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BARGAINING COSTS AND FAILURES IN THE SEALED-BID DOUBLE AUCTION

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ABSTRACT

This paper analyzes bilateral bargaining in the sealed-bid double auction with bargaining costs. There exists a multiplicity of equilibria to this game, all of which have unsatisfactory properties. Since anything seems possible, we focus on the completely mixed strategy equilibria (C.M.S.) but find that such equilibria require that the negotiator with the higher bargaining cost receive higher profits. Allowing the bargaining process to be dynamic does not entirely solve the problem because the offers in the dynamic game can demonstrate *chaotic* behavior. Moreover when failure costs are low there exist many infinite horizon C.M.S. equilibria. One feature of the C.M.S equilibrium is the existence of a significant probability of delay which is consistent with empirical reality. Finally, if there is asymmetric information over bargaining costs. the negotiator with the higher bargaining costs obtains lower profits. Thus, asymmetric cost information leads to more plausible properties for most bargaining equilibria.

I. Introduction

The Stahl-Rubinstein model of bargaining posits a game in which offers and counter-offers are are tendered until an agreement is reached. The surplus over which the players are bargaining shrinks with successive offers--based on individual discount rates. In the perfect equilibrium of this game the player making the first offer receives a potentially significant advantage and there are no bargaining failures--agreement occurs in the first period. The first period agreement result is frequently criticized from an empirical point of view due to the numerous observations of bargaining delays and failures (see, for example, Tracy [1986] and Card [1988]). In addition, it is hard to describe institutional arrangements in the field that allow for endogenous first mover advantages. Extensions to the Stahl-Rubinstein model of bargaining where there is asymmetric information concerning individual values or discount rates show that delays in agreements can occur (see Cho [1988]). Nonetheless, these bargaining "failures" are efficient because delay serves the purpose of information transmission. Delay allows for the separation of types provided that the discount rate is not arbitrarily close to one.

To remedy the first mover advantage, analysts have investigated simultaneous move bargaining games. In particular, a sealed-bid double auction for bilateral trading has been developed and analyzed (see Myerson and Satterthwaite [1983], Chatterjee and Samuelson [1983], Leininger, Linhart, and Radner [1989], Satterthwaite and Williams [1989], and Broman [1989]). This mechanism is usually described as a game of incomplete information in which the buyer and seller of an object simultaneously submit offers. There is asymmetric information about buyer's reservation price and the seller's cost of production. Trade occurs when the buyer's offer exceeds the seller's offer, and the price paid by the buyer is the average of the two offers. This mechanism has a multiplicity of Bayes-Nash equilibria, some with low efficiency properties (see Leininger, Linhart and Radner [1989]). Thus, it seems that uncertainty concerning the players' valuations is enough to generate delay, failures, or other inefficiencies in bargaining. This paper complements this recent research of the sealed-bid double auction by introducing delay costs and multiple bargaining periods.

We shall begin by analyzing the sealed-bid double auction with complete information. In addition, if the parties do not come to an agreement each must pay a cost of $c > 0$. The cost c can be interpreted as a direct loss out of current earnings from not agreeing. The cost becomes sunk if there is disagreement but can be avoided if the parties reach an accord. For example, strikes are disadvantageous to both parties: the firm relinquishes profits and the workers loose income. In pretrial negotiations the plaintiff and defendant often pay incremental lawyer fees until a settlement is reached. A further empirical example of such institutions is analyzed in Rosenthal [1988] where owners of marshland contemplate draining the marsh and enjoying the returns to better pasture. For the marsh to be drained, however, the owners must agree on a rule to divide the surplus. Every time a proposal fails the owners collectively forego the rent to the surplus and pay legal fees to draw up a new proposal. In the dynamic version of this game, disagreement does not affect the size of the surplus because all the cost of failure are borne incrementaly and out-of-pocket.

The sealed-bid double auction (under complete information) has a multiplicity of Nash equilibria and thus to make any headway we must confront the problem of equilibrium selection. We focus our attention on completely mixed strategy (C.M.S.) equilibria since alternative selection criteria such as focal equilibria and trembling hand perfect equilibria are not robust in both payoffs and conjectures. Of course, when mixing is considered failure and delay become nontrivial and the expected value of the game is very sensitive to the underlying parameters of the bargaining environment. The next section will formally describe the bargaining model for the case of discrete values. Later sections will extend this model to both dynamic and incomplete information environments.

II. The One-Shot Game

The bargaining process can be reduced to the strategic interaction between a buyer and a seller negotiating the terms of trade for an object. We assume the buyer has a reservation price of 1, and the seller's reservation price is 0. If the buyer and seller do not come to an agreement each will incur a cost of $c>0$. Notice that this game has a simple definition of efficient outcomes: trade occurs in the first period and joint profits are equal to 1. This model conforms to an agency model of bargaining where a union and management board negotiate a wage contract via intermediaries. Thus, we have a bilateral monopoly situation where the players must solve a coordination problem with short commitment.

The sealed-bid double auction requires each player to simultaneously submit an offer price. Let s denote the offer price for the seller and b the offer price of the buyer. An agreement is reached if $s \leq b$. If an agreement is reached the settlement price occurs half way between the offers, i.e., the seller receives $\frac{(b+s)}{2}$ and the buyer receives $1-\frac{(b+s)}{2}$. If $s>b$, then the parties have not reached an agreement and each player pays c. The model presented below can be extended to the case of multiple parties or where the settlement price is given by $s + k(b-s)$ if $b \geq s$ and $k \in [0,1]$.

II.1 *The Model*

Suppose that buyer and seller offers are limited to a finite set $\mathbb{Z} \subseteq [0,1]$. Without loss of generality, the elements of $\mathbb{Z}$, denoted by z_i, will be indexed from 0 to n such that $z_i = \frac{i}{n}$, i.e., the offer grid is defined by $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$. Thus, the actions are given by the seller selecting a $z_i \varepsilon \mathbb{Z}$ and the buyer selecting $z_j \varepsilon \mathbb{Z}$. The payoffs for the seller can thus be written as:

$$\begin{array}{ll} \frac{(z_i+z_j)}{2} & \text{if } i \leq j \\ -c & \text{if } i > j \end{array}$$

In a similar fashion we can describe the payoffs of the buyer as:

$$\begin{array}{ll} 1-\frac{(z_i+z_j)}{2} & \text{if } i \leq j \\ -c & \text{if } i > j \end{array}$$

This payoff structure can easily be represented in matrix form where the column index is the catalog of buyer offers and the row index is the catalog of seller offers. We shall denote the seller payoff matrix as $\mathcal{A}$. In particular, $\mathcal{A}$ is a $(n+1) \times (n+1)$ matrix with entries:

$$\begin{array}{ll} a_{ij} = \frac{(i+j)}{2n} & \text{for } i \leq j \\ a_{ij} = -c & \text{for } i > j \end{array}$$

Similarly, the buyer payoff matrix $(\mathcal{B})$ is a $(n+1) \times (n+1)$ matrix with entries

$$\begin{array}{ll} b_{ij} = 1-\frac{(i+j)}{2n} & \text{for } i \leq j \\ b_{ij} = -c & \text{for } i > j \end{array}$$

A <u>strategy</u> for the seller is a probability distribution over $\mathbb{Z}$, $P_s : \mathbb{Z} \mapsto [0,1]$ where $P_s(z_i) = p_{si}$ with $\sum_k p_{sk}=1$ and $p_{sk} \geq 0$. Similarly, a <u>strategy</u> for the buyer is a probability distribution P_b: $\mathbb{Z} \mapsto [0,1]$ for the buyer offers $z_0, \ldots, z_n$.

II.2 *Equilibrium*

An <u>equilibrium</u> in this game is simply a pair of functions (P_s^*, P_b^*) such that P_s^* maximizes the seller's expected profits π_s;

$$\pi_s\,(P_s|P_b^*) = P_s\,A\,(P_b^*)^T$$
subject to $p_{si} \geq 0$ and $\sum p_{si} = 1$

and P_b^* maximizes

$$\pi_b\,(P_b|P_s^*) = P_s^*\,B\,(P_b)^T$$
subject to $p_{bi} \geq 0$ and $\sum p_{bi} = 1$

where ^{T} denotes the matrix transpose.

We now state an obvious result for mixed strategy equilibria (indifference in payoff for all equilibrium messages).

Lemma 1: $$\mathcal{A}\,(P_b^*)^T = [\,\pi_{si}] = \begin{cases} k & \text{iff } p_{si}^* > 0 \\ < k & \text{if } p_{si}^* = 0 \end{cases}$$

$$(P_s^*)\,\mathcal{B} = [\,\pi_{si}] = \begin{cases} l & \text{iff } p_{bi}^* > 0 \\ < l & \text{if } p_{bi}^* = 0 \end{cases}$$

Let $S=\{z_i : p_{si}^* > 0\}$ and $B=\{z_j : p_{bj}^* > 0\}$, then:

Lemma 2: $S = B$.

Lemma 2 shows that there must be consistency in the offers of each "rival" or else one side can "take advantage" of the other in its mixed strategy. Lemma 2 also tells us that there are 2^{n+1}-1 non-empty subsets of $\mathbb{Z}$ that are potential equilibria to this game. Notice that Lemma 1 and 2 together tells us that we need only consider the reduced matrices of $\mathcal{A}$ and $\mathcal{B}$ in which the rows (and corresponding columns) of $z_k \in \mathbb{Z}\backslash S$ are eliminated. For each subset Ω of $\mathbb{Z}$, let $\mathcal{A}_\Omega$ and $\mathcal{B}_\Omega$ denote the reduced matrices to the subset Ω. From Lemma 1 we need only consider the equations:

$$\mathcal{A}_\Omega\,(P_b^*)^T = K \qquad (2.1)$$
$$(P_s^*)\,\mathcal{B}_\Omega = L \qquad (2.2)$$

Where K and L are the appropriate dimensioned matrix of constants k and l respectively. Suppose $\mathcal{A}_\Omega^{-1}$ and $\mathcal{B}_\Omega^{-1}$ exist so that

$$(P_b^*)^T = \mathcal{A}_\Omega^{-1}\,K \qquad (2.3)$$
$$(P_s^*) = L\,\mathcal{B}_\Omega^{-1} \qquad (2.4)$$

, Since $P_b^*(k)$ is a linear function of k and $\sum p_{bi}^*(k) = 1$ (there is only one k that will solve this problem). Thus, for each k and l there is a unique pair ($P_b^*(k)$, $P_s^*(l)$) that solve (2.3) and

(2.4). All proofs in this paper will be supplied in the Appendix.

Theorem 1: There exists 2^{n+1}-1 equilibria to the sealed-bid double auction under certainty with offer grid of finesse n and failure cost c>0.

Theorem 1 is an extension of the result found in Broman [1989]. The number of pure strategy equilibria is n (each element in $\mathbb{Z}$ is an equilibrium). It is also the case that there is one equilibrium which spans the entire offers of $\mathbb{Z}$ (the completely mixed strategy equilibrium). The probability for the mixed strategy equilibrium that spans $\mathbb{Z}$ (for the seller) is given by the algorithm (see Appendix for derivation):

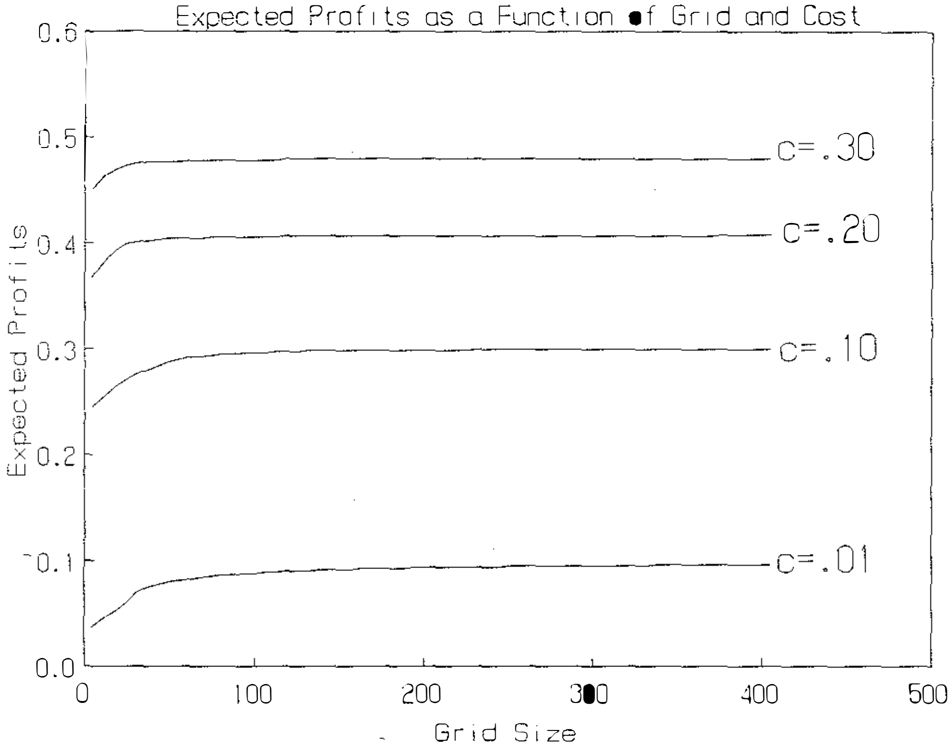
$$p_i = \frac{1}{2(n-i+nc)} \prod_{j=1}^{i-1} \frac{n-j+nc+\frac{1}{2}}{n-j+nc} p_0$$

$$p_0 = \prod_{j=1}^{n} \frac{n-j+nc}{n-j+nc+\frac{1}{2}}$$

The limit as $n \to \infty$ of the above probabilities exists with $p_i \to 0$ and $p_0 \to a(c) > 0$. Figure 1 supplies a graph for different values of c as n gets large.

