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EQUILIBRIA RESISTANT TO MUTATION

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Abstract

The paper requires that equilibrium behavior for two person symmetric games be resistant to genetic evolution. In particular the paper assumes that the evolution of genotypes selecting a behavioral rule can be described according to some generalization of the replicator model. This paper defines an equilibrium concept, 'evolutionary equilibrium', which is defined as the limit of stationary points of the evolutionary process as the proportion of the population that mutates goes to zero. Then the set of evolutionary equilibria, as defined in the paper, is a nonempty subset of the set of perfect equilibria (and thus of the set of Nash equilibria) and a superset of the set of regular equilibria and the set of ESS.

EQUILIBRIA RESISTANT TO MUTATION

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1 Introduction

Many games cannot be solved in pure strategies. The solution is then for one of the players to play a mixed strategy; in other words to select a strategy according to a specific randomization device. Numerous scholars (Rubinstein (1988), for instance) find troublesome that optimal behavior should follow by chance. Thus it seems useful to give a different interpretation of game theory. Instead of imagining two specific players confronting the game, suppose that there is an infinite population of potential players. At each period, players are randomly and anonymously matched. Each player plays a pure strategy. Equilibria in mixed strategies are then interpreted as equilibria where the population is not homogeneous [1].

One of the justification for assuming that people use optimal strategies is the belief that such strategies are 'evolutionary stable'. If (i) particular strategies are transmitted genetically and (ii) evolutionary laws select for optimal strategies, then people will act as optimizers. In most games there are no optimal strategies since best responses depend on what other players choose. However there may be modes of behavior that will persist and are immune to genetic drift. This paper examines a specific model for the study of

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[1]Harsanyi (1973) gives a similar interpretation of mixed strategies. Harsanyi justifies mixed strategies by players' uncertainty on their opponents payoffs (or types). In particular the payoff function for player i of type ξ_i is $u_i + \epsilon\xi_i$. A type knows his own value of ξ_i and knows the distribution function for ξ_j over the set of possible types Ξ_j; in particular, all players know that the expected value of ξ_j is zero. Then in equilibrium each type selects a pure strategy and each player plays the mixed strategy determined by the pure strategy played by each type and the distribution of types. Furthermore a mixed strategy equilibrium in the complete information game is the limit as ϵ goes to zero of equilibria of incomplete information games.

genetic evolution which is called 'the replicator model' [2]. In particular the paper seeks to define and characterize equilibria of the evolutionary process.

There is a very large literature in game theory which discusses different definitions of equilibria. The equilibrium concept which is used by economists is called 'Nash Equilibrium' although game theorists have given numerous examples where the equilibrium concept is inadequate [3]. A strengthening of the notion of Nash equilibrium has led to the concepts of 'perfect equilibrium' and 'proper equilibrium' [4]. This paper examines the relationship between equilibria of the genetic process and existing equilibrium concepts in game theory [5].

The replicator model describes the distribution of strategies in the population in terms of a differential equation. Thus in order to examine the dynamic equilibria of the replicator model it is necessary to solve a system of nonlinear differential equations which is usually done trough numerical simulations. Alternatively there are static equilibrium concepts that can be defined for the replicator model and which can be solved analytically. A particularly well known static equilibrium concepts is Maynard-Smith and Price's (1973) evolutionary stable strategy (which is denoted in this paper by ESS). This paper defines a different static equilibrium concept called 'evolutionary equilibrium'. The equilibrium concept is based on an arbitrarily small proportion of genes mutating towards a 'random' strategy.

The second section of the paper describes the replicator model and its relationship with the concepts of Nash and perfect equilibrium. The third section of the paper defines and establishes formal properties of an evolutionary equilibrium. In particular this section proves that an evolutionary equilibrium exists for a large class of payoff matrices. The fourth section analyzes the relationship between evolutionary equilibria and other equilibrium concepts in game theory: the set of evolutionary equilibria is a subset of the set of perfect equilibria and a superset of the set of regular equilibria. Throughout the paper definitions are indicated by italics.

2 The replicator model

Suppose that individuals from a large population are paired randomly. Each individual selects a strategy $i \in \{1, \ldots, n\}$. The scalar x_i is the proportion of individuals who select strategy i and the column vector $x = (x_1, \ldots, x_n)$ describes the proportion of the

[2] The following authors discuss the properties of the replicator model: Hines (1987), Hofbauer (1981), Schuster et al. (1981), Taylor and Jonker (1978), Zeeman (1979).

[3] See for instance van Damme (1987).

[4] These equilibrium concepts were introduced in Selten (1975) and Myerson (1978).

[5] The following paper discuss similar issues: Crawford (1988), Friedman (1988), Nachbar (1990), Samuelson (1988). The following papers discuss these issues in some more specialized contexts: Axelrod and Hamilton (1981), Boyd and Lorberbaum (1987), Crawford (1989).

population that adopts each possible strategy. Thus $x \in \Delta^n$ where

$$\Delta^n \equiv \{x \in \Re^n_+ : \sum_{i=1}^{n} x_i = 1\}.$$

If an individual selects strategy i and is matched with an individual that has selected strategy j, a_{ij} ($a_{ij} \geq 0$) individuals will adopt strategy i in the next period. The matrix A, where

$$A \equiv \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

is called the payoff matrix for the evolutionary game. Then the proportion of the population adopting strategy i at time $t+1$ is

$$x_i^{t+1} = x_i^t \frac{(Ax^t)_i}{x^t \cdot Ax^t}$$

where $(Ax)_i = \sum_{j=1}^{n} a_{ij}x_j$ and $x \cdot Ax = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$. The next proposition shows that the law of motion is well defined.

