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ON BAYESIAN IMPLEMENTABLE ALLOCATIONS*

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Abstract

This paper identifies several social choice correspondences which are and are not fully implementable in economic environments when agents are incompletely informed about the environment. We show that in contrast to results in the case of complete information, neither efficient allocations nor core allocations define implementable social choice correspondences. We also identify conditions under which the Rational Expectations Equilibrium correspondence is implementable. We extend the concepts of fair allocations and Lindahl allocations to economies with incomplete information, and show that envy-free allocations and Lindahl allocations are implementable under some conditions while fair allocations are not.

I. Introduction

The problem of full implementation can be stated as follows. Given a set of "desirable" allocations, under what circumstances can we construct a non-cooperative game whose equilibrium outcomes coincide with those allocations ? The primary reason for posing such a question is that in most economic settings, the desirability of an allocation is a function of the parameters of the economic environment. Since these parameters involve personal characteristics (e.g. preferences) of the individuals in the economy, the game must be constructed so that as these personal characteristics change, the equilibrium behavior of the individuals also changes in the desired way.

Until recently, the only positive results obtained on full implementation assumed that all individuals had complete information, i.e. that the personal characteristics of all individuals were publicly known. A characterization of allocations implementable via Nash equilibria was given by Maskin [1977], who showed that a condition called monotonocity was necessary and essentially sufficient for implementation. A drawback of the assumption of complete information, besides its being unrealistic, is that it is hard to justify the use of a decentralized mechanism to get individuals to reveal "private" characteristics which are already known (see Postlewaite [1986] for a further critique along these lines).

Recent work on full implementation has focussed on economies with incomplete information. Postlewaite and Schmeidler [1986] showed that an incomplete information version of Maskin's condition was necessary for implementation via Bayesian equilibrium. In a recent paper (Palfrey and

Srivastava [1985]), we prove that an extended version of that condition, called *Bayesian Monotonicity*, is necessary for implementation, and together with a self-selection condition, is sufficient for implementation. In this paper, we identify several important economic allocations which are and are not implementable.

There are many similarities between the formal structure of the complete information Nash equilibrium approach (Hurwicz [1979a, 1979b], Schmeidler [1980], Postlewaite [1986], Hurwicz, Maskin and Postlewaiate [1984]) and the incomplete information approach. Both approaches define a class of environments and associate to each environment a set of feasible allocations. With incomplete information, individuals are also endowed with a partition of the class of environments and a prior over the environments. When a state of the economy is realized, each individual observes an event from his partition and revises his priors according to Bayes' rule. This information structure is common knowledge, and, for the purposes of this paper, is assumed to be known by the planner. The complete information model is a special case of the above formulation: the partition is the complete information partition of the environments and consequently priors do not play a role; this degenerate information structure is also implicitly common knowledge among the individuals and known to the planner as well.

A social choice function for an incomplete information economy is defined as a mapping which associates with each environment ("state") in the class of environments a feasible allocation. One may think of a social choice function, then, as defining a state-contingent allocation. A Social Choice

Correspondence for an incomplete information economy is a collection of these state-contingent allocations (Postlewaite and Schmeidler [1986]). In contrast, a social choice correspondence in the complete information setup is usually defined as a mapping which assigns to each environment a set of (desirable) feasible allocations. The reason for the difference is that with incomplete information, the relevant notion of an allocation is a state-contingent allocation. To avoid confusion, we will refer to the incomplete information correspondence as a <u>social choice set</u>, since it is simply a set of feasible state contingent allocations.

To study whether social choice sets can be attained as equilibrium outcomes of non-cooperative games, we use the notion of (full) Bayesian implementation. Bayesian implementation is a property defined on social choice sets just as Nash implementation is a property defined on social choice correspondences. Nash implementation is a special case of Bayesian implementation when all individuals in the economy have complete information. In that case, the set of all (measurable) selections from a given social choice correspondence in a complete information economy defines a social choice set for that economy.

Some of the most important results from the complete information approach involve identifying particular correspondences which are Nash implementable. They include, under appropriate regularity conditions:

(1) the Pareto correspondence

(2) the Core correspondence

(3) the Walrasian correspondence

(4) the Fair correspondence

(5) the Lindahl correspondence.

The purpose of this paper is to perform an analogous identification for incomplete information economies. However, in order to do so, we need to define a point-set mapping on a domain of incomplete information economies which associates a social choice set to each economy in the domain. This maps each economy into a set of state-contingent allocations in much the same way as a social choice correspondence associates each complete information environment to a set of (noncontingent) allocations. We then ask what mappings of this type are implementable everywhere in the domain.

We find that the set of Rational Expectations Equilbria are implementable in domains of incomplete information economies in which individuals are risk averse, information is non-exclusive, individual endowments are state independent, and the equilibria are interior to consumption sets. Core allocations and efficient allocations, defined in several ways following Holmstrom and Myerson [1983], are <u>not</u> implementable. Envy-free allocations and Lindahl allocations (appropriately modified to account for incomplete information) are implementable. Fair allocations (envy-free and efficient) are not implementable.

Finally, we would like to stress the fundamental difference between the approach this paper takes and related research which has investigated achievability (or truthful implementation) rather than full implementation. Achievability does not require that the set of equilibrium outcomes of a mechanism coincide with the social choice set or even that they are contained in the social choice set. Rather, an allocation is achievable if it is an equilibrium outcome to some game. The work of Myerson [1979] and Harris and Townsend [1981] and most subsequent work (in auction theory, for example) has relied heavily on the revelation principle either to identify the set of achievable allocations in a particular setting or to characterize "truth-telling" mechanisms to achieve allocations with particular properties. Importantly, as pointed out first by Dasgupta, Hammond, and Maskin [1979] for complete information settings and later by Postlewaite and Schmeidler [1986] with incomplete information, the revelation principle applies only to achievability, not to (full) implementation.

