DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

SINCERE VOTING IN MODELS OF LEGISLATIVE ELECTIONS*

David Austen-Smith
University of Rochester and
California Institute of Technology

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Abstract

An assumption of sincere voting for one's most preferred candidate is frequently invoked in models of electoral competition in which the elected legislature consists of more than a single candidate or party. Voters, however, have preferences over policy outcomes -- which are determined by the ex post elected legislature -- and not over candidates per se. This observation provokes the following question. For what methods of translating election results into legislative policy outcomes is sincere voting rational in the legislative election? This paper provides the answer. One of the principal implications is that for sincerity to be rational, there necessarily exists a candidate for office whose electoral platform is the final legislative outcome, <u>whether or not that candidate is elected to the legislature</u>.

Introduction

Two papers published recently in the Review have developed formal models of multi-candidate electoral competition in which the elected legislature consists of more than a single candidate or party (Greenberg and Weber, 1985; Greenberg and Shepsle, 1987). Greenberg and Weber analyse the fixed-standard method. In this scheme, there is a prespecified number of votes, m, such that the legislature consists of all those candidates winning at least m votes in the election. Greenberg and Shepsle examine the fixed-number method. Here, the size of the legislature, K, is predetermined, and these seats go to the candidates with the K-highest number of votes. Both papers share the following four assumptions:

(A.1) Candidates are identified with possible outcomes;

(A.2) The set of possible outcomes is isomorphic to a compact subset of the real line;

(A.3) Individual voters have single-peaked preferences over the set of possible outcomes;

(A.4) Given a set of proposed alternatives, each individual votes for his or her most preferred alternative in that set.

For our purposes, the first three assumptions are harmless; the last is not. (A.3) implies that individuals care about outcomes and not candidates per se, but the mechanism by which final outcomes are generated from any elected legislature is left unspecified. To be fair, Greenberg and Weber informally defend (A.4) by assuming "that all of the elected candidates form a committee (or the cabinet) with each having one vote and decisions are made according to some majority rule" (p.698). Although this is a well-specified outcome function, it nevertheless turns out (as we shall see) to be problematic for their model. Greenberg and Shepsle offer no defense at all.

By (A.3), individuals are interested in legislative outcomes and not in legislative composition for its own sake. If individuals are presumed rational, then voting behavior will be directed toward promoting their most favored outcome. Call any mechanism which specifies a final policy outcome, conditional on the policy positions of the elected candidates and, possibly, on individuals' electoral voting behavior, a legislative outcome function (LOF). Then it is natural to ask: "Given individual voters are rational and assuming (A.1) - (A.3), for what class of legislative outcome functions does

(A.4) necessarily constitute rational behavior?". In the next section, this question is made precise and answered.

The implications of the answer for the models of Greenberg et.al. -- and others which assume sincere voting over lists of candidates for multi-member legislatures (e.g. Sugden, 1984) -- are striking. The main result (loosely stated) is as follows. Suppose the election scheme is defined for all possible sets of candidates, nontrivial (i.e. not all candidates running for office necessarily get elected), and anonymous (i.e. does not depend on the names of candidates). (The fixed-standard and the fixed-number schemes, for example, satisfy these criteria.) Then there exists no anonymous LOF such that sincere voting constitutes rational behavior at the election stage.[1]

In other words, for the sincere voting assumption to be justified on rationality grounds, legislative outcomes must depend on the entire list of candidate platforms and electoral votes, and not simply on the platforms of the candidates elected to office. In particular, there must exist a candidate for office, c, whose electoral platform is the final legislative outcome <u>whether or not c is elected to the legislature</u>. Thus the LOF described by Greenberg and Weber is insufficient to defend the assumption of sincere voting successfully.

Model and Results

Let $N = \{1, \dots, n\}$ be the finite set of voters, and let the set of possible outcomes be X. Assume X is a closed interval [(A.2)]. Let **U** be the set of symmetric and strictly single-peaked preference orderings (i.e. no flat spots) on X [(A.3)]. For any $U_i \in \mathbf{U}$, let $x_i = \text{argmax.}_X U_i(\cdot)$: x_i is individual i's ideal point in X. A preference profile for N can then be summarized by a list $\mathbf{x} = (x_i)_N \in X^n$. Individuals' preferences are common knowledge. Let $C = \{y_1, \dots, y_c\} \in \mathbb{C}$ denote an arbitrary set of candidates, where $\mathbb{C}$ is the set of finite subsets of X [(A.1)]. For any $i \in N$ and set of candidates C, i's (pure) voting strategy is a function:

$$\sigma_i : X \times C \rightarrow C,$$

where $\sigma_i(x_i, C) = y_k$ means individual i, with ideal point x_i in X, casts his vote for candidate $y_k \in C$.