Proposition 1 *If $x^0 \in \Delta^n$ then for $t = 1, 2, \dots,$ $x^t \in \Delta^n$.*

The last expression can be rewritten as:

$$x_i^{t+1} - x_i^t = x_i^t \frac{(Ax^t)_i - x^t \cdot Ax^t}{x^t \cdot Ax^t}.$$

or by dropping the time subscripts

$$\Delta x_i = x_i \frac{(Ax)_i - x \cdot Ax}{x \cdot Ax} \equiv R_i^D(x). \tag{1}$$

The system of difference equations $\Delta x = R^D(x)$ is called the replicator model in discrete time; when we need to specify the payoff function, A, the replicator model model is denoted by R_A^D. Let $\Delta t = \frac{1}{x \cdot Ax}$ be the time interval between periods; then

$$\Delta x_i = x_i[(Ax)_i - x \cdot Ax]\,\Delta t$$

By letting $\Delta t \to 0$ the last expression can be written as:

$$\dot{x}_i = x_i[(Ax)_i - x \cdot Ax] \equiv R_i^C(x)^6. \tag{2}$$

[6] Hofbauer (1981) noticed that by setting $b_{ij} = a_{ij} - a_{nj}$ and letting $y_i = x_i/x_n$, expression (2) can be rewritten as

$$\dot{y}_i = y_i(b_{in} + \sum_{j=1}^{n-1} b_{ij}y_j) \;\; (i \in \{1, \dots, n-1\})$$

which is the Volterra-Lotka equation.

The system of differential equations $\dot{x} = R^C(x)$ is called the replicator model in continuous time; again when we need to specify the payoff function, A, the replicator model is denoted by R^C_A. Let the function $X : \Re^n \times \Re_+ \rightarrow \Re^n$ be such that for all $t \in \Re_+$ and $x^0 \in \Delta^n$,

$$\frac{dX(t,x^0)}{dt} = R^C(X(t,x^0)) \text{ and } X(0,x^0) = x^0$$ [7].

The next proposition states that this system too is well defined.

Proposition 2 *If $x^0 \in \Delta^n$ and $t \in \Re_+$ then $X(t,x^0) \in \Delta^n$.*

The next theorem describes one of the most important properties of the replicator model.

Theorem 1 *If (x,x) is a Nash equilibrium of the normal game (A, A^T) then x is a stationary point of the replicator model.*

Unfortunately there are stationary points of the replicator model that are not Nash equilibria of the normal game.

A partial converse of the theorem was given by Bomze (1986).

Theorem 2 *(Bomze) (i) If x is a stable stationary point* [8] *of the replicator model in continuous time then (x,x) is a Nash equilibrium. (ii) If x is an asymptotically stable stationary point* [9] *of the replicator model in continuous time then (x,x) is an isolated perfect equilibrium.*

Unfortunately, asymptotically stable stationary points are difficult to characterize and do not always exist. This paper further characterizes the relationship between the replicator model and game theory. The idea is to look at stationary points of the replicator model that are resistant to mutation.

[7] The existence and uniqueness of X follow from the differentiability of R^C.

[8] An equilibrium $\tilde{x}$ is *stable* if given any positive scalar ϵ, there is a positive scalar δ such that for all strategies x in the ball centered at $\tilde{x}$ and with radius δ, $x \in B(\tilde{x},\delta) \bigcap \Delta^n$, and for all positive t, $X(x,t) \in B(\tilde{x},\epsilon) \bigcap \Delta^n$.

[9] An equilibrium $\tilde{x}$ is *asymptotically stable* if it is stable and if δ can be chosen such that

$$\forall x \in B(\tilde{x},\delta),\ \lim_{t\rightarrow\infty} X(x,t) = \tilde{x}$$

3 Evolutionary equilibrium

This paper examines a class of laws of motion somewhat more general than the replicator model. Let

$$\mathcal{H}_A = \{H : \Delta^n \to T\Delta^n \,|\, \exists L : T\Delta^n \to T\Delta^n \text{ such that } L \text{ is continuously differentiable, } L_i(y)y_i \geq 0,\ (L_i(y)y_i = 0 \Leftrightarrow y_i = 0),\ H = L \circ R_A^C\}$$

where $T\Delta = \{x \in \Re^n \,|\, \sum_i x_i = 0\}$. In order to make the notation less cumbersome, when no confusion can arise the subscript A in $\mathcal{H}_A$, is dropped. Clearly if $L(x) \equiv I_{n\times n}$ then $H = R^C$ and $H \in \mathcal{H}$. If $L(x) \equiv \frac{1}{xAx} I_{n\times n}$ then $H = R^D$ and $H \in \mathcal{H}$ [10]. Suppose that the strategies selected by the offsprings are subject to mutation. Assume that the set of possible mutation rates is

$$\mathcal{M}^n = \{m : \Delta^n \to T\Delta^n \,|\, m \text{ is bounded, continuously differentiable and } (\forall A \subset \{1,\ldots,n\}) \sum_{i\in A} x_i = 1 \Rightarrow \sum_{i\in A} m_i(x) \leq 0\}.$$

Again, when it does not lead to confusion, the subscript n is dropped.

For an evolutionary game with payoff matrix A, the generalized replicator model $H \in \mathcal{H}$, and the mutation function $m \in \mathcal{M}$, define an *evolutionary system* by the following differential equation:

$$\begin{aligned} \Delta x &= (1-\mu)H(x) + \mu m(x) \quad \text{(discrete version)} \\ \dot{x} &= (1-\mu)H(x) + \mu m(x) \quad \text{(continuous version).} \end{aligned}$$

A vector $\tilde{x}$ is an *evolutionary equilibrium* for the payoff function A and the generalized replicator model $H \in \mathcal{H}$ if for every function m in $\mathcal{M}$ there is a scalar $\mu' \in (0,1)$ and a vector valued function $x : (0,\mu') \to \Delta^n$ such that for all $\mu \in (0,\mu')$,

$$(1-\mu)H(x(\mu)) + \mu m(x(\mu)) = 0$$

and $\lim_{\mu\downarrow 0} x(\mu) = \tilde{x}$. [11]

The following theorem by Jiang Jia-He (1963) (which generalizes the better known theorem by Fort (1950)) is used in the proof of the existence of evolutionary equilibria.

