DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125

WHEN IS THE CORE EQUIVALENCE THEOREM VALID?

Charalambos D. Aliprantis
California Institute of Technology
Indiana University - Purdue Univeristy at Indianapolis

Owen Burkinshaw
Indiana University - Purdue University at Indianapolis

SOCIAL SCIENCE WORKING PAPER 707

September 1989

WHEN IS THE CORE EQUIVALENCE THEOREM VALID?

Charalambos D. Aliprantis and Owen Burkinshaw

ABSTRACT

In 1983 L. E. Jones exhibited a surprising example of a weakly Pareto optimal allocation in a two consumer pure exchange economy that failed to be supported by prices. In this example the price space is not a vector lattice (Riesz space). Inspired by Jones' example, A. Mas-Colell and S. F. Richard proved that this pathological phenomenon cannot happen when the price space is a vector lattice. In particular, they established that (under certain conditions) in a pure exchange economy the lattice structure of the price space is sufficient to guarantee the supportability of weakly Pareto optimal allocations by prices—i.e., they showed that the second welfare theorem holds true in an exchange economy whose price space is a vector lattice. In addition, C. D. Aliprantis, D. J. Brown and O. Burkinshaw have shown that when the price space of an exchange economy is a certain vector lattice, the Debreu–Scarf core equivalence theorem holds true, i.e., the sets of Walrasian equilibria and Edgeworth equilibria coincide. (An Edgeworth equilibrium is an allocation that belongs to the core of every replica economy of the original economy.) In other words, the lattice structure of the price space is a sufficient condition for avoiding the pathological situation occuring in Jones' example.

This work shows that the lattice structure of the price space is also a necessary condition. That is, "optimum" allocations in an exchange economy are supported by prices (if and) only if the price space is a vector lattice. Specifically, the following converse-type result of the Debreu–Scarf core equivalence theorem is established: *If in a pure exchange economy every Edgeworth equilibrium is supported by prices, then the price space is necessarily a vector lattice.*

WHEN IS THE CORE EQUIVALENCE THEOREM VALID?

Charalambos D. Aliprantis and Owen Burkinshaw*

An answer to the question of the title is provided. It is shown that under some conditions the core equivalence theorem holds true (if and) only if the price space is a vector lattice.

1. INTRODUCTION

Many of the current models of economies with infinitely many commodities require that the commodity space is a Riesz space and that the commodity-price dual pair is a Riesz dual system. Recently, there has been interest in how much these requirements can be relaxed. It is our purpose to examine how the basic welfare theorems relate to the lattice structure of the commodity and price spaces. We consider economies where the commodity space E is a vector lattice with a locally convex topology and the price space is the topological dual E' with a generating positive cone. The main result of this paper asserts that if the core equivalence theorem holds true for such economies, then the price space must be a vector lattice. This shows that the lattice structure of the commodity-price duality is essential for the validity of the fundamental theorems of welfare economics.

In this paper, we shall employ the mathematics of Riesz spaces. We shall follow the notation and terminology of [5]. Let us briefly mention a few things about Riesz spaces. A **Riesz space** (or a **vector lattice**) is a partially ordered vector space which is also a lattice—in the sense that for each pair of vectors x and y the supremum (least upper bound) and the infimum (greatest lower bound) exist; using standard lattice terminology we shall write

$$x \vee y = \sup\{x,y\} \quad \text{and} \quad x \wedge y = \inf\{x,y\}.$$

As usual, the positive cone of a Riesz space E will be denoted by E^+, i.e., $E^+ = \{x \in E\colon\ x \geq 0\}$—the symbol $\Re$ will denote the set of real numbers. For an arbitrary element x of a Riesz space, its *positive part*, its *negative part*, and its *absolute value* are defined by the formulas

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad \text{and} \quad |x| = x \vee (-x).$$

* Research of both authors was supported in part by a Chrysler Corporation grant to IUPUI.

A vector subspace F of a Riesz space E is said to be a **Riesz subspace** whenever for each $x, y \in F$ the supremum $x \vee y$ (taken in E) belongs to F. A (non-empty) subset A of a Riesz space is said to be a *solid set* whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of a Riesz space is referred to as an **ideal**. An ideal is always a Riesz subspace but the converse is not true. A linear functional $f\colon \longrightarrow \Re$ on a Riesz space E is said to be *order bounded* whenever it carries order bounded subsets of E onto bounded subsets of $\Re$; every set of the form $[x, y] = \{z \in E\colon\ x \leq z \leq y\}$ is known as an *order interval* and subsets of order intervals are called *order bounded sets*. Every positive linear functional f, i.e., any linear functional f that satisfies $f(x) \geq 0$ for each $x \geq 0$, is necessarily order bounded. The set of all order bounded linear functionals on a Riesz space is a vector space which is also a partially ordered vector space under the ordering $f \geq g$ whenever $f(x) \geq g(x)$ for all $x \in E^+$. The partially ordered vector space of all order bounded linear functionals on a Riesz space E is called the **order dual** of E and is denoted by $E^\sim$. It turns out that the the order dual $E^\sim$ of a Riesz space E is also a Riesz space whose lattice operations are given by

$$f \vee g(x) = \sup\{f(y) + g(z)\colon\ y,\ z \in E^+ \ \text{ and } \ y + z = x\}$$

and

$$f \wedge g(x) = \inf\{f(y) + g(z)\colon\ y,\ z \in E^+ \ \text{ and } \ y + z = x\}$$

for all $f,\ g \in E^\sim$ and all $x \in E^+$.

Definition 1.1. *A* **Riesz dual system** $\langle E, E' \rangle$ *is a dual system such that*

1) E *is a Riesz space;*
2) E' *is an ideal of the order dual* $E^\sim$ *separating the points of* E*; and*
3) *the duality function* $\langle \cdot, \cdot \rangle$ *is the natural one, i.e.,*

$$\langle x, x' \rangle = x'(x)$$

holds for all $x \in E$ *and all* $x' \in E'$.