Figure 1

Expected Profits as a Function of Grid and Cost



Unfortunately, extending the model to the case of continuous (or measureable) offer sets does not allow for completely mixed strategy equilibrium, even if positive probability can be placed on single offers. This condition follows from the expected profit function of each player. Let f be the density of seller offers and g the density of buyer bids. Suppose f,g> 0 (bounded and continuous) almost everywhere on [0,1], then profits are defined by:

$$\pi_s(f,\alpha)= \qquad \int_0^1 f(x) \left\{ \int_x^1 g^*(y)\frac{(x+y)}{2} \ dy \ - c\int_0^x g^*(y) \ dy \right\} \ dx$$

$$\pi_b(g,\beta)= \qquad \int_0^1 g(y) \left\{ \int_0^y f^*(x)[1-\frac{(x+y)}{2}] \ dx \ - c\int_y^1 f^*(x) \ dx \right\} \ dy$$

For f^*,g^* to be a Nash equilibrium we must have that for all offers x and y:

$$\left\{ \int_x^1 g^*(y)\frac{(x+y)}{2} \, d\beta^*(y) \ + \int_0^x g^*(y)(-c) \, d\beta^*(y) \right\} \ = k \tag{2.5}$$

$$\left\{ \int_0^y f^*(x)[1-\frac{(x+y)}{2}] \ d\alpha^*(x) \ + \int_y^1 f^*(x)(-c) \, d\alpha^*(x) \right\} \ = l \tag{2.6}$$

The equations admit an integral (differential) equation--expand (2.6) and integrate by parts and solve the resulting integral equation to obtain:

$$F^*(y \mid c) = \frac{(c+l)}{(1+c\text{-}y)} - \frac{1}{2(1+c\text{-}y)} \cdot \left\{ e^{-\int_0^y \frac{1}{(1+c-s)} ds} \right\} \cdot \left\{ \int_0^y \frac{(c+l)}{(1+c\text{-}s)} \cdot \left\{ e^{\int_0^s \frac{1}{2(1+c-x)} dx} \right\} ds \right\} \tag{2.7}$$

but $F^*(0) = 0 \rightarrow l=-c$ which is not possible. It is obvious that this same contradiction will hold for any measureable subset of [0,1]. Adding positive probability to single offers only complicates the form (2.7), but the only solution is where $f^*,g^* = 0$ on measurable sets of offers. An example for the case where positive probability is placed on the 0 offer of the seller is given in the Appendix. For the remainder of this paper we will assume that bargaining institution restricts offers to a discrete grid and dispense with any further discussion of the continuous offer case.

II.3 *Equilibrium Selection*

To arrive at an optimal strategy (what to do in the one-shot game) the player must conjecture about the likelihood that his opponent will restrict his offer set to some subset of $\mathbb{Z}$. Thus, an equilibrium in conjectures is a set of of restrictions (conjectures) on $\mathbb{Z}$ where both players "agree" on their opponents choice of Ω.

Definition : A *conjecture* is a mapping C: $\mathbb{Z} \mapsto \Re^{n+1}$, where $C(\mathbb{Z})_i$ is the conjecture that your opponent includes z_i in his offer set. A pair of conjectures $[C(\mathbb{Z})_b, C(\mathbb{Z})_s]$ are said to be *symmetric conjectures* when $C(\mathbb{Z})_{bj} = C(\mathbb{Z})_{sn-j}$.

Suppose the seller comes to a conclusion about $C(\mathbb{Z})_{si}$, then the seller should realize, due to the symmetry of the payoffs, that the buyer would set $C(\mathbb{Z})_{bn-i}=C(\mathbb{Z})_{si}$. Symmetric conjectures only support offer sets that are symmetric around $\frac{1}{2}$. Thus, the set of potential equilibria with symmetric conjectures is $2^{\frac{n}{2}}- 1$ if n is even, $2^{\frac{n+1}{2}} - 1$ if n is odd.

Proposition 1: The only symmetric equilibrium in conjectures that is trembling hand perfect is where both buyer and seller evenly split the surplus, i.e., the offers are $\frac{1}{2}$ and $\frac{1}{2}$.

Although this selection seems very appealing, it is not robust to changes in the bargaining grid and is insensitive to asymmetries in costs. For example, when $\frac{1}{2}$ is not part of the offer set then this refinement clearly does not apply. Indeed there no trembling hand perfect symmetric equilibria in conjectures for this case. Furthermore, any of the pure strategies would seem hard to support without some sort of preplay communication. On the other hand, any mixed strategy based on a subset of offers will not be trembling hand perfect. If we restrict our attention to equilibria that result in symmetric payoffs, then symmetric pairs of offers will generate such payoffs (there are $\frac{n}{2}$ of these equilibria including the C.M.S. equilibrium). However, given that players plan to mix they have acknowledged that they have some uncertainty as to the strategy used by their counterpart. If players cannot rule out the possibility that an offer will be made they should utilize C.M.S.. The equilibrium yielded by C.M.S. is unique and worth a study.

We now turn our attention to some comparative static results and investigate the case of known but differential bargaining costs.

II.4 *Comparative Statics*

Using the C.M.S equilibrium we shall now investigate its expected profit/efficiency properties for differential cost structures and offer grids.

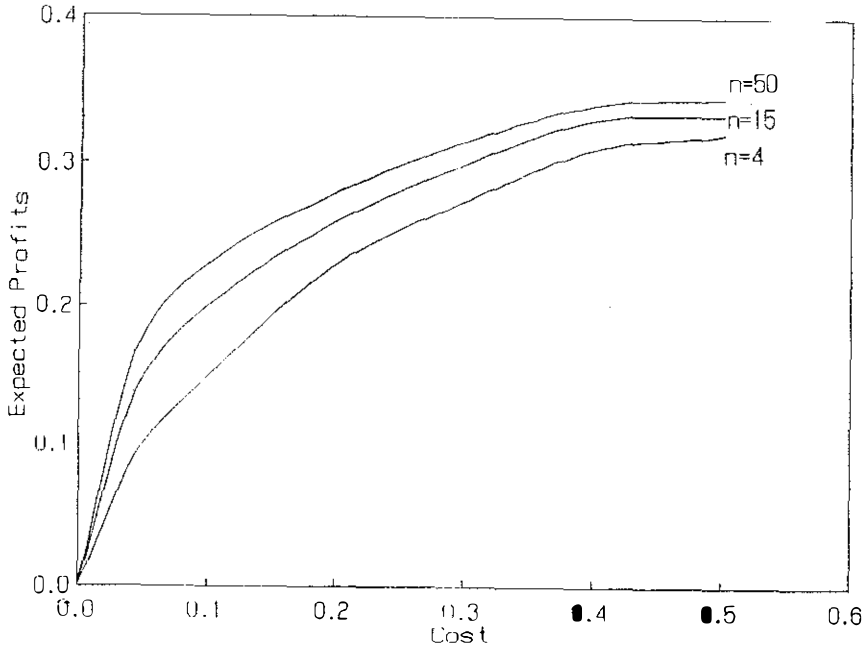
Corollary 1: The probability that the seller (buyer) submits the lowest (highest) offer is increasing and concave in costs.

Corollary 2: Expected profits are increasing and concave in c ($\partial\pi/\partial c > 0$ and $\partial^2\pi/\partial c^2$. Thus, the efficiency of the C.M.S. equilibrium increases as cost is increased.

Figure 2 shows the efficiency of the C.M.S. equilibria for various values of c and n respectively. While the results seem intuitive given the structure of the mechanism, the following corollary provides a counterintuitive result of the C.M.S..equilibrium.

Figure 2

Expected Profits as a Function of Cost and Grid Size



Corollary 3: Let c_s be bargaining cost of the seller and c_b the cost of the buyer. Then, for $c_s>c_b$ ($c_s<c_b$) we have $\pi_s>\pi_b$ ($\pi_s<\pi_b$)--if the failure cost of the seller is larger than that of the buyer then all symmetric mixed strategy equilibria are such that the expected profits of the seller are greater than those of the buyer.

The logic of the result in Corollary 3 follows from the fact that each player's equilibrium mixed strategy depends on the failure cost of his counterpart; if your rival has a lower failure cost than you, then in order to make him indifferent you must place a larger probability on your most stubborn offer. This result could be a feature of the one-shot nature of the game, or the fact that we have an environment with complete information. We will examine these issues later. We end this section with an analysis of the two offer case.

Example : the two offer case

The payoff matrix for the seller is:

$$\text{Seller} \quad \begin{array}{c c} & \text{Buyer} \\ & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & 0.5 \\ -c_s & 1 \end{bmatrix} \end{array}$$

In equilibrium, the seller offers 1 with probability $\frac{1}{1+2c_b}$ and 0 with probability $\frac{2c_b}{1+2c_b}$. The buyer offers 0 with $\frac{1}{1+2c_s}$ and 1 with probability $\frac{2c_s}{1+2c_s}$. Hence the probability that a bargain is struck is $1-(\frac{1}{(1+2c_s)(1+2c_b)})$; expected profits are $\frac{c_s}{1+2c_s}$ for each player. The comparative statics are illuminating: as c falls the probability of failure rises and in the limit when there no bargaining costs there are no bargains. Expected profits also fall as c falls and in the limit (costs are zero) profits are zero. Another case of interest is where $c_s=c_b=\frac{1}{2}$ (costs are equal to the surplus), here the probability of failure remains high ($\frac{3}{4}$) and expect profits are only $\frac{1}{4}$. Hence, even when cost are equal to total surplus, fully half the surplus is consumed--in an expected sense--by bargaining.

III. The Dynamic Game

Several problems have emerged from the analysis of the symmetric one-shot game. First, the expected profit of playing the game increases with costs. Second, the player with a higher cost obtains larger profits than the player with lower cost. One possible source of our counterintuitive result may have to do with the restriction that the game end after one period.

This section builds upon the previous ones and investigates the question of bargaining failures in a dynamic context. We will maintain that C.M.S. are chosen when anything is possible (recall that any collection of points or subset of [0,1] will yield an equilibrium in the one-shot game). One may wish to focus on pure strategies, but since in the discrete case the number of bargaining points is arbitrary, it is not clear how the coordination problem would be solved. Given the significant probability of failure in the one-shot game equilibria, it is important to investigate how extending the length of play affects the efficiency of trading. Going from one period to a repeated game often allows simplification of the problem because delay may be used to communicate information, however this is not likely to be the case here because there is nothing to communicate. That is, in the complete information context, there is no information concerning individual payoffs to be discovered from failure or delay in reaching an agreement.

III.1 *The Model*

The dynamic version of the game is similar to the one described in Section II except that if the buyer price is lower than the seller price, the bargain fails and each player must pay c before the next round of offers are made. We will consider both the finite horizon game with a known ending point and the infinite horizon game (both with discounting).

In the finite horizon case, working backwards from any period (t), the buyer and seller will play a game that is similar in payoff to the matricies in Section III, where the entries above the diagonal are the same but the entries below the diagonal are replaced by: $V=\delta\cdot\Pi^{t-1}-c$, where Π^{t-1} is the expected value of the game in period t and δ is the discount rate. Theorem 1 tells us that for each subset of $\mathbb{Z}$ there exists a unique equilibrium to this game.

