Separating metric perturbations in near-horizon extremal Kerr

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Linear perturbation theory is a powerful toolkit for studying black hole spacetimes. However, the perturbation equations are hard to solve unless we can use separation of variables. In the Kerr spacetime, metric perturbations do not separate, but curvature perturbations do. The cost of curvature perturbations is a very complicated metric-reconstruction procedure. This procedure can be avoided using a symmetry-adapted choice of basis functions in highly symmetric spacetimes, such as near-horizon extremal Kerr. In this paper, we focus on this spacetime, and (i) construct the symmetry-adapted basis functions; (ii) show their orthogonality; and (iii) show that they lead to separation of variables of the scalar, Maxwell, and metric perturbation equations. This separation turns the system of partial differential equations into one of ordinary differential equations over a compact domain, the polar angle.

I. INTRODUCTION

Linear metric perturbations are widely used in studying weakly-coupled gravity [1]. For example, it can be applied to investigating the stability of black holes, gravitational radiation produced by material sources moving in a curved background, and so on. In the context of linearized gravity, the equations that describe gravitational perturbations are the linearized Einstein equations (LEE). Although they are linear, the LEE are still difficult to solve unless we can separate variables. In the Kerr spacetime, while in Boyer-Lindquist (BL) coordinates t and ϕ can be separated, r and θ remain coupled due to lack of symmetry [2].

A successful approach towards separating wave equations for perturbations of the Kerr black hole was first developed by Teukolsky [3, 4]. Instead of looking at metric perturbations, Teukolsky adopted the Newman-Penrose (NP) formalism [5] and obtained a separable wave equation for Weyl curvature tensor components Ψ_0 and Ψ_4. The spin-weighted version of this equation, known as the Teukolsky equation, not only works for gravitational perturbations, i.e. tensor fields, but can also be applied to scalar, vector and spinor fields. To obtain the other Weyl scalars and recover the perturbed metric, one has to go through a complicated metric reconstruction procedure. The methods were independently developed by Chrzanowski [6] and by Cohen and Kegeles [7], in which they obtain the perturbed metric via an analogue of Hertz potentials. However, these methods only apply to certain gauge choices and vacuum or highly-restricted source terms [8].

The desire for separable equations, the complication of metric reconstruction along with gauge- and source-restrictions, motivate us to try to develop a new formalism for studying metric perturbations in the Kerr spacetime, in a covariant, gauge-invariant way.

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The metric perturbation equation may not be separable in Kerr, but Schwarzschild perturbations have long been known as separable due to the time translation invariance and spherical symmetry [9–12]. The gauge-independent language of Schwarzschild perturbations was started by Sarbach and Tiglio [13], and brought to fruition by Martel and Poisson [14]. In the Schwarzschild background, metric perturbations are expanded in scalar, vector, and symmetric tensor spherical harmonics. These basis functions naturally lead to separation of variables in the LEE.

Schematically, the separation of variables in some differential equations of motion, such as the scalar wave equation, Maxwell's equations, and the linearized Einstein equations, can all be understood via

$$\mathcal{D}_x\left[\begin{pmatrix}\text{symmetry}\\ \text{adapted}\\ \text{basis}\end{pmatrix}\times\begin{pmatrix}\text{dependence}\\ \text{on rest of}\\ \text{coordinates}\end{pmatrix}\right]=\begin{pmatrix}\text{symmetry}\\ \text{adapted}\\ \text{basis}\end{pmatrix}\times\mathcal{D}_{x'}\left[\begin{matrix}\text{dependence}\\ \text{on rest of}\\ \text{coordinates}\end{matrix}\right].$$

Here $\mathcal{D}_x[\cdot]$ is some isometry-equivariant differential operator. If the argument is decomposed in a natural isometry-adapted basis, then these basis functions pull straight through the differential operator, leaving new operators $\mathcal{D}_{x'}[\cdot]$ which only act on the remaining non-symmetry coordinates.

We show that this type of reduction is true for a special class of Kerr spacetime: the near-horizon extremal Kerr (NHEK). This spacetime was introduced in [15] as an analogue of $AdS_2 \times S^2$. Due to its similarity with the anti-de Sitter space AdS_2, the spacetime has four Killing vector fields that generate the isometry group $SL(2,\mathbb{R}) \times U(1)$. The three dimensional orbit space of the isometry reduces the system of partial differential equations to one of ordinary differential equations, leading to separable equations of motion. This is achieved by expanding unknown tensors into some basis functions adapted to the isometry. In this paper, we (i) construct these basis functions, (ii) prove orthogonality in geodesically-complete coordinates, and (iii) show separation of variables in the differential equations for some physical systems. With these accomplishments, we arrive at a new formalism to deal with (extremal) Kerr perturbation that differs from using metric reconstruction on solutions to the Teukolsky equation. There will be no gauge preference, no complications of solving PDEs but only ODEs. This greatly reduces the amount of work while studying perturbations of extremal black holes, whether in GR or beyond-GR theories.