Definition: Individual $i \in N$ votes sincerely with respect to C if and only if:

$\sigma_i(x_i, C) = y_k \Rightarrow \sim[\exists\, y_{k'} \in C\backslash\{y_k\} \mid U_i(y_{k'}) > U_i(y_k)]$.

Let $\sigma_i^T(x_i, C)$ denote i's sincere voting strategy with respect to C.

Let $\sigma(\mathbf{x}, C) = (\sigma_1(x_1, C), \ldots, \sigma_n(x_n, C))$ denote an arbitrary list of voter strategies (given the profile $\mathbf{x}$ and the set C), and let $\sigma^T(\mathbf{x}, C) = (\sigma_1^T(x_1, C), \ldots, \sigma_n^T(x_n, C))$. Where there is no danger of ambiguity, the dependency of σ etc. on $\mathbf{x}$ and C, will be supressed.

An election rule for C is a mapping,

ε_C: $C^n \rightarrow 2^C\backslash\varnothing$.

The interpretation here is that an election rule takes the respective votes for candidates, and defines which candidates are elected. Let $v_k(\sigma) = |\{i \in N \mid \sigma_i = y_k\}|$. Then, for example, under the fixed-standard method, $\varepsilon_C = \varepsilon_C^{FS}$, studied in Greenberg and Weber,

$\varepsilon_C^{FS}(\sigma) = \{y_k \in C \mid v_k(\sigma) \geq m > 0\}$.

And the fixed-number method, $\varepsilon_C = \varepsilon_C^{FN}$, defined in Greenberg and Shepsle is,

$\varepsilon_C^{FN}(\sigma) = \{y_k \in C \mid v_k(\sigma) \leq v_{k'}(\sigma)$ for at most K-1 candidates $y_{k'} \in C\backslash\{y_k\}\}$.

An election rule ε_C is nontrivial if and only if $\exists \sigma \in C^n$, $\exists y_k \in C$ such that $v_k(\sigma) > 0$ and $y_k \notin \varepsilon_C(\sigma)$. If $\forall C$, $\forall \sigma$, $y_k \in \varepsilon_C(\sigma)$ implies $v_k(\sigma) > 0$, then say that ε_C is E-efficient. If ε_C is symmetric with respect to voters, then ε_C is anonymous. Let $\Sigma = \{\varepsilon_C \mid \varepsilon_C$ is nontrivial, E-efficient and anonymous$\}$. Both ε_C^{FS} (with $m > 1$) and ε_C^{FN} (with $K < n$) are members of Σ.

Call any set $C^* \in 2^C\backslash\varnothing$ determined via an election rule ε_C, a legislature.

Given a set C, a legislative outcome function for C (LOF) is a mapping,

λ_C: $(2^C\backslash\varnothing) \times C^n \rightarrow X$.

For every possible legislature elected from C, the LOF defines the legislative policy outcome. This outcome is not restricted a priori to lie in C. For example, the final outcome may be some weighted average of the elected candidates' platforms. And notice that we allow the LOF to depend on voter strategies as well as on the positions of the elected set of candidates. This, for example, permits successful candidates' vote-shares to matter in legislative decision-making. Of course, the LOF may be constant across $\sigma \in C^n$ for any given $C^* \in 2^C \backslash \varnothing$.

Say λ_C is anonymous if it is symmetric with respect to both candidates and voters. λ_C is L-efficient if $\forall C^* \in 2^C \backslash \varnothing$, $\forall \sigma \in C^n$, $\lambda_C(C^*, \sigma) \in$ co.(C^*), where co.(H) is the convex hull of H (a subset of $\mathbf{R}$). Let $\Lambda = \{\lambda_C \mid \lambda_C$ is L-efficient and anonymous$\}$. In the current setting, restricting attention to Λ is natural (especially in view of (A.1)). For example, the simple majoritarian rule suggested by Greenberg and Weber is,

$$\lambda_C(C^*, \sigma) = \text{median } \{y_k \mid y_k \in C^*\},$$

which is clearly L-efficient and anonymous.