Infinite horizon games are more appealing than games with known-ending points because of the the nature of the process we seek to describe. In this game no surplus is consumed by failure, the mirage of profits remains the same, independent of the number of times the players have failed to come to a bargain. In equilibrium, expected profits must be strictly positive because each player can accept a zero share of the surplus. Any player who agrees to play this game will not want to stop until a bargain occurs. Thus, the only rational reasons for the game to end is a bargain, death, or some other unexpected event. Therefore, the assumption of an uncertain end to the game, or that the game will go on until the players strike a bargain is the most intuitive.

A strategy for the seller will be a function $P_s^t : \mathbb{Z} \times \mathbb{N} \mapsto [0,1]$ where $P_s(z_i,t) = p_{si}^t$ where p_{si}^t is the probability that the seller offers z_i in period t. Thus, $\forall$ t we must have $\sum_i p_{si}^t = 1$ and $p_{si}^t \geq 0$. Similarly, a strategy for the buyer will be a function $P_b^t : \mathbb{Z} \times \mathbb{N} \mapsto [0,1]$ representing the probability that the buyer offers z_j in period t.

V.2 *Equilibrium and Equilibrium Selection*

A sequential equilibrium in this game is a pair of sequences of functions $(\rho_s^{(t)*}, \rho_b^{(t)*})$, where $\rho_s^{(t)*} = \{ P_s^{t*}, P_s^{t-1*} \ldots.\}$ and $\rho_b^{(t)*} = \{ P_b^{t*}, P_b^{t-1*} \ldots.\}$ such for each t, P_s^{t*} maximizes:

$$\pi_s (P_s^t) = P_s^t \, \mathcal{A}(\rho_s^{(t-1)*}, \rho_b^{(t-1)*}) \, (P_b^{t*})^T$$

subject to $\sum p_{si}^t = 1$ and $p_{si}^t \geq 0$

and P_b^{t*} maximizes,

$$\pi_b (P_b^t) = P_s^{t*} \, \mathcal{B} \, (\rho_s^{(t-1)*}, \rho_b^{(t-1)*}) \, (P_b^t)^T$$

subject to $\sum p_{bi}^t = 1$ and $p_{bi}^t \geq 0$

Note: $\mathcal{A}(\rho_s^{(t-1)*}, \rho_b^{(t-1)*})$ is defined to be the matrix of payoffs for the seller that is identical to the matrix $\mathcal{A}$ defined in Section III, except that below the diagonal the entries are the discounted expected returns minus c, if the players use $\rho_s^{(t-1)*}$ and $\rho_b^{(t-1)*}$. $\mathcal{B} \, (\rho_s^{(t-1)*}, \rho_b^{(t-1)*})$ is defined similarly.

The sequences ρ_s^t and ρ_b^t are the mixed strategies of the players that are best responses for each subgame in the dynamic game. The strategies at time t are dependent on future actions because future actions drive the expected future profits and thus the continuation value of the game. Unlike the one-shot game where failure to agree lead to a loss there is no reason for the continuation value of the game (V) to be negative if discounted expected profits are higher than c. Thus, some individual rationality constraints may become binding because individuals have no incentive to demand less than the continuation value of the game. Intuitively, for a given set of offers Ω, higher continuation values in period t decrease the expected returns of period t+1. Lower expected profits in period t+1 decrease the continuation value of period t+2 which will increase the expected return in period t+3 and so on. Thus the set of individually rational offers may change abruptly over time, a fact that poses further equilibrium selection problems.

Given that there may be different sets of acceptable offers at different times, we must select from the set of C.M.S. sequential equilibria the one which seems most plausible for each period. If the set of individually rational offers shrinks for each period of backward induction, it is natural to eliminate the offers that would never be accepted. If the set of individually rational offers expands, the question arises as to which of the C.M.S. equilibria will be selected. We select the C.M.S. sequential equilibria with the largest cardinality. This equilibrium has the property that any offer which is individually rational to accept in period t will be offered in period t with some (however low) positive probability. Moreover, the fact that in the future an offer will be unacceptable, or that it has been unacceptable in the past, should not influence today's decisions. As a result the C.M.S. are the intuitive extension of the equilibria examined in the one-shot game. Selecting the C.M.S. implies that if $\frac{m+1}{n} > v > \frac{m}{n}$, then $\Omega_t=\{ \frac{m+1}{n}, \frac{m+2}{n}, \ldots, \frac{n-m-1}{n}, \frac{n-m-2}{n}\}$.

It is easy to show that there is a function which maps continuation values into equilibrium mixed strategies. Thus, for either the finite or infinite horizon game we can rewrite our equilibrium definition in terms of continuation values instead of probabilities. Let $\Pi(n,m,V)$ be the expected profit from the C.M.S. equilibrium of the one-shot game with a grid of finesse $\frac{1}{n}$, smallest individually rational offer $\frac{m}{n}$, continuation value V.

Definition: An *infinite horizon steady state C.M.S. sequential equilibrium* to the bargaining game is a pair of values (V_s^*, V_b^*) such that :

$$V_s^*= \delta\cdot\Pi_s(n.m,V_s^*) -c \text{ ; where } \frac{m}{n}>V_s^*>\frac{m-1}{n},$$

and

$$V_b^*= \delta\cdot\Pi_b z(n,m,V_b^*) -c \text{ where } \frac{m}{n}>V_b^*>\frac{m-1}{n}.$$

We should note that this definition is not exhaustive of C.M.S. sequential equilibria. In fact any pair of sequences $V_s^*=\{V_s^0,\ldots.V_s^T\}$; $V_b^*=\{V_b^0,\ldots.V_b^T\}$ is an equilibrium if and only if $V_i^{t+1}=\delta\cdot\Pi(n,m,V_i^t)$-c for t in [0,T-1] and i=s,b. and $V_i^0=\delta\cdot\Pi(n,m,V_i^T)$-c.

<u>Proposition</u> <u>2</u>: Over the interval $[-\infty, \frac{m}{n}]$, $\Pi(n,m,V)=\prod_{j=m}^{n-m} \frac{2n-2j-2nV}{2n-(2j-1)-2nV} (\frac{n-m}{n}-V) +V$:

i) is continuous in V, ii) is decreasing in V, iii) is concave in V, and iv) has a limit of $\frac{m}{n}$ when $V \to \frac{m}{n}$ and equals $\frac{1}{2}$ when $V \to -\infty$.

In other words the profit function is very well behaved. Proposition 2 will be enough to guarantee that the infinite horizon equilibria exist. However, before discussing that issue we turn to the analysis of the finite horizon problem.

III. 3 *Equilibrium, Efficiency, and Convergence in the Finite Horizon Game*

The significant probability of failure found in the one-shot game suggest that it may be individually rational for individuals to consume a greater part of the surplus attempting to appropriate the rest. This section makes clear that the incentives for stubbornness remain strong in most dynamic situations. Notice that if $\frac{1}{2}$ is not part of the bargaining grid there will not exist a fully efficient symmetric equilibrium. So as not to entirely rule out efficient equilibria outright, we shall assume throughout that $\frac{1}{2}$ is part of the offer grid.

Lemma 3: If $c>\frac{1}{n}$ there exist no C.M.S. equilibrium such that the game achieves full efficiency, regardless of the number of periods played.

The interpretation of Lemma 3 is rather simple. Mixed strategies imply a positive probability of failure, however, inefficiency in a dynamic game may not be significant provided that the costs of failure are small enough because future profits constrain the subset of individually rational offers Ω_t^*. If future profits are high enough a unique individually rational offer may subsist, however the range of costs where an efficient solution may occur is arbitrarily small even if players are perfectly patient.

Proposition 3: $\forall$ n and $\delta>0$ there exists $\hat{c}$ such that $\delta\cdot\Pi(n,0,-\hat{c})-\hat{c}>0$.

Thus in any two period game the players will randomize on a different set in the first period than in the second. If c and δ are small enough it is not rational for the seller to accept an offer of 0 because his continuation value is positive. Thus, let c' be a solution to $\delta\cdot\Pi(n,0,-c') = c'$; such a solution exists since $\Pi(n,0,-c)$ is bounded above and δ is less than 1. Furthermore, since $\Pi(n,0,-c)$ is strictly concave and $c'<\frac{1}{2}$, c' is unique.

Theorem 5: For n=2, the game achieves full efficiency in two periods if $c<c'$. For $n>2$, the game does not achieve full efficiency in two periods.

Taken together, Lemma 3 and Theorem 5 suggest that except for n=2 (where the discontinuities are most severe) the efficiency of the game will depend on bargaining costs. If costs are high ($c>c'$) the one-shot game is more efficient than any multistage sealed-bid double auction. If costs are small ($c<\frac{1}{n}$) then the game may achieve full efficiency if the horizon is long. In the intermediate case ($c'>c>\frac{1}{n}$) the game will not achieve full efficiency and expected profits will not be larger than the one-shot game provided the horizon is short.

Beyond the question of efficiency, the model forces us to confront the problem of convergence. Built into the discrete problem are a set of discontinuities that may lead to chaotic continuation values. Within the interval $[0,c']$ there exist costs such that the finite game forces each player to dramatically alter his strategy from one period to the next, even if many periods of play remain and players are not perfectly patient. As Figure 3 shows, for $n=10$, $c=0.28$, $\delta=1.0$ and $t=1$ to $T=600$, each period of play has a specific continuation value and these values follow a chaotic pattern. A finite game will converge to the infinite horizon steady state based on Ω_m if and only if $\frac{m}{n}>V^t>\frac{m-1}{n}$ and $\frac{m}{n}>V^{t+1}>\frac{m-1}{n}$. This condition does not hold in general; thus it is not possible to give either conditions for chaotic behavior or convergence simply based on n, c and δ. Indeed, for a wide range of costs the process fails to stabilize. Figure 4 provides a graph of expected profits for various cost at 100 and 200 periods to the end ($T=600$). For costs between .15 and .30 the process exhibits major discontinuities in expected profits.

The chaotic nature of the strategies displayed in Figure 3 could in fact be related to the convergence of the finite game to an infinite horizon equilibrium which is not a one period steady state but rather a pair of sequences. This possibility remains unexplored, because of the complexity of the strategies involved. Moreover, as the next section will show, many infinite-horizon-steady-state equilibria exist making the exploration of even more complex equilibria uninteresting. If the finite game had always converged to one of these, it would have resolved an equilibrium selection problem. That is not the case so we must look elsewhere for an appropriate equilibrium.

Figure 3

Continuation Values 200 periods from Horizon (n=10, c=.28, δ=1 and T=600)