[10] The set $\mathcal{H}$ includes other functions such as $H(x) = \Big((x_1[(Ax)_1 - x\cdot Ax])^3, -(x_1[(Ax)_1 - x\cdot Ax])^3\Big)$ where $L : T\Delta^2 \to T\Delta^2$ is defined as $L(y) = (y_1^3, -y_1^3)$.

[11] Showing that a vector of strategies is an evolutionary equilibrium by using the definition seems quite hard. In many examples a simpler procedure will be the following: (i) first show that there exists an evolutionary equilibrium; this paper gives a sufficient condition for the existence of equilibria, 'nondegeneracy,' which is straightforward to check. (ii) Serially eliminate all strictly dominated strategies. If there is a unique Nash equilibrium that puts positive weight only on undominated strategies then this will be an evolutionary equilibrium. The validity of this procedure is proven in the rest of the paper.

Let X be a compact convex subset of a normed space, let d be a metric defined on X, and let $C(X,X)$ be the set of continuous function with domain and range in X. Then $(C(X,X),\rho)$ is a metric space where

$$\rho(f,g) = \sup_{x\in X} d(f(x),g(x)).$$

Finally, let $F\colon C(X,X) \to X$ be the fixed point correspondence; i.e., for all $f \in C(X,X)$

$$F(f) = \{x \in X \mid f(x) = x\}.$$

A set D is said to be *totally disconnected* if all the connected subsets of D are singletons.

Theorem 3 *(Jiang Jia-He) Suppose $F(f)$ is a totally disconnected set. Then there is a vector p in $F(f)$ that satisfies the following property. For every neighborhood U of p there is an $\epsilon > 0$ such that:*

$$g \in C(X,X) \text{ and } \rho(f,g) < \epsilon \Rightarrow F(g) \bigcap U \neq \emptyset.$$

The vector p described in the theorem is called an essential fixed point.

For all subsets of the strategy set $S \subset \{1,\ldots,n\}$, let $A|_S$ be the matrix $(a_{ij})_{i\in S, j\in S}$. A matrix A is said to be *nondegenerate* if for all $S \subset \{1,\ldots,n\}$ such that $\#(S) \geq 2$, the matrix $A|_S$ is nonsingular [12].

Proposition 3 *For all nondegenerate payoff matrices A there is an evolutionary equilibrium.*

Proof: Fix a payoff matrix A and a generalized replicator function $H \in \mathcal{H}_A$. In order to be able to use Theorem 3 we need to construct a function, H', whose fixed points correspond to the stationary points of H. Let

$$H' : \Delta^n \longrightarrow \Re^n \text{ be defined by } H' \equiv I + H.$$

Notice that fixed points of H' are stationary points for H. Unfortunately the function H' does not map its domain, Δ^n into itself. In order to remedy this problem we extend

[12] The only property used in the paper is that there are finitely many symmetric equilibria in all the submatrices $A|_S$. I think that a necessary and sufficient condition for the latter property is

$$\forall (i,j) \subset \{1,\ldots,n\},\ a_{ij} = a_{ii} \Rightarrow a_{ji} \neq a_{jj}.$$

Notice that either assumption is much weaker than the Lemke and Howson nondegeneracy condition.

the function H' to a domain $E\Delta^n$ which is invariant under the extension, $\tilde{H}$. Specifically let

$$\begin{aligned} M &= \max_{x\in\Delta, i\in\{1,\ldots,n\}}|H_i'(x)|+1; \\ E\Delta^n &= \{x\in\Re^n \mid \sum_{i=1}^n x_i = 1,\ \forall i\in\{1,\ldots,n\}\ x_i\in[-M,M]\}; \end{aligned}$$

$$\begin{aligned} \alpha : E\Delta^n &\rightarrow [0,1) \\ \alpha(x) &\equiv \min\{\alpha\in[0,1) \mid \alpha/n\mathbf{1}+(1-\alpha)x\in\Delta^n\}; \end{aligned}$$

$$\begin{aligned} \tilde{H} : E\Delta^n &\longrightarrow E\Delta^n \\ \tilde{H}(x) &= H'(\alpha(x)/n+(1-\alpha(x))x). \end{aligned}$$

Notice that fixed points of H' are stationary points for H and that all the fixed point of $\tilde{H}$ are in Δ^n (and thus are fixed points of H'). Since by assumption A is nondegenerate there are finitely many stationary points to H and Theorem 3 is applicable. Let $g = (1-\mu)\tilde{H}+\mu(m+I)$. Then for small enough μ, $g : E\Delta^n \rightarrow E\Delta^n$ and $\rho(\tilde{H},g) < \epsilon$. A fixed point of g corresponds to a stationary point of $(1-\mu)H+\mu m$. Thus the set of perturbations allowed in the theorem includes the ones in the definition of evolutionary equilibrium.

