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# STRUCTURAL INSTABILITY OF THE CORE

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## ABSTRACT

Let $\sigma$ be a q-rule, where any coalition of size q, from the society of size n, is decisive. Let $w(\sigma) = 2q - n + 1$ and let W be a smooth "policy space" of dimension w. Let $U(W)^N$ be the space of all smooth profiles on W, endowed with the Whitney topology. It is shown that there exists an "instability dimension" $w^*(\sigma)$ with $2 \leq w^*(\sigma) \leq w(\sigma)$ such that:

(i) if $w \geq w^*(\sigma)$ then the core of $\sigma$ is empty for a dense set of profiles in $U(W)^N$ (i.e., almost always)

(ii) if $w \geq w(\sigma) + 1$ then the cycle set is dense in W, almost always

(iii) if $w > w(\sigma) + 1$ then the cycle set is also path connected, almost always.

The method of proof is first of all to show that if a point belongs to the core then certain generalized symmetry conditions in terms of "pivotal" coalitions of size 2q - n must be satisfied. Secondly, if the dimension is $w(\sigma)$ then it is shown that these symmetry conditions can almost never be satisfied.

# STRUCTURAL INSTABILITY OF THE CORE

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## 1. INTRODUCTION

It is now generally well-known that if the set of alternatives, W, can be represented by a subset of Euclidean space, and individual preferences are smooth, then the direction gradients at a point in the majority rule core must satisfy strong symmetry properties (Plott, 1967). Indeed, if the dimension of W is at least two, and if these properties are satisfied at a point then an arbitrary small perturbation of these preferences (in the $C^1$-topology) is sufficient to destroy the required symmetry. On the other hand when the majority rule core is empty then it will generally be the case that voting trajectories can be constructed throughout the space (McKelvey, 1976, 1979).

Another way of expressing these results is that majority rule, $\sigma_m$, in the spatial context, is <u>classified</u> by an integer $w^*(\sigma_m)$, which is two in the case when the society is of odd size and three if the society is of even size. Let $U(W)^N$ be the set of smooth profiles for the society, N, on W, endowed with the Whitney topology, and let $GO(\sigma_m,W,N,u)$ be the core, or set of unbeaten points in W under the majority rule preference relation $\sigma_m(u)$. Previous results (due to Matthews, 1980; and Schofield, 1983a) have shown essentially that if the dimension of W, or dim(W), is at least $w^*(\sigma_m)$ then $GO(\sigma_m,W,N,u)$ is empty, for any profile u in a dense set in $U(W)^N$. Moreover, if $\dim(W) \geq w^*(\sigma_m) + 1$ then again, for a dense set of profiles in $U(W)^N$, voting cycles can fill up a dense set in W (see Cohen, 1979; Cohen and Matthews, 1980; Matthews, 1982; Schofield, 1983a).

In an earlier paper (Schofield, 1980) it was shown that this theorem was valid for an arbitrary voting rule $\sigma$ (without vetoers) where the <u>instability dimension</u>, $w^*(\sigma)$ of $\sigma$, was bounded above by $(n - 1)$, with n being the size of the society.

In this paper we obtain the same theorem for a social choice mechanism, $\sigma$, (without vetoers) whose set of decisive coalitions contains all coalitions with at least q different members. For such a "q-rule" we obtain a generalized symmetry requirement that must be satisfied at any point in the core (Theorem 3). We use this result to show that the instability dimension, $w^*(\sigma)$, for a q rule is no greater than $w(\sigma) = 2q - n + 1$ (Theorem 1). Since majority rule $\sigma_m$ can be regarded as a q-rule with $(n,q) = (2k - 1,k)$ or $(2k,k + 1)$ depending on whether n is odd or even, this allows us to infer that for majority rule,

$$w^*(\sigma_m) = 2k - (2k - 1) + 1 = 2$$

or

$$w^*(\sigma_m) = 2(k + 1) - 2k + 1 = 3$$

depending on whether n is odd or even. Thus Theorem 1 generalizes the previous results on majority rule. Moreover, Theorem 3 gives an indication how the instability dimension can be computed for any social choice mechanism without vetoers.

We also examine the cycle set, which we label $IC(\sigma,W,N,u)$ or IC. We show that IC, for a q-rule, is almost always open dense in dimension at least $w(\sigma) + 1$, and in dimension at least $w(\sigma) + 2$ is also path connected. The significance of path connectedness is that, with this property, local agenda manipulation can lead from almost any point in W to almost any other point (McKelvey, 1979). This implies that in dimension $w(\sigma) + 2$, a q-rule must almost always fail implementability and "Maskin" monotonicity (see also Maskin, 1977; Maskin, 1979; Dasgupta, Hammond and Maskin, 1979; Ferejohn, Grether and McKelvey, 1982; Moulin, 1983; Schofield, 1984a).

## 2. SOCIAL CHOICE THEORY

In this section we briefly review the social choice definitions that we use.

(i) A binary relation p on W is a subset of W X W. We adopt the notation xpy iff $(x,y) \in p$. A strict preference relation p on W is a binary relation which is irreflexive (i.e., $\nexists x \in W$ s.t. xpx) and asymmetric (xpy $\Rightarrow$ not(ypx) wherever $x,y \in W$). We read "xpy" as "x is preferred to y."

Let B(W) be the set of strict preference relations on W, and define $B(W)^N = B(W) \times \cdots \times B(W)$, n times, to be the set of strict preference profiles for the society N, of size n, on W. That is if $p = (p_1,\ldots,p_n) \in B(W)^N$ then $p_i$



represents i's preference under p.

(ii) A strict preference relation $p \in B(W)$ is said to be cyclic iff there exists a finite subset $X = \{x_1,\ldots,x_r\} \subset W$ such that

$$x_1 p x_2 \cdots x_r p x_1.$$

Such a set X is called a p-cycle. If there exists no p-cycle then p is called acyclic. Write A(W) for the set of acyclic strict preference relations.

(iii) A social preference function (SF) is a function $\sigma : B(W)^N \to B(W)$ which satisfies the independence axiom [see Schofield, 1980, def. 2.2]. For $p \in B(W)^N$, $x\sigma(p)y$ reads "x is socially preferred to y under the social preference function $\sigma$ and profile p."

(iv) The core, or global optima set, of $\sigma$ given a profile $p \in B(W)^N$ is defined to be

$$GO(\sigma,W,N,p) = \{x \in W : \nexists y \in W \text{ s.t. } x\sigma(p)y\}.$$

(v) The global pareto set of a subset $M \subset N$, given the profile $p \in B(W)^N$, is defined to be

$$GO(W,M,p) = \{x \in W : \nexists y \in W \text{ s.t. } xp_iy \ \ \forall i \in M\}.$$

(vi) The global cycle set $GC(\sigma,W,N,p)$ of an SF, $\sigma$, given a profile $p \in B(W)^N$ is defined by $x \in GC(\sigma,W,N,p)$ iff there exists a finite $\sigma(p)$-cycle X such that $x \in X$.

(vii) Each SF, $\sigma$, defines a class $\mathbb{D}_\sigma$ of subsets of N, called <u>$\sigma$-decisive coalitions</u>, where $M \varepsilon \mathbb{D}_\sigma$ iff wherever $p \varepsilon B(W)^N$ and $xp_iy$, $\forall i \varepsilon M$, then $x\sigma(p)y$.

(viii) An SF, $\sigma$, is called a <u>voting rule</u> iff for any $x,y \varepsilon W$ and $p \varepsilon B(W)^N$ then $x\sigma(p)y$ implies there exists $M \varepsilon \mathbb{D}_\sigma$ such that $xp_iy$ for all $i \varepsilon M$. A voting rule called a <u>q-rule</u> and written $\sigma_q$ iff it is the case that $M \varepsilon \mathbb{D}_\sigma$ whenever $|M| \geq q$, where $n/2 < q \leq n - 1$.