Most examples of dual systems employed in economics are Riesz dual systems. Here are a few examples of Riesz dual systems.

a) $\langle L_p(\mu), L_q(\mu) \rangle$, $1 < p,\ q < \infty$; $\frac{1}{p} + \frac{1}{q} = 1$;
b) $\langle \ell_p, \ell_q \rangle$, $1 \leq p,\ q \leq \infty$; $\frac{1}{p} + \frac{1}{q} = 1$;
c) $\langle L_\infty(\mu), L_1(\mu) \rangle$ and $\langle L_1(\mu), L_\infty(\mu) \rangle$, μ a σ-finite measure; and
d) $\langle C(\Omega), \mathrm{ca}(\Omega) \rangle$, Ω a Hausdorff compact topological space.

The dual system $\langle \mathrm{ca}(\Omega), C(\Omega) \rangle$ is not (in general) a Riesz dual system. A **symmetric Riesz dual system** is a Riesz dual system $\langle E, E' \rangle$ such that each order interval of E is weakly compact (i.e., $\sigma(E, E')$-compact). The term "*symmetric Riesz dual system*" is justified from the fact that in a Riesz dual system $\langle E, E' \rangle$ the order intervals of E are weakly compact if and only if E is (under its natural embedding) an ideal of $(E')^\sim$, and so $\langle E', E \rangle$ is also a Riesz dual system.

A locally solid Riesz space (E, τ) is a Riesz space E equipped with a locally solid topology τ (a linear topology τ is said to be *locally solid* whenever it has a base at zero consisting of solid neighborhoods). The topological dual E' of a locally solid Riesz space (E, τ) is always an ideal of the order dual $E^{\sim}$, and hence E' is always a Riesz subspace of the order dual. A *locally convex-solid Riesz space* (E, τ) is a Riesz space E equipped with a Hausdorff linear topology τ that has a base at zero consisting of solid and convex neighborhoods—the topology τ is referred to as a *locally convex-solid topology.* It turns out that a linear topology on a Riesz space space is locally convex-solid if and only if it is generated by a family of lattice seminorms. (A seminorm q on a Riesz space is said to be a *lattice seminorm* whenever $|x| \leq |y|$ implies $q(x) \leq q(y)$.) If (E, τ) is a locally convex-solid Riesz space and E' denotes its topological dual, then $\langle E, E' \rangle$ is a Riesz dual system—as a matter of fact, every Riesz dual system $\langle E, E' \rangle$ can be obtained in this manner, i.e., if $\langle E, E' \rangle$ is a Riesz dual system, then there exists a Hausdorff locally convex-solid topology τ on E consistent with the duality $\langle E, E' \rangle$ such that E' is the topological dual of (E, τ).

For a given Riesz dual system $\langle E, E' \rangle$ there are two distinguishable consistent locally convex-solid topologies on E. They are the absolute weak and absolute Mackey topologies. The *absolute weak topology* $|\sigma|(E, E')$ is the locally convex-solid topology on E of uniform convergence on the order intervals of E' and is generated by the family of lattice seminorms $\{q_{x'}\colon\ x' \in E'\}$, where

$$q_{x'}(x) = |x'|(|x|)$$

for all $x \in E$ and all $x' \in E'$. *The absolute Mackey topology* $|\tau|(E, E')$ is the locally convex-solid topology on E of uniform convergence on the convex, solid, and $\sigma(E', E)$-compact subsets of E'. We have the following inclusions

$$\sigma(E, E') \subseteq |\sigma|(E, E') \subseteq |\tau|(E, E') \subseteq \tau(E, E') .$$

A locally convex-solid topology τ on E is consistent with the duality $\langle E, E' \rangle$ if and only if $|\sigma|(E, E') \subseteq \tau \subseteq |\tau|(E, E')$ holds.

Since the introduction of Riesz spaces (vector lattices) to general equilibrium theory by C. D. Aliprantis and D. J. Brown [1], several authors have utilized the lattice structure of Riesz spaces and obtained equilibrium results; see [2, 3, 7, 8, 9, 10, 12, 13]. Among the major contributions in this quest were the works of A. Mas-Colell [8, 9]. In [8] A. Mas-Colell introduced two important notions for an exchange economy; the notion of uniform properness for preferences and the closedness condition. We shall discuss these two notions below; for a complete discussion and proofs see Chapter 3 of [4].

Definition 1.2. *Let $\succeq$ be a preference relation defined on the positive cone C of a partially ordered topological vector space. Then the preference $\succeq$ is said to be* **uniformly proper** *whenever there exists a neighborhood V of zero and some vector $v > 0$ (called a vector of uniform properness) such that*

$$x - \alpha v + z \succeq x \quad \textit{in} \quad C \quad \textit{with} \quad \alpha > 0 \quad \textit{implies} \quad z \notin \alpha V .$$

In the above definition, if the vector v and the neighborhood V are needed to be emphasized, then we shall say that the preference $\succeq$ is (v, V)-uniformly proper; likewise, if the vector v must be emphasized, then we shall simply say that $\succeq$ is v-uniformly proper. The reader should keep in mind that a preference $\succeq$ is uniformly proper if and only if there exists an open convex cone Γ satisfying

1) $\Gamma \cap (-E^+) \neq \emptyset$; and
2) $(x + \Gamma) \cap \{y \in E^+ \colon\ y \succeq x\} = \emptyset$ for all $x \in E^+$.

For details and more about uniformly proper preferences see [4, Section 3.2]. Now let us introduce the economic model that will be the subject of our discussion.