Fix $m \in \mathcal{M}$. Since all the conditions are satisfied we use Theorem 3 to prove the existence of an evolutionary equilibrium. Thus there is an $\tilde{x}$ such that for every $\epsilon > 0$ there is a μ_ϵ and a function

$$x_\epsilon(\mu) : (0,\mu_\epsilon) \rightarrow \Delta^n$$

such that $\forall\mu \in (0,\mu_\epsilon)$

$$(1-\mu)H(x(\mu))+\mu m(x_\epsilon(\mu)) = 0$$

and $\sup_{\mu\in(0,\mu_\epsilon)}|x_\epsilon(\mu)-\tilde{x}| < \epsilon$. Then since there are finitely many fixed points of H there is an $\epsilon' > 0$ such that $B(\tilde{x},\epsilon')\bigcap F(H) = \{\tilde{x}\}$. For $\mu \in (0,\mu_{\epsilon'})$ let $x(\mu) = x_{\epsilon'}(\mu)$. Then $x : (0,\mu_{\epsilon'}) \rightarrow \Delta$ is such that

$$(1-\mu)H(x(\mu))+\mu m(x(\mu)) \equiv 0,$$

and $x(\mu) \rightarrow \tilde{x}$. Therefore, $\tilde{x}$ is an evolutionary equilibrium. ∎

4 Relationship between evolutionary equilibrium and other equilibrium concepts

The rest of the paper relates evolutionary equilibria to other game theoretic equilibria; i.e., equilibrium concepts that are derived from assumptions on the type of beliefs individuals have and on Bayesian maximization. There are three reasons for being interested in these relationships. (i) Showing that an evolutionary equilibrium corresponds to a game theoretic equilibrium allows us to argue that individuals act 'as if' they are Bayesian maximizers. (ii) There are ways of computing game theoretic equilibria that can be used to compute evolutionary equilibria. (iii) Requiring that a game theoretic equilibrium be evolutionary stable refines the set of equilibria.

The equilibrium concepts that are analyzed are following: (1) Nash equilibrium, (2) perfect equilibrium, (3) strict dominance solvability, (4) regular equilibrium, (5) proper equilibrium, (6) strictly proper equilibrium, (7) ESS, (8) essential equilibrium.

4.1 Nash equilibrium

The concept of Nash equilibrium is the most widely used equilibrium concept in game theory although it is often considered to be too weak (see however Bernheim (1984) and Pearce (1984)).

Proposition 4 *An evolutionary equilibrium, $\tilde{x}$, is a symmetric Nash equilibrium, $(\tilde{x}, \tilde{x})$.*

Proof: Let $\tilde{x}$ be an evolutionary equilibrium. (i) Suppose that there is a strategy, say 1, such that $\tilde{x}_1 = 0$. In order to prove that $\tilde{x}$ is a Nash equilibrium it suffices to show that $(A\tilde{x})_1 \leq \tilde{x}A\tilde{x}$. Let m be such that $m_1(x) > 0$ for all x in a neighborhood of $\tilde{x}$. Then since $\tilde{x}$ is an evolutionary equilibrium there is a $\mu' > 0$ and a function $x : (0, \mu') \to \Delta$ such that for every μ in $(0, \mu')$,

$$(1 - \mu)H(x(\mu)) + \mu m(x) = 0.$$

This implies that for all μ in $(0, \mu')$, $(Ax(\mu))_1 - x(\mu)Ax(\mu) < 0$. Thus $(A\tilde{x})_1 \leq \tilde{x}A\tilde{x}$. (ii) Suppose that $\tilde{x}_i > 0$ and $\tilde{x}_j > 0$. Then, $(A\tilde{x})_i - \tilde{x}A\tilde{x} = 0$ and $(A\tilde{x})_j - \tilde{x}A\tilde{x} = 0$. Thus $(A\tilde{x})_i = (A\tilde{x})_j$. Therefore $\tilde{x}$ is a Nash equilibrium.

∎

In the next section we show that not all perfect equilibria are evolutionary equilibria and thus that not all Nash equilibria are evolutionary equilibria.

4.2 Perfect equilibrium

There are several ways in which perturbations have been introduced in solution concepts. Evolutionary equilibria consider perturbations in the law of motion; essential equilibria (which are analyzed later in this section) consider perturbation in the payoff function;

finally, perfect equilibria consider equilibria that are 'resistant' to some perturbation of the strategy set.

The next lemma is used in showing the relationship between perfect and evolutionary equilibria.

Lemma 1 *Let $\tilde{x}$ be an evolutionary equilibrium. Then $\tilde{x}_i = 0$ if strategy i is weakly dominated*[13].

Proof: Suppose that strategy 1 weakly dominates strategy 2. Suppose that $\tilde{x}$ is an evolutionary equilibrium and $\tilde{x}_2 > 0$. Let $m \in \mathcal{M}$ be such that in a neighborhood of $\tilde{x}$ (where $x_2 > 0$)

$$m_1(x) \equiv 1, \; m_2(x) \equiv -1.$$

Then since $\tilde{x}$ is an evolutionary equilibrium there exists a constant μ' and a function $x : (0, \mu') \longrightarrow \Delta$ such that for $\mu \in (0, \mu')$

$$(1-\mu)H_1(y) + \mu = 0 \tag{3}$$
$$(1-\mu)H_2(y) - \mu = 0 \tag{4}$$

and $\lim_{\mu \to 0} x(\mu) = \tilde{x}$. Condition (3) implies that

$$x(\mu)Ax(\mu) > (Ax(\mu))_1;$$

condition (4) implies that for all $\mu \in (0, \mu')$

$$(Ax(\mu))_2 > x(\mu)Ax(\mu).$$

Thus (3) and (4) combined give

$$(Ax(\mu))_2 > (Ax(\mu))_1$$

which contradicts the assumption of weak domination. ∎

Proposition 5 *An evolutionary equilibrium is a symmetric perfect equilibrium.*

Proof: The result follows from previous lemma and van Damme (1987, Theorem 3.2.2): for a two person finite normal game an equilibrium is perfect if only if every weakly dominated strategy is played with probability 0. ∎

[13]A strategy i is *weakly dominated* if there exists a strategy j such that the payoff to j is at least as great as the payoff for using i regardless of what the other players strategy and strictly better for some vector of strategy.