We organize the paper as follows. In Sec. II we review the NHEK limit of the Kerr black hole, and elaborate on the structure of NHEK's isometry Lie group $SL(2,\mathbb{R})\times U(1)$. In Sec. III, we construct the highest weight module for NHEK's isometry group, and obtain the scalar/vector/symmetric tensor basis functions. In Sec. IV we present a proof of orthogonality for the basis functions in global coordinates. In Sec. V we show that with these bases, we can separate variables in the scalar Laplacian, Maxwell system, and linearized Einstein equation. Finally we conclude and discuss future work in Sec. VI.

II. KERR AND THE NHEK LIMIT

In this paper we choose geometric units ($G=c=1$) and signature $(-+++)$ for our metric g on the spacetime manifold $\mathcal{M}$. A rotating, asymptotically-flat black hole in vacuum general relativity is described by the Kerr metric [16]. For simplicity we will set the mass to $M=1$. In BL coordinates (t,r,θ,ϕ) the line element of the Kerr black hole is given by [17]

$$\mathrm{d}s^2 = -\frac{\Delta}{\Sigma}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi)^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)\,\mathrm{d}\phi - a\,\mathrm{d}t\right]^2, \tag{1}$$

where $\Delta = r^2 - 2r + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. The ranges of the BL coordinates are given by $t\in(-\infty,+\infty)$, $r\in(0,+\infty)$, $\theta\in[0,\pi]$, $\phi\in[0,2\pi)$. In this paper we focus on the near-horizon region of extremal Kerr, $a=M=1$. This region is usually described by the scaling coordinates (T,Φ,R) introduced in [15], which are related to the BL coordinates via

$$t = \frac{2T}{\lambda}\,, \qquad \phi = \Phi + \frac{T}{\lambda}\,, \qquad r = 1+\lambda R\,. \tag{2}$$

We also introduce a new coordinate u for the polar angle via $u=\cos\theta$. The NHEK limit is then obtained by taking the $\lambda\to 0$ limit of the extremal Kerr metric in these coordinates, which yields the line element

$$\mathrm{d}s^2 = 2\Gamma(u)\Big[-R^2\,\mathrm{d}T^2 + \frac{\mathrm{d}R^2}{R^2} + \frac{\mathrm{d}u^2}{1-u^2} + \Lambda(u)^2(\mathrm{d}\Phi + R\,\mathrm{d}T)^2\Big], \tag{3}$$

where $\Gamma(u) = (1+u^2)/2$ and $\Lambda(u) = 2\sqrt{1-u^2}/(1+u^2)$. This metric is interpreted on the region $T\in(-\infty,+\infty)$, $\Phi\in[0,2\pi)$, $R\in(0,+\infty)$, $u\in[-1,1]$.

From now on we will refer to (T,Φ,R,u) as *Poincaré coordinates*. The T,R-coordinates of NHEK are similar to the Poincaré coordinates on the two dimensional anti-de Sitter space AdS_2, which only cover a subspace of the global spacetime called the *Poincaré patch*. In particular, the $u=\pm1$ submanifolds are both precisely AdS_2. We can make this metric geodesically complete by defining the *global coordinates* (τ,φ,ψ,u) according to [15]

$$T = \frac{\sin\tau}{\cos\tau-\cos\psi}\,, \quad R = \frac{\cos\tau-\cos\psi}{\sin\psi}\,, \quad \Phi = \varphi + \ln\left|\frac{\cos\tau - \sin\tau\cot\psi}{1+\sin\tau\csc\psi}\right|, \tag{4}$$

where $\tau\in(-\infty,+\infty)$, $\psi\in[0,\pi]$, $\varphi\sim\varphi+2\pi$. The NHEK metric in global coordinates is

$$\mathrm{d}s^2 = 2\Gamma(u)\Big[(-\mathrm{d}\tau^2+\mathrm{d}\psi^2)\csc^2\psi + \frac{\mathrm{d}u^2}{1-u^2} + \Lambda(u)^2(\mathrm{d}\varphi - \cot\psi\,\mathrm{d}\tau)^2\Big]. \tag{5}$$

The NHEK spacetime has four Killing vector fields (KVFs), which generate the isometry group $G\equiv SL(2,\mathbb{R})\times U(1)$. The four generators in Poincaré coordinates are given by

$$\begin{aligned} H_0 &= T\,\partial_T - R\,\partial_R\,, \\ H_+ &= \partial_T\,, \\ H_- &= (T^2+\frac{1}{R^2})\,\partial_T - 2\,TR\,\partial_R - \frac{2}{R}\,\partial_\Phi\,, \\ Q_0 &= \partial_\Phi\,. \end{aligned} \tag{6}$$