Given a finite set of candidates C in X, a LOF λ_C, an election rule ε_C, and a vector of voting strategies σ, the final legislative outcome is given by $\lambda_C(\varepsilon_C(\sigma), \sigma) \in X$. Define the mapping,

$$\gamma_C : C^n \rightarrow X$$

by setting $\gamma_C(\sigma) = \lambda_C(\varepsilon_C(\sigma), \sigma)$, $\forall \sigma \in C^n$. If $\varepsilon_C \in \Sigma$ and $\lambda_C \in \Lambda$ then γ_C is anonymous (i.e. symmetric with respect to voters), and efficient (i.e. $\gamma_C(\sigma) = y$ implies $y \in$ co.(C)).

For any individual i, $U_i(y)$ is i's payoff from outcome $y \in X$. Given C and γ_C, define i's indirect utility by,

$$u_i(\sigma) \equiv U_i(\gamma_C(\sigma)), \; i \in N.$$

Write $\sigma_{-i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ... , \sigma_n)$.

Definition: The sincere voting strategy σ_i^T is weakly dominant under γ_C for i iff,

$$u_i(\sigma_i^T, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}), \; \forall \sigma_i \neq \sigma_i^T, \forall \sigma_{-i}.$$

If for all profiles $\mathbf{x} \in X^n$, $\sigma_i^T(x_i, C)$ is weakly dominant under γ_C for all individuals $i \in N$, say that γ_C is straightforward.

With the above framework, the question posed in the Introduction can be stated precisely. Given an arbitrary set of candidates C and an election rule $\varepsilon_C \in \Sigma$, for what subset $\Lambda(\varepsilon_C)$ of Λ is the following true:

$\lambda \in \Lambda(\varepsilon_C) \Leftrightarrow \gamma_C$ is straightforward and $\gamma_C(\cdot) = \lambda_C(\varepsilon_C(\cdot), \cdot)$?

A first step toward the answer is given by Proposition 1. The argument for this result, given in the Appendix, is due to Kim Border.

Proposition 1: Assume $\varepsilon_C \in \Sigma$, $\lambda_C \in \Lambda$, and $\gamma_C(\cdot) = \lambda_C(\varepsilon_C(\cdot), \cdot)$. Suppose γ_C is straightforward. Then, $\forall \sigma(\mathbf{x}, C) \in C^n$, $\gamma_C(\sigma(\mathbf{x}, C)) \in C$.

Suppose γ_C is defined as in Proposition 1. Then the result says that if γ_C is straightforward, we can without loss of generality write, $\gamma_C : C^n \to C$. In other words, under anonymity and efficiency, for sincere voting [(A.4)] to be rational for all individuals at the electoral stage, the LOF must select an election platform offered by one of the candidates. So, for example, either bargaining between elected candidates, leading to a compromise policy outcome lying between some pair of electoral platforms, must be ruled out or sincere voting is not rational.

In view of Proposition 1, we can now apply a theorem of Moulin [see also Border and Jordan (1983)]. For the framework developed above, the result is:[2]

Theorem [Moulin, 1980]: Assume $\varepsilon_C \in \Sigma$, $\lambda_C \in \Lambda$, and $\gamma_C(\cdot) = \lambda_C(\varepsilon_C(\cdot), \cdot)$. Then the following two statements are equivalent:

(1) γ_C is straightforward;

(2) $\exists$ (n-1) real numbers $a^c{}_1, \dots, a^c{}_{n-1} \in C \cup \{-\infty, +\infty\}$ such that, $\forall \sigma(\mathbf{x}, C) \in C^n$,

$$\gamma_C(\sigma(\mathbf{x}, C)) = \text{median}\{\sigma_1, \dots, \sigma_n, a^c{}_1, \dots, a^c{}_{n-1}\}.$$

Therefore, if γ_C is defined as in Proposition 1 and is straightforward, then the composition of the election rule and the LOF must reduce to a rule based on an order statistic.

Typically, election rules and legislative outcome functions are defined independently of the list of candidates competing for legislative office. This is certainly true for those rules studied by Greenberg et.al., and for many others. The only case (of which I am aware) in which such independence might be violated is the practice of using plurality voting for pairwise contests, but some other method (e.g. Borda rule) in multi-candidate contests. (Of course, the composition of any set of candidates for legislative office will depend on which particular election rules etc. are in force.) Appropriate notions of candidate-independence for election rules and LOFs can be defined implicitly through γ_C.