The next example shows that not all symmetric perfect equilibria are evolutionary equilibria. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The vector $(\tilde{x}, \tilde{x})$ where $\tilde{x} = (0, 1, 0)$ is a symmetric perfect equilibrium but $\tilde{x}$ is not an evolutionary equilibrium (pick $m(x) = (1, -1, 0)$ for every x in a neighborhood of $\tilde{x}$).

4.3 Strict dominance solvability

The next proposition says that an evolutionary equilibrium is resistant to the elimination of dominated strategies. Thus restricting the replicator model to rationalizable strategies will not reduce the set of evolutionary equilibria.

Proposition 6 *Suppose that strategy j dominates strategy i and suppose that $\tilde{x}$ is an evolutionary equilibrium for the payoff matrix A. Let A_{-i} be the payoff matrix where the i^{th} row and column have been deleted. Then $\tilde{x}_i$ is an evolutionary equilibrium for the payoff matrix A_{-i}.*

Proof: Let $m' \in \mathcal{M}^{n-1}$ be a mutation function. Let $m \in \mathcal{M}^n$ be such that

$$m_j(x) = \begin{cases} m'_j(x) & \text{if } j \neq i \\ 0 & \text{if } j = i. \end{cases}$$

Since $\tilde{x}$ is an evolutionary equilibrium then for small enough $\mu' > 0$ there is a function $x : (0, \mu') \to \Delta^n$ such that $x(\mu) \to \tilde{x}$ and for all $\mu \in (0, \mu')$, $(1-\mu)H_A(x(\mu)) + \mu m(x(\mu)) = 0$. Since $\tilde{x}$ is a Nash equilibrium and since strategy i dominates strategy j, then $\tilde{x}A\tilde{x} \geq (A\tilde{x})_j > (A\tilde{x})_i$. Then there is a small enough $\mu'' > 0$ such that for all $\mu \in (0, \mu'')$, $x(\mu)Ax(\mu) > (Ax(\mu))_i$. Since $m_i(x) = 0$ then $x_i(\mu) = 0$. Therefore, for all $\mu \in (0, \mu'')$, $(1-\mu)H_{A_{-i}}(x_{-i}(\mu)) + \mu m'(x_{-i}(\mu)) = 0$, and $\tilde{x}_{-i}$ is an evolutionary equilibrium for A_{-i}. ■

An equilibrium is *strictly dominance solvable* if it is can be obtained by reducing the game to a single cell by iterated deletion of dominated strategies.

Proposition 7 *A strictly dominance solvable equilibrium is an evolutionary equilibrium*[14].

Proof: Suppose that $\tilde{x}$ is an evolutionary equilibrium, strategy 1 dominates strategy 2 in the normal game $A_{\{1,\ldots,n\}-\{4\}}$, and strategy 3 dominates strategy 4 in the game A. By Lemma 1, $\tilde{x}_4 = 0$. Suppose $\tilde{x}_2 > 0$. Choose the function m such that for every x in a neighborhood of $\tilde{x}$, $m_1(x) = 1$, $m_2(x) = -1$. Then for small enough μ, $(Ax(\mu))_2 > (Ax(\mu))_1$. But this is impossible since $x_4(\mu) \to 0$. Thus $\tilde{x}_2 = 0$. ■

[14]Thus an evolutionary equilibrium is ecologically solvable as defined by Nachbar (1990).

4.4 Regular equilibrium

The concept of regular equilibrium was introduced by Harsanyi (1973). The following description of the equilibrium is taken from van Damme (1987) although it is simplified by looking at two person symmetric games. Let $z = (x, y)$ be a vector of strategies for the game (A, A^T). Let $k \in \text{supp}(x)$, let $l \in \text{supp}(y)$ and let $m = (k, l)$. Then let $F(x|k)$ be such that:

$$(\forall i \neq k)\ F_i(x|k) = x_i[(Ay)_i - (Ay)_k] \text{ and } F_k(x|k) = \sum_{i=1}^{n} x_i - 1.$$

Similarly let $F(y|l)$ be such that

$$(\forall i \neq l)\ G_i(y|l) = y_i[(Ax)_i - (Ax)_l] \text{ and } G_l(y|l) = \sum_{i=1}^{n} y_i - 1.$$

Finally let

$$H(z|m) = (F(x|k), G(y|l))^T \text{ and } J(\tilde{z}|m) = \frac{\partial H(z|m)}{\partial z}\ |_{z=\tilde{z}}.$$

Then $\tilde{z}$ is a *regular equilibrium* if for some $m \in \text{supp}(x) \times \text{supp}(y)$, $H(z|m) = 0$ and det $J(z|m) \neq 0$.

Intuitively, a regular equilibrium is one for which the best response mapping is continuously differentiable at a neighborhood of the Nash equilibrium.

Proposition 8 *A symmetric regular equilibrium is an evolutionary equilibrium.*

Proof: Van Damme (1987, Theorem 9.4.3) proves that a Nash equilibrium $(\tilde{x}, \tilde{x})$ is regular if and only if $dR^C/dx|_{x=\tilde{x}}$ is nonsingular. Notice that if $dR^C/dx|_{x=\tilde{x}}$ is nonsingular and μ is small enough then

$$\frac{d}{dx}[(1-\mu)L(R^C(x)) + \mu m(x)]|_{\mu=0,x=\tilde{x}}$$

is nonsingular. Therefore if $(\tilde{x}, \tilde{x})$ is a regular equilibrium then by the implicit function theorem $\tilde{x}$ is an evolutionary equilibrium. ∎

The next example shows that not all games with nondegenerate payoff matrices A have a regular equilibrium.