H_0 is the infinitesimal generator of *dilation*, which leaves the metric invariant under $R\to cR$ and $T\to T/c$ for some constant $c\in(0,+\infty)$. Q_0 is the generator of the rotation along Φ which generates the $U(1)$ group. H_+ is the time translation generator inherited from Kerr. The four generators form a *representation* ρ_P of the Lie algebra $\mathfrak{g}\equiv\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{u}(1)$,

$$\begin{aligned} [H_0\,,H_\pm] &= \mp H_\pm\,, \\ [H_+\,,H_-] &= 2\,H_0\,, \\ [H_s\,,Q_0] &= 0\,. \qquad (s=0,\pm) \end{aligned} \tag{7}$$

In global coordinates, we can similarly obtain four (different) generators that are KVFs of the NHEK spacetime,

$$\begin{aligned} L_\pm &= ie^{\pm i\tau}\sin\psi(-\cot\psi\partial_\tau \mp i\partial_\psi + \partial_\varphi), \\ L_0 &= i\partial_\tau, \\ W_0 &= -i\partial_\varphi. \end{aligned} \tag{8}$$

This is a different representation, ρ_g. But since it is still a Lie algebra representation, they satisfy the same commutation relations as in Eq. (7) with all H's replaced by L's, and Q_0 replaced W_0.

We say that the group G acts on the manifold $\mathcal{M}$ by translation, $G \circlearrowleft \mathcal{M}$. That is, every element $g\in G$ determines an isomorphism $\phi_g:\mathcal{M}\to\mathcal{M}$, and these isomorphisms, under composition, form a representation of the group G. There is an induced action on the space of functions/vector fields/forms/tensors/etc. living on $\mathcal{M}$ by pullback under the map ϕ_g [18]. We call the pullback ϕ_g^*, overloading this symbol to mean the pullback from sections of any tensor bundle to itself. In this way, the group also acts on all spaces of (p,q)-tensors.

Studying the neighborhood of the identity $e \in G$, we get the induced action of the Lie algebra $\mathfrak{g}$ on these same tensor bundles. The infinitesimal version of a pullback of a tensor field is the Lie derivative of that field [18]. Thus the induced action of $\mathfrak{g}$ on tensors is Lie derivation along the representation of the Lie algebra element. That is, given a representation as tangent vector fields $\rho : \mathfrak{g} \to \mathfrak{X}(\mathcal{M})$, for some algebra element $\alpha \in \mathfrak{g}$, the induced action of α on a tensor $\mathbf{t}$ is via the Lie derivative,

$$\alpha \cdot \mathbf{t} = \mathcal{L}_{\rho(\alpha)} \mathbf{t} \,. \tag{9}$$

One of the crucial algebra elements we will need is the Casimir element of the $\mathfrak{sl}(2,\mathbb{R})$ factor. Let $h_0, h_\pm \in \mathfrak{g}$ be the algebra elements whose representations are respectively $\rho_P(h_0) = H_0$, etc. Then the Casimir element of the $\mathfrak{sl}(2,\mathbb{R})$ factor, in this basis, is proportional to

$$\Omega \equiv h_0(h_0 - 1) - h_- h_+ \,, \tag{10}$$

which commutes with every element of $\mathfrak{g}$. Under the Poincaré-coordinates representation ρ_P, the Casimir acts on tensors via

$$\Omega \cdot \mathbf{t} = \left(\mathcal{L}_{H_0}(\mathcal{L}_{H_0} - \mathrm{id}) - \mathcal{L}_{H_-}\mathcal{L}_{H_+} \right) \mathbf{t} \,. \tag{11}$$

By construction, the differential operator on the right hand side of Eq. (11) commutes with $\mathcal{L}_X$, where X is one of $\{H_0, H_\pm, Q_0\}$. Similarly, under the global-coordinates representation ρ_g, the Casimir acts as in Eq. (11), but with H's replaced with L's; and this operator will similarly commute with $\mathcal{L}_X$ where X is one of $\{L_0, L_\pm, W_0\}$.

III. THE HIGHEST (LOWEST) WEIGHT METHOD

In this section we construct the scalar, vector and symmetric tensor bases for NHEK's isometry group $SL(2,\mathbb{R}) \times U(1)$. First we briefly review the formalism of finding basis functions adapted to the isometry group in Schwarzschild spacetime. By drawing analogy to the Schwarzschild case and further utilizing the *highest (lowest) weight method* for non-compact groups, we will be able to construct unitary representations of NHEK's isometry group.

A. Review: Unitary representations of $SO(3)$ in Schwarzschild

The full spacetime manifold of Schwarzschild spacetime is $\mathcal{M}_{\mathrm{Sch}} = M^2 \times S^2$. The two dimensional submanifold M^2 is the $(\bar{t},\bar{r})$-plane, and S^2 is the unit two-sphere coordinated by $(\bar{\theta},\bar{\phi})$. Here $(\bar{t},\bar{r},\bar{\theta},\bar{\phi})$ are the usual Schwarzschild coordinates. Part of the isometry group of Schwarzschild is $SO(3)$, which acts on the S^2 factors. The three generators of the group are simply the rotations along each Cartesian axis, i.e. $J_x, J_y, J_z \in \mathfrak{so}(3)$. The Casimir operator of $\mathfrak{so}(3)$ is given by $J^2 = J_x^2 + J_y^2 + J_z^2$.