Definition 1.3. *An* **exchange economy** *is a pair*

$$(\langle E, E' \rangle, \{(\succeq_i, \omega_i)\colon\ i = 1, \ldots, m\})$$

that satisfies the following properties:

i) *The dual pair $\langle E, E' \rangle$ is a Riesz dual system that describes the commodity-price duality; E is the price space and E' is the price space*[1]*. The evaluation $\langle x, p \rangle$ will be denoted—as usual—by $p \cdot x$, i.e., $\langle x, p \rangle = p \cdot x$. (The linear functionals of E will be referred to in general as prices.)*

ii) *There are m consumers indexed by i each of whom has E^+ as his consumption set and has an initial endowment $\omega_i > 0$. The total endowment will be denoted by ω, i.e.,*

$$\omega = \sum_{i=1}^{m} \omega_i .$$

iii) *The taste of each consumer i is represented by a preference relation (i.e., by a reflexive, complete, and transitive relation) on E^+. It is assumed that each preference relation $\succeq_i$ is*
   a) *monotone; i.e., $x \geq y$ in E^+ implies $x \succeq_i y$;*
   b) *convex; i.e., the set $\{y \in E^+ \colon\ y \succeq_i x\}$ is convex for each $x \in E^+$;*
   c) *continuous for some locally convex-solid topology τ on E consistent with the duality $\langle E, E' \rangle$; and*
   d) *has ω as an extremely desirable bundle, i.e.,*

$$x + \alpha\omega \succ_i x$$

holds for all $x \in E^+$ and all $\alpha > 0$.

Recall that an **allocation** is an m-tuple $(x_1, \ldots, x_m)$ such that $x_i \in E^+$ holds for each i and $\sum_{i=1}^{m} x_i = \omega$. A non-zero price p is said to **support** an allocation $(x_1, \ldots, x_m)$ whenever

$$x \succeq_i x_i \text{ in } E^+ \text{ implies } p \cdot x \geq p \cdot x_i .$$

[1] Later, this restriction will be weakened. Specifically, we shall also assume that $\langle E, E' \rangle$ is a dual system such that E is a Riesz space and E' is a vector subspace of the order dual $E^{\sim}$.

Supporting prices are necessarily positive prices—and hence, they belong to the order dual $E^{\sim}$. To see this, let a price p support an allocation $(x_1, \ldots, x_m)$ and let $x \geq 0$. Since $\succeq_1$ is monotone, we see that $x_1 + x \succeq_1 x_1$ and so $p \cdot (x_1 + x) \geq p \cdot x_1$ from which it follows that $p \cdot x \geq 0$.

An allocation $(x_1, \ldots, x_m)$ is said to be **weakly Pareto optimal** whenever there is no other allocation $(y_1, \ldots, y_m)$ satisfying $y_i \succ_i x_i$ for each consumer i. The first fundamental theorem of welfare economics can be now formulated as follows.

Theorem 1.4. (The First Welfare Theorem) *If an allocation $(x_1, \ldots, x_m)$ is supported by a price p with $p \cdot \omega \neq 0$, then the allocation $(x_1, \ldots, x_m)$ is weakly Pareto optimal.*

Proof. Let a price p with $p \cdot \omega \neq 0$ support an allocation $(x_1, \ldots, x_m)$. Assume by way of contradiction that the allocation $(x_1, \ldots, x_m)$ is not weakly Pareto optimal. So, there exists another allocation $(y_1, \ldots, y_m)$ satisfying $y_i \succ_i x_i$ for each i. Clearly, $p \cdot y_i \geq p \cdot x_i$ holds for each i and from the identity $\sum_{i=1}^m y_i = \sum_{i=1}^m x_i = \omega$, it follows that $p \cdot y_i = p \cdot x_i$ for each i.

From $\sum_{i=1}^m x_i = \omega$ and $p \cdot \omega \neq 0$, we see that there exists some k with $p \cdot x_k > 0$. By the continuity of the preference $\succeq_k$ there exists some $0 < \varepsilon < 1$ such that $\varepsilon y_k \succ_k x_k$. Therefore, we have

$$p \cdot x_k = p \cdot y_k > \varepsilon p \cdot y_k = p \cdot (\varepsilon y_k) \geq p \cdot x_k \,,$$

which is impossible. This contradiction shows that the allocation $(x_1, \ldots, x_m)$ is weakly Pareto optimal. ■

The converse statement of the preceding theorem is known as the Second Welfare Theorem. Using the notion of uniform properness A. Mas-Colell [8] established the following version of the second welfare theorem.

Theorem 1.5. (The Second Welfare Theorem) *If in an exchange economy preferences are monotone, convex and uniformly proper, then every weakly Pareto optimal allocation can be supported by a non-zero price.*

Moreover, if for each i we pick a convex solid τ-neighborhood V_i of zero and a vector $v_i > 0$ that satisfy the definition of uniform properness for $\succeq_i$, then every weakly Pareto optimal allocation can be supported by a price $p > 0$ that satisfies

$$p \cdot \Big(\sum_{i=1}^m v_i\Big) = 1 \quad \textit{and} \quad |p \cdot z| \leq 1 \quad \textit{for all} \quad z \in V = \bigcap_{i=1}^m V_i \,.$$

An allocation $(x_1, \ldots, x_m)$ is said to be a **quasiequilibrium** whenever there exists a non-zero price $p \in E'$ such that

$$x \succeq_i x_i \text{ in } E^+ \text{ implies } p \cdot x \geq p \cdot \omega_i \,.$$

It is easy to see that if an allocation $(x_1, \ldots, x_m)$ is a quasiequilibrium with respect to a price p, then $p \cdot x_i = p \cdot \omega_i$ holds for each i and so the price p supports the allocation $(x_1, \ldots, x_m)$.

In view of the first welfare theorem (Theorem 1.4), we know that a quasiequilibrium is necessarily a weakly Pareto optimal allocation. This observation tells us where our search for quasiequilibria should be confined, i.e., the quasiequilibria are among the weakly Pareto optimal allocations. Following this method, A. Mas-Colell [8] was able to prove the existence of quasiequilibria by introducing the notion of closedness for an exchange economy. We shall discuss this condition next.