Let

$$(A, A^T) = \begin{pmatrix} 2,2 & 2,2 \\ 2,2 & 1,1 \end{pmatrix}.$$

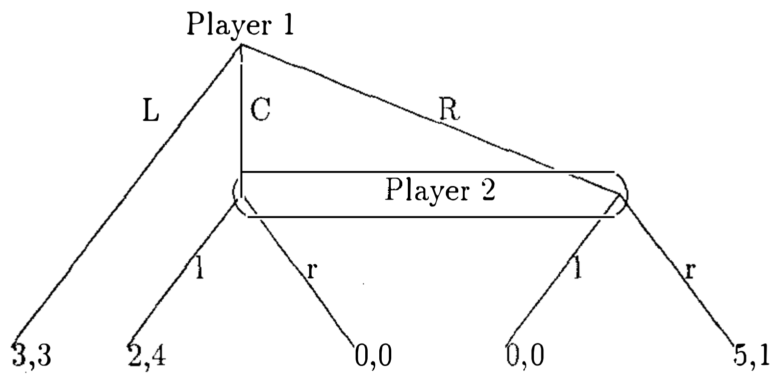

Figure 1: Game with a proper equilibrium which is eliminated by forward induction.

Clearly the matrix is nondegenerate and the only perfect equilibrium is 'top', 'left'. The Jacobian of the best response function (as defined by Harsanyi (1973)) at the equilibrium point is

$$\det \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 0$$

and thus the game has no regular equilibria.

4.5 Proper equilibrium

The game in Figure 1 is used by Tan and Werlang (1988, p.172) to show the insufficiency of the concept of proper equilibrium [15]. There are two proper equilibria in the game: Rr and Ll. By a forward induction argument Tan and Werlang argue that since C is dominated by L, C should never be played and therefore l should never be employed [16].

Therefore properness allows unreasonable equilibria, such as Ll. Are Rr and Ll evolutionary equilibria? We can construct a symmetric game by assuming that two individuals are randomly assigned to the roles of player 1 and player 2. Figure 2 shows the extensive form for such a game.

An evolutionary game is constructed by normalizing this extensive form.

[15] $\tilde{x}$ is proper if and only if there exist $\{\epsilon_i\}$ and $x(\epsilon_i)$ such that $x_i(\epsilon_i) \leq \epsilon_i$ if i is not a best response and $\tilde{x}_j(\epsilon_i) \leq \epsilon_i \tilde{x}_k(\epsilon_i)$ if j is weakly dominated by k and $\lim \tilde{x}(\epsilon_i) = \tilde{x}$.

[16] Ll corresponds to the equilibrium where player 2 warns player 1 that he will play l. Then player 1 has the choice of playing L and receiving 3, playing C and receiving 2, and playing R and receiving 0. If player 2 gets to move he realizes that player 1 did not believe in his bluff. Then player 2 is better off not to follow with his threat and play r.

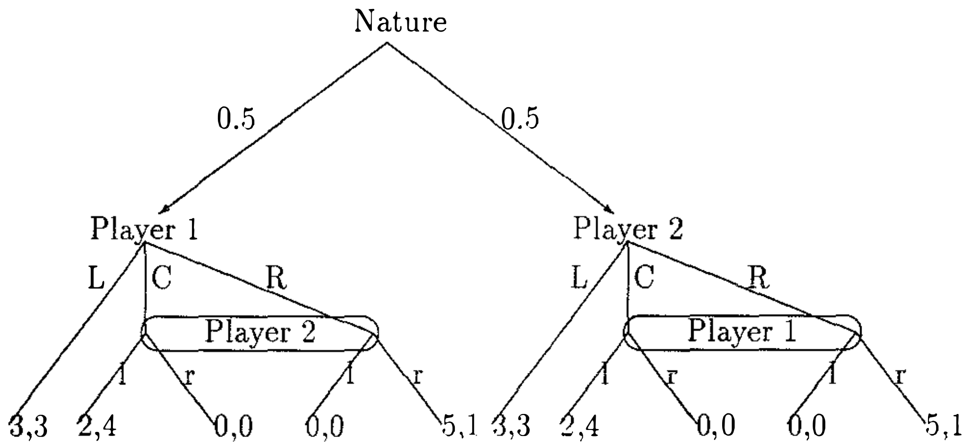

Figure 2: Symmetrization of extensive game in Figure 1.

Rr	6	1	5	0	8	3
Rl	5	0	9	4	8	3
Cr	1	3	0	2	3	5
Cl	0	2	4	6	3	5
Lr	4	3	3	3	6	6
Ll	3	7	7	7	6	6

Notice that the matrix is degenerate. Rr is clearly a strict Nash equilibrium and thus a regular equilibrium (Van Damme (1987), Theorem 2.3.3) and thus an evolutionary equilibrium. Notice that $\tilde{x} = (0,0,0,0,0,1)$ is a symmetric proper equilibrium [17]. Suppose that $\tilde{x}$ is an evolutionary equilibrium. Then for all x in a neighborhood of $\tilde{x}$ let

$$m(x) = (1,0,0,0,1,-2).$$

Choose μ small enough so that $x(\mu)Ax(\mu) > 5.5$ and $x_6 > 0.9$. Then $x_2(\mu) = x_3(\mu) = x_4(\mu) = 0$. The assumptions on m also give that $x_1(\mu) > 0$ and $(Ax)_6 > xAx > (Ax)_5$ which is impossible given the payoff function. Thus $\tilde{x}$ is not an evolutionary equilibrium.

4.6 Strictly Perfect Equilibrium

A Nash equilibrium is strictly perfect if it is resistant to all perturbations of the strategy set. The concept of strict perfect equilibrium resembles the concept of evolutionary equilibrium but as the following examples illustrates not every nondegenerate game has a symmetric strictly perfect equilibrium.

[17] Just set $x_i = \frac{\epsilon^2}{1+\epsilon+4\epsilon^2}$ if $i \neq 5,6$, $x_5 = \frac{\epsilon}{1+\epsilon+4\epsilon^2}$, and $x_6 = \frac{1}{1+\epsilon+4\epsilon^2}$.

Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then the game (A, A^T) has no symmetric strictly perfect equilibrium but has a unique evolutionary equilibrium $\tilde{x} = (1, 0, 0)$ [18].

4.7 ESS

The most widely used equilibrium concept in evolutionary game theory is the the concept of evolutionary stable strategy (ESS). While evolutionary equilibria consider dynamic perturbation, ESS considers stable perturbation. A strategy $x \in \Delta^n$ is an (*ESS*) if for any other strategy $y \in \Delta^n - \{x\}$ there is an ϵ' such that for all $\epsilon \in (0, \epsilon')$

$$xA(\epsilon y + (1-\epsilon)x) > yA(\epsilon y + (1-\epsilon)x).$$

Thus contrary to the notion of evolutionary equilibrium, ESS considers mutation is a static framework. This condition can be rewritten in the following way. Strategy x is an ESS if for all strategies y different than x one of the two conditions holds:

$$\begin{array}{ll} (i) & xAx > yAx \\ (ii) & xAx = yAx \text{ and } xAy > yAy. \end{array}$$

The next proposition relates ESS to the replicator model in continuous time.

Proposition 9 *(Zeeman) An ESS is an asymptotically stable stationary point of the replicator model R^C.*

Unfortunately the requirements of ESS and asymptotically stable stationary points seem too strict as the following example shows. Let

$$A = \begin{pmatrix} \epsilon & 1 & -1 \\ -1 & \epsilon & 1 \\ 1 & -1 & \epsilon \end{pmatrix}$$

[18]Proof: The vector $(\tilde{x}, \tilde{x})$ is the unique symmetric perfect equilibrium, A is nondegenerate and thus $\tilde{x}$ is an evolutionary equilibrium. Consider the perturbation $\frac{1}{2\epsilon+\epsilon^2}(\epsilon, \epsilon^2, \epsilon)$. Suppose $x(\epsilon) \to (1, 0, 0)$. Then $x_1(\epsilon) + x_2(\epsilon) \geq x_1(\epsilon) + x_3(\epsilon)$; i.e., $x_2(\epsilon) \geq x_3(\epsilon)$. This implies that $x_1(\epsilon) + x_3(\epsilon) \geq x_1(\epsilon) + x_2(\epsilon)$, or $x_2(\epsilon) = x_3(\epsilon)$. But this is possible only if $x_1(\epsilon) - x_3(\epsilon) \geq x_1(\epsilon) + x_3(\epsilon)$. Contradiction.

where $\epsilon \in (0, 1/3)$. The only Nash equilibrium is $\tilde{x} = (1/3, 1/3, 1/3)$. The Hessian of the law of motion is negative definite therefore: (1) $\tilde{x}$ is not asymptotically stable (2) and thus $\tilde{x}$ is not an ESS (3) and the game has not asymptotically stable stationary points and no ESS; (4) since the determinant of the Hessian is nonzero, $\tilde{x}$ is regular (5) and thus an evolutionary equilibrium.

Suppose that $\epsilon < 0$. Then $\tilde{x}$ is an ESS and is thus an asymptotically stable stationary point for the replicator model. The replicator model in discrete time in not stable at $\tilde{x}$ since one of the eigenvalues of the linearized system is greater than one. Thus ESS are not necessarily asymptotically stable point of the replicator model in discrete time.

Proposition 10 *A hyperbolic stationary point of the replicator model in continuous time is an evolutionary equilibrium.*

Proof: A stationary point is hyperbolic if and only if all the eigenvalues are negative. Thus the proposition follows from the implicit function theorem. ∎

Proposition 11 *An ESS is an evolutionary equilibrium for the law of motion R^C.*

Proof: Let $\tilde{x}$ be an ESS. van Damme (1987, Theorem 9.2.8, 9.4.8) shows that there is an open ball U centered at $\tilde{x}$ such that the function

$$\begin{aligned} V : U &\rightarrow \Re \\ V(x) &\equiv \prod_i x_i^{\tilde{x}_i} \end{aligned}$$

is a Lyapunov function and such that $\tilde{x}$ is the only fixed point of R^C in U. Take c to be large enough so that $V^{-1}(c) \subset U$. Let X be the solution of the differential equation $\dot{x} = R^C$. For $x \in U$, let $F(x) \equiv X(1, x)$. Then F is continuous and maps $V^{-1}(c)$ into $V^{-1}(c)$. Then by an argument similar to the one in proposition 3 one can show that there $\tilde{x}$ is an evolutionary equilibrium for R^C. ∎

Finally notice that not all ESS are regular equilibria since the game matrix discussed in Section 4.4 has no regular equilibria but has $(1, 0)$ as the unique ESS.

4.8 Essential equilibrium

A Nash equilibrium (x, y) is essential [19] for a game (A, B) if for an arbitrarily small perturbation of the payoff matrix (A', B') there is a Nash equilibrium to (A', B') close

[19] The equilibrium concept is defined by Wu Wen-Tsun and Jiang Jia-He (1962).

to (x, y). This notion predates the concepts of Hyperstable equilibrium introduced by Kohlberg and Mertens (1986) [20].

A symmetric Nash equilibrium $(\tilde{x}, \tilde{x})$ for the game (A, A^T) is *symmetric essential* if for any symmetric game with payoffs close enough to A there is a symmetric Nash equilibrium close enough to $(\tilde{x}, \tilde{x})$. The next propositions characterize the set of symmetric essential equilibria.