(x) If $\mathbb{D}$ is a class of subsets of N, then the <u>collegium</u> $K(\mathbb{D})$ of $\mathbb{D}$ is defined to be

$$K(\mathbb{D}) = \bigcap_{M\varepsilon\mathbb{D}} M.$$

(xi) An SF, $\sigma$, is said to be <u>collegial</u> iff $K(\mathbb{D}_\sigma) \neq \Phi$. If $K(\mathbb{D}_\sigma) = \Phi$ then $\sigma$ is said to be <u>non-collegial.</u>

(xii) For any class $\mathbb{D}$ of subsets of N, define the <u>Nakamura number</u> $v(\mathbb{D})$ by

$$v(\mathbb{D}) = \infty \text{ iff } K(\mathbb{D}) \neq \Phi$$

$$v(\mathbb{D}) = \min\{|\mathbb{D}'| : \mathbb{D}' \subset \mathbb{D} \text{ and } K(\mathbb{D}') = \Phi\} \text{ if } K(\mathbb{D}) = \Phi.$$

For an SF, $\sigma$, define $v(\sigma) = v(\mathbb{D}_\sigma)$.

As an example of a q-rule, consider the rule given by

$$q = k \text{ when } n = 2k - 1$$

$$q = k + 1 \text{ when } n = 2k$$



This rule is called <u>majority rule</u> and written $\sigma_m$.

It is easy to show that except for the case $(n,q) = (4,3)$ the Nakamura number for majority rule $\sigma_m$ is 3. For the q-rule $\sigma$ given by $(n,q) = (4,3)$ we obtain $v(\sigma) = 4$.

An important question in social choice concerns the existence of the core for $\sigma(p)$. It is well known that if W is a finite set then

$$GC(\sigma,W,N,p) \cup GO(\sigma,W,N,p) \neq \Phi.$$

Indeed this will also be true if W is a compact topological space and preferences are continuous (Walker, 1977). Thus in this case the question of existence of a core is effectively equivalent to the question of non-existence of $\sigma$-cycles.

Suppose we let

$$GO(W,\mathbb{D},p) = \bigcap_{M\varepsilon\mathbb{D}} GO(W,M,p)$$

when $\mathbb{D}$ is a family of coalitions. Then clearly, for any SF,

$$GO(\sigma,W,N,p) \subset GO(W,\mathbb{D}_\sigma,p)$$

with equality if $\sigma$ is a voting rule. If $p \varepsilon A(W)^N$ then $GO(W,M,p) \neq \Phi$ for each $M \varepsilon N$. Thus if $K(\mathbb{D}_\sigma) \neq \Phi$ and $\sigma$ is a voting rule, then the core will be non-empty. On the other hand if $K(\mathbb{D}_\sigma) = \Phi$ and W is a finite set of cardinality $|W| \geq v(\sigma)$ then a result of Nakamura (1978) shows that there exists $p \varepsilon A(W)^N$ such that $GO(\sigma,W,N,p) = \Phi$ and so $GC(\sigma,W,N,p) \neq \Phi$. If $|W| \leq v(\sigma)$ then for every $p \varepsilon A(W)^N$ it is the case that $GC(\sigma,W,N,p) = \Phi$ and so $GO(\sigma,W,N,p) \neq \Phi$.

It can be shown that there is a parallel result in the case that W is admissible (i.e., a compact convex subset of Euclidean space). Let $C(W)^N$ be the set of all preference profiles on W such that each individual preference is convex. Then results by Greenberg (1979), Schofield (1984a) and Strnad (1984) show that $GO(\sigma,W,N,p) \neq \Phi$ for all $p \in C(W)^N$ iff $\dim(W) \leq v(\sigma) - 2$. As a consequence the existence of a core cannot be guaranteed when $\dim(W) \geq v(\sigma) - 1$.

In other words if $\dim(W) \geq v(\sigma) - 1$ then it is always possible to find a profile $p \in C(W)^N$ such that $GO(\sigma,W,N,p)$ is empty. The result proved in this paper is that if $\dim(W) \geq w^*(\sigma)$ then $GO(\sigma,W,N,p)$ is empty for "almost every" smooth profile p. Of course this means that $w^*(\sigma) > v^*(\sigma)$, where $v^*(\sigma) = v(\sigma) - 2$ is the <u>stability dimension</u> for $\sigma$. For example, if $\sigma_q$ is a q-rule then $v^*(\sigma_q)$ is the greatest integer which is strictly less than $q/n - q$ (Greenberg, 1979; Schofield, 1983b). It is easy to show that $2q - n + 1 > v^*(\sigma_q)$.

## 3. SINGULARITY THEORY

In this paper we shall be concerned with the existence of a core when preferences can be represented by smooth utility functions. In this case the class of preferences has a topology and we enquire whether the subset of profiles which has a non-empty core can have non-empty interior.

We restrict attention to the case when W is a smooth manifold of dimension $w = \dim(W)$. Assume that for each $i \in N$, there is a smooth ($C^\infty$) utility function



$$u_i : W \to \mathbb{R}$$

which represents i's preference $p_i$. That is to say $u_i(x) > u_i(y)$ iff $xp_iy$, for any $x,y \in W$. We write $u = (u_1,\ldots,u_n) : W \to \mathbb{R}^n$ for a smooth profile for N, and write $U(W)^N$ for the set of all smooth profiles when this set is endowed with the Whitney $C^\infty$-topology (Golubitsky and Guellemin, 1973, and Hirsch, 1976). A subset U of $U(W)^N$ is <u>residual</u> iff U is the countable intersection of a family of open dense subsets of $U(W)^N$. Since $U(W)^N$ is a Baire space, a residual subset U of $U(W)^N$ is dense in $U(W)^N$: indeed if W is compact then U will be open dense. A property O of profiles is called <u>generic</u> iff $U_O = \{u \in U(W)^N : u \text{ satisfies } O\}$ contains a residual subset of $U(W)^N$. If O is generic we shall also say O is true <u>almost always</u> (abbreviated to a.a.).

Given the profile $u \in U(W)^N$ the differential $du_i(x)$ at x of the $i^{th}$ component $u_i$ is a linear map $du_i(x) : T_xW \to \mathbb{R}$ where $T_xW$ is the tangent space at x, and $T_xW$ is linearly isomorphic to $\mathbb{R}^w$. With respect to a coordinate chart at x, therefore, the Jacobian $J_u(x) = (du_i(x))_{i \in N}$ may be regarded as a linear transformation $J_u(x) : \mathbb{R}^w \to \mathbb{R}^n$ and thus identified, with respect to the appropriate basis, with a (wxn) matrix.

When $\sigma$ is a social preference function we shall write $\sigma(u)$ for the preference relation obtained when preference is represented by $u \in U(W)^N$. In the obvious way write $GO(W,M,u)$ and $GO(\sigma,W,N,u)$ for the global pareto set of M and core of $\sigma$, given u. We may approximate the core by the "critical optima set" defined as follows.

Definition 1

(i) Let $u \in U(W)^N$, and let $J_u(x) : T_xW \to \mathbb{R}^n$ be a representation of the Jacobian at x. For any coalition $M \subset N$ let $J_u^M(x) : T_xW \to \mathbb{R}^m$, where $|M| = m$, be the obvious restricted Jacobian. Let $Pos_m = \{y \in \mathbb{R}^m : y_i > 0, \ \forall i \in M\}$. Then define the critical pareto set $IO(W,M,u)$, of M by $x \in IO(W,M,u)$ iff there exists no $v \in T_xW$ such that $J_u^M(x)(v) \in Pos_m$. Define the singularity set, $\Lambda(W,M,u)$, of M by

$$x \in \Lambda(W,M,u) \text{ iff rank } J_u^M(x) < \min\{m,w\}.$$

(ii) Let $\mathbb{D}$ be a class of subsets of N. Define

$$IO(W,\mathbb{D},u) = \bigcap_{M \in \mathbb{D}} IO(W,M,u)$$

and

$$\Lambda(W,\mathbb{D},u) = \bigcap_{M \in \mathbb{D}} \Lambda(W,M,u)$$

(iii) If $\sigma$ is a social preference function and $\mathbb{D}_\sigma$ is the set of $\sigma$-decisive coalitions define the critical optima set and singularity set of $\sigma(u)$, respectively by

$$IO(\sigma,W,N,u) = IO(W,\mathbb{D}_\sigma,u)$$

$$\Lambda(\sigma,W,N,u) = \Lambda(W,\mathbb{D}_\sigma,u).$$

It is well known (Smale, 1973) that for any $M \subset N$, and any $u \in U(W)^N$.