Definition: Let $\gamma_C : C^n \to X$, and let $\mathbf{C}^*$ be an arbitrary subset of $\mathbf{C}$. Then, γ_C is $\mathbf{C}^*$-independent if and only if, $\forall(\mathbf{x}, C) \in X^n \times \mathbf{C}^*$, $\gamma_C(\sigma(\mathbf{x}, C)) = \gamma(\sigma(\mathbf{x}, C))$.

Proposition 2: Assume $\forall C \in \mathbf{C}$: $\varepsilon_C \in \Sigma$, $\lambda_C \in \Lambda$, $\gamma_C(\cdot) = \lambda_C(\varepsilon_C(\cdot), \cdot)$, and γ_C is straightforward. If γ_C is $\mathbf{C}^*$-independent for some $\mathbf{C}^*$ in $\mathbf{C}$, then $\exists C^o = \{c_1, \dots, c_{n-1}\}$ in $\mathbf{R} \cup \{-\infty, +\infty\}$ such that:

(1) $C \in \mathbf{C}^* \Rightarrow C \supseteq (C^o \cap \mathbf{R})$,

(2) $\forall(\mathbf{x}, C) \in X^n \times \mathbf{C}^*$, $\gamma_C(\sigma(\mathbf{x}, C)) = \text{median}\{\sigma_1, \dots, \sigma_n, c_1, \dots, c_{n-1}\}$.

Proof: By Proposition 1 and Moulin's Theorem, for each C, $\exists\, a^c{}_1, \dots, a^c{}_{n-1} \in C \cup \{-\infty, +\infty\}$ such that, $\forall \sigma(\mathbf{x}, C) \in C^n$, $\gamma_C(\sigma(\mathbf{x}, C)) = \text{median}\{\sigma_1, \dots, \sigma_n, a^c{}_1, \dots, a^c{}_{n-1}\}$. By $\mathbf{C}^*$-independence, the (n-1) real numbers, $a^c{}_1, \dots, a^c{}_{n-1}$, must be independent of $C \in \mathbf{C}^*$. The result follows. ||

This result says that the only election rules and LOFs that are both candidate-independent on $\mathbf{C}^*$ and lead to rational sincere voting, must involve a (possibly empty) set of "phantom candidates" who always compete and who are endowed with at least one "vote", irrespective of the voting strategies of individuals in N. For example, suppose $|C^o \cap \mathbf{R}| = 1$. Then under the assumptions of Proposition 2, there must exist an alternative $y_o \in X$ -- the status quo, for example -- such that (1) any admissible set of candidates C in an election includes y_o, and (2) y_o is not the final legislative outcome if and only if there is a distinct candidate y_k in C such that all individuals prefer y_k to y_o.

In general, however, admissible sets of candidates do not include such predetermined and especially favored alternatives. This observation motivates the main result of the paper, a straightforward consequence of Proposition 2.

Corollary 1: Assume $\gamma_C(\cdot) = \lambda_C(\varepsilon_C(\cdot), \cdot)$ and, $\forall C \in \mathbf{C}$, $\varepsilon_C \in \Sigma$, $\lambda_C \in \Lambda$. Then γ_C cannot be both straightforward and $\mathbf{C}$-independent (i.e. $\mathbf{C}^*$-independent with $\mathbf{C}^* = \mathbf{C}$).

Proof: By Proposition 2, if γ_C is both straightforward and $\mathbf{C}$-independent, then $C^o \cap \mathbf{R} = \varnothing$. Therefore, $c_t \in \{-\infty, +\infty\}$, all $t = 1, \ldots, n\text{-}1$. Since ε_C is nontrivial, $\exists(\mathbf{x}, C) \in X^n \times \mathbf{C}$ such that $|\varepsilon_C(\sigma(\mathbf{x}, C))| \equiv |C^*| < |C|$. By Proposition 1, $\lambda_C(\varepsilon_C(\cdot), \cdot) \in C$. In particular, because λ_C is L-efficient, $\lambda_C(C^*, \sigma(\mathbf{x}, C)) \in C^*$. However, $\forall \varepsilon_C \in \Sigma$ we can choose $(\mathbf{x}, C)$ so that:

$$\text{median}\{\sigma_1, \ldots, \sigma_n, c_1, \ldots, c_{n-1}\} \in C \backslash C^*.$$

But by Proposition 2, $\gamma_C(\cdot) = \text{median}\{\sigma_1, \ldots, \sigma_n, c_1, \ldots, c_{n-1}\}$: a contradiction. ||