The paper is organized as follows. Definitions and the basic model are laid out in Section II. Section III establishes the non-implementability of efficient allocations. The implementability of Rational Expectations Equilibria is analyzed in Section IV. Sections V and VI address envy-free and fair allocations, and public goods economies, respectively.

II. The Model and Definitions

This Section contains our basic model and definitions of feasibility and implementation as they apply to our specific setting. These are adapted from Palfrey and Srivastava [1985]. In that paper, full implementation of a set of environment contingent (state-contingent) allocations was defined relative to a given differential information economy. In this paper, we define an extended notion of implementation, which we call global implementation. This concept is convenient for assessing the implementability of social choice sets across a large domain of differential information economies, in the spirit of earlier work on Nash implementation.

A. Definitions

First we define a differential information economy (henceforth referred to as any economy) and a domain of economies. An economy is a finite set of economic environments (states), S, $S=\{1,2,...,S\}$, a set of individuals, I, indexed by i, and an aggregate endowment, $\bar{w} \in R^L$, $\bar{w} \gg 0$. In environment s, agent i has utility function $U^i(\cdot,s)$ and endowment $w^i(s)$. We assume that U^i is strictly increasing and bounded below. Consumption sets are the nonnegative orthant, and we normalize $U^i(0,s) = 0$ for all i and s.

Let A be the set of all feasible allocations,

$$A = \{z \in R_+^{LI} \;\Big|\; \sum_{i=1}^{I} z^i \le \bar{w} \}, \text{ and let}$$

$$X = \{ x: S \to A \}$$

be the set of feasible environment-contingent allocations. A Social Choice Set is a subset $F \subset X$.

The information available to agent $i \in I$ is given by a partition Π^i of S; we denote by $E^i(s)$ the event in Π^i which contains s. Each agent also has a prior distribution over environments, denoted by q^i, and we assume that $q^i(s) > 0$ for all i,s. An <u>economy</u> d is a collection $<S,I,\hat{w},U,\Pi,q>$. A <u>domain</u>, D, is a collection of economies.

We assume that the partitions (Π^i), the priors, (q^i), and the set of state dependent utility functions, $(U^i(\cdot,s))$, are all common knowledge. Given $E^i(s) \in \Pi^i$, the expected utility of x^i is

$$V^i(x,s) = \sum_{t \in E^i(s)} q^i(t|E^i(s))U^i(x^i(t),t).$$

where

$$q^i(t|E^i(s)) = \begin{cases} 0 & \text{if } t \notin E^i(s) \\ \dfrac{q^i(t)}{q^i(E^i(s))} & \text{if } t \in E^i(s) \end{cases}$$

If i has complete information, i.e. $E^i(s) = \{s\}$ for all s, then, at any state s, the expected utility of x^i to i reduces to $U^i(x^i(s),s)$.

A mechanism for an economy is a pair (M,g), where $M = M^1 \times M^2 \times \ldots \times M^I$, $g: M \to A$. We denote by g^i the i'th component vector of g. A strategy for agent i is $\sigma^i: \Pi^i \to M^i$. Given an environment s, let $E(s) = (E^1(s),\ldots,E^I(s))$, $\sigma(E(s)) = (\sigma^1(E^1(s)),\ldots,\sigma^I(E^I(s)))$, $\sigma = (\sigma^1,\ldots,\sigma^I)$, and $\sigma^{-i} = (\sigma^1,\ldots,\sigma^{i-1},\sigma^{i+1},\ldots,\sigma^I)$. Given a strategy σ, let $g(\sigma) = (g(\sigma(E(1)),\ldots,g(\sigma(E(S))))$.

<u>Definition 1</u>: σ^i is a <u>best response</u> to σ^{-i} if for all $s \in S$,

$V^i(g(\sigma^{-i},\sigma^i),s) \geq V^i(g(\sigma^{-i},\hat{\sigma}^i),s)$ for all $\hat{\sigma}^i: \Pi^i \to M^i$.

<u>Definition 2</u> : σ is an <u>equilibrium</u> to (M,g) if σ^i is a best response to σ^{-i} for all i.

B. <u>Implementation</u>

Fix an economy $d = <S,I,\hat{w},U,\Pi,q>$. We can now define (full) implementation of a social choice set in the economy.

<u>Definition 3</u> : A mechanism (M,g) <u>implements</u> F if

(i) For any $x \in F$, there exists an equilibrium σ to (M,g) such that $g(\sigma) = x$.

(ii) If σ is an equilibrium to (M,g) then $g(\sigma) \in F$.

If there exists a mechanism (M,g) which implements F, then F is said to be <u>implementable.</u> We turn next to the Bayesian monotonicity condition which is necessary for implementation. Notation is considerably simplified if we make the following assumption of no redundant states:

<u>Assumption</u>: (No Redundant States) For all s, $\bigcap_i E^i(s) = \{s\}$

<u>Definition 4</u> : A collection of functions $\alpha = (\alpha^1,\ldots,\alpha^I)$, with $\alpha^i: \Pi^i \to \Pi^i$, is <u>compatible with Π</u> if for all $(E^1,\ldots,E^I)$ such that $E^i \in \Pi^i$ for all i, $\bigcap_i E^i \neq \emptyset$ implies $\bigcap_i \alpha^i(E^i) \neq \emptyset$.