$$GO(W,M,u) \subset IO(W,M,u).$$

Thus when $\sigma$ is a SF, we obtain

$$GO(\sigma,W,N,u) \subset GO(W,\mathbb{D}_\sigma,u) \subset IO(\sigma,W,N,u).$$

We write $\partial W$ for the boundary of W, noting that $\partial W$ will itself be a manifold of dimension $\dim(W) - 1$. We also let $\text{Int } W = W \backslash \partial W$. It is known that if $x \in \text{Int } W$ then $x \in IO(W,M,u)$ iff there exists a "semipositive" solution $\lambda = (\lambda_i : i \in M)$, with $\sum_{i \in M} \lambda_i = 1$, to the equation $\sum_{i \in M} \lambda_i du_i(x) = 0$. Consequently if $w \geq |M| = m$ and the boundary $\partial W$ of W is empty then

$$IO(W,M,u) \subset \Lambda(W,M,u).$$

Thus when $w \geq \max\{|M| : M \in \mathbb{D}\}$ and $\partial W = \Phi$ it is the case that

$$GO(\sigma,W,N,u) \subset IO(\sigma,W,N,u) \subset \Lambda(W,\mathbb{D}_\sigma,u).$$

If we can show that generically it is the case that $\Lambda(W,\mathbb{D}_\sigma,u) = \Phi$ then this will show that, when W has no boundary, both $IO(\sigma,W,N,u)$ and $GO(\sigma,W,N,u)$ are generically empty. To proceed in this fashion we make use of the Thom Singularity Theorem. Consider any coalition $M \subset N$, where $|M| = m$.

Write $$\Lambda(W,M,u) = \bigcup_{s>0} \Lambda_s(W,N,u)$$

where $$\Lambda_s = \Lambda_s(W,M,u) = \{x \in W : \text{rank } J_u^M(x) = m - s\}$$

Then generically the dimension of $\Lambda_s$ is given by $d_s = \dim[\Lambda_s] = w - s(w - m + s)$, whenever $w \geq m$. In particular $d_1 = m - 1$. Moreover, $\Lambda(W,M,u)$ is a stratified manifold consisting generically of an $(m - 1)$ dimensional manifold and lower dimensional submanifolds $\Lambda_s$, for $s \geq 2$. We shall write $\dim[\Lambda(W,M,u)]$ for the dimension of the highest dimensional component $\Lambda_1$ of $\Lambda(W,M,u)$. As Smale has shown, under certain regularity assumptions, $IO(W,M,u)$ will also be a stratified manifold with $\dim[IO(W,M,u)] \leq m - 1$. Moreover, if $M_1, M_2$ are two subsets of N then an argument based on the genericity of transversal intersection allows us to obtain a bound on the dimension of the intersection of the two singularity sets. This gives the following theorem (see Schofield, 1980).

Singularity Theorem

Let W be a smooth manifold of dimension w, and let N be a society of size n. Under the stated dimension constraints, the following two properties are true, for almost all $u \in U(W)^N$.

(i) $\dim[\Lambda(W,M,u)] \leq m - 1$ for any $M \subset N$, with $|M| = m$, whenever $w \geq m$.

(ii) $\dim[\Lambda(W,M_1,u) \cap \Lambda(W,M_2,u)]$

$$\leq \max\{\dim\Lambda(W,M_1,u) + \dim\Lambda(W,M_2,u) - w, m_{12} - 1\},$$

where $m_{12} = |M_1 \cap M_2|$, whenever $w \geq \max\{m_1,m_2\}$.

Moreover, the optima set may be substituted for the singularity set in statements (i) and (ii) whenever W has empty boundary.

We shall refer to this theorem as ST.

Suppose that $\sigma$ is a q-rule. Define $w(\sigma) = 2q - n + 1$. We now state our main core theorem.



Theorem 1

Let $\sigma$ be a q-rule for a society N and W a smooth manifold. Then there exists an integer $w^*(\sigma)$ with $2 \leq w^*(\sigma) \leq w(\sigma)$ such that

(i) $\{u \in U(W)^N : \text{Int } W \cap IO(\sigma,W,N,u) = \Phi\}$ is residual in $U(W)^N$ whenever $\dim(W) \geq w^*(\sigma)$.

(ii) $\{u \in U(W)^N : IO(\sigma,W,N,u) = \Phi\}$ is residual in $U(W)^N$ whenever $\dim(W) \geq w^*(\sigma) + 1$.

Previously (Schofield, 1980) it was shown that this theorem was true for an arbitrary non-collegial social preference function $\sigma$, where the "instability dimension" $w^*(\sigma)$ was shown to be bounded above by $w(\sigma) = n - 1$

Suppose we define

$$O(\sigma,W) = \{u \in U(W)^N : IO(\sigma,W,N,u) \neq \Phi\}$$

By Theorem 1, for any smooth manifold W without boundary, of dimension at least $w^*(\sigma)$ it is the case that $U(W)^N \backslash O(\sigma,W)$ is residual, and thus dense. Suppose that $u \in O(\sigma,W)$. Then any neighborhood U of u in $U(W)^N$ must intersect $W \backslash O(\sigma,W)$. That is to say, in any neighborhood U of u, there exists $u' \in U$ such that $IO(\sigma,W,N,u') = \Phi$. For this reason we shall say that the optima set is structurally unstable in dimension $w^*(\sigma)$.

We also obtain results on the generic existence of σ-cycles. First of all note that at a point $x \in W$, the differential $du_i(x)$ may be represented, with respect to a coordinate chart, as a vector $\nabla u_i(x) \in \mathbb{R}^W$, called the direction gradient. For convenience we shall write $\nabla u_i(x)$ as $p_i(x)$, and call

$$p(x) = (p_1(x),\dots,p_n(x)) \in (\mathbb{R}^W)^n$$

a profile of direction gradients for u at x.

Definition 2

Let $u \in U(W)^N$, where W is a smooth manifold.

(i) At a point $x \in W$, for coalition $M \subset N$, let

$$p_M(x) = \mathrm{Con}(\{p_i(x) : i \in M\}) \in \mathbb{R}^W$$

be the generalized direction gradient for M. (Here Con(A) is the convex hull of A in $\mathbb{R}^W$.)

(ii) Let $\mathbb{D}$ be a family of subsets of N. At $x \in W$, define

$$p_{\mathbb{D}}(x) = \bigcap_{M \in \mathbb{D}(x)} p_M(x) \in \mathbb{R}^W$$

where

$$\mathbb{D}(x) = \{M \in \mathbb{D} : 0 \notin p_M(x)\}.$$

(iii) Let σ be an SF, with $\mathbb{D}$ its family of decisive coalitions and define

$$p_\sigma(x) = p_{\mathbb{D}}(x) \in \mathbb{R}^W$$

to be the generalized direction gradient of σ(u) at x. Define the critical cycle set IC(σ,W,N,u) by

$$x \in IC(\sigma,W,N,u) \text{ iff } p_\sigma(x) = \Phi.$$

(iv) Define the local cycle set LC(σ,W,N,u) as follows $x \in LC(\sigma,W,N,u)$ iff for any neighborhood U of x there exists a finite σ(u)-cycle X with $X \subset U$.

(v) A path connected component W′ of W is a subset W′ of W such that for any two points $x,y \in W'$ there exists a continuous function $C : [0,1] \to W'$ whose image lies in W′, such that $C(0) = x$ and $C(1) = y$.



Note that though the profile p(x) of direction gradients at a point $x \in W$ is dependent on the actual representation chosen, the condition $p_{\mathbb{D}}(x) = \Phi$ is independent of the representation. Thus IC(σ,W,N,u) is well defined. From previous results (Schofield, 1978, 1984c) it is known that

$$IC(\sigma,W,N,u) \subset LC(\sigma,W,N,u) \subset \mathrm{clos}\ IC(\sigma,W,N,u)$$

where clos(A) means the closure of A in W. It is evident that $LC(\sigma,W,N,u) \subset GC(\sigma,W,N,u)$. Our second theorem states that, under a certain dimension constraint, the critical cycle set will be open dense and path connected. In this case for almost all pairs of points x,y, say, in W there will exist a path between x and y, within IC(σ,W,N,u) with the property that the path can be arbitrarily closely approximated by a series of coalition manipulations.