Proposition 12 *(Bomze) A regular equilibrium is a symmetric essential equilibrium.*

Proposition 13 *(van Damme) An ESS is a symmetric essential equilibrium.*

Proposition 14 *Restrict the set of mutation function of the form $m_i(x) = x_i[(Cx)_i + xCx]$. Then a symmetric essential equilibrium is an evolutionary equilibrium.*

Proof: Suppose $\tilde{x}$ is a symmetric essential equilibrium of the game (A, A^T). Let $m_i(x) = x_i[(Cx)_i + xCx]$ and let $A_\mu = (1-\mu)A + \mu C$. Then

$$\begin{aligned}\dot{x}(\mu) &= (1-\mu)x_i(\mu)[(Ax(\mu))_i - x(\mu)Ax(\mu)] + \mu x_i(\mu)[(Cx(\mu))_i + x(\mu)Cx(\mu)] \\ &= x_i(\mu)[(A_\mu x(\mu))_i - x(\mu)A_\mu x(\mu)] = 0\end{aligned}$$

if $(x(\mu), x(\mu))$ is a Nash equilibrium of the game (A_μ, A_μ^T). But since $\tilde{x}$ is a symmetric essential equilibrium then for any perturbation of the payoff there is a Nash equilibrium arbitrarily close to $\tilde{x}$. Therefore $\tilde{x}$ is an evolutionary equilibrium. ∎

[20] A subset, H, of the set of Nash equilibria for the game (A, B) is hyperstable if it is minimal according to the following condition: given any small perturbation of the payoff matrix, (A', B'), there is a Nash equilibrium to (A', B'), (x', y'), close to the set H.

References

Axelrod, R. and W. Hamilton (1981). "The evolution of cooperation." *Science*, 211, 1390-1398.

Bernheim, B. D. (1984). "Rationalizable strategic behavior." *Econometrica*, 53, 1007-1029.

Boyd R. and J. Lorberbaum (1987). "No pure strategy is evolutionary stable in the repeated Prisoner's Dilemma game." *Nature*, 237, 58-59.

Brown and von Neumann (1950). "Solutions of games by differential equations." In *Contributions to the Theory of Games, I*, Annals of Mathematical Studies. Princeton: Princeton University Press.

Bomze, I. M. (1986). "Non-cooperative 2-person games in biology: a classification." *International Journal of Game Theory*, 15, 31-59.

Crawford, V.P, (1988). "Learning and mixed-strategy equilibria in evolutionary games." Discussion Paper 88-53, University of California, San Diego.

Crawford, V. P. (1989). "An "evolutionary" explanation of Van Huyck, Battalio, and Beil's experimental results on coordination." University of California, San Diego.

Damme, E.E.C. van (1987). *Stability and Perfection of Nash Equilibria.* Berlin: Springer-Verlag.

Fort, M. K. (1950). "Essential and non-essential fixed points." *American Journal of Mathematics*, 72, 315-322.

Friedman, Daniel (1988). "On evolutionary games in economics." University of California, Santa Cruz.

Harsanyi, John C. (1973). "Games with randomly disturbed payoffs: a new rationale for mixed-strategy equilibrium points." *Int. J. Game Theory*, 2, 1-23.

Hines, W. G. S. (1987). " Evolutionary stable strategies: a review of basic theory." *Theoretical Population Biology*, 31, 195-272.

Hofbauer, J (1981). "On the occurrence of limit cycles in the Volterra-Lotka equation." Nonlinear Analysis, 5, 1003-1007.

Jiang, Jia-He (1963). "Essential component of the set of fixed points of the multivalued mappings and its application to the theory of games." *Scientia Sinica*, 12, 951-964.

Kohlberg, E. and J. F. Mertens (1986). "On the strategic stability of equilibria." *Econometrica*, 54, 5, 1003-1037.

Lemke, C. E. and J. T. Howson (1964). "Equilibrium points of bimatrix games." *J. Soc. Indust. Appl. Math.*, 12, 413-423.

Maynard Smith, J. and G. R. Price (1973), "The logic of animal conflicts," *Nature*, 246, 15-18.

Myerson, R. B. (1978). "Refinements of the Nash equilibrium concept." *International Journal of Game Theory*, 7, 73-80.

Nachbar, John (1990). "An ecological approach to economic games." *International Journal of Game Theory*, forthcoming.

Pearce, D. G. (1984). "Rationalizable strategic behavior and the problem of perfection." *Econometrica*, 52, 1029-1051.

Rubinstein (1988). "Comments on the interpretation of game theory." STICERD discussion paper no. 88/181, London School of Economics.

Samuelson, Larry (1988). "Evolutionary foundations of solution concepts for finite, two player, normal-form games." In *Theoretical Aspects of Reasoning about Knowledge*, ed. M. Y. Vardi. Los Altos: Morgan Kaufmann, 211-225.

Schuster, P., K. Sigmund, J. Hofbauer, and R. Wolff (1981). "Selfregulation of behavior in animal societies. Part I: symmetric contests." *Biological Cybernetics*, 40, 1-8.

Selten, R. (1975). "Reexamination of the perfectness xoncept for equilibrium points in extensive Games." *International Journal of Game Theory.* Vol. 4, 25-55.

Tan, Tommy C. and S. Werlang (1988). "A guide to knowledge and games." In *Theoretical Aspects of Reasoning about Knowledge*, ed. M. Y. Vardi. Los Altos: Morgan Kaufmann, pp. 163-177.

Taylor, P. D. and Jonker L. B. (1978). "Evolutionarily stable strategies and game dynamics." *Mathematical Biosciences*, 40, 145-156.

Wu, Wen-Tsun and Jiang Jia-He (1962). "Essential equilibrium point of n-person noncooperative games," *Scientia Sinica*, 11, 1307-1322.

Zeeman, E.C. (1979). "Population dynamics from game theory." *Proc. Int. Conf. Global Theory of Dynamical Systems.* Lecture Notes in Mathematics, vol 819. Berlin: Springer Verlag.