So, in answer to the original question, Proposition 2 shows that the subset of LOFs, $\Lambda(\varepsilon_C)$, for which it is true that,

$$\lambda \in \Lambda(\varepsilon_C) \Leftrightarrow \gamma_C \text{ is straightforward and } \gamma_C(\cdot) = \lambda_C(\varepsilon_C(\cdot), \cdot),$$

is empty when $\varepsilon_C \in \Sigma$, $\lambda_C \in \Lambda$. Given anonymity, efficiency, and candidate-independence, this result implies that if sincere voting is rational at the election stage, then <u>either</u> the election rule must be trivial (i.e. $\forall C \in \mathbf{C}$, $\forall \sigma \in C^n$, $\varepsilon_C(\sigma) = \{y_k \in C \mid v_k(\sigma) > 0\}$), <u>or</u> the LOF must be defined on the entire list of candidate platforms, C, and not only on the positions of the ex post elected candidates, C*. Hence, if the election rule is nontrivial and sincere voting is rational, it is possible, as claimed in the Introduction, for the final policy outcome to be the electoral platform of a candidate who is not elected to the legislature. (Example 2, in the next section, illustrates this possibility.)

In view of these observations, Corollary 1 can alternatively be expressed as a possibility result. For any $C \in \mathbf{C}$, define $\gamma_C^* : C^n \to X$, and suppose γ_C^* is anonymous (i.e. symmetric with respect to voters), and efficient (i.e. $\gamma_C^*(\sigma) = y$ implies $y \in \text{co.}(C)$). Then, applying earlier results:

Corollary 2: [γ_C* straightforward and $\mathbf{C}$-independent] $\Leftrightarrow \exists$ an order statistic ρ on N such that, $\forall(\mathbf{x}, C) \in X^n \times \mathbf{C}$,

$$U_{i(\rho)}(y_k) > U_{i(\rho)}(y_{k'}), \forall y_{k'} \in C\backslash\{y_k\} \;\Rightarrow\; y_k = \gamma_C{}^*(\sigma(\mathbf{x}, C)),$$

where $i(\rho) \in N$ is the individual with the $\rho^{th.}$-ranked ideal point.

Under the premise of Corollary 2 there are no predetermined candidates such as the status quo. In this case (given anonymity, efficiency, and $\mathbf{C}$-independence), if sincere voting is invariably a rational strategy for individuals, there must exist some individual i* -- identified by the relative position of his ideal point x_{i^*} (e.g. the median voter) -- such that if any candidate y_k adopts $y_k = x_{i^*}$ as her electoral platform, then x_{i^*} will <u>necessarily</u> be the final policy outcome, <u>whether or not</u> y_k <u>is elected to the legislature</u>. Once x_{i^*} is adopted by some candidate for office then, ceteris paribus, all voters are indifferent over all possible compositions of the legislature. So for other candidates to have an incentive to enter the election, their payoffs must depend on factors other than influencing the legislative outcome. Specifically, being an elected member of the legislature per se must be of value.

Two Examples

The following examples illustrate the main points of the previous discussion. Let N = {1, ... , 7} and X = [0, 1]. Assume the election rule is the fixed-number method, $\varepsilon_C = \varepsilon_C{}^{FN}$, with the size of the legislature K set equal to 2. The LOF is: the final outcome is a weighted average of the two elected candidates' positions, with the weights being given by relative vote shares. This LOF is both L-efficient and anonymous. Assume individuals have symmetric single-peaked preferences on X with ideal points described by $\mathbf{x} = (x_1, \ldots, x_7)$.

<u>Example 1</u>: $\mathbf{x}$= (0, 1/5, 38/70, 7/10, 4/5, 9/10, 1), and C = {1/5, 4/5}. In the terminology of Greeenberg and Shepsle, C constitutes a 2-equilibrium under sincere voting in which i = 1, 2 vote

for y_1, and i = 3, 4, 5, 6, 7 vote for y_2. So $C_1{}^* = C_1$ here. The final outcome is (2/7)·(1/5) + (5/7)·(4/5) = 44/70. However, given other individuals' vote sincerely, if i = 3 votes strategically for y_1, the final outcome is 38/70. This clearly makes 3 better off.[3]

Example 2: $\mathbf{x} = (0, 1/5, 3/10, 19/35, 7/10, 4/5, 1)$, and $C = \{1/5, 4/5, 19/35\}$. Under sincere voting, i = 1, 2, 3 vote for y_1, i = 5, 6, 7 vote for y_2, and y_3 receives 4's vote. Since there can only be two candidates elected, $C^* = \{y_1, y_2\}$, as in Example 1. (If y_1 and y_2 alone were candidates, this would again constitute a 2-equilibrium.) The final outcome under the LOF is: (3/7)·(1/5) + (3/7)·(4/5) = 3/7. However, given others' voting strategies, if individual 4 votes strategically for candidate y_2 then the final outcome is 19/35. And this yields a higher payoff to 4 than sincere voting.