<u>Theorem 2</u>

Let σ be a q-rule on a smooth manifold W and let $w(\sigma) = 2q - n + 1$ as before. Then

(i) If $\dim(W) \geq w(\sigma) + 1$ there exists a residual set $U_1$ in $U(W)^N$ such that for any $u \in U_1$, IC(σ,W,N,u) is an open dense set, and consists of a finite number of path connected components.

(ii) If $\dim(W) \geq w(\sigma) + 2$ then there exists a residual set $U_2 \subset U(W)^N$ such that for any $u \in U_2$, IC(σ,W,N,u) is open dense <u>and</u> path connected in W.

The next three sections are devoted to the proof of the two theorems. In Section 4 we show that any point, x, in IO(σ,W,N,u) may be characterized by a certain symmetry conditions on the profile of direction gradients p(x). In Section 5 the singularity theorem is generalized to show that the required symmetry conditions will always fail for a residual set of profiles, when the dimension of the manifold is at least w(σ). This gives Theorem 1. Finally in Section 6, this result is generalized to give Theorem 2.



## 4. SYMMETRY CONDITIONS FOR A σ-EQUILIBRIUM

As we have observed if $x \in \text{Int } W \cap IO(\sigma,W,N,u)$ then for each $M \in \mathbb{D}_\sigma$ it is the case that $\sum_{i \in M} \lambda_i p_i(x) = 0$ for some $\lambda = (..\lambda_i.. : i \in M) \in \mathbb{R}^m$ with $\lambda_i \geq 0$ $\forall i \in M$, and $\sum_{i \in M} \lambda_i = 1$. On the other hand if there exists some vector $v \in \mathbb{R}^W$ such that $p_i(x) \cdot v > 0$ for all $i \in M$, and $x \in \text{Int } W$, then x cannot belong to IO(σ,W,N,u). In this section we examine the symmetry requirements on a "profile" $p \in (\mathbb{R}^W)^n$ which are necessary for $p = p(x) = (p_1(x),\ldots,p_n(x))$ to be the profile of direction gradients at a point $x \in IO(\sigma,W,N,u)$.

<u>Definition 3</u>

(i) For any vector $y \in \mathbb{R}^W$ define

$$H^+(y) = \{z \in \mathbb{R}^W : z \cdot y > 0\}$$
$$H^-(y) = \{z \in \mathbb{R}^W : z \cdot y < 0\}$$
$$H^0(y) = \{z \in \mathbb{R}^W : z \cdot y = 0\}$$

to be the positive and negative open half spaces, and the normal hyperplane, respectively, associated with y.

(ii) Let $p = (p_1,\ldots,p_n) \in (\mathbb{R}^W)^n$ be a profile of vectors in $\mathbb{R}^W$, for a society N of size n. For each $M \subset N$, define

$$p_M = \text{Con}(\{p_i : i \in M\}), p_M^- = \text{Con}(\{-p_i : i \in M\})$$

$$H^+(p_M) = \bigcap_{i \in M} H^+(p_i) \subset \mathbb{R}^W, H^-(p_M) = \bigcap_{i \in M} H(-p_i).$$

The half-space $H^+(p_i)$ is called the $i^{th}$ preference half space, and the set $H^+(p_M)$ is called the <u>preference cone of coalition M</u>. For any non-zero vector $y \in \mathbb{R}^W$ let

$$N_p(y) = \{i \in N : p_i \in H^+(y)\}$$

be the subset of N which is <u>effective</u> for y, given p.

(iii) If $\sigma$ is a voting rule, with decisive coalitions $\mathbb{D}_\sigma$, say $p \in (\mathbb{R}^W)^n$ is a <u>σ-equilibrium</u> iff for no $y \in \mathbb{R}^W$ does $N_p(y) \in \mathbb{D}_\sigma$.

(iv) Given a family $\mathbb{D}$ of subsets of N, define two families of subsets of N, called $E(\mathbb{D})$ and $E'(\mathbb{D})$, whose members are called <u>pivotal subgroups</u>, as follows:

(a) $M \in E(\mathbb{D})$ iff $\forall L \subset N\backslash M$ either $M \cup L \in \mathbb{D}$ or $N\backslash L \in \mathbb{D}$.

(b) $M \in E'(\mathbb{D})$ iff $\forall L \subset N\backslash M$ and any $i \in N$ either $(M \cup L)\backslash\{i\} \in \mathbb{D}$ or $(N\backslash L)\backslash\{i\} \in \mathbb{D}$.

Note that if $p(x) = (p_1(x),\ldots,p_n(x))$ is a profile of direction gradients at a point $x \in W$, then a coalition M is effective at x iff $p_i(x) \in H^+(y)$ for all $i \in M$, and some $y \in \mathbb{R}^W$. But then $p_i(x) \cdot y > 0$ or $y \in H^+(p_i(x))$ $\forall i \in M$. Hence $H^+(p_M(x)) \neq \Phi$. Alternatively $M \subset N_{p(x)}(y)$ for $y \in \mathbb{R}^W$.

To examine the critical optima set, we deduce those conditions on a profile of vectors such that no decisive coalition is effective.

<u>Theorem 3</u>

Let $\sigma$ be a voting rule and $\mathbb{D}$ the set of decisive coalitions. Let $p = (p_1,\ldots,p_n) \in (\mathbb{R}^W)^n$ be a profile of vectors.



(i) If p is a σ-equilibrium and $p_i \neq 0$ $\forall i \in N$, then for each $M \in E(\mathbb{D})$ there exists $j_M \in N\backslash M$ such that $\{p_i : i \in M \cup \{j_M\}\}$ is a linearly dependent set.

(ii) If p is a σ-equilibrium and $p_k = 0$ for exactly one member $k \in N$, then, for all $M \in E'(\mathbb{D})$ there exists $j_M \in N\backslash M\backslash\{k\}$ such that $\{p_i : i \in M \cup \{j_M\}\}$ is a linearly dependent set.

<u>Proof</u>

(i) Pick any $M \in E(\mathbb{D})$. If $\{p_i : i \in M\}$ are linearly dependent then for any $j \in N\backslash M$, $\{p_i : i \in M \cup \{j\}\}$ are linearly dependent, and we are finished. Suppose therefore that $\{p_i : i \in M\}$ are linearly independent, and that for every $j \in N\backslash M$, $p_j \notin \text{Span}(\{p_i : i \in M\})$. Here $\text{Span}(\{p_i : i \in M\})$ is the vector subspace of $\mathbb{R}^W$ spanned by $\{p_i : i \in M\}$. By definition $w > |M| = m$. Since $\{p_i : i \in M\}$ are linearly independent, these vectors belong to a hyperplane in $\mathbb{R}^W$ of dimension at most $w - 1$. That is to say there exists $x \in \mathbb{R}^W$ such that

$$p_i \in H^0(x) \quad \forall i \in M$$
$$p_j \notin H^0(x) \quad \forall j \in N\backslash M.$$

But then

$$x \in H^0(p_i) \quad \forall i \in M, \text{ and } x \in \bigcap_M H^0(p_i).$$

Moreover $H^0(p_i) = \partial H^+(p_i)$, the boundary of $H^+(p_i)$ and so

$x \in \partial \bigcap_M H^+(p_i)$.

Thus $x \in \partial H^+(p_M)$. In identical fashion $x \in \partial H^-(p_M)$. For $j \in N\backslash M$, $p_j \notin H^0(x)$ so $x \notin H^0(p_j)$ and so either $x \in H^+(p_j)$ or $x \in H^-(p_j)$. Let

$$L_1 = \{j \in N : x \in H^+(p_j)\}$$
$$L_2 = \{j \in N : x \in H^-(p_j)\}.$$

For each $j \in N\backslash M$, $x \notin H^0(p_j)$ and thus there exists a neighborhood $U_j(x)$ of $x$ such that $U_j(x) \cap H^0(p_j) = \Phi$. In particular

$$j \in L_1 \text{ iff } U_j(x) \subset H^+(p_j)$$

and

$$j \in L_2 \text{ iff } U_j(x) \subset H^-(p_j).$$

Let $U(x) = \bigcap_{N\backslash M} U_j(x)$.