Although this definition appears to be quite formidable, it captures a very simple idea. Each α^i can be thought of as a reporting strategy for agent i, the interpretation being that when i has observed $E^i(s)$, he acts as if he observed $\alpha^i(E^i(s))$. Consider a direct mechanism, i.e. $M^i = \Pi^i$ for all i. In this case, the set of all possible α^i coincides with the set of all strategies available to agent i. Suppose now that when s occurs, each i reports $\alpha^i(E^i(s))$ instead of $E^i(s)$. Then, there are two possibilities: either $\bigcap_i \alpha^i(E^i(s)) = \emptyset$ or $\bigcap_i \alpha^i(E^i(s)) \neq \emptyset$. In the former case, the reports of the agents are called incompatible while in the latter case, they are called compatible. If the reports are incompatible, the planner can infer that some agent must be lying about his event, while if the reports are compatible, the planner cannot tell whether anyone is lying. It is possible for the planner to provide effective incentives to prevent any equilibrium which involves incompatible reports. This allows us to restrict attention to compatible reporting strategies, where compatibility is formalized in the above definition.

Since we assume that for all s, $\bigcap_i E^i(s) = \{s\}$, we have $\bigcap_i \alpha^i(E^i) \neq \emptyset$ if and only if it is a singleton. Accordingly, for any α compatible with Π, we define

$\alpha(s) = \bigcap_i \alpha^i(E^i(s))$, $\quad x_\alpha(t) = x(\alpha(t))$, $\quad x_\alpha = (x_\alpha(1), \ldots, x_\alpha(S))$.

Definition 5 : F satisfies Bayesian Monotonicity (BM) if for all α compatible with Π, If

(i) $x \in F$

(ii) For all i, for all s, for all $y : S \to A$,

$$V^i(x,\alpha(s)) \geq V^i(y,\alpha(s)) \Rightarrow V^i(x_\alpha,s) \geq V^i(y_\alpha,s)$$

Then $x_\alpha \in F$.

A detailed interpretation of this condition is given in Palfrey and Srivastava [1985]. Here, we remark that in (ii) of the condition, the first

inequality is satisfied only if x is an equilibrium outcome to some mechanism and the second is satisfied only if x_α is an equilibrium outcome to the same mechanism. The condition then states that if whenever x is an equilibrium outcome so is x_α, then x_α must lie in the social choice set.

BM is a necessary for implementation, as are certain self-selection conditions. For the purposes of this paper, we will assume a condition which guarantees that self-selection will hold for all social choice sets in an economy. It is called Non-Exclusivity of Information (NEI), and is due to Postlewaite and Schmeidler [1986].

Assumption (NEI): $E^i(s) \supset \bigcap_{j \neq i} E^j(s)$ for all s,i.

Notice that NEI together with the assumption of no redundant states implies:

$$\bigcap_{j \neq i} E^j(s) = \{s\} \qquad \text{for all i,s.}$$

We will repeatedly apply the following theorem which is a special case of a more general theorem proved in Palfrey and Srivastava [1985]:

Theorem 1 : (a) If F is implementable, then F satisfies BM.

(b) If $I \geq 3$, F satisfies BM, NEI holds, and $F \neq 0$, then F is implementable.

By $F \neq 0$ we mean that $x \in F \Rightarrow x^i(s) \neq 0$ for all i,s.

C. Global implementation relative to a domain of economies

Let D be a domain. Define a Social Choice Correspondence (SCC) on D, F, as a set-valued function which assigns to every economy $d \in D$ a social choice set $F(d) \subset X$.

Definition 6: An SCC F is Globally Implementable Relative to Domain D if, for all $d \in D$, $F(d)$ is implementable.

The following two domains will be referred to frequently in the rest of the paper:

1. The NEI domain with risk averse preferences:

$$D_1 = \{ d \mid S \text{ is finite ; NEI is satisfied ; } w^i(s) = w^i \gg 0 \text{ for all } i,s \text{ ; } U^i(\cdot,s) \text{ concave for all } i \}$$

2. The complete information neoclassical domain:

$$D_2 = \{E_3, E_4, \ldots, E_I, \ldots\}$$

where E_I includes all environments with I individuals each of whom has a continuous quasi-concave increasing utility function in each state and all agents have complete information.

The first domain is an important subset of the set of all economies. In all economies in this domain, as pointed out in Palfrey and Srivastava [1986], any singleton state-contingent allocation can be achieved (truthfully implemented) by a "forcing" type of mechanism, but only social choice sets satisfying Bayesian Monotonicity are fully implementable.

The second domain is even more restrictive informationally but is of special interest because of its extensive use in the literature on

implementation via Nash equilibrium in complete information economies. It is well known for example that the SCC which assigns to each economy its set of (ex post) Pareto optimal state contingent allocations is globally implementable relative to D_2. Another important SCC which is known to be globally implementable relative to D_2 is the constrained Walrasian equilibrium correspondence. (This is equivalent to the Walrasian correspondence except in some cases where the equilibrium occurs on the boundary of the feasible set; see Hurwicz, Maskin, and Postlewaite [1984]). Also, Thomson [1979] has shown that fair allocations are Nash implementable. Lindahl equilibrium allocations have been shown to be Nash implementable in public goods economies with linear production technologies. In contrast, we are interested in identifying what interesting SCC's are globally implementable relative to domains in which individuals do not have complete information. These include domains such as D_1.

