refining Nash's original formulation, applying those refinements to situations of far greater complexity than the simple games we describe here, and testing ideas in empirical contexts. If there is a critique that can be directed against Sun Tzu's analysis of war, though, it is that he seems to pay too little heed to the necessity of resolving the dilemma of circular reasoning, and as a consequence he fails to infer this concept of an equilibrium as a solution to interdependent decision making. Put differently, he fails to take full account of the possibility that, in addition to the king he is advising, enemy kings have also read **The Art of War**, and the fact of this common knowledge is itself common knowledge. In that event, only the notion of an equilibrium can be used to formulate strategic plans and to resolve circular reasoning.

Despite this criticism, however, we can find within game theory itself a reason for supposing that this failure does not necessarily negate the value of Sun Tzu's advice and analysis. Specifically, one fact about Nash equilibria that applies to the types of games that especially concerned Sun Tzu -- games of pure conflict in which one person's gain is another person's loss (called zero-sum or constant-sum games)-- is that decision makers are assured of achieving an equilibrium if they abide by a simple rule of thumb in choosing their actions. Suppose a person assumes that his opponent is as intelligent as he is and is capable of anticipating his thoughts. In this event, the decision maker whose actions we are studying should "assume the worst" -- should assume that regardless of what action he takes, his opponent will take best advantage of him. Barring the assumption that one's opponent is somehow less capable than oneself -- always a dangerous supposition and likely to lead to unpleasant surprises--

a person should then choose the strategy that maximizes one's minimum gain, or equivalently, minimizes one's maximum loss (called a minmax strategy).

Applying this argument to the game in Figure 5, notice that the minimum gain person 1 associates with A is 1, from B it is 5, and from C it is 2. Thus, person 1 should choose B. Similarly, the minimum gain to person 2 from D is 1, for E it is 4, and from F it is 0. Thus, person 2 should choose E. More interestingly, notice now that this reasoning leads to the joint choice of (B,E), which is the game's Nash equilibrium. Hence, in the case of games of pure conflict, a prudent strategy leads to actions that are consistent with the strategic imperatives proscribed by game theory.

Although we cannot find any direct reference in Sun Tzu to the notion of an equilibrium as a means of terminating circular strategic reasoning, it is not unreasonable to suppose that he nevertheless grasped the essence of a strategy designed to minimize one's losses, but which would nevertheless take advantage of an unskilled opponent:

> *Anciently the skillful warriors first made themselves invincible and awaited the enemy's moment of vulnerability.* (IV,1)
> *Therefore the skilled commander takes up a position in which he cannot be defeated and misses no opportunity to master his enemy.* (IV,13)

And to cite a previously quoted passage,

> *Invincibility depends on one's self; the enemy's vulnerability on him. It follows that those skilled in war can make themselves invincible but cannot cause an enemy to be certainly vulnerable. Therefore it is said that one may know how to win, but cannot necessarily do so.* (IV,2,3,4)

Sun Tzu, then, takes us at least part of the way towards the notion of an equilibrium in that, if both sides to a conflict heed the advice of adopting strategies that minimize one's potential losses against a skilled opponent but that are also viable against a less skilled player, then an equilibrium prevails. Thus, as long as the game under consideration is constant- or zero-sum, and as long as an equilibrium exists in simple strategies, Sun Tzu provides us with the requisite tools for achieving an equilibrium and for taking advantage of an opponent who fails to act accordingly.

## 5. Mixed Strategies

One consequence of the preceding discussion is that whether or not a game is characterized by perfect or imperfect information matters little if there is a dominant choice or a well defined equilibrium for a game. Players can "solve" the game and arrive at determinate outcomes. However, military conflicts typically have a different strategic character. Specifically, they generally do not have an equilibrium in simple strategies. Thus, we must be careful about our arguments for the relevance of the notion of an equilibrium, because if games do not always have equilibria then this non-existence precludes the possibility of scientific generality.

With this in mind, let us return to the game in Figure 2, which we used to illustrate circular reasoning and which, at first glance, does not appear to possess an equilibrium. Recall our suggestion that person 1 might decide, out of exasperation, that person 2 will make a random choice. We tentatively rejected this idea because such an assumption did not avoid the circular reasoning we sought to avoid. However, notice that we assumed that person 2 used a particular lottery, 50-50, and we did not check whether all such lotteries led us in a cycle.