Now $x \in \partial H^+(p_M) \cap \partial H^-(p_M)$
and so $U(x) \cap H^+(p_M) \neq \Phi$ and $U(x) \cap H^+(p_M) \neq \Phi$.
Choose $x_1 \in U(x) \cap H^+(p_M)$, $x_2 \in U(x) \cap H^-(p_M)$.
However,

$$U(x) \subset \bigcap_{L_1} H^+(p_j)$$

and

$$U(x) \subset \bigcap_{L_2} H^-(p_j).$$



Thus

$$x_1 \in H^+(p_M) \cap H^+(p_{L_1})$$

and

$$x_2 \in H^-(p_M) \cap H^-(p_{L_2}).$$

Hence

$$\forall i \in M \cup L_1,\ x_1 \in H^+(p_i) \text{ or } p_i \in H^+(x_1)$$

and $\forall i \in M \cup L_2$, $x_2 \in H^-(p_i)$ or $p_i \in H^+(-x_2)$. Thus $N_p(x_1) = M \cup L_1$ and $N_p(-x_2) = N \cup L_2$. By definition $\{M,L_1,L_2\}$ is a disjoint partition of N. By assumption, $M \in E(\mathbb{D})$ and so for any $L_1 \subset N\backslash M$ either $M \cup L_1 \in \mathbb{D}$ or $N\backslash L_1 = M \cup L_2 \in \mathbb{D}$. But both $M \cup L_1$ and $M \cup L_2$ are effective. Thus either $M \cup L_1$ or $M \cup L_2$ is both effective and decisive. Hence the profile $p = (p_1,\ldots,p_n)$ cannot be a $\sigma$-equilibrium. Thus if $p = (p_1,\ldots,p_n)$ is a $\sigma$-equilibrium, then for any $M \in E(\mathbb{D})$, there exists a $j_M \in N\backslash M$ such that $\{p_i : i \in M \cup \{j_M\}\}$ is a linearly dependent set.

(ii) Suppose that $p_k = 0$ for exactly one $k \in N$. Repeat the proof of (i) for $N' = N\backslash\{k\}$, to show that for $M \subset N'$, there exists a partition $\{M,L_1,L_2\}$ for $N'$ such that both $M \cup L_1$ and $M \cup L_2$ are effective.

Q.E.D.

We now introduce a number of integers that will prove useful in classifying voting rules, and in particular the class of all q-rules. Since we assume that $n/2 < 1 < n - 1$ for a q-rule, then we also assume that $n \geq 3$ and $q \geq 2$.

<u>Definition 4</u>

For a voting rule, $\sigma$, let

$$e(\sigma) = \min\{|M| : M \in E(\mathbb{D}_\sigma)\} \text{ and}$$
$$e'(\sigma) = \min\{|M| : M \in E'(\mathbb{D}_\sigma)\}.$$

If $(n,q)$ are integers such that $n/2 < q < n$ define

$e(n,q) = 2q - n - 1$, $e'(n,q) = 2q - n$.

For any $q \leq n - 1$, let $s(n,q)$ be the greatest integer such that $s(n,q) \leq \frac{n-2}{n-q}$. For a general voting rule, $\sigma$, let $s(\sigma) = s(n,q)$ where $q = \min\{|M| : M \in \mathbb{D}_\sigma\}$. Let $v(n,q)$ be the greatest integer which is strictly less than $\frac{q}{n-q}$.

From Schofield (1984d) it is known that if $\dim(W) \leq s(\sigma)$ then the $\sigma$-core can be structurally stable. Thus we require that $s(\sigma) < w^*(\sigma)$ for an arbitrary voting rule. Moreover, for a q-rule, $\sigma$, it can be shown that the Nakamura number $v(\sigma)$ satisfies $v(\sigma) = v(n,q) + 2$ (Schofield, 1983b). Since we shall show that $w^*(\sigma) \leq e'(n,q) + 1$ for a q-rule, we need to demonstrate that $v(n,q) \leq s(n,q) < e'(n,q) + 1$. The following two lemmas show that this is the case.



<u>Lemma 1</u>

If $\sigma$ is a q-rule then $e(\sigma) = e(n,q)$ and $e'(\sigma) = e'(n,q)$. If $\sigma$ is majority rule, with n odd, then $e(\sigma) = 0$, $e'(\sigma) = 1$, whereas if n is even then $e(\sigma) = 1$, $e'(\sigma) = 2$.

<u>Proof</u>

Suppose that $|M| = 2q - n - 1$, and $L_1 \subset N \backslash M$. If $|L_1| \geq n + 1 - q$ then $|M \cup L_1| \geq q$ and so $M \cup L_1 \in \mathbb{D}_\sigma$.

If $|L_1| \leq n - q$, then $|N \backslash L_1| \geq q$ and so $M \cup (N \backslash M \backslash L_1) \in \mathbb{D}_\sigma$. Thus $|M| \geq 2q - n - 1 \Rightarrow M \in E(\mathbb{D})$. Clearly if $|M| \leq 2q - n - 2$, then there exists $L_1 \subset N \backslash M$ such that $|L_1| = n + 1 - q$ yet $|M \cup L_1| = q - 1$ so $M \cup L_1 \notin \mathbb{D}_\sigma$ and $|N \backslash L_1| = q - 1$ so $N \backslash L_1 \notin \mathbb{D}_\sigma$. Thus $M \in E(\mathbb{D}) \Rightarrow |M| \geq 2q - n - 1$. Thus $e(n,q) = 2q - n - 1$. In identical fashion, when $p_k = 0$, let $N' = N \backslash \{k\}$ so $|N'| = n' = n - 1$. Then

$$e'(n,q) = e(n',q) = 2q - n' - 1 = 2q - (n - 1) - 1$$
$$= 2q - n.$$

Finally if $\sigma$ is majority rule with n odd then

$$e(\sigma) = e(2k - 1, k) = 2k - (2k - 1) - 1 = 0, \text{ and } e'(\sigma) = 1.$$

If n is even then $e(\sigma) = e(2k, k + 1) = 2k + 2 - 2k - 1 = 1$ and $e'(\sigma) = 2$

Q.E.D.

<u>Lemma 2</u>

For any q-rule, $\sigma$ (with $n/2 < q < n$) then $1 \leq v(n,q) \leq s(n,q) < e'(n,q) + 1 = 2q - n + 1$. If $\sigma$ is majority rule then $s(\sigma) = 1$ if n is odd and $s(\sigma) = 2$ if n is even.

<u>Proof</u>

By previous results (Schofield, 1983b) $v(\sigma) = 2 + v(n,q)$, where $v(n,q)$ is the greatest integer which is strictly less than $\frac{q}{n-q}$. Thus $q = v(n,q)(n-q) + r$ where $0 < r \leq (n-q)$. Moreover,

$$s(n,q) \leq \frac{n-2}{n-q} = 1 + \frac{q}{n-q} - \frac{2}{n-q}$$

$$= 1 + v(n,q) + \frac{r-2}{n-q} .$$

Since $n - q \geq 1$, we obtain $v(n,q) \leq s(n,q)$. Clearly $v(n,q) \geq 1$. To show that $s(n,q) < 2q - n + 1$, consider first of all the case $(n,q) = (2k-1,k)$ and $k \geq 2$. Then $s(n,q) \leq \frac{2k-3}{k-1} = 2 - \frac{1}{k-1}$ and so $s(n,q) = 1$. Moreover, $e'(n,q) = 2$. If $(n,q) = (2k,k+1)$ then $s(n,q) = \frac{2k-2}{k-1} = 2$. Moreover, $e'(n,q) + 1 = 3$. Finally, let $q = n - 1$. Clearly $s(n,q) = n - 2$ while $e'(n,q) + 1 = n - 1$. Moreover, $s(n,-)$ is a monotonically increasing but convex function of q, while $e'(n,-)$ is a linear function of q. Thus $s(n,q) < e'(n,q) + 1$ for all $q \in (n/2, n-1]$.

Q.E.D.