For the rest of the discussion in this section, we shall assume that each preference relation $\succeq_i$ is represented by a utility function u_i. A *feasible allocation* is an m-tuple $(x_1, \ldots, x_m)$ such that $x_i \in E^+$ holds for each i and $\sum_{i=1}^m x_i \leq \omega$. A *utility allocation* is any vector of the form $\big(u_1(x_1), \ldots, u_m(x_m)\big)$ where $(x_1, \ldots, x_m)$ is a feasible allocation. The set of all utility allocations is referred to as the **utility space** of the economy and is denoted by $\mathbf{U}$, i.e.,

$$\mathbf{U} = \Big\{\big(u_1(x_1), \ldots, u_m(x_m)\big)\colon\ (x_1, \ldots, x_m) \text{ is a feasible allocation}\Big\}\ .$$

The utility space enjoys several interesting properties; for details see [4, Section 3.5]. An exchange economy is said to satisfy the **closedness condition** whenever its utility space is a closed (or, equivalently, a compact) subset of $\Re^m$.

The following remarkable theorem was proven by A. Mas-Colell in [8].

Theorem 1.6. (Mas-Colell) *If an exchange economy satisfies the closedness condition, preferences are uniformly proper and the total endowment is extremely desirable by each consumer, then the economy has a quasiequilibrium.*

There is one more important companion theorem to the welfare theorems. It is the core equivalence theorem of G. Debreu and H. E. Scarf [6]. The theorem (whose origins go back to Y. Edgeworth) asserts—under some appropriate hypotheses—that an allocation is a Walrasian equilibrium if and only if it belongs to the core of every r-fold replica of the economy.

The definitions of the notions mentioned above are as follows.

a) An allocation $(x_1, \ldots, x_m)$ is said to be a **Walrasian equilibrium** whenever there exists a non-zero price p such that each x_i is a maximal element in the budget set $\mathcal{B}_i(p) = \{x \in E^+\colon\ p \cdot x \leq p \cdot x_i\}$.

b) An allocation $(x_1, \ldots, x_m)$ is said to be a **core allocation** whenever it cannot be blocked by any coalition, i.e., whenever there is no allocation $(y_1, \ldots, y_m)$ and no coalition S such that
   1) $\sum_{i\in S} y_i = \sum_{i \in S} \omega_i$; and
   2) $y_i \succ_i x_i$ for each $i \in S$.

c) The r-fold replica of an economy is a new economy with the following characteristics:
   α) $\langle E, E' \rangle$ is also the Riesz dual system of the r-fold replica economy;
   β) The economy has mr consumers indexed by (i, j) $(i = 1, \ldots, m$; $(j = 1, \ldots, r)$ such that each consumer (i, j) has an initial endowment $\omega_{ij} = \omega_i$ and a preference $\succeq_{ij} = \succeq_i$. The consumers $(i, j)(j = 1, \ldots, r)$ are the consumers of "*type* i" and, of course, they are exact replicas of consumer i.

Every allocation $(x_1, \ldots, x_m)$ can be considered as an allocation of every r-replica economy by letting $x_{ij} = x_i$ for each (i, j). Following [2] and [3], we shall say that an allocation is an *Edgeworth equilibrium* whenever it belongs to the core of every r-fold replica of the economy. The core equivalence theorem of G. Debreu and H. E. Scarf [6] is also known as the Third Fundamental Theorem of Welfare Economics. It was extended to the infinite dimensional case by C. D. Aliprantis, D. J. Brown, and O. Burkinshaw [2, Theorem 4.18, p. 1133] as follows.

Theorem 1.7. (The Third Welfare Theorem) *If preferences are uniformly proper and $\omega \gg 0$, then an allocation is a Walrasian equilibrium if and only if it is an Edgeworth equilibrium.*

The proofs of Theorems 1.3, 1.4, 1.5, and 1.7 utilize the lattice structures of the commodity and price spaces. *How far can we relax the lattice structures of these spaces?* We shall discuss this question—and provide some answers—in the next two sections.

2. THE SECOND WELFARE THEOREM

The purpose of this section is to discuss the lattice structure of the price space in order for the second welfare theorem to be valid. Accordingly, we shall consider the commodity-price duality $\langle E, E' \rangle$ when is described by a dual system where E is a Riesz space and E' is a vector subspace of the order dual $E^\sim$. Surprisingly enough, when E' does not inherit the lattice structure of $E^\sim$ (i.e., when E' is not a Riesz subspace of $E^\sim$), then a weakly Pareto optimal allocation need not be supported by prices. The following example of L. E. Jones [6] will clarify the situation.

Example 2.1. (Jones) Consider the commodity space $E = L_1[0,1]$. Since E is a Banach lattice its order dual coincides with its norm dual and so $E^\sim = E' = L_\infty[0,1]$. (Keep in mind that $L_\infty[0,1]$ is a Riesz space under the pointwise lattice operations.) The price space E' is taken to be the vector space of all continuously differentiable functions on $[0,1]$, i.e., $E' = C^1[0,1]$. Clearly, E' is a vector subspace of $E^\sim$ but it is not a Riesz subspace—the pointwise supremum of two differentiable functions need not be a differentiable function. Note also that the natural ordering of $C^1[0,1]$ (which is the same as the one induced by the ordering from $L_\infty[0,1]$) makes $C^1[0,1]$ a partially ordered vector space with a generating cone. (Recall that the cone C of a partially ordered vector space E is said to be *generating* whenever for each $x \in E$ there exist $y, z \in C$ such that $x = y - z$.)