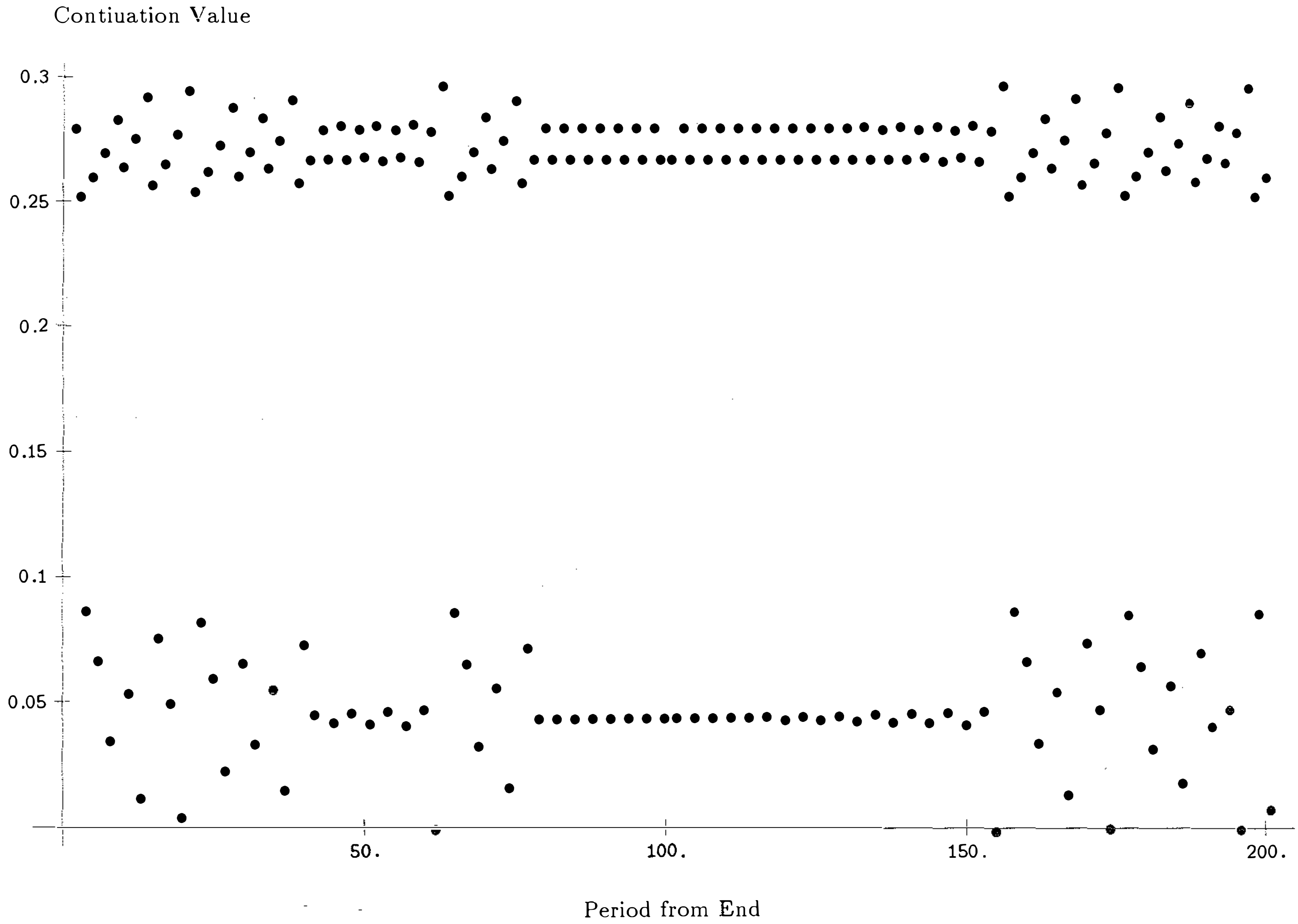
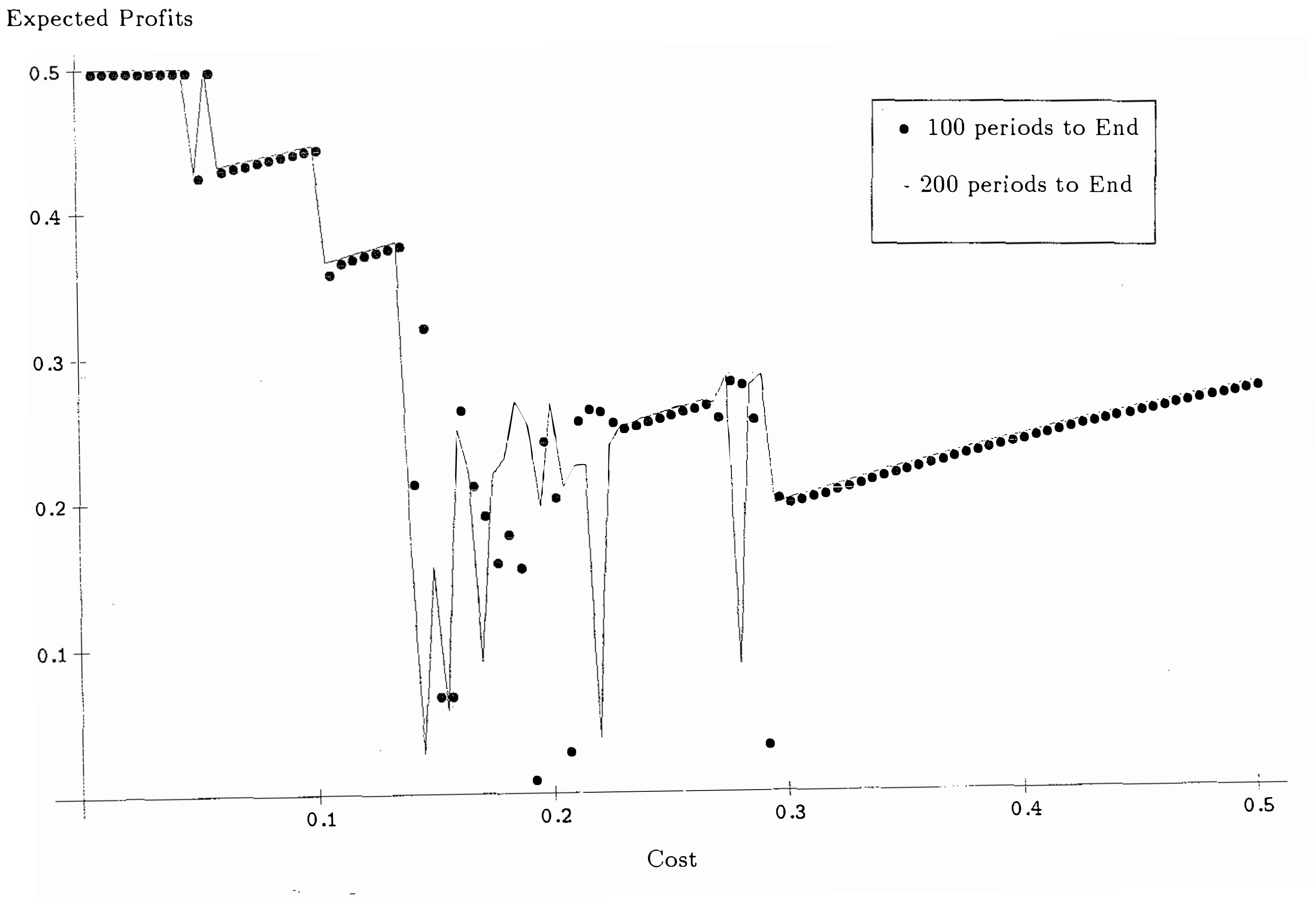


Figure 4

Expected Profits as a Function of Cost (n=10, c=.28, δ=1 and T=600)



III.4 *Equilibrium and Efficiency in the Infinite Horizon Game*

Infinite horizon equilibria are simply the steady states for continuation values. Recall that c' is the cost such that the continuation value of the game is zero if one period remains. For the infinite horizon game to be more efficient than the one-shot game the continuation value of the game must be positive. Thus c' allows us to characterize the efficiency of the infinite horizon equilibrium when costs are "high".

<u>Proposition 4</u>: For every n, δ and $c>c'$ there exist a <u>unique</u> V^* and thus a unique infinite horizon equilibrium to the bargaining game.

The bargaining game does not achieve full efficiency in an infinite horizon for $c>c'$. In addition, for $c>c'$ the dynamic game is less efficient than the one-shot game. The result is not surprising since the one-shot game is the equivalent of committing not to return to the bargaining table in case of failure. As a result, equilibrium strategies must be less stubborn, and hence the probability of failure falls.

<u>Theorem 6</u>: If $c< c'$, at least one infinite horizon equilibrium exists. Furthermore, the number of infinite horizon equilibria is at most $\frac{n}{2}+1$.

Given c, a C.M.S. infinite horizon steady state equilirium based on Ω_m will exist only if $d\cdot\Pi(n,m,V)-c>\frac{m-1}{n}$. Since $\Pi(n,m,V)$ is maximized over the interval $[\frac{m-1}{n},\frac{m}{n}]$ at the point $V=\frac{m-1}{n}$, a necessary condition for the existence of such equilibria is $\Pi(n,m,\frac{m-1}{n})-c>\frac{m-1}{n}$.

<u>Proposition 5</u>:
If , for a given c, $\Pi(n,m,\frac{m-1}{n})-c<\frac{m-1}{n}$ then $\Pi(n,m+1,\frac{m}{n})-c<\frac{m}{n}$. Thus, for a given c, if a C.M.S. infinite horizon steady state equilibrium based on Ω_m does not exists then no such equilibrium based on Ω_{m+1} will exist.

<u>Proposition 6</u>: If $c< \frac{3}{5n}$ then the infinite horizon game and the finite horizon game have an efficient C.M.S. equilibrium .

We conclude this section will analysis of the two offer case for the infinite horizon game.

Example (Two offer case):

The seller would like to maximize (at t = 0):

$$V_s^0(p)=pq(0.5)+(1-p)q+(1-p)(1-q)(\delta\cdot V_s^1-c) \tag{3.1}$$

where p is the probability that the seller offers 0 and q is the probability that the buyer offers 1, and V^1 is the continuation value of the game if there is a bargaining failure. Differentiating (3.1) yields

$$\frac{\partial V_s^0}{\partial p} = q(0.5) - q - (1 - q)(\delta V_s^1 - c) = 0$$

or

$$-\frac{q}{2(1-q)} = \delta\cdot V_s^1 - c \tag{3.2}$$

Equation (3.3) leads to the conclusion that the discounted continuation value of the game ($\delta\cdot V_s^1$) must always be less than c. Now, in equilibrium p=q and $V_s^1 = V_s^0$ so that substituting (3.2) into (3.1) we find:

$$-\delta q^2 + q(1 + \delta + 2c) - 2c = 0 \tag{3.3}$$

Equation (3.3) is a quadratic equation (in q) that is negative at q=0 and positive at q=1; the maximum of the function occurs at $q = \frac{1+\delta+2c}{2\delta} > 1$. Thus, (3.4) is strictly increasing on the interval [0,1] and therefore there is a unique solution to this problem. We now state some results we can derive directly from (3.3):

(1) As costs (c) increase the probability of failure $(1 - q)^2$ falls so the game ends sooner.

Reason: $\partial q/\partial c = \dfrac{(2 - 2q^*)}{(1 + \delta + 2c - \delta 2q^*)} > 0 \rightarrow$ q increases and thus $(1 - q)\downarrow$.

(2) As the discount rate rises the probability of failure increases.

Reason: $\partial q/\partial \delta = \dfrac{(q^{*2} - q^*)}{(1 + \delta + 2c - \delta 2q^*)} < 0$

(3) When $\delta=0$ we return to the one-shot game and $q = \frac{2c}{1 + 2c}$.

Thus, in the complete information game (with an infinite horizon and two offers) the unique symmetric equilibrium has delay ($q < 1$) and this delay rises as the cost of bargaining falls and the impatience of the players rises. Unless there is a more "suitable" negotiating mechanism or institution there may be no way for the players to resolve their differences instantaneously. Most of the surplus is in fact consumed in the bargaining process.

IV. Asymmetric and Uncertain Costs

The symmetric cost case, although informative, cannot provide the whole answer to the issue of negotiation failure. Clearly cost are not strictly symmetric. One disturbing phenomenon in this setting is that profits increase with costs. Thus, the higher cost player has a higher expected profit than the lower cost player. We now extend the model in Section II by having each player's cost drawn independently from a common distribution.

IV.1 *The Model*

Suppose c is drawn from a common density function $f(\cdot)$, with cumulative density $F(\cdot)$ on the support $[\underline{c},\bar{c}]$. A strategy for the seller is a function $P_s:[\underline{c},\bar{c}]\times\mathbb{Z}\mapsto[0,1]$ where $P_s(c,z_i) = p_{si}(c)$ and $p_{si}(c)$ is the probability that a seller of type c offers z_i. Similarly, a strategy for the buyer is a function $P_b : [\underline{c},\bar{c}] \times \mathbb{Z} \mapsto [0,1]$ where $p_{bj}(c)$ is the probability that a buyer of type c bids z_j.

IV.2 *Equilibrium*

The profits for a buyer of type $\hat{c}$ of offering $\frac{i}{n}$ given the strategy of the seller are:

$\pi_{ib}(c|\hat{c}) = \sum_{j=0}^{i} p_{sj}(c)\,\frac{2n-2i-i+j}{2n} - \sum_{j=i+1}^{n} p_{sj}(c)\,\hat{c}$, and thus the expected profits from offering $\frac{i}{n}$ are:

$$\pi_{ib}(P_s|\,\hat{c}) = \sum_{j=0}^{i} \frac{2i+n-j-i}{2n}\left\{\int_{\underline{c}}^{\bar{c}} p_{sj}(c)\,f(c)\,dc\right\} - \hat{c}\cdot\sum_{j=i+1}^{n}\int_{\underline{c}}^{\bar{c}} p_{sj}(c)\,f(c)\,dc \qquad (4.1)$$

For a seller of type c' the expected profits of an offer of $\frac{n-j}{n}$ can be described by:

$$\pi_{js}(P_b|\,c') = \sum_{i=j}^{n} \frac{2j+n-j-i}{2n}\left\{\int_{\underline{c}}^{\bar{c}} p_{bi}(c)\,f(c)\,dc\right\} - c'\cdot\sum_{i=0}^{j-1}\int_{\underline{c}}^{\bar{c}} p_{bi}(c)\,f(c)\,dc \qquad (4.2)$$

Thus, an equilibrium is a pair of functions $P_b^*(c)$, $P_s^*(c)$ such that $P_b^*(c)$ maximizes $P_b(c)\,\Pi(P_s^*(c))$, where $\Pi(P_s^*(c))$ is a 1xn vector whose i^{th} entry is $\pi_{ib}(P_s|c)$; $P_s^*(c)$ maximizes $P_s(c)\,\Pi(P_b^*(c))$, where $\Pi(P^*{}_b(c))$ is a 1xn vector whose j^{th} entry is $\pi_{sj}(P_b|\,c)$.