In any space that $SO(3)$ acts upon, we can look for bases of functions which simultaneously diagonalize J^2 and J_z—that is, they are eigenfunctions of both operators. In the space of complex functions on the unit sphere, these eigenfunctions turn out to be the spherical harmonic functions $Y^{\mu,\nu}$, where μ,ν label the functions (they are not tensor indices). The even/odd parity vector harmonics, $Y_A^{\mu,\nu}, X_A^{\mu,\nu}$, and tensor harmonics, $Y_{AB}^{\mu,\nu}, X_{AB}^{\mu,\nu}$, are also simultaneous eigenfunctions of J^2 and J_z (where now A,B are (co-)tangent indices on S^2). All of the scalars, vectors, and tensors here have eigenvalue $-\mu(\mu+1)$ for the operator J^2, and eigenvalue $i\nu$ for J_z.

Under any rotation, scalar spherical harmonics with different values of μ may not rotate into each other. In this sense, the function space has been split up into diagonal blocks labeled by μ. We say that each μ block "lives in" or "transforms under" a representation of $SO(3)$. Among physicists it is common to say that the space labeled by μ *is* a representation, but technically it is an $SO(3)$-module.

We have not yet imposed regularity or tried to make these representations unitary. Let us define the raising and lowering operators $J_\pm = J_x \pm i J_y$, which increase/decrease the ν index (eigenvalue of $-iJ_z$) by one. A *highest weight* state is one which is annihilated by the raising operator, $J_+ f = 0$, and similarly a lowest weight state is annihilated by the lowering operator. For spherical harmonics, we find that the highest weight condition imposes that $\nu = \mu$, and $Y^{\mu,\mu}$ is annihilated by J_+. Similarly, the lowest weight condition imposes that $\nu = -\mu$.

From the representation theory of compact simple Lie groups, irreducible unitary representations must be finite-dimensional [19]. Therefore, if we start with a highest-weight state $Y^{\mu,\mu}$, after a finite number of applications of the lowering operator, we must end on a lowest-weight state $Y^{\mu,-\mu}$. This gives us the condition that $2\mu+1$ is a positive integer, or $\mu = 0, \frac{1}{2}, 1, \ldots$. Periodicity in the azimuthal angle $\bar{\phi}$ gives the condition that ν must be an integer m. This gives the ordinary spherical harmonics $Y^{l,m}$. The same arguments apply to the vector and tensor representations.

Since these bases are adapted to the isometry group of Schwarzschild, they readily lead to a separation of variables in the linearized Einstein equations [14].

B. Unitary representations of $SL(2,\mathbb{R}) \times U(1)$ in NHEK

We now apply the highest/lowest weight formalism to NHEK. In the Schwarzschild spacetime, the orbit space of the isometry $SO(3)$ is S^2, therefore we expect a $2+2$ decomposition of the whole manifold. Similarly, in the NHEK spacetime, the isometry group $SL(2,\mathbb{R}) \times U(1)$

acts on the three dimensional hypersurfaces Σ_u of constant polar angle θ (or u). This enables us to perform a $3+1$ decomposition of the spacetime. In both cases, we can simultaneously diagonalize some algebra elements, including the Casimir, in various tensor spaces.

However there is an important difference between the two spacetimes. In the NHEK case, we encounter the non-compact group $SL(2,\mathbb{R})$. It is known that for non-compact simple Lie groups like $SL(2,\mathbb{R})$, the only irreducible unitary finite-dimensional representation is the trivial representation [19]. As a result, one can find two distinct unitary representations of $SL(2,\mathbb{R})\times U(1)$: the *highest weight module* or the *lowest weight module*. Both of them are infinite dimensional representations in the NHEK case. For compact groups like $SO(3)$, these two modules coincide.

Our method to find the general (scalar, vector, and symmetric tensor) basis functions ξ associated with the highest weight module of NHEK's isometry can be summarized into four steps. Notice that the method presented here is not restricted to NHEK spacetime. For instance it can also be applied to finding the basis functions in near-horizon near-extremal Kerr (near-NHEK) which has the same isometry group as NHEK's [20]. This will be left for future work. For readers who are more interested in what the bases of NHEK's isometry look like either in Poincaré or global coordinates, the explicit expressions are given in App. A.

a. Orbit space. The orbit of a point $p\in\mathcal{M}$ is $Gp=\{\phi_g(p)|g\in G\}$, all points which are related to p by an $SL(2,\mathbb{R})\times U(1)$ transformation. Gp is a 3-dimensional submanifold of $\mathcal{M}$, and the collection of all the orbit spaces forms a foliation. In this case, each leaf Σ_u is a surface of constant θ (or u). Thus we can perform a $3+1$ decomposition of the spacetime, and look for basis functions of $SL(2,\mathbb{R})\times U(1)$ acting on a hypersurface Σ_u.