## 5. GENERIC NON-EXISTENCE OF OPTIMA

We turn now to the proof of Theorem 1 by a generalization of the Singularity Theorem. As before, when $\mathbb{D}$ is a family of subsets of N let

$$\Lambda(W,\mathbb{D},u) = \bigcap_{M \in \mathbb{D}} \Lambda(W,M,u).$$

We seek a bound for the dimension

$$d(\mathbb{D}) = \dim[\Lambda(W,\mathbb{D},u)].$$

If we can show that $d(\mathbb{D}) < 0$ almost always (a.a.) then this will imply that $\Lambda(W,\mathbb{D},u) = \Phi$. A lower bound on $d(\mathbb{D})$ is given by

$$d(\mathbb{D}) \geq k(\mathbb{D}) - 1 \text{ a.a.}$$

where $k(\mathbb{D}) = |K(\mathbb{D})|$ and $K(\mathbb{D})$ is the collegium of $\mathbb{D}$. To see this observe that if $\{p_i(x) : i \in K(\mathbb{D})\}$ are linearly dependent, then so is $\{p_i(x) : i \in M\}$ for each $M \in \mathbb{D}$. Thus $\Lambda(W,K(\mathbb{D}),u) \subset \Lambda(W,\mathbb{D},u)$. By ST(i), $\dim[\Lambda(W,K(\mathbb{D}),u)] = k(\mathbb{D}) - 1$ a.a. and so $d(\mathbb{D}) = \dim[\Lambda(W,\mathbb{D},u)] \geq k(\mathbb{D}) - 1$ a.a. The next theorem uses ST to obtain an upper bound on $d(\mathbb{D})$ when $\mathbb{D}$ is a subset of the family of subsets

$$\mathbb{D}_r' = \{M \subset N : |M| = r\}.$$

For $\mathbb{D} \subset \mathbb{D}_r'$, say $\mathbb{D}$ satisfies the <u>intersection property</u> iff for any $\mathbb{D}' \subset \mathbb{D}$ it is the case that $k(\mathbb{D}') \leq r - |\mathbb{D}'| + 1$.

Theorem 4

Let W be a smooth manifold of dimension w, and let $\mathbb{D}$ be a subfamily of $\mathbb{D}_r'$ which satisfies the intersection property. If $w \geq r$ then it is almost always the case that

$$\dim[\Lambda(W,\mathbb{D},u)] \leq r - |\mathbb{D}|.$$

Proof

We proceed by induction on the cardinality of subfamilies of $\mathbb{D}$. Let $H_s$ be the induction hypothesis: for any subfamily $\mathbb{D}'$ of $\mathbb{D}$ with $|\mathbb{D}'| = s$, then

$$d(\mathbb{D}') = \dim[\Lambda(W,\mathbb{D}',u)] \leq r - s, \text{ a.a.}$$

$H_1$ can be shown directly. Choose any $M \in \mathbb{D}$, and let $\mathbb{D}' = \{M\}$. Then $|\mathbb{D}'| = 1$ and $\mathcal{K}(\mathbb{D}') = M$ and so $k(\mathbb{D}') = |M| = r$. By ST(i)

$$d(\mathbb{D}') \leq r - 1, \text{ a.a.}$$

Now we show $H_2$. Let $\mathbb{D}' = \{M_1,M_2\}$. By the intersection property $k(\mathbb{D}') = |M_1 \cap M_2| \leq r - 1$. By ST(ii),

$$d(\mathbb{D}') \leq \max\{2(r - 1) - w,\ k(\mathbb{D}') - 1\}, \text{ a.a.}$$

But $w \geq r$ and so $2(r - 1) - w \leq r - 2$. Thus $d(\mathbb{D}') \leq r - 2$, a.a.

Assume $H_s$ is true for some s with $2 \leq s < |\mathbb{D}|$. Consider $\mathbb{D}' \subset \mathbb{D}$ with $|\mathbb{D}'| = s$. Choose $M \in \mathbb{D}\backslash\mathbb{D}'$, and let $\mathbb{D}'' = \mathbb{D}' \cup \{M\}$, so $|\mathbb{D}''| = s + 1$. By $H_s$, $d(\mathbb{D}') \leq r - s$. By ST(ii),



$$d(\mathbb{D}'') \leq \max\{d(\mathbb{D}') + (r - 1) - w,\ k(\mathbb{D}'') - 1\} \text{ a.a.}$$

Again $w \geq r$ and so

$$d(\mathbb{D}') + (r - 1) - w \leq (r - s) + (r - w) - 1 \leq r - (s + 1) \text{ a.a.}$$

By the intersection property $k(\mathbb{D}'') \leq r - |\mathbb{D}''| + 1$ and so $k(\mathbb{D}'') - 1 \leq r - (s + 1)$. Thus $d(\mathbb{D}'') \leq r - (s + 1)$ a.a. Hence $H_x \Rightarrow H_{s+1}$. Consequently

$$d(\mathbb{D}) \leq q - |\mathbb{D}| \text{ a.a.}$$

Q.E.D.

Note that if $\mathbb{D} \subset \mathbb{D}_r'$ and there is a subfamily $\mathbb{D}'$ of $\mathbb{D}$, where $\mathbb{D}'$ satisfies the intersection property (even though $\mathbb{D}$ might not) then immediately by Theorem 4 we see that

$$d(\mathbb{D}) \leq d(\mathbb{D}') \leq r - |\mathbb{D}'|.$$

By a similar method to the proof of Theorem 4 it is possible to show that for any family of subsets of N, that if $\dim(W) \geq n - 1$ then

$$d(\mathbb{D}) \leq k(\mathbb{D}) - 1 \text{ a.a.}$$

See Schofield (1980). These two results then give the following corollary.

Corollary 1

Let W be a smooth manifold of dimension w.

(i) If $w \geq r$ and $r \leq n - 1$ then

$$\Lambda(W, \mathbb{D}_r, u) = \Phi \text{ a.a.}$$

where as before $\mathbb{D}_r = \{M \subset N : |M| \geq r\}$.

(ii) If $w \geq n - 1$ then $\Lambda(W, \mathbb{D}, u) = \Phi$ a.a. for any non-collegial family $\mathbb{D}$ of subsets of N.

Proof

(i) Since $r \leq n - 1$ there exists some subset R of N with $|R| = r + 1$. Let

$$\mathbb{D}_r' = \{M \subset R : |M| = r\}.$$

Clearly $|\mathbb{D}_r'| = r + 1$. Moreover $\mathbb{D}_r' \subset \mathbb{D}_r$ and so

$$\Lambda(W, \mathbb{D}_r, u) \subset \Lambda(W, \mathbb{D}_r', u).$$

But $\mathbb{D}_r'$ satisfies the intersection property, and so by Theorem 4,

$$d(\mathbb{D}_r') \leq r - (r + 1) < 0 \text{ a.a.}$$

Thus

$$\Lambda(W, \mathbb{D}_r', u) = \Phi \text{ a.a.}$$

and hence

$$\Lambda(W, \mathbb{D}_r, u) = \Phi \text{ a.a.}$$

(ii) Since $\mathbb{D}$ is non-collegial $k(\mathbb{D}) = 0$ and thus by Schofield (1980)



$$\dim[\Lambda(W, \mathbb{D}, u)] = < 0 \text{ a.a.}$$

or

$$\Lambda(W, \mathbb{D}, u) = \Phi \text{ a.a.}$$

Q.E.D.

Corollary 1(i) essentially shows that Theorem 1 is valid with the instability dimension $w^*(\sigma)$ bounded above by q for the case of a q-rule. We now sharpen this result by showing that indeed $w^*(\sigma)$ is bounded above by $2q - n + 1$. For the voting rule, $\sigma$, define the partition $\{IO_r(\sigma, W, N, u)\}_{r=1}^{n}$ of $IO(\sigma, W, N, u)$ by $x \in IO_r(\sigma, W, N, u)$ iff $x \in IO(\sigma, W, N, u)$ and $|\{i \in N : p_i(x) = 0\}| = r$. Suppose now that $x \in IO_r(\sigma, W, N, u)$ for $r \geq 2$. Then $x \in \Lambda(W, \mathbb{D}, u)$ where $\mathbb{D} = \{\{i\}, \{j\}\}$ for $i \neq j$. By Corollary 1, if $\dim(W) \geq 1$ then $d(\mathbb{D}) < 0$ a.a. and so $IO_r(\sigma, W, N, u) = \Phi$ a.a. Thus we see that

$$IO(\sigma, W, N, u) = IO_0(\sigma, W, N, u) \cup IO_1(\sigma, W, N, u) \text{ a.a.}$$

Corollary 2

Let $\sigma$ be a q-rule, and let W be a smooth manifold of dimension w, without boundary. If $2q - n + 1 \leq w \leq n - 2$ then $IO(\sigma, W, N, u)$ is almost always empty.