Since $\pi_{ib}(P_s|\,\hat{c})$ and $\pi_{js}(P_b|\,c')$ are linear functions of $\hat{c}$ and c' respectively, equations (4.1) and (4.2) can be rewritten as:

$$\pi_{ib}(\hat{c}) = \hat{A}_i - \hat{B}_i\cdot\hat{c} \qquad (4.3)$$

$$\pi_{js}(c') = A'_j - B'_j\cdot c' \qquad (4.4)$$

Notice that $\pi_{ib}(\cdot)$ and $\pi_{js}(\cdot)$ are linearly decreasing functions of c. If $\pi_{is}(c)<\pi_{js}(c)$ and i>j then a seller of type c will have a dominant strategy of playing i with zero probability, and all sellers with higher cost will have same dominant strategy. If $f(\cdot)$ is continuous, then for almost every c one $\pi_{sj}(c)$ will be larger than all the other $\pi_{si}(c)$, so almost all players will have pure strategies.

Proposition 7: The best response function is decreasing in c; if $\pi_i(c) > \pi_k(c)$ for k>i, then $\forall\ \hat{c} > c$, $\pi_i(\hat{c}) > \pi_k(\hat{c})$ for k>i.

Thus, if a player of type c makes a demand of $\frac{i}{n}$, all players of type $\hat{c} > c$ will make a demand of no more than $\frac{i}{n}$. Hence, we can restrict our attention to strategies where the demanded share of the surplus decreases with cost. Such strategies are completely defined by a vector of "cut-off points", $\Gamma = [\gamma_1, \gamma_2, \ldots, \gamma_n]$ where $\underline{c} \le \gamma_i \le \bar{c}$ for the buyer and a vector of "cut-off points", $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ where $\underline{c} \le \lambda_i \le \bar{c}$ for the seller. A seller (buyer) with a cost of c between λ_i and λ_{i+1} will demand $\frac{i}{n}$. Thus, a cut-off point will be a function $\lambda_i : \Re^n \mapsto \Re$ and thus $\Lambda : \Re^n \mapsto \Re^n$.

We note that best responses do not require that every offer be submitted with positive probability, so that offers may be skipped. However, the buyer and seller profit functions are continuous in c and the best response function is the maximum of $\pi_{ib}(\hat{c})$ over Ω. Since the maximum of continuous function is continuous, $\lambda_i^*(\cdot)$ is continuous. Thus, an equilibrium to the asymmetric information game is defined as a pair of functions (Λ^*, Γ^*) such that $\Lambda^*(\Gamma') = \Lambda'$ and $\Gamma^*(\Lambda') = (\Gamma')$. We shall say that a symmetric equilibrium to this game occurs iff $\Lambda^*(\Lambda') = \Lambda'$.

Proposition 8:

A) Symmetric equilibria exist.

B) $\forall$ $f(\cdot)$ and every pair of symmetric offers $\exists$ a unique equilibrium

C) In any symmetric equilibrium $\pi_b(c) > \pi_s(c')$ if $c < c'$.

Theorem 7: If c is drawn from a uniform distribution over the interval $[\underline{c}, \bar{c}]$, $\underline{c} > 0$, then $\exists$ a unique equilibrium set of cut-off points $\{\lambda_1^*, \ldots, \lambda_n^*\}$ such that $\underline{c} < \lambda_1^* < \ldots < \lambda_n^* < \bar{c}$.

Part 3 of Proposition 8 shows that, unlike the certainty case, higher cost players obtain lower expected returns from the game. However, failures in the negotiation process do not appear to be caused by the uncertainty of bargaining costs. Delay occurs because players compete for a larger share of the surplus. Hence the process still results in inefficient outcomes. It is clear that these same results will hold for the special case in which one of the players cost is known to all with certainty while the other player draws his cost from a known nondegenerate distribution.

VII. Conclusion

This paper has attempted to analyze bilateral bargaining under the sealed-bid double auction when there is complete information concerning individual payoffs and bargaining costs. Unfortunately, in this environment there exists a multiplicity of equilibria. All of the equilibria suffer from unsatisfactory properties; the pure strategy equilibria require coordination and are not sensitive to bargaining costs (including focal equilibria--equal split); mixed strategy equilibria are not trembling hand perfect. Because the traders are arguing over surplus, it is rational to accept any offer, thus we focused on completely mixed strategy equilibria but find that such equilibria require that the negotiator with the higher bargaining cost receive higher profits.

Allowing the bargaining process to be dynamic does not entirely solve the problems of the one-shot game. The continuation value of the game and thus the offers in the dynamic game can demonstrate chaotic behavior because of the discontinuities of the bargaining grid. Moreover when the failure costs are less than half the surplus, there may exist multiple infinite horizon C.M.S. equilibria. When costs are more than half the surplus, however, the dynamic game is well behaved (the finite game converges to the unique steady state equilibrium) but the multistage game equilibria are less efficient than one-shot game equilibrium.

These results seem rather negative and not very reassuring. Yet, unlike the Stahl-Rubinstein model we find that there is significant probability of delay, a property of the equilibrium which is consistent with empirical reality. As bargaining costs fall, the mixed strategy equilibrium approaches a pure strategy in which failure occurs with probability one. This suggests that failure costs play an important role in bringing the parties closer to agreement, and that one should focus on the determinants of bargaining costs to understand the success or failure of negotiations. In the dynamic version of this game efficiency can be increased (provided $c<c'$) and inefficiency conditions are parameter sensitive. Thus, the comparative static results of the model have some appeal. Recently, Currie and McConnell [1990] have supplied a comparative (empirical) study of compulsory arbitration with right to

strike rules and found that while arbitration increases the number of disputes, the total delay costs are lower under arbitration with no significant difference in the final wage package. These results suggest that bargaining costs and delays are significant factors in bargaining. We have made bargaining costs exogenous in our model, but it is clear that these costs are artifacts of the bargaining institution and as such should be explicitly model as part of the game. We leave this point as a next step in the research of the sealed-bid double auction.

Appendix

Proofs

Proof of Theorem 1:

All that is required is to show that $\mathcal{A}_\Omega$ is invertible because that will lead to a unique solution for P_b. Note that $\mathcal{A}_\Omega$ is a matrix with an upper triangle of positive constants and a lower triangle with -c. The determinant of $\mathcal{A}_\Omega$ will be a polynomial of degree $|\Omega|$-1. In particular, the determinant of $\mathcal{A}_\Omega$ ($\det[\mathcal{A}_\Omega]$) is given by:

$$\det[\mathcal{A}_\Omega] = \sum_\pi (\operatorname{sgn} \pi)\, a_{1\pi(1)} \cdots a_{m\pi(m)}, \tag{A}$$

where $m=|\Omega|$ and π are a permutations on the elements of $\mathcal{A}_\Omega$. We shall now show that $\det[\mathcal{A}_\Omega] > 0$. The permutation $\pi(i)=i$ results in the product of the diagonal elements, $D \geq 0$. Next, the permutation where two of the elements of $\mathcal{A}_\Omega$ are reversed, i.e. $\pi(i)=j$ and $\pi(j)=i$, results in an odd number of inversions $\rightarrow(\operatorname{sgn} \pi) = -1$. From Equation A, this term in the sum is given by $(-1)\cdot -c \cdot b(\pi)$ where $b(\pi)$ is a nonnegative number. At least one of the two elements permutations will be strictly positive. Continuing with the permutations where three element positions are reversed the sgn π for these will 1. These permutations will be given by $(1)\cdot -(c)^2$ $b^2(\pi)$ where $b^2(\pi)$ is a nonnegative number. At least one of these will be strictly positive. We can continue this argument to obtain elements of the form $(-1)^{\alpha(\pi)}\cdot(-c)^{\beta(\pi)}\cdot b^m(\pi)$ where $b^m(\pi)>0$, and $\alpha(\pi)$ and $\beta(\pi)$ are either both odd or both even. Thus, $\det[\mathcal{A}_\Omega]$ will be the sum of nonegative terms (at least one of which is strictly positive) so that $\det[\mathcal{A}_\Omega] >0$. $\mathcal{B}_\Omega$ has the same properties as $\mathcal{A}_\Omega$.

Since the above arguement holds for any subset Ω of $\mathbb{Z}$ we have 2^n -1 Nash equilibria to this game.□

Algorithm for Computing P_s^*:

The difference in expected profits to the buyer from offering z_i versus z_{i+1} is: $\pi_i^{\,b}$ $\pi_{i+1}^{\,b} = p_{n-i}^{\,s}\,(\frac{i}{n}+c) - \frac{1}{2n}\sum_{j=0}^{n-i-1} p_j^{\,s}$. The incentive compatibility constraints require that the expected profits of all offers must be equal when players mix. Thus $\pi_i^{\,b}$ - $\pi_{i-1}^{\,b}=0$, which is equivalent to: $p_{n-i}^{\,s} = \frac{1}{(2n-2i+2nc)} \sum_{j=0}^{n-i-1} p_j^{\,s}$. Note that if $c=-\frac{i}{n}$ the equation is undefined but we assumed that c was positive. We can write down what the equilibrium probabilities are solely as a function of $p_n{}^s$ as follows:

$$p_0{}^s = p_0{}^s$$

$$p_1{}^s = \frac{1}{(2n-2+2nc)} \; p_0{}^s$$

$$p_2{}^s = \frac{1}{(2n-4+2nc)} \; p_0{}^s \; + p_1{}^s \;\; or \;\; p_2{}^s = \frac{1}{(2n-4+2nc)} \, \frac{2n-1+2nc}{2n-2+2nc} \;\; p_{\bullet}{}^s$$

similarly

$$p_3{}^s = \frac{1}{2n-6+2nc} \, \frac{2n-3+2nc}{2n-4+2nc} \, \frac{2n-1+2nc}{2n-2+2nc} \;\; p_0{}^s$$

by induction

$$p_i{}^s = \frac{1}{2n-2i+2nc} \prod_{j=1}^{i-1} \frac{2n-(2j-1)+2nc}{2n-2j+2nc} \; p_0{}^s.$$

Now we can solve for $p_{\bullet}$ because all the probabilities must add up to one or $\prod_{j=1}^{n} \frac{2n-(2j-1)+2nc}{2n-2j+2nc} \cdot$ $p_{\bullet}=1.$ Hence $p_{\bullet} = \prod_{j=1}^{n} \frac{2n-2j+2nc}{2n-(2j-1)+2nc}$ which is always defined for $c>0$.

$\boxed{\text{Example of } F^* \text{ for } F^*(0)=a>0}$

We can obtain the following equations to solve for our distribution F^*:

$$l = \int_0^y f^*(x)[1-\frac{x+1}{2}]dx + \frac{a}{2}$$

$$l = \int_0^y f^*(x)[1-\frac{x+1}{2}]dx - c\int_y^1 f^*(x)\, dx \; + a[1-\frac{y}{2}]$$

so that

$$\int_0^{\infty} F^*(s)\, ds = 2l - a \quad \text{and}$$

$$F^*(y) = \frac{l + c - a(\frac{y}{2})}{(1 + c - y)} \quad - \quad \frac{1}{2(1 + c - y)} \int_0^y F^*(s)\, ds$$

Thus obtaining two equations and two unknowns (a,l):

$$\frac{l + c}{1 + c} = a$$

$$\int_0^1 F^*(s)\, ds = 2l + a$$

which has only one solution: $a = l = 1$, but this cannot be an equilibrium.