While the LOF described is relatively special , it is easy to check that for Example 2, unless $\lambda_C((y_1, y_2), \cdot) = x_4$, individual 4 will never wish to vote sincerely for y_3 (given sincere voting by the remaining individuals). The same conclusion holds if the election rule $\varepsilon_C{}^{FN}$ is replaced by the fixed-standard rule, $\varepsilon_C = \varepsilon_C{}^{FS}$, with the standard m = 3. Thus the example illustrates why Greenberg's and Weber's defense of (A.4) is not quite sufficient. For the defense, they invoke the LOF which selects the median of the elected candidates' positions. With the two-candidate legislature $C_2{}^* = \{y_1, y_2\} \neq C$, any alternative $y^* \in [y_1, y_2]$ is a median outcome: but only a final outcome of $\lambda_C(\cdot, \cdot) = y^* = x_4$ can support the assumption of sincere voting at the electoral stage.

Conclusion

This note argues against the assumption of sincere voting in models of multi-member legislative elections. Instead, the complete legislative game -- election rule and legislative outcome function -- should first be explicitly defined, and then (rational) individuals' voting behavior deduced. The use of sincere voting in equilibrium may be a property of the game.

From a normative perspective, requiring any legislative game to be structured to induce sincere voting in equilibrium is considerably less demanding than requiring it to insure sincere voting everywhere. And recent work in implementation theory shows that multi-stage games -- such as the legislative election games discussed here -- are powerful instruments for generating truthful (equilibrium) behavior by players in the game (cf. Moore and Repullo, 1986). Nevertheless, the results reported above say that such structures for legislative election models must have the order statistic property described in Moulin's theorem (1980).

Appendix

Proof of Proposition 1: First observe that since the domain of γ_C is finite, so is the range of γ_C. Since γ_C is efficient, the result is trivial if $|C| = 1$. Let $|C| \geq 2$ and suppose the proposition is false. Then $\exists \mathbf{x} \in X^n$ such that $\gamma_C(\sigma^T(\mathbf{x}, \cdot)) = z \in X \backslash C$. Because $|C| \geq 2$ and ε_C and λ_C are '-efficient, $\exists y_1, y_2 \in C, \exists \mathbf{x}' \in X^n$ such that $\sigma^T(\mathbf{x}', \cdot) = (y_2, y_2, \ldots, y_2)$, and (1) $y_1 < z < y_2 = \gamma_C(\sigma^T(\mathbf{x}', \cdot))$, (2) $\sim\exists y_k \in C\backslash\{y_1, y_2\}$ such that $y_k \in (y_1, y_2)$. (Note that the sincere strategy profile at $\mathbf{x}'$ does not entail that all individuals share a common ideal point in X.) Since λ_C is L-efficient, $\sigma^T(\mathbf{x}, \cdot)$ must be such that $\sigma_i{}^T(x_i, \cdot) \geq y_2$ for some $i \in N$, and $\sigma_i{}^T(x_i, \cdot) \leq y_1$ for some $i \in N$. Let $N(\mathbf{x}, y_1) = \{i \in N \mid \sigma_i{}^T(x_i, \cdot) = y_1\}$ and $N(\mathbf{x}, y_2) = \{i \in N \mid \sigma_i{}^T(x_i, \cdot) = y_2\}$. Without loss of generality, assume $N(\mathbf{x}, y_1) \cup N(\mathbf{x}, y_2) = N$, and relabel N (if necessary) so that $N(\mathbf{x}, y_1) = \{1, \ldots, h\}$, $h < n$. Let $X_1 = [y_1, [y_1+y_2]/2)$, and $X_2 = ([y_1+y_2]/2, y_2]$. Then we can pick $\mathbf{x} = (x_1, x_2, \ldots, x_h, x_{h+1}', x_{h+2}', \ldots, x_{n-1}', x_n')$, where $x_i' \in X_2\ \forall i = h+1, \ldots, n$. For r = 0, 1, ..., h, define the preference profile:

$$\mathbf{x}^{h-r} = (x_1, x_2, \ldots, x_{h-r-1}, x_{h-r}, x_{h-r+1}', \ldots, x_n'), \text{ where } x_i' \in X_2\ \forall i.$$