So suppose more generally that person 2 chooses between alternatives C and D with probabilities p and 1-p, and suppose also that person 1 chooses between A and B with probabilities q and 1-q. Notice now that if person 1, given 2's probabilities of p and 1-p, is not indifferent between the lotteries provided by A and B, then he should switch to one or the other with certainty. That is, person 1 would be willing to stay with q and 1-q if and only if the expected value he associates with A equals the expected value he associates with B, as those expected values are determined by person 2's strategy. For the game in Figure 2, this equality requires that

$$4p + 2(1-p) = p + 3(1-p)$$

which we can solve, and conclude that $p = 1/4$. The same argument applies, of course, to person 1 -- 1 should be willing to give A and B some weight if and only if 2 is indifferent between C and D, which requires that,

$$q + 4(1-q) = 3q + 2(1-q),$$

which we can solve to give $q = 1/2$.

The conclusion we reach here, then, is that if person 1 chooses randomly between A and B, and if person 2 chooses C with probability 1/4 and D with probability 3/4, then neither person has any incentive to shift unilaterally to any other lottery (including degenerate lotteries in which A or B and C or D are chosen with certainty).

Thus, for the situation portrayed in Figure 2, there exists a mixed strategy Nash equilibrium.

What gives this notion of a mixed strategy equilibrium special relevance is the important theorem proved by Von Neumann and Morgenstern (1944), which establishes that every n-person game in which each decision maker has a finite number of choices, has at least one equilibrium in either mixed or pure strategies.[3] Thus, the potential scientific generality of the Nash equilibrium concept is established.

However, notice that a mixed strategy (1) minimizes one's vulnerability to an equally strategic opponent, and (2) takes advantage of an opponent who errs. Moreover, the use of a random device in particular ensures that one's tactics do not fall into a pattern that an opponent can detect. Of course, the notion of a mixed strategy equilibrium might, at first glance, seem to be little more than a cute mathematical trick. However, rather than being a mathematical trick, Sun Tzu seems to have anticipated its advantages,

> *Therefore, when I have won a victory, I do not repeat my tactics but respond to circumstances in an infinite variety of ways.* (VI,26)
> *He [the general] changes his methods and alters his plans so that people have no knowledge of what he is doing.* (XI,45)
> *He alters his camp-sites and marches by devious routes. and thus makes it impossible for others to anticipate his purpose.* (XI,46)

Admitting mixed strategies into one's arsenal of choices has the consequence of taking a finite number of pure strategies and rendering one's choices infinite in number.

> *The musical notes are only five in number but their melodies are so numerous that one cannot hear them all. The primary colors are only five in number but their combinations are so infinite that one cannot visualize them all. The flavors are only five in number but their blends are so various that one cannot taste them all. In battle there are only the normal and extraordinary forces, but their combinations are limitless; none can comprehend them all.* (V,8-11)

[3] By pure strategies we mean the strategies formed by a direct analysis of the game's structure, whereas mixed strategies correspond to lotteries over these pure strategies.

Moreover, the special character of a mixed strategy is that even after pure choices are revealed, an opponent cannot be certain that a choice that appears inferior in the short term is not part of a grander, more all-encompassing plan -- they preclude the possibility that an enemy can infer a future choice with certainty based on previous actions:

> *It is according to the enemy's shapes that I lay the plans for victory, but the multitude does not comprehend this. Although everyone can see the outward aspects, none understands the way in which I have created victory. Therefore, when I have won a victory, I do not repeat my tactics but respond to circumstances in an infinite variety of ways.* (VI,25,26)

We cannot say whether Sun Tzu ever explicitly advocated the use of random devices to disguise strategic intent, nor do the commentators on his writings offer clarification. Nevertheless, we can offer an interpretation of his distinction between normal (*cheng*) and extraordinary forces (*ch'i*) that renders the notion of a mixed strategy a center-piece of his analysis. Specifically, if we associate the concept of a normal force with a pure (non-random) strategy and that of an extraordinary force with that of a mixed strategy, we can give fuller meaning to Sun Tzu's admonition that:

> *Now the resources of those skilled in the use of extraordinary forces are as infinite as the heavens and earth; as inexhaustible as the flow of the great rivers. For they end and recommence; cyclical, as are the movements of the sun and moon. They die away and are reborn; recurrent as are the passing seasons.* (V,6,7)

Having uncovered evidence that Sun Tzu appreciated the role of mixed strategies, we can nevertheless offer one criticism of his analysis. Recall that in the previous section we also cited evidence that Sun Tzu understood the role of minmax pure strategies. The difficulty, however, is that if a game possesses an equilibrium only in mixed strategies, minmax pure strategies cannot yield an equilibrium and, thus, they cannot terminate cyclical reasoning. And unfortunately, what we cannot discover in **The Art of War** are any clear guidelines for ascertaining when a strategist should choose minmax strategies and when he should abide by mixed strategies. In this sense, then, Sun Tzu's analysis is incomplete.