Proof

Since $\sigma$ is a q-rule, we assume $n/2 < q \leq n - 1$, with $n \geq 3$. If $q = n - 1$ then $2q - n + 1 = n - 1$, so we may assume $q \leq n - 2$. Let

$r = 2q - n + 1$, so that $2 \leq r \leq n - 3$. Suppose that $x \in IO(\sigma,W,N,u)$. By Corollary 1,

$$x \in IO_0(\sigma,W,N,u) \cup IO_1(\sigma,W,N,u) \text{ a.a.}$$

There are two cases to consider:

(i) $x \in IO_1(\sigma,W,N,u)$. Suppose, without loss of generality, that $p_n(x) = 0$. Define $\mathbb{D}_r' = \{M \subset N\backslash\{n\} : |M| = r\}$. Since $r \leq n - 3$, there exists a coalition $R \subset N\backslash\{n\}$ with $|R| = r + 1$. Let $\mathbb{D}_r'' = \{M \subset R : |M| = r\}$. Clearly $\mathbb{D}_r''$ satisfies the intersection property and $\mathbb{D}_r'' \subset \mathbb{D}_r'$. Moreover, $|\mathbb{D}_r''| = r + 1$. Just as in Corollary 1(i) therefore

$$\Lambda(W,\mathbb{D}_r',u) \subset \Lambda(W,\mathbb{D}_r'',u) = \Phi \text{ a.a.}$$

Hence it is almost always the case that at each point $x \in W$ there exists some subset $V$ of $N\backslash\{n\}$, with $|V| = r$, such that $\{p_i(x) : i \in V\}$ are linearly independent. Now define $\mathbb{D} = \{M \subset V : |M| = r - 1\}$. By Lemma 2, $r - 1 = 2q - n = e(n,q)$, and so $\mathbb{D} \subset E'(\mathbb{D}_\sigma)$ where $\mathbb{D}_\sigma$ is the family of $\sigma$-decisive coalitions. By Theorem 3, for each $M \in \mathbb{D}$ there exist $j(M) \in (N\backslash\{n\})\backslash M$ such that $\{p_i(x) : i \in M \cup \{j(M)\}\}$ are linearly dependent. Let $M' = M \cup \{j(M)\}$, and define $\mathbb{D}' = \{M' : M' = M \cup \{j(M)\} \text{ and } M \in \mathbb{D}\}$. Thus $x \in \Lambda(W,\mathbb{D}',u)$. Note first of all that $|M'| = r$ for each



$M' \in \mathbb{D}'$ and that $|\mathbb{D}'| = r$. Since the vectors $\{p_i(x) : i \in V\}$ are linearly independent it is evident that $\mathbb{D}$ and thus $\mathbb{D}'$ satisfy the intersection property. By Theorem 4

$$\dim[\Lambda(W,\mathbb{D}',u)] \leq r - |\mathbb{D}'| = 0, \text{ a.a.}$$

But $x \in IO_1(\sigma,W,N,u)$ and so

$$x \in \Lambda(W,\{n\},u) \cap \Lambda(W,\mathbb{D}',u).$$

By ST(ii) it is almost always the case that

$$\dim[\Lambda(W,\{n\},u) \cap \Lambda(W,\mathbb{D}',u)]$$
$$\leq (1 - 1) + 0 - w < 0 \quad \text{since} \quad w \geq r \geq 2.$$

Hence $IO_1(\sigma,W,N,u) = \Phi$ a.a.

(ii) $x \in IO_0(\sigma,W,N,u)$. Assume therefore that $p_i(x) \neq 0$, $\forall i \in N$. Again without loss of generality assume that there exists a subset $V \subset N$ with $|V| = r$ such that $\{p_i(x) : i \in V\}$ are linearly independent. Define

$$\mathbb{D} = \{M \subset V : |M| = r - 2\}$$

and proceed as in (i). Again by Theorem 3, $\mathbb{D} \subset E(\mathbb{D}_\sigma)$, and so for each $M \in \mathbb{D}$ there exists $j(M) \in N\backslash V$ such that $\{p_i(x) : i \in M \cup \{j(M)\}\}$ are linearly dependent. Note that if $r = 2$ then this implies that $p_i(x) = 0$, for some $i \in N$, contradicting the assumption that $x \in IO_0(\sigma,W,N,u)$. We may therefore assume that $r \geq 3$. Define

$$\mathbb{D}' = \{M' : M' = M \cup \{j(M)\} \text{ for } M \in \mathbb{D}\}.$$

Clearly $x \in \Lambda(W,\mathbb{D}',u)$. But again both $\mathbb{D}$ and $\mathbb{D}'$ satisfy the intersection property, and hence by Theorem 4,

$$\dim[\Lambda(W,\mathbb{D}',u)] \leq r - |\mathbb{D}'| \text{ a.a.}$$

Since $|V| = r$, $|\mathbb{D}'| = |\mathbb{D}| = C(r,r-2)$ the number of combinations of $(r - 2)$ objects from $r$ objects. Moreover, $r \geq 3$ implies that $|\mathbb{D}'| \geq r$. This implies that $\Lambda(W,\mathbb{D}',u) = \Phi$ a.a. Thus $IO_0(\sigma,W,N,u) = \Phi$ a.a.

Q.E.D.

<u>Corollary 3</u>

Let $\sigma$ be a q-rule. Let $w(\sigma) = 2q - n + 1$. Then

$$\text{Int } W \cap IO(\sigma,W,N,u) = \Phi \text{ a.a.}$$

whenever $W$ is a smooth manifold of dimension $w \geq w(\sigma)$.

<u>Proof</u>

Suppose first of all that the boundary of $W$ is empty. If $\dim(W) \geq n - 1$, then Corollary 1 applies, since $\sigma$ is non-collegial, to give $IO(\sigma,W,N,u) = \Phi$ a.a. If $2q - n + 1 \leq \dim(W) \leq n - 2$, then Corollary 2 applies to give the same result. Hence $\dim(W) \geq 2q - n + 1$ implies $IO(\sigma,W,N,u) = \Phi$ a.a. If $W$ has a non-empty boundary, then $\dim(W) \geq 2q - n + 1$ implies $IO(\sigma,\text{Int } W,N,u/\text{Int } W) = \Phi$



a.a. and so $\text{Int } W \cap IO(\sigma,W,N,u) = \Phi$ a.a.

Q.E.D.

<u>Proof of Theorem 1</u>

By Corollary 3, we see that part (i) of Theorem 1 is valid wherever $\dim(W) \geq w(\sigma)$. Define $w^*(\sigma)$ to be the smallest integer with $w^*(\sigma) \leq w(\sigma)$ such that Theorem 1(i) is valid. To see that $w^*(\sigma) \geq 2$ proceed as follows.

As we have noted, by Schofield (1984d), if $\dim W \leq s(n,q)$ then

$$O(\sigma,W) = \{u \in U(W)^N : IO(\sigma,W,N,u) \neq \Phi\}$$

has a non-empty interior in $U(W)^N$. By Lemma 2, for any $q \in (n/2,n-1)$,

$$s(n,q) < 2q - n + 1.$$

Thus $w^*(\sigma)$ must satisfy

$$s(n,q) < w^*(\sigma) \leq 2q - n + 1.$$

In particular, $s(n,q) \geq 1$ and so $w^*(\sigma) \geq 2$.