The rest of the paper addresses the following questions. First, is there a natural analogue to Pareto optimality which defines a globally implementable SCC in a large domain of differential information economies? Second, is there a natural analogue to the Walrasian correspondence which defines a globally implementable SCF relative to a large domain? Third, can we adapt the definition of fair allocations to economies with incomplete information in a way such that they are globally implementable? Fourth, what is the appropriate (in this sense) extension of the complete information Lindahl equilibrium?

III. Efficient Allocations

In this Section, we examine whether efficient allocations are globally implementable relative to D_1. Following Holmstrom and Myerson [1983], we distinguish between three notions of optimality, namely ex-ante, interim, and ex-post optimality. We will show that the SCC mapping economies into the set of optimal state-contingent allocations, variously defined, is not globally implementable. This result is surprising, and shows that the the intuition behind the implementability of the Pareto set in complete information settings (domain D_2) does not extend to the case of incomplete information, at least for the three notions of optimality we consider. In our example showing that interim efficient allocations cannot always be implemented, the allocation we consider is also durable in the sense of Holmstrom and Myerson [1983], so it follows that durable allocations are not globally implementable either.

Definition 7: (a) An allocation $x:S \to A$ is ex ante efficient if there does not exist $y:S \to A$ such that

$$\sum_{t \in S} q^i(t)U^i(y^i(t),t) \geq \sum_{t \in S} q^i(t)U^i(x^i(t),t)$$ for all i with strict inequality for at least one i.

(b) An allocation $x:S \to A$ is interim efficient if there does not exist $y:S \to A$ such that $V^i(y,s) \geq V^i(x,s)$ for all i and s, with strict inequality for at least one i and s.

(c) An allocation $x:S \to A$ is ex-post efficient if there does not exist $y:S \to A$ such that $U^i(y^i(s),s) \geq U^i(x^i(s),s)$ for all i and s, with strict inequality for at least one i and s.

Example 1: (Ex-ante efficient allocations are not globally implementable relative to D_1)

Consider the following economy. There are two completely informed agents, one good, and two environments (states). The parameters are as follows : $\bar{w}=(5,5)$, $U^1(x,s) = s \cdot \log(x)$, $s=1,2$, $U^2(x,s) = \log(x)$, $q^1=q^2=(.5,.5)$. Let F be the social choice set consisting of all ex-ante efficient allocations. We will show that F does not satisfy BM. The following allocation is an element of F, since, as pointed out in Palfrey and Srivastava [1985], it arises from ex-ante trading in a complete contingent claims market for a particular profile of endowments.

	State 1	State 2
Agent 1	5/2	10/3
Agent 2	5/2	5/3

Consider $\alpha=(\alpha^1,\alpha^2)$ with $\alpha^i(s)=\{1\}$ for both i and for both s. Then, α is compatible with Π since both agents are completely informed, and $\alpha(s)=\{1\}$ for all t. Since utility is increasing, the inequality in part (ii) of the definition of BM is trivially satisfied. For example, for s=2, the inequality for agent 1 reads

$$U^1(x^1(1),1) \geq U^1(y^1(1),1) \text{ implies } U^1(x^1(1),2) \geq U^1(y^1(1),2).$$

BM then requires that (x(1),x(1)) be an element of F. However, (x(1),x(1)) is not ex-ante efficient and since BM is necessary for implementation, we conclude the ex-ante efficient SCC is not implementable.

The reason why F fails to satisfy BM is intuitive. Ex-ante allocations provide insurance, while the structure of the game analyzed in this paper requires trades to be made after agents receive their private information. Since all agents are completely informed, there is no practical possibility for insurance. It is therefore not particularly surprising that ex-ante efficient allocations can fail to be implementable. We note that this example does not depend on their being only one good.

Example 2: (Interim efficient allocations are not globally implementable relative to D_1)

There are four agents, a single good, and three equally likely states. Agents 1 and 2 are completely informed, and have strictly increasing preferences over the good in each state. Agents 3 and 4 each have the partition $\{\{1\},\{2,3\}\}$, i.e. they cannot distinguish states 2 and 3. Their preferences are given by : $U^i(x^i(s),s) = \beta^i(s)\log(x^i(s))$, with $\beta^3(1) = 0.5$, $\beta^3(2) = 0.25$, $\beta^3(3) = 0.75$, and $\beta^4(1) = 0.5$, $\beta^4(2) = 0.75$, $\beta^4(3) = 0.25$. The aggregate endowment in each state is 4. The following allocation is interim-efficient:

	State 1	State 2	State 3
Agent 1	1	1	1
Agent 2	1	1	1
Agent 3	1	.5	1.5
Agent 4	1	1.5	.5

Consider the following α compatible with Π : $\alpha^i(E^i(s)) = E^i(1)$, all i,s. In this case, α maps all states onto state 1, so that $\alpha(s) = \{1\}$ for all s. The inequality condition in BM is trivially satisfied for agents 1 and 2 since they have state independent preferences. The condition for agent 3 reads:

$.5\log(1) \geq .5\log(y(1))$ implies

$.5[.25\log(1)]+.5[.75\log(1)] \geq .5[.25\log(y(1))]+.5[.75\log(y(1))]$.

Both inequalities reduce to the same expression, so the hypotheses of BM are satisfied for agent 3. Since the interim preferences of agent 4 are also state independent, the inequality condition also holds for agent 4. BM then requires that the state 1 allocation in each state, which is the initial endowment in each state, say w, be interim efficient. However, it is clear that x interim dominates w, and this social choice set therefore does not satisfy BM. Thus, the interim-efficient SCC is not globally implementable even if the domain is restricted to NEI information structures and state independent endowments.