Proof of Proposition 1:

We know that $\Omega = \{\frac{1}{2}\}$ is a symmetric equilibrium in conjecture. When a buyer contemplates his equilibrium strategy given that the other player plays a given equilibrium strategy based on Ω he is indifferent between making all offers in Ω. If we assume that his opponent can tremble between a mixed strategy over Ω and making any offer $\omega \varepsilon \Omega$ the buyer strictly prefers offering ω with probability one. In fact ω is a best response to any equilibrium strategy as long as $\omega \in \Omega$. Hence, given Ω, only those equilibria where both player adopt pure strategies are trembling hand perfect. Among the pure strategy equilibria only $\Omega = \{\frac{1}{2}\}$ is symmetric in conjectures.□

Proof of Corollary 1:

It is obvious that $\frac{\partial^2 p_0}{\partial c^2} > 0$.

$$\frac{\partial^2 p_0}{\partial c^2} = 4p_0\left\{ \left[\sum_{i=0}^{n-1} \frac{1}{(2i+2c)(2i+1+2c)}\right]^2 - \sum_{i=0}^{n-1}\frac{4i+1+4c}{(2i+2c)^2(2i+1+2c)^2}\right\}$$

$$= 8p_0\left\{ \left[\sum_{i=0}^{n-1} \frac{1}{(2i+2c)(2i+1+2c)} \sum_{j=i+1}^{n-1} \frac{1}{(2i+2c)(2i+1+2c)} - \sum^{n-1}\frac{2i+2c}{(2i+2c)^2(2i+1+2c)^2}\right\} \equiv Q_i.$$

$$Q_i < 0 \text{ iff } \frac{1}{(2i+2c)(2i+1+2c)} \sum_{j=i+1}^{n-1} \frac{1}{(2i+2c)(2i+1+2c)} < \frac{2i+2c}{(2i+2c)^2(2i+1+2c)^2}$$

$$\text{or } A_i \equiv (2i+1+2c) \sum_{l=i+1} \prod_{j \neq l} (2j+2c)(2j+1+2c) < \prod_{j=i+1}^{n-1} (2j+2c)(2j+1+2c) \equiv B_i$$

Notice that $Q_{n-1} < 0$ and

$$A_{n-2} = (2n-3+2c)\,(2n-4+2c) < (2n-1+2c)\,(2n-2+2c) = B_{n-2}$$

Assume $A_i < B_i$ then

$$B_{i-1}=(2i+2c)\ (2i+1+2c)\prod_{j=i+1}^{n-1}(2j+2c)(2j+1+2c)\ =(2i+2c)\ (2i+1+2c)B_i$$

$$A_{i-1}=(2i-1+2c)\ \sum_{l=i}\prod_{j\neq l}(2j+2c)(2j+1+2c)$$

$$A_{i-1}=(2i-1+2c)\left\{(2i+2c)(2i+1+2c)\sum_{l=i+1}\prod_{j\neq l}(2j+2c)(2j+1+2c)+\prod_{j=i+1}^{n-1}(2j+2c)(2j+1+2c)\right\}$$

$$=(2i-1+2c)\left\{\ (2i+2c)A_i+\prod_{j=i+1}^{n-1}(2j+2c)(2j+1+2c)\right\}$$

$$=(2i-1+2c)\left\{\ (2i+2c)A_i+B_i\right\}$$

$<\ (2i-1+2c)\ (2i+1+2c)\quad B_i\ <\ B_{i-1}$ because $2i-1\geq 1$. Thus, $A_{i-1}<B_{i-1}$ and hence $Q_i<0\ \forall\ i$.

Therefore, $\dfrac{\partial^2 p_0}{\partial c^2}=\displaystyle\sum_{j=0}^{n-1} Q_i<0$ □

Proof of Corollary 2:

For the seller we have

$$p_i=\frac{1}{(2n-2i+1+2c)}\prod_{j=i+1}^{n}\frac{2n-2j+2c}{(2n-2j+1+2c)}\qquad and\qquad p_{i-1}=\frac{(2n-2i+2c)}{(2n-2i-1+2c)}p_i$$

$$\frac{\partial p_{i-1}}{\partial c}=\frac{\partial p_i}{\partial c}\ \frac{(2n-2i+2c)^2}{(2n-2i+1+2c)(2n-2i-1+2c)}+2p_i\ \frac{(2n-2i+1+2c)-(2n-2i+2c)}{(2n-2i-1+2c)^2}$$

$$\frac{\partial p_i}{\partial c}\ \frac{(2n-2i+2c)^2}{(2n-2i+1+2c)(2n-2i-1+2c)}+\frac{2p_i}{(2n-2i-1+2c)^2}$$

So $\dfrac{\partial p_i}{\partial c}>0$ implies $\dfrac{\partial p_{i-1}}{\partial c}>0$. Buyer expected profits are given by:

$$\Pi^b{}_0=\sum_{i=0}^{n}(0.5-\tfrac{i}{n})p^s{}_i\quad so\ that\quad \frac{\partial\Pi^b{}_0}{\partial c}=\sum_{i=0}^{n}(0.5-\tfrac{i}{n})\frac{\partial p^s{}_i}{\partial c}$$

We know $\exists$ j be such that $\dfrac{\partial p^s{}_i}{\partial c}<0$ if $i\geq j$ and $\dfrac{\partial p^s{}_i}{\partial c}>0$ if $i<j$, hence,

$$(0.5-\tfrac{i}{n})\frac{\partial p^s{}_i}{\partial c}<(0.5-\tfrac{j}{n})\frac{\partial p^s{}_i}{\partial c}\ \text{if } i>j \text{ and } (0.5-\tfrac{i}{n})\frac{\partial p^s{}_i}{\partial c}>(0.5-\tfrac{j-1}{n})\frac{\partial p^s{}_i}{\partial c}\ \text{if } i<j-1.$$

Since $\displaystyle\sum_{i=0} p^s{}_i=1$ we have $\displaystyle\sum_{i=0}\frac{\partial p^s{}_i}{\partial c}=0\rightarrow\sum_{i=0}^{j-1}\frac{\partial p^s{}_i}{\partial c}=-\sum_{i=j}^{n}\frac{\partial p^s{}_i}{\partial c}$

$\frac{\partial \Pi^b{}_0}{\partial c} = \sum_{i=0}^{n} (0.5 - \frac{i}{n}) \frac{\partial p^s{}_i}{\partial c} > \sum_{i=0}^{j-1} \frac{\partial p^s{}_i}{\partial c} (0.5 - \frac{j-1}{n}) - \sum_{i=0}^{j-1} \frac{\partial p^s{}_i}{\partial c} (0.5 - \frac{j}{n}) = \sum_{i=0}^{j-1} \frac{\partial p^s{}_i}{\partial c} \frac{1}{n} > 0$. We now show that expected profits are concave in c.

We shall show that $\frac{\partial p_i}{\partial c} > 0$ implies $\frac{\partial^2 p_i}{\partial c^2} < 0$

$$\frac{\partial p}{\partial c} = 2p_i \left\{ \sum_{i+1}^{n-1} \frac{1}{(2n-2j+2c)(2n-2j+1+2c)} - \frac{1}{(2n-2i+1+2c)} \right\} = 2 p_i [A-B] \text{ and}$$

$$\frac{\partial^2 p}{\partial c^2} = 4p_i \left\{ \left[\sum_{j=i+1}^{n-1} \frac{1}{(2i+2c)(2i+1+2c)} - \frac{1}{(2n-2i+1+2c)} \right]^2 - \sum_{j=i+1}^{n-1} \frac{4i+1+4c}{(2i+2c)^2(2i+1+2c)^2} + \frac{1}{(2n-2i+1+2c)^2} \right\}$$

$$= 4p_i \left\{ [A-B]^2 + B^2 - D \right\} = 4p_i \left\{ A^2 + 2B^2 - 2AB - D \right\}$$

since $\frac{\partial p_i}{\partial c} > 0 \rightarrow A > B$ and hence $2B^2 - 2AB < 0$ (by the same argument as in the concavity of p_i $D > A^2$),

so $\frac{\partial^2 p}{\partial c^2} < 0$ *if* $\frac{\partial p_i}{\partial c} > 0$. Notice that $\frac{\partial p_{i-1}}{\partial c} = \frac{\partial p_i}{\partial c} \frac{(2n-2i+2c)^2}{(2n-2i+1+2c)(2n-2i-1+2c)} + \frac{2p_i}{(2n-2i-1+2c)^2}$

$$\frac{\partial^2 p_{i-1}}{\partial c^2} = \frac{\partial^2 p_i}{\partial c^2} \frac{(2n-2i+2c)^2}{(2n-2i+1+2c)(2n-2i-1+2c)} + 8 \frac{\partial p_i}{\partial c} \frac{1}{(2n-2i-1+2c)^2} - 8 \frac{p_i}{(2n-2i-1+2c)^3}$$

so that $\frac{\partial^2 p_i}{\partial c^2} < 0 \rightarrow \frac{\partial^2 p_{i-1}}{\partial c^2} < 0$

Now

$$\Pi^b{}_0 = \sum_{i=0}^{n} (0.5 - \frac{i}{n}) p^s{}_i \rightarrow \frac{\partial \Pi^b{}_0}{\partial c} = \sum_{i=0}^{n} (0.5 - \frac{i}{n}) \frac{\partial p^s{}_i}{\partial c} \rightarrow \frac{\partial^2 \Pi^b{}_0}{\partial c^2} = \sum_{i=0}^{n} (0.5 - \frac{i}{n}) \frac{\partial^2 p^s{}_i}{\partial c^2} .$$

$\exists$ *j such that* $\frac{\partial^2 p^s{}_i}{\partial c^2} > 0$ *if* $i \geq j$ *and* $\frac{\partial^2 p^s{}_i}{\partial c^2} < 0$ *if* $i < j$, *so that* $(0.5 - \frac{i}{n}) \frac{\partial^2 p^s{}_i}{\partial c^2} < (0.5 - \frac{j}{n}) \frac{\partial^2 p^s{}_i}{\partial c^2}$ *if* $i > j$ *and*

$(0.5 - \frac{i}{n}) \frac{\partial^2 p^s{}_i}{\partial c^2} > (0.5 - \frac{j-1}{n}) \frac{\partial^2 p^s{}_i}{\partial c^2}$ *if* $i < j$. Again $\sum_{i=0} p^s{}_i = 1$ *so* $\sum_{i=0} \frac{\partial^2 p^s{}_i}{\partial c^2} = 40$, *so that* $\sum_{i=0}^{j-1} \frac{\partial^2 p^s{}_i}{\partial c^2} = -\sum_{i=j}^{n} \frac{\partial^2 p^s{}_i}{\partial c^2}$

Hence $\frac{\partial^2 \Pi^b{}_0}{\partial c^2} = \sum_{i=0}^{n} (0.5 - \frac{i}{n})\frac{\partial^2 p^s{}_i}{\partial c^2} < \sum_{i=0}^{j-1} \frac{\partial^2 p^s{}_i}{\partial c^2} (0.5 - \frac{j-1}{n}) - \sum_{i=0}^{j-1} \frac{\partial^2 p^s{}_i}{\partial c^2} (0.5 - \frac{j}{n}) =$

$\sum_{i=0}^{j-1} \frac{\partial^2 p^s{}_i}{\partial c^2} \frac{1}{n} < 0$ □

Proof of Corollary 3:

Since profits are increasing in c it follows directly that the player with the higher cost will obtain a larger expected profit relative to the player with the smaller bargaining cost.□

Proof of Proposition 2:

We can characterize expected profits of this game as follows: $\Pi\ (n,m,V) = (1-p_m)V + p_m\frac{n-m}{n}$,

rearranging terms leads to $\Pi\ (n,m,V)=V + p_m(\frac{n-m}{n}-V)$, but , $p_m=\prod_{j=m}^{n-m} \frac{2n-2j-2nV}{2n-(2j-1)-2nV}$. Thus we have,

$$\Pi\ (n,m,V) = \prod_{j=m}^{n-m} \frac{2n-2j-2nV}{2n-(2j-1)-2nV} (\frac{n-m}{n}-V) + V$$

i) The profit function will be continuous if p_m is continuous. Since p_m a product of ratios whose numerators and denominators are all continuous in V, with denominators that are strictly positive for all V less than m/n, p_m is continuous in V.