b. Highest weight states. Second, we simultaneously diagonalize $\{\mathcal{L}_{Q_0},\mathcal{L}_{H_0},\Omega\}$ in the space of scalar, vector, and symmetric tensor functions. We label the eigenstates by m,h,k respectively,

$$\begin{aligned}\mathcal{L}_{Q_0}\,\xi^{(m\,h\,k)}&=im\,\xi^{(m\,h\,k)}\,,\\ \Omega\,\xi^{(m\,h\,k)}&=h(h+1)\,\xi^{(m\,h\,k)}\,,\\ \mathcal{L}_{H_0}\,\xi^{(m\,h\,k)}&=(-h+k)\,\xi^{(m\,h\,k)}\,.\end{aligned}\tag{12}$$

We also impose the highest weight condition, $k=0$,

$$\mathcal{L}_{H_+}\,\xi^{(m\,h\,0)}=0\,.\tag{13}$$

The solutions $\xi^{(m\,h\,0)}$ that satisfy both Eq. (12) and (13) are the highest weight basis functions. At each point on Σ_u, the spaces of scalars, vectors, and symmetric tensors have dimensions 1, 3, and 6. Thus the space of solutions of this system of equations is a linear vector space of dimension 1, 3, and 6 for scalars, vectors, and symmetric tensors, for each choice of (m,h). Correspondingly, for each (m,h), there will be 1, 3, and 6 free coefficients c_β for the solution, with β ranging over the appropriate dimensionality.

c. Descendants. Next, we obtain basis functions with arbitrary weight by applying the lowering operator $\mathcal{L}_{H_-}$ to the highest weight states k times, i.e.

$$\xi^{(m\,h\,k)}=(\mathcal{L}_{H_-})^k\,\xi^{(m\,h\,0)}.\tag{14}$$

d. Lifting to the whole manifold. Finally, we promote the basis functions living on Σ_u to functions living on the whole manifold $\mathcal{M}$ by sending all unknown constant coefficients c_β to be unknown smooth functions $c_\beta(u)$. While lifting the vector and tensor bases from Σ_u to $\mathcal{M}$, i.e. $V_i\to V_a$ and $W_{ij}\to W_{ab}$, we also set all their projections on u-direction to be zero, i.e. $V_u=0$, $W_{iu}=W_{ui}=W_{uu}=0$.

To obtain the basis functions in global coordinates, one just replaces H_s by L_s, where $s=0,\pm$, and Q_0 by $-iW_0$ in steps b and c. To construct the lowest weight modules of NHEK's isometry group, one should instead impose the lowest weight condition $\mathcal{L}_{H_-}\,\xi^{(m\,h\,0)}=0$ in step b. All descendant states will then be obtained by applying the raising operator $\mathcal{L}_{H_+}$ on the lowest weight states. In Poincaré coordinates, we focus on the basis functions that form the highest weight module because their expressions are simpler. In global coordinates, we show both representations explicitly in App. A 2 a and A 2 b. Unless otherwise specified, our basis functions will refer to those obtained using the highest weight method.

Let us remark on the allowed values of m, h, k. It is straightforward to see $k\in\mathbb{Z}$ since it labels the weight and $m\in\mathbb{Z}$ due to the periodic boundary conditions for the azimuthal angle. In order to have a unitary representation of the isometry group, there are conditions on h as well. For the scalar case, for instance, if we apply the raising operator on a scalar in the highest weight module, we get

$$\mathcal{L}_{H_+}\,F^{(m\,h\,k)}=k(k-1-2h)\,F^{(m\,h\,k-1)}.\tag{15}$$

A nontrivial unitary representation of NHEK's isometry group then requires $k-1-2h\neq 0$, otherwise there would be a lowest weight state that would lead to a finite-dimensional (and hence non-unitary) representation. The same conclusion holds for either the vector or the tensor bases. The values of h also depend on the regularity conditions we impose. For instance, in global coordinates, the highest weight scalar basis is proportional to

$$F^{(m\,h\,0)}\propto(\sin\psi)^{-h}\exp[i(h\tau+m\varphi)+m\psi].\tag{16}$$

Regularity at the boundaries $\psi=0$ and $\psi=\pi$ requires $h\leq 0$. Another example is given in Sec. V B when we solve for the free massless scalar wave equation in the NHEK spacetime, where h must take on some fixed values due to the regularity conditions for spheroidal harmonics.