Example 3 : (Ex-post efficient allocations are not globally implementable relative to D_1)

There are three possible states, two goods, and four agents, with $\bar{w}=(6,6)$. Agents 1 and 2 are completely informed, while $\Pi^3=\Pi^4=\{\{1,2\},\{3\}\}$. Thus, agents 3 and 4 can only distinguish state 3. The priors of agent 3 are given by $q^3(1|\{1,2\})=q^3(2|\{1,2\})=0.5$, and $q^4=q^3$. Preferences of all agents are Cobb-Douglas, given by $U^i(x,s)=\beta^i(s)\log(x_1)+(1-\beta^i(s))\log(x_2)$. The β parameters are:

	State 1	State 2	State 3
Agent 1	.25	.25	.25
Agent 2	.30	.30	.30
Agent 3	.25	.75	.50
Agent 4	.75	.25	.50

The following allocation is ex-post efficient, being the state-by-state Walrasian allocations when $w^1=w^2=(2,2)$, $w^3=w^4=(1,1)$:

	State 1	State 2	State 3
Agent 1	(1.43,2.31)	(1.43,2.31)	(1.43,2.31)
Agent 2	(1.71,2.15)	(1.71,2.15)	(1.71,2.15)
Agent 3	(0.71,1.15)	(2.15,0.39)	(1.43,0.77)
Agent 4	(2.14,0.39)	(0.71,1.15)	(1.43,0.77)

Since agents 1 and 2 are completely informed and have state independent preferences, the inequality in condition (ii) of BM is satisfied for them for any allocation and for any admissible α. Therefore, we need to check the inequality condition only for agents 3 and 4. Consider, then, the following: $\alpha^1(\{s\}) = \{3\}$, all s, $\alpha^2(\{s\}) = \{3\}$, all s, $\alpha^3(\{1,2\}) = \{3\}$, $\alpha^3(\{3\}) = \{3\}$. This α is compatible with Π, and for all s, $\alpha(s)=3$. We now show that condition (ii) of BM is satisfied for agent 3. A similar argument shows that it holds for agent 4. If s=1 or s=2, the inequality condition for agent 3 is:

$$.5\log(1.43) + .5\log(.77) \geq .5\log(y\ ) + .5\log(y\ )$$

implies

$$.5[.25\log(1.43)+.75\log(.77)] + .5[.75\log(1.43)+.25\log(.77)] \geq$$
$$.5[.25\log(y_1)+.75\log(y_2)]+.5[.75\log(y_1)+.25\log(y_2)]$$

This holds for all (y_1,y_2) since both inequalities reduce to the same expression. The same argument applies to agent 4. BM then requires that the state 3 allocations be ex-post efficient in both states 1 and 2. However, x(1) ex-post dominates x(3) in state 1. Thus, ex-post efficient allocations are not implementable in D_1.

Collectively, these examples show that in general, it is impossible to design a mechanism all of whose equilibria are efficient. This suggests that it may be inappropriate to presume that the agents commit to a specific mechanism prior to the occurrence of the state, since there can be collective gains to renegotiating the mechanism once everyone observes their private information. This is related to the problem of durability and endogenous mechanism choice, the implications of which for full implementation are unknown. The problem illustrated by these examples may be solvable if agents are not committed to a specific mechanism prior to the realization of the state, since the process of choosing a mechanism can lead to transmission of information across agents. A detailed discussion of problems arising due to information leakage in endogenous mechanism choice problems is given in Crawford [1985].

Finally, we note that consistent with the three notions of optimality, we can define three notions of the core. The examples above also show that core allocations, variously defined, are not generally implementable. This again contrasts with the case of complete information.

IV. Market Equilibria

In this Section, we examine whether market determined outcomes yield SCC's which are globally implementable relative to D_1. As stated before, in the case of complete information, it is well known that the (constrained) Walrasian correspondence is globally implementable (relative to D_2). With incomplete information, the most frequently studied notion of market equilibrium is that of a Rational Expectations Equilibrium (REE), and we turn next to the question: relative to which domains is the REE SCC globally implementable?

An REE is defined as follows. Given a price function $p: S \to R^L$, let $E^i(p,s) = \{ t \in E^i(s) \mid p(t) = p(s) \}$. This event represents the information of i at s given prices p. Notice that if prices are fully revealing, then $E^i(p,s) = \{s\}$. At s, individual i chooses $z^i \in R^L$ to maximize

$$\sum_{t \in E^i(p,s)} q^i(t \mid E^i(p,s)) U^i(z^i,t) \text{ subject to } p(s)\cdot[z^i - w^i(s)] \leq 0.$$

Let $x^i(s)$ denote the demand correspondence of i at s. An REE is a price function p such that when agents solve the above problem, markets clear. Let (x,p) denote an REE. Consider the REE social choice set

$F = \{ x: S \to A \mid \exists\, p: S \to R^L \text{ such that } (x,p) \text{ is an REE } \}$. Our main result about REE's is the following:

Theorem 2: Let $\mathcal{D}_1$ be a subset of economies in D_1 such that for all $d \in \mathcal{D}_1$ for all $i \in I$, $s \in S$

$$\{ z \in R^L_+ \mid U^i(z,s) \geq \delta \} \subset R^L_{++} \text{ for all } \delta > 0.$$

The REE SCC is globally implementable relative to $\mathcal{D}_1$.

Proof: Let $x \in F$, and let p be the associated price. The hypotheses of the theorem ensure that $p(s) \gg 0$ for all s and that $x^i(s) \gg 0$ for all i and s.