ii) Let us first evaluate the first partial of p_m with respect to V

$$\frac{\partial p_m}{\partial V} = \frac{(\prod_{j=1}^{n} (2n-(2j-1)-2nV)(\sum_{j=1}^{n} \prod_{i\neq j} (2n-2i-2nV)}{(\prod_{j=1}^{n} 2n-(2j-1)-2nV)^2} \frac{(\prod_{j=1}^{n} 2n-2j-2nV)(\sum_{j=1}^{n} \prod_{i\neq j} 2n-(2i-1)-2nV)}{(\prod_{j=1}^{n} 2n-(2j-1)-2nV)^2}$$

or

$$\frac{\partial p_m}{\partial V} = -\ 2n \frac{\sum_{i=1}^{n} (\prod_{j\neq i} 2n-(2j-1)-2nV)(\prod_{j\neq i} 2n-2j-2nV)}{(\prod_{j=1}^{n} 2n-(2j-1)-2nV)^2}$$

$$\frac{\partial p_m}{\partial V} = -\ 2n\ p_m \sum_{j=1}^{n-1} \frac{1}{(2j+1)-2nV)(2j-2nV)} < 0.$$

The same method can be used to show that for all other offers in $[\frac{m}{n}, \frac{n-m}{n}]$

$$\frac{\partial p_i}{\partial V} > 0 \text{ for } i>1$$

Now recall that by incentive compatibility $\pi_n=\pi_0$. Note that the coefficients of the first row of the payoff matrix are all positive, independent of the value of v. Moreover p_m is the coefficient of the last entry in each row. In the first row the last entry is always $\frac{1}{2}$-- corresponding to buyer sending 1 and seller sending 0. In the first row all entries are in $[0,\frac{1}{2}]$; putting less weight on $\frac{1}{2}$ while increasing the weight of all other offers must decrease profits. Thus p_m decreases when V goes up, and so do profits.

iii) This follows directly from the fact that $\frac{\partial^2 \Pi(n,m,V)}{\partial c^2}<0$ (see Corollary 1).

iv) When V=m/n we have $\Pi(n,m,V) = \prod_{j=m}^{n-m} \frac{n-j-m}{n-j-1-m}\left(\frac{n-2m}{n}\right) + \frac{n}{m}$. The last term of this product (j=n-m) is zero so that $\Pi(n,m,V) = \frac{m}{n}$.

Recall that $p_m=\prod_{j=m}^{n-m} \frac{2n-2j-2nV}{2n-(2j-1)-2nV}$. Now as V goes to minus infinity $\frac{2n-2j-2nV}{2n-(2j-1)-2nV}$ goes to 1 for each j. So p_{m+1} goes to 1 this implies that $\pi_m(V)$ goes to $\frac{1}{2}$ when v goes to $-\infty$. Because we look at mixed strategy equilibria the profits of offering $\frac{m}{n}$ are equal to expected profits.□

Proof of Lemma 3:

The game achieves full efficiency if and only if the players each send $\frac{1}{2}$. If they fully randomize this can only occur if all the other offers are dominated or if V= $\delta\Pi$-c>$\frac{1}{2}$-$\frac{1}{n}$. Now, $\Pi(n,m,V)<\frac{1}{2}$ because of individual rationality (since the seller offers $\frac{m}{n}$ with positive probability and the expected profits from such an offer are at most $\frac{1}{2}$. In a mixed strategy equilibrium, expected profits are equal for all offers, thus profits are at most $\frac{1}{2}$). So for the game to achieve full efficiency c cannot be greater than $\frac{1}{n}$.□

Proof of Proposition 3:

First evaluate $\Pi(n,0,-c)$ -c at c= $\frac{1}{n}$. It is easy to show that $p_0(\frac{1}{n})>\frac{1}{2}$ so $\Pi(n,0,\frac{1}{n})$ -c $>\frac{n-3}{2n}$. In fact, $\delta\cdot\Pi(n,0,\frac{1}{n})$ -c>0 for $\delta>\frac{2}{n-1}$. Because profits are linear in δ, $[\frac{2}{n-1}, 1] \subseteq \Delta(n)$.□

Proof of Theorem 5:

The proof involves four steps. We must eliminate the possibility that $V>\frac{n-1}{2n}$ in the second period. So we must investigate whether it is possible for $\Pi(n,0\text{-}c)$ than $\frac{n-1}{2n}$ -c .

1) if $c>\frac{1}{n}$ use Corollary 3.

2) for n>3 if $c<\frac{1}{n}$ then $p_0<\frac{1}{2}$ and $p_0(n,\frac{1}{n})<p_0(4,\frac{1}{4})$ for n>4. Now, $p_0(4,\frac{1}{4})\simeq 0.2251$

<0.5 so $\Pi(n,0,-c)\ -c=(1-p_0(n,\frac{1}{n}))\text{-}c+p_0(n,\frac{1}{n})\ -c<\frac{1}{2}-\frac{3}{2}c$. Thus, $\frac{1}{2}-\frac{3}{2}c>\frac{1}{2}-\frac{1}{n}$ iff $c<\frac{2}{3n}$. So the

two period game will not achieve full efficiency if c is less than $\frac{2}{3n}$.

3) if $c<\frac{1}{2n}$ note that $p_0(n,\frac{1}{2n})<p_0(4,\frac{1}{8})<\frac{3}{10}$ for all n>4.

Thus $\Pi(n,0,-c)\ -c<\frac{3}{10}-\frac{17}{10}c$

but $\frac{3}{10}-\frac{17}{10}c<\frac{1}{2}-\frac{1}{n}$ for all c>0 if n>4

If n=4 $\frac{1}{2}-\frac{1}{n}=\frac{1}{4}$ and $\frac{3}{10}-\frac{17}{10}c>\frac{1}{4}$ iff $c<\frac{1}{34}<\frac{1}{32}=\frac{1}{8n}$

$p_0(4,\frac{1}{32})<0.09$ so for $c<\frac{1}{32}$

$\Pi(n,0,c)\ -c<\frac{1}{10}-\frac{19}{10}c$ which is always less that $\frac{1}{4}$. Again if $c<\frac{1}{2n}$ the two period game does not acheive full efficiency.

4) All that remains unaccounted for are the $c\in[\frac{1}{2n},\frac{2}{3n}]$

Note again that $p_0(n,\frac{2}{3n})<p_0(4,\frac{2}{12})<\frac{1}{3}$ for all n>4.

$c>\frac{1}{2n}$ thus $\Pi(n,c)\ -c<\frac{1}{3}-\frac{5}{6n}$

$\frac{1}{3}-\frac{5}{6n}-\frac{1}{2}+\frac{1}{n}=-\frac{1}{6}+\frac{1}{6n}<0$ for all n>1.

Thus for all n>3 the game never achieves full efficiency in two stages. For n =3, $\frac{1}{2}$ is not in the offer set so the game cannot achieve full efficiency over symmetric mixed strategies. □

Proof of Proposition 4:

We shall show that $\Pi^1>\Pi^t>\Pi^2$. Note that $\Pi^t(n,c)=\Pi^1(n,V)=\Pi^1(n,\delta\cdot\Pi^{t+1}(n.c)\text{-}c)$. Now note that $\delta\cdot\Pi^{t+1}(n.c)>0$ so V>-c which implies that $\Pi^t(n,c)<\Pi^1(n,c)$ because profits are increasing in costs and therefore decreasing in V and for t>2 $\Pi^{t-1}(n,c)<\Pi^1(n,c)$. This leads to $V(c,\Pi^{t-1})<V(\Pi^1)$ and thus $\Pi^t(n,c)>\Pi^2(n,c)$. So Π^∞ must belong to $[\Pi^2, \Pi^1]$ if it exists. A solution exists because $\Pi(n,0,V)$ is continuous for $\Re^+$. Uniqueness is easy because $\Pi(n,V)$ is strictly increasing in c and thus decreasing in V. Thus there must exist a V such that $V=\Pi(n,0,V)$-c.□

Proof of Theorem 6:

The proof of the Theorem follows from our equilibrium selection. Note first that given a cost c players will use the mixed strategy based on Ω_m if and only if $\frac{m}{n} > V > \frac{m-1}{n}$; in this range $\Pi(n,m,V)$ is a continuous decreasing function of V thus $\delta\Pi(n,m,V)$-c will also be a decreasing continuous function of V. So there is at most one solution to the problem $\delta\Pi(n,m,V)-c-V=0$ which is the fixed point condition for an infinite horizon equilibrium. A solution to the problem will exist if $\delta\cdot\Pi(n,m,\frac{m-1}{n})-c > \frac{m-1}{n}$ but $\Pi(n,m,\frac{m-1}{n}) < 1/2$ so a solution may not exist except for m=0. There may also exist solutions for m=k, (k$\in$ $\{\frac{1}{n};\ldots;\frac{1}{2}\}$ provided that c< $\frac{n-k+1}{2n}$.□

Proof of Proposition 5:

$$\Pi(n,m,\tfrac{m-1}{n}) = (1-p_m(\tfrac{m-1}{n}))\tfrac{m-1}{n} + p_m\tfrac{n-m}{n}$$

$$\Pi(n,m,\tfrac{m-1}{n}) = \tfrac{m-1}{n} + p_m(\tfrac{m-1}{n})\tfrac{n-2m+1}{n}$$

$$\Pi(n,m+1,\tfrac{m}{n}) = \tfrac{m}{n} + p_{m+1}(\tfrac{m}{n})\tfrac{n-2m-1}{n}$$

$$p_{m+1}(\tfrac{m}{n}) = \prod_{m+1}^{n-m+2} \frac{2i-2m}{(2i+1-2m)} = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n-4m+4}{2m-4m+5}$$

$$p_m(\tfrac{m-1}{n}) = \prod_{m}^{n-m+1} \frac{2i-2m}{(2i+1-2m)} = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n-4m+4}{2m-4m+5}\cdot\frac{2n-4m+6}{2n-4m+7}\cdot\frac{2n-4m+8}{2n-4m+9}$$

Let $k=n-2m$ so that

$$p_{m+1}(\tfrac{m}{n}) = p_m(\tfrac{m-1}{n})\cdot\frac{2k+7}{2k+6}\cdot\frac{2k+9}{2k+8}$$

If $\Pi(n,m,\frac{m-1}{n}) = \frac{m-1}{n} + p_m(\frac{m-1}{n})\frac{k+1}{n} < \frac{m-1}{n} + c$ *and* $p_m(\frac{m-1}{n})\frac{k+1}{n} > p_{m+1}(\frac{m}{n})\frac{k-1}{n}$ *then*

$$\Pi(n,m+1,\tfrac{m}{n}) = \tfrac{m}{n} + p_{m+1}(\tfrac{m}{n})\tfrac{k-1}{n} < \tfrac{m}{n} + c$$

or $\frac{2k+7}{2k+6}\cdot\frac{2k+9}{2k+8}\, k-1 < k+1$, *or*

$(2k+7)(2k+9)(k-1)<(2k+6)(2K+8)(k+1)$ which is equivalent to

$4k^3+28k^2+41k-63 < 4k^3+32k^2+76k+48$ □

Proof of Proposition 6:

Let us look at the case where only three offers are still rational ($\Omega=\{\frac{n-2}{2n}; \frac{1}{2}; \frac{n+2}{2n}\}$, $V>\frac{n-4}{2n}$ and $d=1$). Then $\Pi(V,c) = p_0(V)\cdot(\frac{n+2}{2n}-V) + V$. For an efficient C.M.S. Equilibrium it must be rational for the seller to refuse any offer less than $\frac{1}{2}$. So $\Pi(V,c)$-c must be greater than $\frac{n-2}{2n}$, or $p_0(V)\cdot(\frac{n+2}{2n}-V) + V - c > \frac{n-2}{2n}$. Because Proposition 2 showed that $\Pi(V,c)$ is decreasing in V, we only need to evaluate $\Pi(V)$ at $(\frac{n-4}{2n})$, that yields the condition: $p_0(\frac{n-4}{2n})\cdot(\frac{n+2}{2n}-\frac{n-4}{2n}) + \frac{n-4}{2n} - c > \frac{1}{2}-\frac{1}{n}$. That condition is equivalent to $(3\cdot p_0(\frac{n-4}{2n})-1)\cdot(\frac{1}{n}) > c$. Now

$p_0(\frac{n-4}{2n}) = \frac{8}{15}$, that implies that c must be less than $\frac{3}{5n}$ for the game to have a C.M.S.E. that is efficient. It is easy to check that if Ω is of cardinality greater than 3 then c must be even smaller. □

Proof of Proposition 7:

The best response function is decreasing in c iff $c>c'$ and $\forall$ $j>i$ $\pi_{bi}(c') \geq \pi_{bi}(c')$ implies $\pi_{bi}(c) \geq \pi_{bi}(c)$.