IV. ORTHOGONALITY IN GLOBAL COORDINATES

In this section we present a proof that all the scalar, vector, and symmetric tensor basis functions of NHEK's isometry group, when given in global coordinates, form orthogonal basis sets. In this proof we will use the vector basis functions defined on Σ_u as an example. That is, they are functions of τ, φ, ψ. As we shall see, lifting to the whole manifold $\mathcal{M}$ and extending the proof to the scalar and symmetric tensor cases will be straightforward.

Let us introduce the metric induced on the hypersurface Σ_u as γ_{ij}, and D is the unique torsion-free Levi-Civita connection that is compatible with γ. Here Latin letters in the middle of the alphabet (i, j, k) denote 3-dimensional tangent indices on Σ_u. Consider the vector basis function $\mathbf{u}^{(m\,h\,k)}(\tau, \varphi, \psi)$ and $\mathbf{v}^{(m'\,h'\,k')}(\tau, \varphi, \psi)$. We would like to show bases with different m, h, k are orthogonal,

$$\langle \mathbf{u}, \mathbf{v} \rangle \equiv \int_{\Sigma_u} \mathrm{dVol}\, \overline{u_i^{(m\,h\,k)}} v^i_{(m'\,h'\,k')} \propto \delta_{m,m'} \delta_{h,h'} \delta_{k,k'} \,. \tag{17}$$

Here the overbar denotes complex conjugation, and the volume element is given by

$$\int_{\Sigma_u} \mathrm{dVol} = \lim_{T\to\infty} \int_{-T}^{T} \mathrm{d}\tau \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\psi \sqrt{-\gamma} \,, \tag{18}$$

where γ is the determinant of the three dimensional metric, and in these coordinates $\sqrt{-\gamma} = 2 \csc^2 \psi \sqrt{1-u^4}$. In order to prove Eq. (17) we first note the basis components $v_j^{(m\,h\,k)}$ in global coordinates have the following τ and φ dependence,

$$v_j^{(m\,h\,k)} \sim \exp\left(im\varphi\right) \exp\left[i(h-k)\tau\right]. \tag{19}$$

This dependence on τ and φ is the same for the scalar and tensor basis components. Once we integrate over φ and τ in Eq. (17), the integral will be proportional to $\delta_{m,m'}\delta_{h-k,h'-k'}$. Notice that the boundaries $\tau \to \pm\infty$ are oscillatory, so the τ integral needs to be regulated in the same way as Fourier integrals.

Now we only need to show bases with different weight k are orthogonal. Once this is done we will recover Eq. (17). For simplicity, from now on we only track the k-index in the vector bases. Recall that we obtain the lower weight bases by applying the lowering operator order by order,

$$\langle \mathbf{u}^{(k)}, \mathbf{v}^{(k')} \rangle = \langle \mathbf{u}^{(k)}, \mathcal{L}_{L_-} \mathbf{v}^{(k'-1)} \rangle \,. \tag{20}$$

Now we attempt to "integrate by parts" with the Lie derivative,

$$\langle \mathbf{u}^{(k)}, \mathcal{L}_{L_-} \mathbf{v}^{(k'-1)} \rangle = \int_{\Sigma_u} \mathcal{L}_{L_-} \left(\overline{u_i^{(k)}} v^i_{(k')} \right) \mathrm{dVol} - \langle \overline{\mathcal{L}_{L_-}} \mathbf{u}^{(k)}, \mathbf{v}^{(k'-1)} \rangle, \tag{21}$$

$$= \int_{\Sigma_u} \mathcal{L}_{L_-} \left(\overline{u_i^{(k)}} v^i_{(k')} \right) \mathrm{dVol} + \langle \mathcal{L}_{L_+} \mathbf{u}^{(k)}, \mathbf{v}^{(k'-1)} \rangle, \tag{22}$$

where in the last line we used the fact that $\overline{L_+} = -L_-$. Note that this relationship does not hold between $H_\pm$, so this type of proof will not work in Poincaré coordinates.

We would like to discard the first term on the RHS of Eq. (21), which would show that $\mathcal{L}_{L_+}$ and $\mathcal{L}_{L_-}$ are adjoints of each other. We can do this by converting the Lie derivative into a covariant derivative and then a total divergence. Since $L_\pm$ are KVFs, they are automatically divergence-free, so we can pull them inside the covariant derivative:

$$\int_{\Sigma_u} \mathrm{dVol}\, \mathcal{L}_{L_-} \left(\overline{u_i^{(k)}} v^i_{(k')} \right) = \int_{\Sigma_u} \mathrm{dVol}\, L_-^j D_j \left(\overline{u_i^{(k)}} v^i_{(k')} \right) = \int_{\Sigma_u} \mathrm{dVol}\, D_j \left(L_-^j \overline{u_i^{(k)}} v^i_{(k')} \right) \,. \tag{23}$$

This step is identical if we are considering scalars/vectors/tensors, since the argument of the Lie derivative has all indices contracted. Using Stokes' theorem, the integral of the total derivative becomes a boundary integral, evaluated at $\psi = 0, \pi$. This boundary contribution vanishes for $h < -1$.