Let α be compatible with Π, and suppose that the BM inequality condition is satisfied. Suppose that x_α is not an REE. Given some $s \in S$, let $s' = \alpha(s)$, and recall that $E^i(p,s') = \{t' \in E^i(s') \mid p(t') = p(s')\}$. Note that strict concavity of the utility function implies that $x^i(t') = x^i(s')$ for all $t' \in E^i(p,s')$, and so prices and demands are constant across $E^i(p,s')$.

Define $E^i(p,s) = \{ t \in E^i(s) \mid p(\alpha(t)) = p(s') \}$.

Note (1) if $t \in E^i(p,s)$, then $\alpha(t) \in E^i(p,s')$, and

(2) if $t \in E^i(s)$, $t \notin E^i(p,s)$, then $\alpha(t) \notin E^i(p,s')$.

Since x_α is not an REE, then there exists a state s, an agent i, and an allocation z^i such that $p(\alpha(s))\cdot[z^i - w^i] \leq 0$, and

$$(3) \quad \sum_{t \in E^i(p,s)} q^i(t \mid E^i(p,s)) U^i(z^i,t) > \sum_{t \in E^i(p,s)} q^i(t \mid E^i(p,s)) U^i(x^i_\alpha(s),t)$$

Our strong interiority assumptions allow z^i to be chosen such that $0 \ll z^i \ll \hat{w}$.

Let $s' = \alpha(s)$.

Since $x^i(t') = x^i(s')$ for all $t' \in E^i(p,s')$, (1) implies that $x^i_\alpha(t) = x^i_\alpha(s)$ for all $t \in E^i(p,s)$. Thus, (3) can be written as

$$(4) \quad \sum_{t \in E^i(p,s)} q^i(t \mid E^i(p,s)) U^i(z^i,t) > \sum_{t \in E^i(p,s)} q^i(t \mid E^i(p,s)) U^i(x^i_\alpha(t),t).$$

Define y^i as follows

$$y^i(t') = \begin{cases} z^i & \text{if } t' \in E^i(p,s') \\ x^i(t') & \text{otherwise.} \end{cases}$$

Points (1) and (2) above imply that for all $t \in E^i(p,s)$, $y^i_\alpha(t) = z^i$, and for all $t \in E^i(s)$, $t \notin E^i(p,s)$, $y^i_\alpha(t) = x^i_\alpha(t)$. Inequality (4) implies that

$$V^i(y_\alpha,s) > V^i(x_\alpha,s).$$

Let $y^j(t') = [\bar{w} - y^i(t')]/[I-1]$ for all $j \neq i$. Then, since all the $x^i(t')$ and z^i are strictly positive and strictly less than $\bar{w}$, y is feasible, i.e. $y(t') \in A$ for all t'. The inequality condition of BM then implies that $V^i(y,s') > V^i(x,s')$. From (2) above,

$$\sum_{t'\in E^i(p,s')} q^i(t'|E^i(p,s'))U^i(y^i(t'),t') > \sum_{t'\in E^i(p,s')} q^i(t'|E^i(p,s'))U^i(x^i(t'),t').$$

Note that by construction, prices are non-revealing over the event $E^i(p,s')$, and that $x^i(t') = x^i(s')$ for all $t' \in E^i(p,s')$. Finally, note that z^i is feasible at $E^i(p,s')$, which contradicts the assumption that $x^i(s')$ is utility maximizing at $E^i(p,s')$. Therefore, the rational expectations equilibrium social choice set satisfies BM. Our assumptions guarantee that all allocations are interior to the feasible set, so that Theorem 1 applies to show that the REE SCC is globally implementable relative to D_1. ■

The next example shows that the curvature assumption on indifference curves cannot be dispensed with. The assumption rules out equilibria on the boundary of the feasible set, which can lead to pathologies of the sort discussed in Hurwicz, Maskin, and Postlewaite [1984].

Example 4: The economy consists of 3 agents, 3 goods and 3 equally likely states. Endowments are given by: $w^1(s) = (0,0,10)$ for all s, $w^2(s) = (1,0,0)$ for all s, and $w^3(s) = (0,1,0)$ for all s. The utility functions of the agents are given by:

	State 1	State 2	State 3
Agent 1	$x_1+x_2+2x_3$	$.2x_1+x_2+x_3$	$x_1+.2x_2+x_3$
Agent 2	x_3	$5x_3$	x_3
Agent 3	x_3	x_3	$5x_3$

Agents 2 and 3 are completely informed, while the information of agent 1 is $\Pi^1=\{\{1\},\{2,3\}\}$. The following prices and allocations form the unique REE for this example :

	State 1	State 2	State 3
Prices	(.5,.5,1)	(.2,1,1)	(1,.2,1)
Agent 1	(1,1,9)	(1,1,8.8)	(1,1,8.8)
Agent 2	(0,0,.5)	(0,0,.2)	(0,0,1)
Agent 3	(0,0,.5)	(0,0,1)	(0,0,.2)

To show that this SCC is not implementable, consider α^i such that $\alpha^i(E^i(s)) = E^i(1)$ for all s. This α is clearly compatible with Π. To see that the BM inequality is satisfied, note that it is trivially satisfied for agents 2 and 3. For agent 1, it reduces to showing that:

$$20 \geq y_1 + y_2 + 2y_3 \text{ implies } 10.2 \geq .6y_1 + .6y_2 + y_3$$

It is straightforward to verify that this condition is satisfied for all feasible y. BM then requires that the state 1 allocation be an REE in both states 2 and 3. Since the REE is unique and the allocations are different across the states, it follows that BM is violated.

Next, we show that the assumption of state independent individual endowments cannot, in general, be relaxed.