To prove part (ii), suppose that $W$ is a smooth manifold without boundary. Then $W = \text{Int } W$, and the result follows by part(i). On the other hand if $W$ is a smooth manifold with boundary then $W = \text{Int } W \cup \partial W$, where $\partial W$, the boundary of $W$, is itself a smooth manifold of dimension $\dim(W) - 1$ without boundary. Since $\dim(W) - 1 \geq w^*(\sigma)$ by assumption, then by part (i)

$$U_1 = \{u \in U(W)^N : IO(\sigma,\partial W,N,u/\partial W) = \Phi\}$$

is residual, where $u/\partial W$ is the smooth profile restricted to $\partial W$. Again by part (i)

$$U_2 = \{u \in U(W)^N : IO(\sigma,\text{Int } W,N,u/\text{Int } W) = \Phi\}$$

is residual where $u/\text{Int } W$ is the smooth profile restricted to Int W. But

$$U_1 \cap U_2 = \{u \in U(W)^N : IO(\sigma,W,N,u) = \Phi\}$$

is itself residual whenever $\dim(W) \geq w^*(\sigma) + 1$. This proves the result.

Q.E.D.

It is as yet an open question whether $w^*(\sigma) = w(\sigma)$ or $w^*(\sigma < w(\sigma)$. This turns on whether or not a structurally stable core can exist in dimension $2q - n$.

## 6. GENERIC DENSENESS OF THE CYCLE SET

To prove Theorem 2 we make use of a procedure introduced in Schofield (1983a).

<u>Lemma 4</u>

Let $\sigma$ be a q-rule on a smooth manifold W, without boundary, of dimension $w = w(\sigma) + 1$. Then there exists a residual set $U_1$ in $U(W)^N$ such that for any $u \in U_1$,



$$W\backslash IC(\sigma,W,N,u) \subset \bigcup_r S_r(u)$$

where each $S_r(u)$ is a subset of the singularity manifold associated with at most $w(\sigma) + 1$ individuals, and of dimension at most $w(\sigma)$.

<u>Proof</u>

There exists a residual set $A \subset U(W)^N$, such that for each $u \in A$,

(i) for each $i \in N$, $\Lambda(W,\{i\},u)$ is zero dimensional

(ii) for each pair $\{i,j\}$, $\Lambda(W,\{i,j\},u)$ is of dimension one.

Since $w(\sigma) + 1 \geq 3$, the set

$$V(u) = W\backslash \bigcup \Lambda(W,\{i\},u)\backslash \bigcup_{i,j} \Lambda(W,\{i,j\},u)$$

is open dense in W. Moreover, if $x \in V(u)$ and $p_\sigma(x) \neq \Phi$, then for any $v \in p_\sigma(x)$ it is the case that $p_k(x) = \lambda v$, for $\lambda \in \mathbb{R}$, for at most one $k \in N$. Suppose therefore that $x \in V(u)\backslash IC(\sigma,W,N,u)$. By definition $p_\sigma(x) \neq \Phi$. Choose $v \in p_\sigma(x)$. By definition, for all $M \in \mathbb{D}_\sigma$, either $0 \in p_M(x)$ or $v \in p_M(x)$. Introduce a new player, labelled $(n + 1)$ with $p_{n+1}(x) = -v$. Define a new voting rule $\sigma'$ whose decisive coalitions, $\mathbb{D}_{\sigma'}$, are given by $M \in \mathbb{D}_\sigma$ iff $M \cup \{n + 1\} \in \mathbb{D}_{\sigma'}$. Clearly if $M \in E(\mathbb{D}_\sigma)$ then $M \cup \{n + 1\} \in E(\mathbb{D}_{\sigma'})$. Moreover, $v \in p_\sigma(x)$ iff $0 \in p_M(x)$ for all $M \in \mathbb{D}_{\sigma'}$. Thus $v \in p_\sigma(x)$ iff $\{p_1(x),\ldots,p_{n+1}(x)\}$ is a $\sigma'$-equilibrium. Just as in the proof of Corollary 2, define the family $\mathbb{D}'$ of subsets of $N \cup \{n + 1\}$ by $M' \in \mathbb{D}'$ iff $M' = M \cup j(M)$ where $|M| = 2q - n + 1$, $j(M) \notin M$ and $\{p_i(x) : i \in M'\}$ is a linearly dependent set. Since $x \in V(u)$ it is the case that $p_i(x) = 0$ for no $i = 1,\dots,n + 1$. From Theorem 3, it follows that

$$x \in \Lambda(W,\mathbb{D}',u).$$

The first possibility is that for some $M' \in \mathbb{D}'$ it is the case that $(n + 1) \notin M'$. Since $|M'| = 2q - n + 2$, it is almost always the case that x must belong to a singularity set of dimension $e'(n,q) + 1 = w(\sigma)$ which was to be proved. The second possibility is that for each $M' \in \mathbb{D}'$ it is the case that $(n + 1) \in M'$.

Since we assume that $q \leq n - 1$, then $2q - n + 2 \leq n$. Thus there exist $M_1', M_2' \in \mathbb{D}'$ with $M_1' = \{n + 1\} \cup M_1$, $M_2' = \{n + 1\} \cup M_2$ and $|M_1 \cup M_2| = 2q - n + 2$. Thus $v \in \mathrm{Span}(\{p_i(x) : i \in M_1\}) \cap \mathrm{Span}(\{p_i(x) : i \in M_2\})$. But then $0 \in \mathrm{Span}(\{p_i(x) : i \in M_1 \cup M_2\})$. Hence $x \in \Lambda(W,M_1 \cup M_2,u)$. Since $|M_1 \cup M_2| = w(\sigma) + 1$, it is almost always the case that x belongs to a singularity manifold of dimension at most $w(\sigma)$.

Q.E.D.

<u>Proof of Theorem 2</u>

(i) Let $\dim(W) \geq w(\sigma) + 1$. By Lemma 4, there is a residual set $U_1$ s.t. if $u \in U_1$ and $x \in \mathrm{Int}\, W \setminus IC(\sigma,W,N,u)$ then x belongs to a singularity manifold $S_r(u)$ of dimension at most $w(\sigma)$. By Milnor (1958), $S_r(u)$ is of measure zero, and moreover $\mathrm{Int}\, W \setminus S_r(u)$ is open dense in Int W. Since N is finite, there can only be a finite number of singularity manifolds.



Thus, it is almost always the case that $IC(\sigma,W,N,u)$ is open dense and can be partitioned into a finite number of path connected components. This proves Theorem 2(i).

(ii) Assume $\dim(W) \geq w(\sigma) + 2$. It is almost always the case that $IC(\sigma,W,N,u)$ is open dense, by (i). But in this dimension range, an open subset of W cannot be separated by submanifolds of dimension $w(\sigma)$. Thus $IC(\sigma,W,N,u)$ almost always consists of a single open dense path connected component.

Q.E.D.

## 7. CONCLUSION

The results presented here indicate how to compute $w^*(\sigma)$ for an arbitrary non-collegial voting rule. Let $\mathbb{D}_\sigma$ be the decisive coalitions and $E'(\mathbb{D}_\sigma)$ the family of pivotal subgroups. Let $\mathbb{D}$ be any subfamily of $E'(\mathbb{D}_\sigma)$, and define

$$m(\mathbb{D}) = \max\{|M| : M \in \mathbb{D}\}.$$

Suppose that $K(\mathbb{D}) = \Phi$ and that $m(\mathbb{D}) + 1 \leq |\mathbb{D}|$. Then precisely as in the proof of Corollary 2, it is the case that if $\dim(W) \geq m(\mathbb{D}) + 1$ then

$$\dim[\Lambda(W,\mathbb{D}',u)] \leq m(\mathbb{D}) + 1 - |\mathbb{D}| \leq 0 \text{ a.a.}$$

where $\mathbb{D}' = \{M \cup \{j(M)\} : M \in \mathbb{D}\}$.

This can be used to show, just as in the proof of Corollary 2, that the optima set is empty a.a. Thus

$$w^*(\sigma) \le \min_{\mathbb{D}}\{m(\mathbb{D})\} + 1$$

where $\mathbb{D}$ can be any subfamily of $E'(\mathbb{D}_\sigma)$ satisfying

(i) $\mathcal{K}(\mathbb{D}) = \Phi$ and (ii) $m(\mathbb{D}) + 1 \le |\mathbb{D}|$.



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