We repeat the procedure of extracting lowering operators from the ket as in Eq. (21), and arrive at

$$\langle \mathbf{u}^{(k)}, \mathbf{v}^{(k')} \rangle = \langle \left(\mathcal{L}_{L_+}\right)^{k'} \mathbf{u}^{(k)}, \mathbf{v}^{(0)} \rangle \,. \tag{24}$$

Recall that the vector basis terminates at the highest weight. Therefore when $k' > k$, $\left(\mathcal{L}_{L_+}\right)^{k'} \mathbf{u}^{(k)}$ will vanish. Similarly when $k' < k$, we can extract all lowering operators from the bra and raise the weight of the states in the ket, which will terminate upon raising the highest weight state. Therefore the vector bases with different weights k, k' are orthogonal.

Since we have also proved that vector bases with different m and $h-k$ are orthogonal, the proof of orthogonality for vector bases is done. It may not be obvious that the proof holds unaltered for scalars/vectors/tensors. In all the relevant steps above, we have noted where each argument works for each of the three types of fields.

Therefore we arrive at the conclusion that the scalar, vector, and symmetric tensor bases in global coordinates form orthogonal basis sets. □

V. SEPARATION OF VARIABLES

In this section we show that with the scalar, vector, and tensor bases we have obtained, it is possible to separate variables for many physical systems in NHEK spacetime. One can show that all conclusions in this section hold for both Poincaré coordinates and global coordinates. In global coordinates the results are in general more complicated. Therefore for concreteness all results in this section are given in Poincaré coordinates.

The main result of this section can be summarized with the schematic equation:

$$\mathcal{D}_x\left[\left(\begin{smallmatrix} SL(2,\mathbb{R})\times U(1) \\ \text{structure }(T,\Phi,R)\end{smallmatrix}\right)^{(m,h,k)}\times\left(\begin{smallmatrix} u\text{ (or }\cos\theta) \\ \text{dependence}\end{smallmatrix}\right)\right]= \left(\begin{smallmatrix} SL(2,\mathbb{R})\times U(1) \\ \text{structure }(T,\Phi,R)\end{smallmatrix}\right)^{(m,h,k)}\times\mathcal{D}_u^{(m,h)}\left[\begin{smallmatrix} u\text{ (or }\cos\theta) \\ \text{dependence}\end{smallmatrix}\right].$$

Here, $\mathcal{D}_x$ is an $SL(2,\mathbb{R})\times U(1)$-equivariant differential operator, which takes derivatives in the T,Φ,R,u directions. We completely specify the T,Φ,R dependence by being in a certain irrep of $SL(2,\mathbb{R})\times U(1)$ labeled by (m,h,k). Then the $SL(2,\mathbb{R})\times U(1)$ structure factors straight through the differential operator $\mathcal{D}_x$, leaving a new differential operator $\mathcal{D}_u^{(m,h)}$ which only takes u derivatives. This greatly simplifies computations, since the partial differential equations have been converted into ordinary differential equations (ODEs). Because of the $SL(2,\mathbb{R})\times U(1)$-invariance, notice that $\mathcal{D}_u^{(m,h)}$ only depends on m and h, which label the irrep, and not on k, which labels the descendant number within the irrep.

A. Covariant differentiation preserves isometry group irrep labels

Let us first make a general statement about how the presence of a group of isometries acting on the manifold can be useful in separation of variables. The conclusions obtained in this subsection will also justify our motivations of finding group representations for NHEK's isometry. Consider a manifold $\mathcal{M}$ with metric g_{ab}, metric-compatible connection ∇, and an isometry Lie group G acting on the manifold. Let $\alpha_{(i)}\in\mathfrak{g}$ be a basis for the Lie algebra, with representation $\{X_{(i)}\}$ on the manifold. Further, let $c^{(i)(j)}$ be the inverse of the Killing form of the Lie algebra in this basis [19]. Then we also have a quadratic Casimir element, which acts on any tensor $\mathbf{t}$ as

$$\Omega\cdot\mathbf{t}\equiv\sum_{i,j}c^{(i)(j)}\mathcal{L}_{X_{(i)}}\mathcal{L}_{X_{(j)}}\mathbf{t}\,. \tag{25}$$

Irreps of G will be labeled by eigenvalues λ_i of *some* of the KVFs, and the eigenvalue ω of the Casimir Ω.

First, we need a lemma on the commutation relation of manifold isometries and covariant derivatives,

$$\left[\mathcal{L}_{X_{(i)}},\nabla_a\right]\mathbf{t}=0, \tag{26}$$

where $\mathbf{t}$ can be a scalar, vector, or tensor. To prove Eq. (26), one can start by showing the commutation relations for $\mathbf{t}$ being a 0-form (which follows immediately from Cartan's magic formula for a 0-form) and a one-form, then use the Leibniz rule to generalize the relations to the vector and tensor cases. Eq. (26) says that the operator ∇_a is $SL(2,\mathbb{R})\times U(1)$-*equivariant*: that is, its action commutes with left-translation by the group [18].