Then $\mathbf{x}^h \equiv \mathbf{x}$, and $\text{Lim.}_{r \to h}(\mathbf{x}^{h-r}) = \mathbf{x}'$. Since $\gamma_C(\sigma^T(\mathbf{x}, \cdot)) = z$ and $\gamma_C(\sigma^T(\mathbf{x}', \cdot)) = y_2 > z$, $\exists r^*$, $0 < r^* \leq h$, such that $\gamma_C(\sigma^T(\mathbf{x}^{h-r^*+1}, \cdot)) = z$ and $\gamma_C(\sigma^T(\mathbf{x}^{h-r^*}, \cdot)) = z^o \in X\backslash\{z\}$. Since λ_C is L-efficient, $z^o \leq y_2$. Suppose $z^o < z$, and let $x_n' = y_2 \in X_2$. Then, by anonymity of γ_C,

$$\gamma_C((y_1, \sigma_{-n}{}^T(\mathbf{x}^{h-r^*}, \cdot))) = \gamma_C(\sigma^T(\mathbf{x}^{h-r^*+1}, \cdot)) = z > z^o.$$

Since n's preferences are strictly single-peaked and $x_n' = y_2$,

$$u_n(y_1, \sigma_{-n}{}^T(\mathbf{x}^{h-r^*}, \cdot)) > u_n(y_2, \sigma_{-n}{}^T(\mathbf{x}^{h-r^*}, \cdot)) = u_n(\sigma^T(\mathbf{x}^{h-r^*}, \cdot))$$

which contradicts straightforwardness of γ_C. Hence, $z^o \in (z, y_2]$. Suppose $z \geq [y_1+y_2]/2$ and choose $x_{h-r^*+1}' \equiv x_j' \in (z, [z+z^o]/2)$, a subset of X_2. By anonymity,

$$\gamma_C((y_1, \sigma_{-j}{}^T(\mathbf{x}^{h-r^*}, \cdot))) = \gamma_C(\sigma^T(\mathbf{x}^{h-r^*+1}, \cdot)) = z < z^o.$$

And by single-peakedness,

$$u_j(y_1, \sigma_{-j}{}^T(\mathbf{x}^{h-r^*}, \cdot)) > u_n(y_2, \sigma_{-j}{}^T(\mathbf{x}^{h-r^*}, \cdot)) = u_j(\sigma^T(\mathbf{x}^{h-r^*}, \cdot)),$$

contradicting the straightforwardness of γ_C: so $z < [y_1+y_2]/2$. But then, by symmetry, we can repeat the previous argument, mutatis mutandis, to conclude $z < [y_1+y_2]/2$ implies γ_C is not straightforward: do this by picking $x_1 = y_1$, $x_i \in X_1$, all $i = 1, \ldots, h$, and an appropriate sequence of preference profiles $(\mathbf{x}^s)_{s \to \underline{s}}$ such that $\mathbf{x}' \equiv \mathbf{x}^0$ and $\mathrm{Lim.}_{s \to \underline{s}} (\mathbf{x}^s) = \mathbf{x}$. This then yields a contradiction of the original supposition that $\exists \mathbf{x} \in X^n$ such that $\gamma_C(\sigma^T(\mathbf{x}, \cdot)) = z \in X \backslash C$. ||

Footnotes

*Much of this paper is a result of the efforts of John Ledyard to understand what I was trying to say about the problem. I am extremely grateful for his help, and for his insistence that I look for a theorem and not simply a set of examples. I am also grateful to Kim Border for providing a proof for one of the results. Despite their input, I retain all responsibility for any remaining errors and ambiguities.

1. Relaxing the assumption of anonymity does little to ameliorate the difficulties, discussed below, with assuming (A.4) generally.

2. In Moulin (1980), the theorem is stated and proved assuming $C \equiv \mathbf{R}$. However, as Moulin remarks in footnote 2 (p.445), the result carries over directly to the case of C being a compact subset of **R**.

3. For a similar example, see Ursprung (1980).

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