Example 5: There are three possible environments (states), two goods, and three agents. Agents 1 and 2 are completely informed, while $\Pi^3=\{\{1,2\},\{3\}\}$. Thus, agent 3 can only distinguish state 3. Agent 3's priors are such that $q^3(1|\{1,2\})=q^3(2|\{1,2\})=0.5$. Preferences of all agents are Cobb-Douglas, given by $U^i(x,s)=\beta^i(s)\log(x_1)+(1-\beta^i(s))\log(x_2)$. Initial endowments and the β parameters are:

	State 1		State 2		State 3	
	w	β	w	β	w	β
Agent 1	(1,3)	.25	(3,1)	.25	(2,2)	.25
Agent 2	(3,1)	.30	(1,3)	.30	(2,2)	.30
Agent 3	(1,1)	.25	(1,1)	.75	(1,1)	.50

The unique REE in this example has fully revealing prices, so that the equilibrium allocations are simply the state-by-state Walrasian allocations. We note that with complete information, this SCC is implementable. The equilibrium prices and allocations are given by:

	State 1	State 2	State 3
Prices	(.36,1)	(.60,1)	(.47,1)
Agent 1	(2.33,2.52)	(1.17,2.09)	(1.56,2.21)
Agent 2	(1.73,1.46)	(1.82,2.51)	(1.88,2.05)
Agent 3	(0.94,1.02)	(2.01,0.40)	(1.56,0.73)

Since agents 1 and 2 are completely informed and have state independent preferences, the inequality in condition (ii) of BM is satisfied for them for any allocation and for any admissible α. Consider the compatible α given by

$\alpha^i(E^i(s)) = E^i(3)$ for all i and s, so that for all s, $\alpha(s)=3$. For agent 3, condition (ii) of BM is satisfied trivially. If s=1 or s=2, the inequality condition is:

$$.5\log(1.56) + .5\log(.73) \geq .5\log(y_1) + .5\log(y_2)$$

implies

$$.5[.25\log(1.56)+.75\log(.73)] + .5[.75\log(1.56)+.25\log(.73)] \geq$$
$$.5[.25\log(y_1)+.75\log(y_2)]+.5[.75\log(y_1)+.25\log(y_2)]$$

This holds for all (y_1,y_2) since both inequalities reduce to the same expression. BM then requires that the state 3 allocations be REE's in both states 1 and 2. Because the REE allocations are unique and different in each state, BM is violated, and the REE SCC is therefore not implementable.

Finally, we note that in the absence of NEI or some other strong informational condition, REE allocations generally violate self-selection conditions which are necessary for implementation. Consequently, they are not, in general, implementable (Blume and Easley [1985]).

V. Fair Allocations

In this Section, we show that non-zero envy-free allocations are globally implementable, where the concept of no-envy used is a straightforward extension of that used in the case of complete information.

Definition 8: An allocation $x:S \to A$ is (interim) envy-free if for all i,j,s,

$$\sum_{t \in E^i(s)} q^i(t|E^i(s))U^i(x^i(t),t) \geq \sum_{t \in E^i(s)} q^i(t|E^i(s))U^i(x^j(t),t).$$

This states that at any state of information, no agent prefers the allocation of any other agent to his own. Thus, this is an interim version of the familiar no-envy condition.

Theorem 3: Non-zero envy-free allocations are globally implementable in D_1.

Proof: Let x be an envy-free fair allocation. Suppose α is compatible with Π, and that all the hypotheses of the BM condition are satisfied. Suppose that x_α is not envy-free. Then, there exists s,i,j such that

$$\sum_{t \in E^i(s)} q^i(t|E^i(s))U^i(x^j_\alpha(t),t) > \sum_{t \in E^i(s)} q^i(t|E^i(s))U^i(x^i_\alpha(t),t).$$

Consider the allocation y where, for all $t \in S$,

$$y^k(t) = \begin{cases} x^k(t) \text{ if } k \neq i,j \\ x^j(t) \text{ if } k = i \\ x^i(t) \text{ if } k = j \end{cases}$$

Then, y is clearly feasible, and by construction,

$$\sum_{t \in E^i(s)} q^i(t|E^i(s))U^i(y^i_\alpha(t),t) > \sum_{t \in E^i(s)} q^i(t|E^i)s))U^i(x^i_\alpha(t),t).$$

The BM inequality then implies that

$$\sum_{t' \in E^i(s')} q^i(t'|E^i(s'))U^i(y^i(t'),t') > \sum_{t' \in E^i(s')} q^i(t'|E^i(s'))U^i(x^i(t'),t'),$$ which

contradicts x being envy-free.

Interim fair allocations, which can be defined as allocations which are both interim efficient and interim envy-free, are not globally implementable in D_1. The reason for this stems from the failure of interim efficient allocations to be globally implementable in D_1.

VI. Public Good Economies

In this Section we consider economies in which there are two goods, a private good, x, and a public good, ξ. The analysis is easily extended to economies with many private and many public goods, as long as the public goods are produced using a linear technology. Each individual in the economy is endowed with $w^i > 0$ units of the private good in each state. Production of each unit of the public good requires one unit of the private good. Thus the aggregate feasibility condition on allocations in the public good economy is

$$\sum_{i=1}^{I} z_i + \zeta = \bar{w}$$

Hence

$$A = \{(z,\zeta) \in R_+^{I+1} \mid \zeta = \sum_{i=1}^{I} (w^i - z^i)\}$$

The set of feasible state contingent allocations is

$$X = \{(x,\xi) : S \to A\}$$

We will denote the private good allocation at s by x(s) and the public good allocation at s by $\xi(s)$.