Assume that $c>c'$ and $\frac{i}{n}$ is a best response for a player of type c' or: $\pi_{bi}(c') \geq \pi_{bj}(c')$ or $\hat{A}_i - \hat{B}_i\cdot c' \geq \hat{A}_j - \hat{B}_j\cdot c'$ $\forall$ j . Furthermore $\forall$ $j>i$ $\hat{A}_i > \hat{A}_j$ and $\hat{B}_i > \hat{B}_j$. But, $\pi_{bi}(c') \geq \pi_{bj}(c')$ and $\hat{A}_i > \hat{A}_j$ and $\hat{B}_i > \hat{B}_j$ imply $\hat{B}_j - \hat{B}_i \leq (\hat{A}_i - \hat{A}_j)c'$. However, if $c>c'$ then $\hat{B}_j - \hat{B}_i \leq (\hat{A}_i - \hat{A}_j)c$, so the best response of a player of type c will be less than or equal to $\frac{i}{n}$.□

Proof of Proposition 8

A) let C be the set defined by $\gamma_i \leq \gamma_j$ iff $i>j$ and $\underline{c} \leq \gamma_j \leq \bar{c}$. Clearly C is bounded and C is closed because it contains its boundary. Moreover it is easy to show that C is convex. $\Gamma^*(\cdot)$ maps C into C continuously. Therefore a fixed point exists. Note that this does not prove that C.M.S. equilibria exist for every Ω and every distribution function.

B) the possible o fers are now 0 and 1 and thus

$$\pi_{1b}(P_s|\,\hat{c}) = 1\{\int_{\underline{c}}^{\bar{c}} p_{s0}(c)\, f(c)\, dc\ - \hat{c}\int_{\underline{c}}^{\bar{c}} p_{s1}(c)\, f(c)\, dc\}$$

$$\pi_{0b}(P_s|\,\hat{c}) = 0.5\{\int_{\underline{c}}^{\bar{c}} p_{s0}(c)\, f(c)\, dc\}$$

however, in equilibrium bbest responses are monotonic so $\exists\ \lambda$ such that

$$\pi_{1b}(P_s|\,\hat{c}) = 1F(\lambda)\ - \hat{c}[1\text{-}F(\lambda)]$$

$$\pi_{0b}(P_s|\,\hat{c}) = 0.5F(\lambda)$$

At a cost of λ the buyer and the seller are indifferent between the offers they make so that:

$F(\lambda)\ \text{-}\lambda[1\text{-}F(\lambda)] = 0.5F(\lambda) \ \rightarrow$

$F(\lambda) = \frac{\lambda}{\lambda+0.5}$. Since $F(\lambda) \uparrow$ and $F(\underline{c}) = 0$; $F(\hat{c})=1$ and $\frac{\lambda}{\lambda + 0.5}$ is decreasing in 1 (and less than 1 at $\bar{c}$) a unique solution will exist.

C) follows directly from Propostion 7.□

Proof of Theorem 7

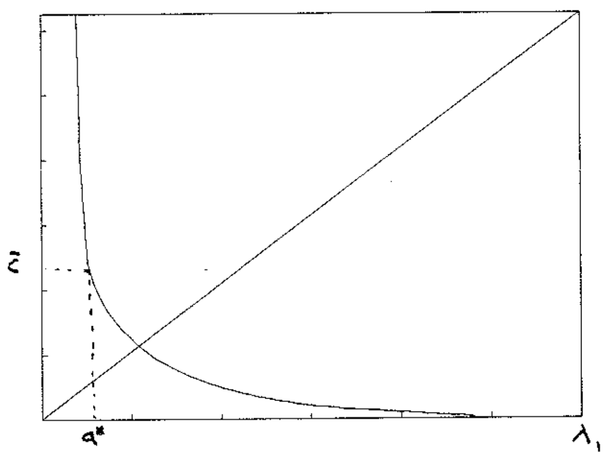
In equilibrium the set of cut-off points must be such that a player's difference in expected profit from offering i versus i+1 is 0. Thus, with a uniform distribution we must solve the following set of i= 1,....n equations:

$$(\lambda_i\ \text{-}\lambda_{i\text{-}1})\lambda_{n\text{-}1+1}\ \text{-}\ (\bar{c}\ \text{-}\ \lambda_i)\frac{1}{2n} = 0 \qquad \text{(i)}$$

where $\lambda_0\ = \underline{c}$. Using equation (1) of the n equations allows to solve for λ_n in terms of λ_1 and obtain:

$$\phi_n(\lambda_1) = \frac{\bar{c}\ \text{-}\ \lambda_1}{2n(\lambda_1\text{-}\underline{c})}.$$

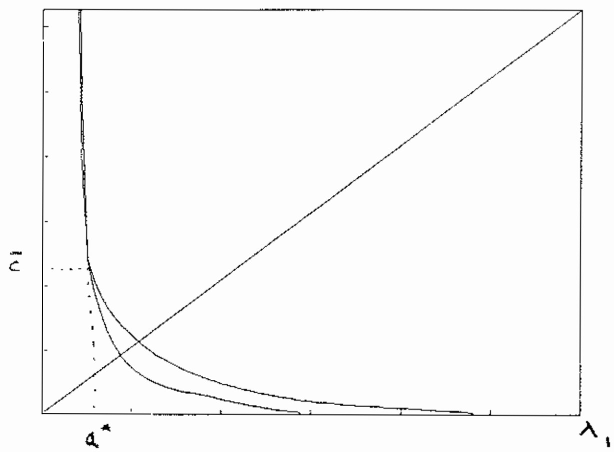
Note that $\phi_n{}'<0$; $\phi_n(\underline{c}) = \infty$; $\phi_n(\bar{c})=0$ and at $\lambda_1 = \frac{\bar{c}(1 + 2n\underline{c})}{(1 + 2n\bar{c})} = a^* < \bar{c}$ we have $\phi_n(a^*) = \bar{c}$.



Next we go to equation n and plug in $\phi_n()$ and solve for λ_{n-1} in terms of λ_1:

$$\phi_{n-1}(\lambda_1) = \phi_n(\lambda_1) - \frac{\bar{c} - \phi_n(\lambda_1)}{2n\lambda_1}.$$

Notice that $\phi_{n-1}'<0$; $\phi_{n-1}(\underset{\cdot}{c}) = \infty$; $\phi_{n-1}(\bar{c}) =- \frac{1}{2n}$; $\phi_n(\lambda_1) - \phi_{n-1}(\lambda_1) = 0$ at a^* and > 0 for $\lambda_1 > a^*$.



Going to equation (2) we find:

$$\phi_2(\lambda_1) = \frac{\phi_{n-1}(\lambda_1)\lambda_1 + \frac{\bar{c}}{2n}}{\phi_{n-1}(\lambda_1) + \frac{1}{2n}} .$$

It is easy to show that $\phi_2(\lambda_1) > \lambda_1$ (for $\lambda_1<\bar{c}$), and $\phi_2(a^*) < \bar{c}$. With these terms and conditions we can continue this process to find:

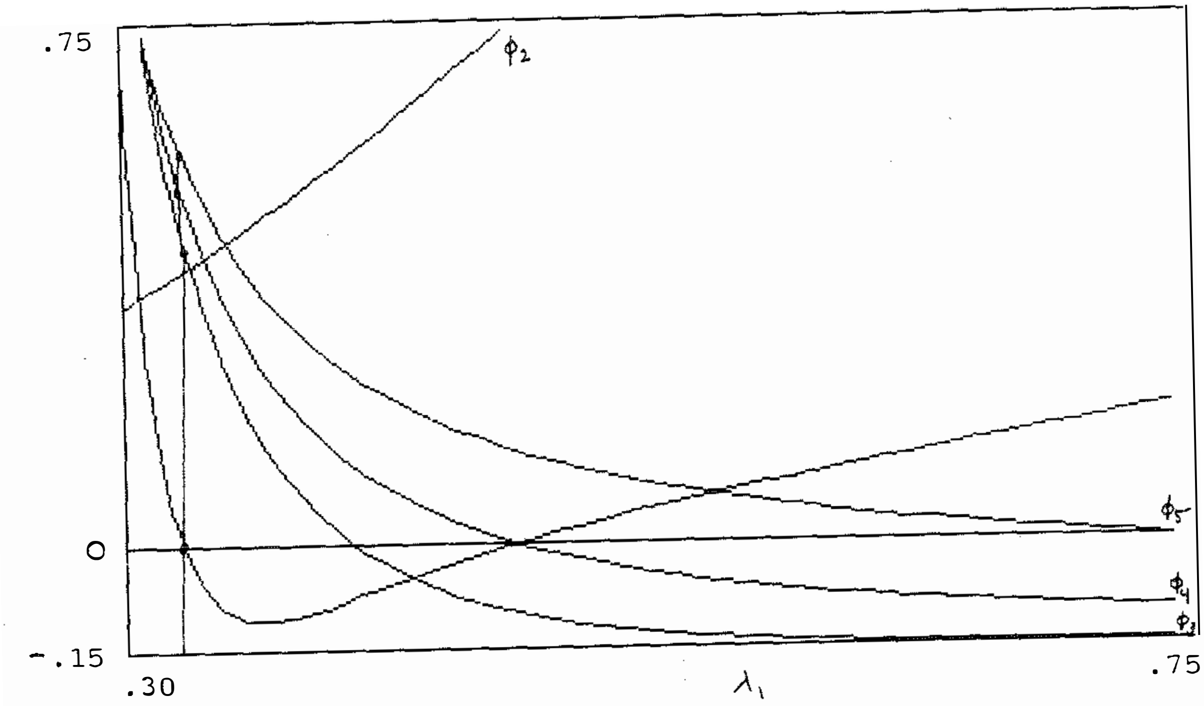
$\lambda_1< \phi_i < \phi_{i+1}<\bar{c}$.

For n odd we will end at the $\frac{n+1}{2}$ equation and find:

$$\phi^*(\lambda_1) = (\phi_{\frac{n+1}{2}}(\lambda_1) - \phi_{\frac{n+1}{2}-1}(\lambda_1))\phi_{\frac{n+1}{2}}(\lambda_1) \quad - \quad (1 - \phi_{\frac{n+1}{2}}(\lambda_1))\frac{1}{2n} = 0 \ .$$

Now, when $\lambda_1 = \underset{\cdot}{c}$, $\phi^* = \infty$; when $\phi_{\frac{n+1}{2}}(\lambda_1) = \phi_{\frac{n+1}{2}-1}(\lambda_1)$, $\phi^* < 0$, after which $\phi^{*\prime}> 0$ with $\phi(\bar{c}) > 0$. Thus, there are two λ_1's that solve the above equation--but only one solves allows for $\lambda_i<\lambda_{i+1}$. The proof is similar for n even. □

Below are the graphs for n=5 and c with the interval [.25,.75].



REFERENCES

1. Broman, E., The Bilateral Monopoly Model: Approaching Certainty Under the Split-the-Difference Mechanism, *J. Econ. Theory* 48 (1989), 134-152.
2. Card, D., Strikes and Wages: A Test of a Signaling Model, N.B.E.R. Working Paper # 2550. (1988)
3. Chatterjee, K., and W. Samuelson, Bargaining Under Incomplete Information, *Oper. Res.* 31 (1983), 835-851.
4. Cho, I., Uncertainty and Delay in Bargaining: Old Question New Approach, mimeo University of Chicago, (1988).
5. Currie, J. and S. McConnell, Wages and Collective Bargaining Legislation: The Case for Compulsory Arbitration, mimeo University of California, Los Angeles (1989).
6. Leininger, W., P. Linhart, and R. Radner, The Sealed-Bid Mechanism for Bargaining with Incomplete Information, *J. Econ. Theory* 48 (1989), 63 -107
7. Myerson, R.B., and M. A. Satterthwaite, Efficient Mechanisms for Bilateral Trading, *J. Econ. Theory* 29 (1983), 265-281.
8. Rosenthal, J.-L., "The Fruits of Revolution, Property Rights, Litigation and French Agriculture; 1700-1860," Caltech PhD dissertation June 1988.
9. Rubinstein, A., Perfect Equilibrium in a Bargaining Model, *Econometrica* 50 (1982), 97-110.
10. Satterthwaite, M. and S. Williams, Bilaterial Trade with the Sealed Bid *k*-Double Auction: Existance and Efficiency, *J. Econ. Theory* 48 (1989), 107-133.
11. Stahl, I., "Bargaining Theory," Stockholm: Economic Research Institute, 1972.
12. Tracy, J., An Investigation into the Determinants of U.S. Strike Activity, *American Econ. Review* 76 (1986) 149-173.