Now consider a two consumer economy with the following characteristics:

1) The commodity-price duality is represented by the dual system $\langle E, E' \rangle$.
2) Consumer 1 has an initial endowment $\omega_1 = \frac{1}{2}\chi_{[0,1]}$ and utility function

$$u_1(x) = \int_0^1 t x(t)\, dt\,.$$

3) Consumer 2 has an initial endowment $\omega_1 = \frac{1}{2}\chi_{[0,1]}$ and utility function

$$u_2(x) = \int_0^1 (1-t)x(t)\,dt\,.$$

Both utility functions (as linear functionals) are weakly continuous, quasiconcave, and strictly monotone. It can be shown that the price $\mathbf{p}$ of $L_\infty[0,1]$ defined by

$$\mathbf{p}(t) = \max\{t,\ 1-t\}$$

is the only—aside from a scalar multiple—price that supports the allocation (x_1, x_2), where

$$x_1 = \chi_{[0,\frac{1}{2}]} \quad \text{and} \quad x_2 = \chi_{(\frac{1}{2},1]}\,.$$

The latter implies that the allocation (x_1, x_2) is Pareto optimal (and hence a weakly Pareto optimal allocation); for details see [4, Example 3.4.5, p. 136]. Since $\mathbf{p} \notin E'$, it follows that the weakly Pareto optimal allocation (x_1, x_2) cannot be price supported with respect to the dual system $\langle E, E'\rangle$. ∎

The preceding example shows that whenever the price space E' is not a Riesz subspace of the order dual $E^\sim$, the second welfare theorem need not be valid. In other words, Example 2.1 suggests that the validity or non validity of the second welfare theorem is associated with the lattice structure of the price space. As a matter of fact, a recent result of A. Mas-Colell and S. F. Richard [10] asserts that the Riesz space structure of the price space E' suffices to guarantee the validity of the second welfare theorem. The theorem can be stated as follows.

Theorem 2.2. (Mas-Colell–Richard) *Assume that an exchange economy with a finite number of agents satisfies the following properties:*

1) *The commodity-price duality $\langle E, E'\rangle$ is defined by a dual system such that E is a Riesz space and E' is a Riesz subspace of the order dual $E^\sim$;*
2) *The order interval $[0, \omega]$ is weakly compact, i.e., $\sigma(E, E')$-compact; and*
3) *Every preference besides being monotone, convex and Mackey continuous, it is also ω-uniformly Mackey proper.*

Then every weakly Pareto optimal allocation $(x_1, \dots, x_m)$ can be supported by a non-zero price $p \in E'$ of the form $p = \bigvee_{i=1}^m p_i$, where the prices $p_i \in E'$ can be chosen to satisfy $p_i \cdot x \geq p_i \cdot x_i$ whenever $x \succeq_i x_i$.

It should be emphasized that in the above theorem the dual system $\langle E, E'\rangle$ is not required to be a Riesz dual system. It is merely required that the price space E' to be a Riesz subspace of the order dual $E^\sim$. Recently, in a remarkable paper, S. F. Richard [11] generalized Theorem 2.2 to production economies—where he also assumed that the linear topology τ is only locally convex and E' is the topological dual of (E, τ).

Let us illustrate Theorem 2.2 (and Theorem 1.6) with a simple example.

Example 2.3. Consider a two consumer pure exchange economy with commodity space $\Re^2$ and the following characteristics.

Consumer 1: Initial endowment $\omega_1 = \left(\frac{3}{4}, \frac{1}{4}\right)$ and utility function

$$u_1(x, y) = 2x + y = (2, 1) \cdot (x, y).$$

Consumer 2: Initial endowment $\omega_2 = \left(\frac{1}{4}, \frac{3}{4}\right)$ and utility function

$$u_2(x, y) = x + 2y = (1, 2) \cdot (x, y).$$

Therefore, the total endowment of the economy is

$$\omega = \omega_1 + \omega_2 = (1, 1).$$

An easy computation shows that the utility space for this exchange economy is the darkened region of the plane shown in Figure 1.

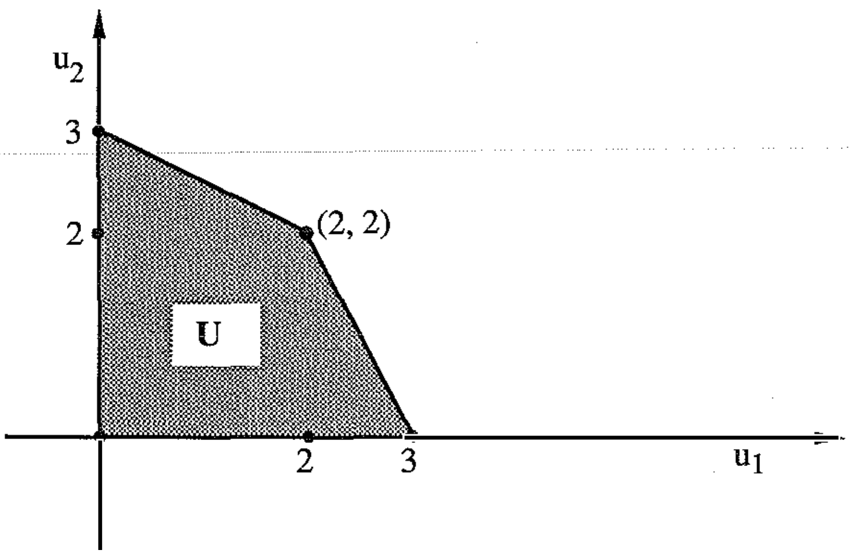

Fig. 1

Thus, the economy satisfies all the hypotheses of Theorem 1.6 and so it must have a quasiequilibrium. The quasiequilibrium corresponds to a utility allocation on the "utility boundary." It turns out that the utility allocation that gives rise to a quasiequilibrium is $(2, 2)$. The quasiequilibrium is $\big((1, 0), (0, 1)\big)$ and the price supporting this quasiequilibrium is $\mathbf{p} = (1, 1)$.

The price $(1, 1)$ is a scalar multiple of the supremum of the prices $(2, 1)$ and $(1, 2)$; we have, of course,

$$(2, 1) \vee (1, 2) = (2 \vee 1, 1 \vee 2) = (2, 2) = 2(1, 1). \quad \blacksquare$$

3. THE THIRD WELFARE THEOREM

In this section, we shall establish that the validity of the core equivalence theorem guarantees that the price space is necessarily a Riesz space. This is the main contribution of this paper to the literature. It simply completes the cycle of ideas according to which the existence of "optimum" allocations is intimately related to the lattice structure of the price space. We are now ready to present the main result of this paper. Keep in mind that by τ we designate a locally convex topology on E which is consistent with the dual pair $\langle E, E' \rangle$.