As before, the specification of an economy also includes its set of individuals, I, economic environments, S, the profile of individual partitions of S, Π, and the profile of individual (strictly positive) priors over the states, q.

With complete information and public goods a natural market equilibrium concept is Lindahl equilibrium. In fact, paralleling the results for private good economies, the Lindahl equilibrium is (fully) implementable via Nash equilibrium with complete information. With incomplete information the appropriate definition of public good equilibrium is not clear. However, in keeping with the parallel to Walrasian equilibrium, we consider <u>Rational Expectations Lindahl Equilibrium</u> (RELE) defined below. It is both the natural extension of REE to public good environments and the natural extension of Lindahl equilibrium to incomplete information economies. While the interpretation in terms of a market-type mechanism is quite awkward, these equilibria do inherit the global implementability properties of REE and (complete information) Lindahl equilibria. Whether these allocations (or REE allocations for that matter) can be implemented by a market-type mechanism is an open question.

Let { $p_x(s)$, $p_\xi^1(s)$, . . ., $p_\xi^I(s)$ } be a private good price function and an I-tuple of individual (Lindahl) public good price functions. For any price function, $p(\cdot)$, let

$$p^i(s) = [p_x(s), p_\xi^i(s)].$$

For any α compatible with Π and any $s \in S$, $s' = \alpha(s)$, let

$$E^i(p^i,s') = \{t' \in E^i(s') \mid p^i(t') = p^i(s')\}$$
$$E^i(p^i,s) = \{t \in E^i(s) \mid p^i(\alpha(t)) = p^i(s')\}$$

Notice that prices may be fully revealing for some individuals and <u>not</u> fully revealing to other individuals, since Lindahl prices will typically be different across individuals. At s, individual i chooses $(z^i,\xi) \in R^2$ to maximize

$$\sum_{t \in E^i(p^i,s)} q^i(t \mid E^i(p^i,s)) U^i(z^i,\zeta,t)$$

subject to

$$p_x(s)(w^i - z^i) \le p_\xi^i(s)$$

Markets must clear and prices must be consistent with profit maximization. A condition for the latter is

$$p_x(s) = \sum_{i=1}^{I} p^i(s) \quad \text{for all } s$$

Definition 9: A Rational Expectations Lindahl Equilibrium (RELE) is a state contingent allocation $(x^*, \xi^*) \in X$ and a collection of price functions $p_x^*(\cdot)$, $p_\xi^*(\cdot), \ldots, p_\xi^{*I}(\cdot)$ such that:

(1) For all i,s

$$x^{*i}(s),\ \xi^*(s) \text{ maximizes } \sum_{t \in E^i(p^i,s)} q^i(t|E^i(p^i,s))U^i(z^i,\xi,t)$$

$$\text{subject to } p_x(s)z^i + p_\xi^i(s)\xi \le p_x(s)w^i$$

(2) $\xi^*(s) = \sum_{i=1}^{I} [w^i - x^{*i}(s)]$ for all s

(3) $\sum_{i=1}^{I} p_\xi^i(s) = p_x(s)$ for all s

The domain of economies we consider here is a simple adaptation of D_1 to allow for public goods. Specifically it is the set of all linear economies with one private and one public good in which aggregate and individual endowments are state independent, utility functions are continuous, strictly increasing and strictly concave, S is finite and NEI holds. Call this domain D_3. The following theorem establishes that with the additional domain restriction to insure that Lindahl allocations will be strictly interior to consumption sets (indifference curves asymptote to the axes), the RELE SCC is globally implementable.

Theorem 4: Let $\mathcal{D}_3$ be the set of economies in D_3 such that

$$\{(z^i,\zeta) \in R_+^2 \mid U^i(z^i,\zeta,s) \ge \delta\} \subset R_{++}^2$$
$$\text{for all } \delta > 0$$

Then the RELE SCC is globally implementable relative to $\mathcal{D}_3$.

Proof: Suppose (x^*, ξ^*) is an RELE. Let α be compatible with Π. Suppose $(x_\alpha^*, \xi_\alpha^*)$ is not an RELE. Then there exists $s, i, (z^i, \zeta)$ such that, denoting $s' = \alpha(s)$,

$$\sum_{t \in E^i(p^i,s)} q^i(t|E^i(p^i,s)[U^i(z^i,\zeta,t) - U^i(x^{*i}(s'),\xi^*(s'),t)] > 0$$

and $p_x(s')z^i + p_\xi^i(s')\zeta \le p_x(s')w^i$.

By strict concavity of U, (z^i,ζ) can be made arbitrarily close to (x^{*i},ξ^*) so that $0 \ll z^i + \zeta \ll \bar{w}$. Hence there exists a (z^{-i},ζ) such that (z,ζ) is feasible. Let

$$y(t') = \begin{cases} (z,\zeta) & \text{if } t' \in E^i(p^i,s') \\ (x^*(t'),\xi^*(t')) & \text{if } t' \notin E^i(p^i,s') \end{cases}$$

The proof continues as in Theorem 2 to establish that the RELE social choice set is monotonic. The assumption on preferences ensures that all RELE allocations are in the interior of the feasible set. This condition also guarantees that private goods are valued enough, so that the proof of Theorem 1 can be easily extended to cover any economy in this domain. Therefore, the RELE SCC is globally implementable on this domain.

It is straightforward to construct examples to show that this result does not hold if w^i is state dependent or if (x^*,ξ^*) is on the boundary of some individual's consumption set. Counterexamples to implementability of efficient allocations (ex ante, interim, or ex post) with public goods are also easily constructed.

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