### 6. Secret Agents

Sun Tzu ends his text with a chapter on secret agents and with the admonition that

> *only the enlightened sovereign and the worthy general who are able to use the most intelligent people as agents are certain to achieve great things. Secret operations are essential in war; upon them the army relies to make its every move.* (XIII,23)

The emphasis Sun Tzu places on secret agents is understandable owing to the enormous strategic advantage to be gained from knowing an opponent's strategy beforehand. In effect, the role of the secret agent is to allow a decision maker to condition has actions on a richer information base and to render moves sequential rather than simultaneous. To see the advantages of this change, consider, for example, the game in Figure 2. If choices are made simultaneously or, equivalently, if no player in the game has perfect information, and if each person uses his equilibrium mixed strategy, then the expected payoff to each person 2.5. On the other hand, as we have already seen, if person 2 moves after 1, but knows person 1's choice beforehand, then the final payoffs are 2 to person 1 and 3 to person 2. Moreover, if person 1 is unaware of 2's better information and continues to employ a mixed strategy, then 2's expected payoff rises higher to 3.5, and person 1's drops to 1.

> *The enemy must not know where I intend to give battle. For if he does not know where I intend to give battle he must prepare in a great many places.* (VI,14)

The role of secret agents, therefore, is clear -- to allow a decision maker to condition on his opponent's choices and to proceed through the play of the game as if it were a sequential game:

> *Now the reason the enlightened prince and the wise general conquer the*

*enemy whenever they move and their achievements surpass those of ordinary men is foreknowledge. What is called 'foreknowledge' cannot be elicited from spirits, nor from gods, nor by analogy with past events, nor from calculations. It must be obtained from men who know the enemy situation.* (XIII,3,4)

For the particular class of games with which Sun Tzu is concerned, choosing one's strategy with foreknowledge of an opponent's choices is always advantageous--regardless of whether the opponent is aware of one's foreknowledge or not. The issue with which Sun Tzu seems to have the greatest difficulty, however, is the possibility that both sides will use agents to feed each other false information. He advises that

*It is essential to seek out enemy agents who have come to conduct espionage against you and to bribe them to serve you. Give them instructions and care for them. Thus double agents are recruited and used.* (XIII,17)

But what of the possibility that double agents are triple agents, and so forth? Once this question is admitted, we return again to the dilemma of circular reasoning that game theory tries to resolve. Now, however, the game is characterized not only by imperfect information but also by <u>incomplete</u> information, and the players must be concerned with what others believe about them -- about their capabilities and their intent -- as a function of their choices. Thus, choices must be selected not only to manipulate outcomes directly, but also to manipulate them indirectly through the manipulation of beliefs. And when all players know that such manipulations are possible, the analysis of strategy becomes doubly complex.

Earlier we interpreted Sun Tzu's admonition that "all warfare is based on deception" as implying that game theory as opposed to simple decision theory is required to understand his analysis. Certainly, mixed strategies are one form of deception. However, it is also evident that deception meant more to Sun Tzu that merely randomizing one's choices:

*when capable, feign incapacity; when active, inactivity. When near, make it appear that you are far away; when far away, that you are near. Offer the enemy bait to lure him; feign disorder and strike him.* (I,18-20)

*Pretend inferiority and encourage his* [the enemy general's] *arrogance.* (I,23)

Once we accept this view of deception, however, a variety of new questions arise such as: Is it possible to deceive an opponent when that opponent is aware of one's intent and opportunities? Can players deceive each other simultaneously when each knows that the other is trying to deceive, and when each knows that the other knows?

Anoth missing element in Sun Tzu's analysis is revealed by his failure to extend his arguments about strategies and his understanding of equilibria to the issue of double agents. In judging Sun Tzu's contribution to our understanding of strategy, however, we should keep in mind that although game theory is a field that has undergone nearly fifty years of development, with hundreds of researchers participating in the process, we have only recently learned how to treat the manipulation of information as part of a player's strategic arsenal (see for example Rasmusen 1989). Understanding the strategic manipulation of information is especially difficult because it deals with the potential for altering an opponent's view of the game that is being played when that opponent knows it is in your interest to thus manipulate. Resolving circular reasoning in this circumstance requires the use of advanced principals of probability theory and mathematics, and so we should not be surprised to learn that Sun Tzu's treatment of information is incomplete. Indeed, we should marvel at the fact that he understood intuitively as much as he did.

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