Theorem 3.1. *Consider a dual system $\langle E, E' \rangle$, where E is a Riesz space and E' is a vector subspace of the order dual $E^{\sim}$ such that*

1) *the order intervals of E are weakly compact*[2]*; and*
2) *the cone $E' \cap E^{\sim}_{+}$ is generating, i.e., every linear functional of E' can be written as a difference of two positive linear functionals of E'.*

Furthermore, assume that whenever (x_1, x_2) is an Edgeworth equilibrium of an arbitrary two consumer exchange economy whose commodity-price duality is described by the dual system $\langle E, E' \rangle$ and each preference $\succeq_i$ is (v_i, V_i)-uniformly τ-proper there exists a non-zero price $p \in E'$ that supports (x_1, x_2) as a quasiequilibrium satisfying

$$p \cdot v = 1 \qquad \text{and} \qquad |p \cdot z| \leq 1 \quad \text{for all} \quad z \in V\,,$$

where $v = v_1 + v_2$ and $V = V_1 \cap V_2$.

Then, the price space E' is a Riesz subspace of the order dual $E^{\sim}$.

In order to establish this result, we need a lemma which is of some independent interest in its own right.

Lemma 3.2. *Consider a two consumer exchange economy with dual system $\langle E, E' \rangle$, where E is a Riesz space and E' is a vector subspace of the order dual $E^{\sim}$. Assume that the preferences are represented by two utility functions of the form*

$$u_1(x) = f(x) \qquad \text{and} \qquad u_2(x) = g(x)\,,$$

where f and g are two positive linear functionals on E.

---

[2] If the positive cone E^+ is in addition weakly closed, then this condition guarantees that E is a Dedekind complete Riesz space. To see this, let $0 \leq x_\alpha \uparrow \leq y$ hold in E. Since the order interval $[0, y]$ is weakly compact and $\{x_\alpha\}$ is a net of $[0, y]$, it follows that $\{x_\alpha\}$ has a convergent subnet to some vector $x \in [0, y]$. Relabelling, we can assume that $x_\alpha \xrightarrow{w} x$ and we claim that $x = \sup\{x_\alpha\}$ holds. From $x_\alpha \geq x_\beta$ (i.e., $x_\alpha - x_\beta \in E^+$) for all $\alpha \geq \beta$ and the fact that E^+ is weakly closed, we see that $x - x_\beta = \lim_\alpha (x_\alpha - x_\beta) \in E^+$, i.e., $x \geq x_\beta$ holds for each β. On the other hand, if $z \geq x_\alpha$ (i.e., $z - x_\alpha \in E^+$) holds for all α, then $z - x = \lim_\alpha (z - x_\alpha) \in E^+$, and so $z \geq x$. Therefore, the vector x is the least upper bound of the net $\{x_\alpha\}$.

If the supremum in the formula

$$f \vee g(\omega) = \sup\{f(y) + g(z)\colon\ y,\ z \in E^+ \ \text{and}\ y + z = \omega\}$$

is attained at (x_1, x_2) [*i.e.*, $x_1,\ x_2 \in E^+$, $x_1 + x_2 = \omega$, *and* $f \vee g(\omega) = f(x_1) + g(x_2)$], *then*

a) $f \vee g(x_1) = f(x_1)$ *and* $f \vee g(x_2) = g(x_2)$*; and*
b) *the price* $f \vee g$ *(which need not lie in* E'*) supports the allocation* (x_1, x_2)*, i.e,* $x \succeq_i x_i$ *implies* $f \vee g(x) \geq f \vee g(x_i)$.

Proof. Assume that $x_1,\ x_2 \in E^+$ satisfy $x_1 + x_2 = \omega$ and $f \vee g(\omega) = f(x_1) + g(x_2)$. To see that $f \vee g(x_1) = f(x_1)$ holds, assume by way of contradiction that $f \vee g(x_1) > f(x_1)$. Then there exist $z_1,\ z_2 \in E^+$ with $z_1 + z_2 = x_1$ and $f(z_1) + g(z_2) > f(x_1)$. This implies

$$\begin{aligned} f \vee g(\omega) &\geq f(z_1) + g(z_2 + x_2) \\ &= f(z_1) + g(z_2) + g(x_2) \\ &> f(x_1) + g(x_2) \\ &= f \vee g(\omega), \end{aligned}$$

which is impossible. Thus, $f \vee g(x_1) = f(x_1)$ and (similarly) $f \vee g(x_2) = g(x_2)$ both hold.

To see that the price $f \vee g$ supports the allocation (x_1, x_2), let $x \succeq_1 x_1$. This means that $f(x) \geq f(x_1)$, and so

$$f \vee g(x) \geq f(x) \geq f(x_1) = f \vee g(x_1).$$

Similarly, $x \succeq_2 x_2$ implies $f \vee g(x) \geq f \vee g(x_2)$. ■

Now to complete the proof of Theorem 3.1, assume that the dual system $\langle E, E' \rangle$ satisfies the properties of the theorem and let $f,\ g \in E'$. It suffices to show that $f \vee g$ (the least upper bound of f and g taken in $E^{\sim}$) belongs to E'. Pick positive linear functionals $f_1,\ f_2,\ g_1,\ g_2 \in E'$ such that $f = f_1 - f_2$ and $g = g_1 - g_2$ and let $h = f_2 + g_2 \in E'$. Then we have $h \geq -f$ and $h \geq -g$ (i.e., $h + f \geq 0$ and $h + g \geq 0$). From the lattice identity $h + f \vee g = (h + f) \vee (h + g)$, we see that it suffices to show that $(h + f) \vee (h + g)$ belongs to E'. Therefore, replacing f and g by $h + f$ and $h + g$ respectively, we can assume from the outset that $f > 0$ and $g > 0$ both hold. In addition, note that if either $f \vee g = f$ or $f \vee g = g$ holds, then there is nothing to prove. Thus, we can also assume that $f \vee g > f$ and $f \vee g > g$ both hold.

Consider now an exchange economy having dual system $\langle E, E' \rangle$ and two consumers with utility functions given by the formulas

$$u_1(x) = f(x) \quad \text{and} \quad u_2(x) = g(x).$$

Clearly, both utility functions (as linear) are uniformly τ-proper. Fix two proper vectors $v_1 > 0$ and $v_2 > 0$ for u_1 and u_2, respectively, corresponding to a τ-neighborhood V of zero and then fix some $\omega > 0$ such that

$$0 < v = v_1 + v_2 \leq \omega, \quad f \vee g(\omega) - f(\omega) > 0, \quad \text{and} \quad f \vee g(\omega) - g(\omega) > 0. \qquad (\star)$$

Since the order interval $[0,\omega]$ is weakly compact, it is easy to see that the supremum

$$f \vee g(\omega) = \sup\{f(y)+g(z)\colon\ y,\ z \in E^+ \text{ and } y+z=\omega\}$$

is attained. That is, there exist $x_1,\ x_2 \in [0,\omega]$ with $x_1+x_2=\omega$ and $f(x_1)+g(x_2) = f \vee g(\omega)$. By Lemma 3.2, we have

$$f \vee g(x_1) = f(x_1) \quad \text{and} \quad f \vee g(x_2) = g(x_2)\,.$$

Next note that $f(x_1) > 0$ and $g(x_2) > 0$ both hold. Indeed, if $f(x_1) = 0$ holds, then

$$f \vee g(\omega) = f(x_1) + g(x_2) = g(x_2) \le g(\omega) \le f \vee g(\omega)\,,$$

and so $f \vee g(\omega) - g(\omega) = 0$, contrary to $(\star)$. Hence, $f(x_1) > 0$ (and similarly) $g(x_2) > 0$. Without loss of generality, we can also assume that

$$g(x_2) \ge f(x_1) > 0\,.$$

Next, pick some $\delta \ge 1$ such that $\delta f(x_1) = f(\delta x_1) = g(x_2)$, and then let

$$y_1 = \delta x_1 \qquad \text{and} \qquad y_2 = x_2\,.$$

Clearly,

$$f \vee g(y_1) = f \vee g(\delta x_1) = \delta(f \vee g)(x_1) = \delta f(x_1) = f(y_1)$$

and

$$f \vee g(y_2) = g(y_2) = f(y_1)\,.$$

Now put $\omega^* = y_1 + y_2 \ge \omega$, and then give the two consumers the initial endowments

$$\omega_1 = \omega_2 = \tfrac{1}{2}(y_1 + y_2)\,.$$

Clearly, $f \vee g(\omega_1) = f \vee g(\omega_2) = f(y_1) = g(y_2)$. Observe that the inequalities

$$\omega = x_1 + x_2 \le y_1 + y_2 = \omega^* \le \delta x_1 + \delta x_2 = \delta\omega$$

guarantee that $A_\omega = A_{\omega^*}$, i.e., the ideals generated by ω and ω^* coincide.

We claim that the allocation (y_1, y_2) is an Edgeworth equilibrium. To see this, note that the price $q = f \vee g \in E^\sim$ satisfies the following property:

$$z \succ_1 y_1 \qquad \text{implies} \qquad q \cdot z > q \cdot \omega_1\,.$$

Indeed, $z \succ_1 y_1$ means $f(z) > f(y_1)$ and so

$$q \cdot z = f \vee g(z) \ge f(z) > f(y_1) = f \vee g(\omega_1) = q \cdot \omega_1\,.$$

Similarly, $z \succ_2 y_2$ implies $q \cdot z > q \cdot \omega_1$. Since (y_1, y_2) is supported by the price q (which is possibly outside of E') as a Walrasian equilibrium, it follows that (y_1, y_2) is an Edgeworth equilibrium.

By our hypothesis, there exists some non-zero price $p_\omega \in E'$ that supports (y_1, y_2) as a quasiequilibrium such that

$$p_\omega \cdot v = 1 \qquad \text{and} \qquad |p_\omega \cdot z| \leq 1 \ \text{ for all } \ z \in V\,.$$

Clearly, $p_\omega \geq 0$. From $p_\omega \cdot y_1 = p_\omega \cdot \omega_1$ and $p_\omega \cdot y_2 = p_\omega \cdot \omega_2$, we see that $p_\omega \cdot y_1 = p_\omega \cdot y_2$. In addition, from

$$1 = p_\omega \cdot v \leq p_\omega \cdot \omega^* = p_\omega \cdot \omega_1 + p_\omega \cdot \omega_2\,,$$

we see that $p_\omega \cdot y_1 = p_\omega \cdot y_2 = p_\omega \cdot \omega_1 = p_\omega \cdot \omega_2 \geq \frac{1}{2}$.

Now fix $0 < y \in E$. Then, we claim that

$$p_\omega \cdot y \geq \tfrac{p_\omega \cdot y_1}{f(y_1)} f(y) \quad \text{and} \quad p_\omega \cdot y \geq \tfrac{p_\omega \cdot y_2}{g(y_2)} g(y)$$

hold. Since both inequalities can be established by similar arguments, we shall establish the validity of the first inequality. We distinguish two cases.

a) Assume that $p_\omega \cdot y = 0$.

This implies that $\lambda y \in \mathcal{B}_1(p_\omega) = \{x \in E^+\colon\ p_\omega \cdot x \leq p_\omega \cdot y_1\}$ for each $\lambda > 0$, and so $y_1 \succeq_1 \lambda y$ holds for each $\lambda > 0$. Therefore,

$$f(y_1) \geq f(\lambda y) = \lambda f(y) \geq 0$$

holds for each $\lambda > 0$, and thus $f(y) = 0$. So, the inequality $p_\omega \cdot y \geq \frac{p_\omega \cdot y_1}{f(y_1)} f(y)$ is trivially true.

b) Assume that $p_\omega \cdot y > 0$.

In this case, we claim that $y_1 \succeq_1 \frac{p_\omega \cdot y_1}{p_\omega \cdot y} y$ holds. Otherwise, $\frac{p_\omega \cdot y_1}{p_\omega \cdot y} y \succ_1 y_1$ and $\frac{p_\omega \cdot y_1}{p_\omega \cdot y} y \in \mathcal{B}_1(p_\omega)$ contradict the maximality of $y_1 \in \mathcal{B}_1(p_\omega)$. Now from $y_1 \succeq_1 \frac{p_\omega \cdot y_1}{p_\omega \cdot y} y$, it follows that $f(y_1) \geq \frac{p_\omega \cdot y_1}{p_\omega \cdot y} f(y)$, or $p_\omega \cdot y \geq \frac{p_\omega \cdot y_1}{f(y_1)} f(y)$.

If we let $\alpha = \frac{p_\omega \cdot y_1}{f(y_1)} = \frac{p_\omega \cdot y_2}{f(y_2)}$, then $p_\omega \geq \alpha f$ holds. Similarly, $p_\omega \geq \alpha g$. Consequently,

$$p_\omega \geq \alpha f \vee \alpha g = \alpha (f \vee g)\,.$$

In particular, we have

$$\begin{aligned} p_\omega \cdot \omega^* = (\alpha f \vee \alpha g)(\omega^*) &= (\alpha f \vee \alpha g)(y_1) + (\alpha f \vee \alpha g)(y_2) \\ &\geq \alpha f(y_1) + \alpha g(y_2) \\ &= p_\omega \cdot y_1 + p_\omega \cdot y_2 = p_\omega \cdot \omega^*\,, \end{aligned}$$

and so $p_\omega \cdot \omega^* = (\alpha f \vee \alpha g)(\omega^*)$ holds. Now if $0 \leq y \leq \omega^*$ holds, then from $p_\omega \cdot y \geq (\alpha f \vee \alpha g)(y)$, $p_\omega \cdot (\omega^* - y) \geq (\alpha f \vee \alpha g)(\omega^* - y)$, and

$$\begin{aligned} p_\omega \cdot \omega^* = p_\omega \cdot y + p_\omega \cdot (\omega^* - y) &= (\alpha f \vee \alpha g)(\omega^*) \\ &= (\alpha f \vee \alpha g)(y) + (\alpha f \vee \alpha g)(\omega^* - y)\,, \end{aligned}$$

we see that

$$p_\omega \cdot y = (\alpha f \vee \alpha g)(y) = \alpha(f \vee g)(y)$$

holds for each $y \in A_{\omega^*} = A_\omega$. Since $1 = p_\omega \cdot v = \alpha(f \vee g)(v)$ holds, it follows that $\alpha = \frac{1}{(f \vee g)(v)} > 0$. Therefore,

$$p_\omega \cdot y = \alpha(f \vee g)(y) \tag{$\star\star$}$$

holds for each $y \in A_\omega$.

Now consider the directed set

$$\Omega = \big\{\omega > 0\colon\ \omega > v,\ \ f \vee g(\omega) > f(\omega),\ \text{ and }\ f \vee g(\omega) > g(\omega)\big\},$$

and note that $\{p_\omega\colon\ \omega \in \Omega\}$ is a net lying in the polar set V° of V. By the classical theorem of L. Alaoglu, the net $\{p_\omega\colon\ \omega \in \Omega\}$ has a weak* limit point in E'. By passing to a subnet, we can assume that $p_\omega \xrightarrow{w^*} p$ holds in E'. From $p_\omega \cdot v = 1$, we see that $p \cdot v = 1$ and so $p \neq 0$. In addition, from $(\star\star)$, it is easy to see that

$$p \cdot y = \alpha(f \vee g)(y)$$

holds for all $y \in E$. Thus, $f \vee g = \frac{1}{\alpha} p \in E'$ holds, and the proof of the theorem is finished.

15

REFERENCES

1. C. D. ALIPRANTIS AND D. J. BROWN, Equilibria in markets with a Riesz space of commodities, *J. Math. Econom.* **11** (1983), 189–207.

2. C. D. ALIPRANTIS, D. J. BROWN, AND O. BURKINSHAW, Edgeworth equilibria, *Econometrica* **55** (1987), 1109–1137.

3. C. D. ALIPRANTIS, D. J. BROWN, AND O. BURKINSHAW, Edgeworth equilibria in production economies, *J. Econom. Theory* **43** (1987), 252–291.

4. C. D. ALIPRANTIS, D. J. BROWN, AND O. BURKINSHAW, *Existence and Optimality of Competitive Equilibria*, Springer-Verlag, Heidelberg & New York, 1989.

5. C. D. ALIPRANTIS AND O. BURKINSHAW, *Positive Operators*, Pure and Applied Mathematics Series # 119, Academic Press, New York & London, 1985.

6. G. DEBREU AND H. E. SCARF, A limit theorem on the core of an economy, *Internat. Econom. Rev.* **4** (1963), 235–246.

7. L. E. JONES, Special problems arising in the study of economies with infinitely many commodities, in: H. F. Sonnenschein, Ed., *Models of Economic Dynamics* (Springer-Verlag Lecture Notes in Economics and Mathem. Systems # 264, Berlin & New York, 1986), 184–205.

8. A. MAS-COLELL, The price equilibrium existence problem in topological vector lattices, *Econometrica* **54** (1986), 1039–1053.

9. A. MAS-COLELL, Valuation equilibrium and Pareto optimum revisited, in: W. Hildenbrand and A. Mas-Colell Eds., *Contributions to Mathematical Economics* (North-Holland, New York, 1986), Chapter 17, pp. 317–331.

10. A. MAS-COLELL AND S. F. RICHARD, A new approach to the existence of equilibria in vector lattices, Preprint, August, 1987.

11. S. F. RICHARD, A new approach to production equilibria in vector lattices, *J. Math. Econom.* **18** (1989), 41–56.

12. N. C. YANNELIS AND W. R. ZAME, Equilibria in Banach lattices without ordered preferences, *J. Math. Econom.* **15** (1986), 85–110.

13. W. R. ZAME, Competitive equilibria in production economies with an infinite-dimensional commodity space, *Econometrica* **55** (1987), 1075–1108.