An important consequence of the commutation relation Eq. (26) is that the Casimir element Ω of the algebra $\mathfrak{g}$ also commutes with the covariant derivative. Simply commute each Lie derivative one at a time, and the coefficients $c^{(i)(j)}$ are constants. As a result,

$$[\Omega,\nabla_a]\,\mathbf{t}=0. \tag{27}$$

Now consider a tensor $\mathbf{t}$ living in an irrep with labels λ_i and ω, meaning

$$\mathcal{L}_{X_{(i)}}\mathbf{t}=\lambda_i\mathbf{t}\,, \tag{28}$$

$$\Omega\cdot\mathbf{t}=\omega\mathbf{t}\,. \tag{29}$$

As an immediate consequence of Eq. (26) and Eq. (27) is that $\nabla\mathbf{t}$ has the same labels λ_i and ω,

$$\mathcal{L}_{X_{(i)}}\nabla\mathbf{t}=\lambda_i\nabla\mathbf{t}\,, \tag{30}$$

$$\Omega\cdot\nabla\mathbf{t}=\omega\nabla\mathbf{t}\,. \tag{31}$$

Thus any differential operator which is built just from ∇_a and the metric g_{ab} can not mix tensors with different irrep labels (λ_i,ω). This even extends to differential operators which include the Levi-Civita tensor ϵ and the Riemann tensor R_{abcd}, because these two objects are also annihilated by all of the $\mathcal{L}_{X_{(i)}}$. As a result, when tensors are decomposed into a sum over irreps with different labels, they will remain separated in the same ways under this type of differential operator. This is the underlying reason why the method of finding the unitary irreps of NHEK's isometry introduced in Sec. III will lead to separation of variables in many physical systems.

B. Scalar Laplacian

As the first example, we look at the massless scalar wave equation $\Box\psi=S$ in NHEK space time, where S is a source term. Since the scalar d'Alembert operator $\Box\equiv\nabla^a\nabla_a$ is built only from g_{ab} and ∇_a, it should commute with Ω and $\mathcal{L}_X$ where X is any KVF. To show this explicitly, note that in Poincaré coordinates, $\Box\psi$ can be written as

$$\Box\psi=\frac{1}{2\Gamma(u)}\left\{(\Omega+\Xi(u)\mathcal{L}^2_{Q_0})\psi+\mathcal{L}_{\partial_u}\left[(1-u^2)\mathcal{L}_{\partial_u}\psi\right]\right\}, \tag{32}$$

where $\Xi(u)\equiv\Lambda(u)^{-2}-1$.

Assume we can decompose an arbitrary scalar field

$\psi(T,\, \Phi,\, R,\, u)$ according to

$$\psi = \sum_{mhk} C_{mhk}(u) F^{(m\,h\,k)}(T,\, \Phi,\, R) \qquad (33)$$
$$= \sum_{mhk} \psi_{mhk}(T,\, \Phi,\, R,\, u),$$

where F is the scalar basis on Σ_u and C_{mhk} are some unknown functions of u. We also decompose the source term using the scalar basis functions via $S = \sum_{mhk} S_{mhk}$. The basis functions $F^{(m,h,k)}$ are eigenfunctions of Ω and $\mathcal{L}_{Q_0}$, and so ψ_{mhk} are also eigenfunctions. Therefore it is straightforward to see that the (T, Φ, R)-dependence in ψ_{mhk} is invariant after applying the scalar box operator. The equation for a specific mode labeled by (m, h, k) becomes

$$S_{mhk} = \Box^{(m,h)} \psi_{mhk} = \frac{1}{2\Gamma(u)} \times \qquad (34)$$
$$\times \left\{ [h(h+1) - m^2 \Xi(u)] \psi_{mhk} + \mathcal{L}_{\partial_u} \left[(1-u^2) \mathcal{L}_{\partial_u} \psi_{mhk} \right] \right\}.$$

This entire equation is proportional to the basis function $F^{(m,h,k)}$, which can thus be divided out, leaving an ODE for one function, $C_{mhk}(u)$.

Specializing to the homogeneous (source-free) case, we find the ODE

$$\frac{d}{du} \left[(1-u^2) \frac{d}{du} C_{mhk} \right] + \left[h(h+1) - \Xi(u) m^2 \right] C_{mhk} = 0\,. \qquad (35)$$

This equation has two regular singularities $u = \pm 1$ and an irregular singularity of rank 1 at $u = \infty$, which falls into the class of confluent forms of Heun's Equation [21]. Explicitly, it is a spheroidal differential equation, whose standard form is

$$\frac{\mathrm{d}}{\mathrm{d}u} \left((1-u^2) \frac{\mathrm{d}\varphi}{\mathrm{d}u} \right) + \left(\lambda + \gamma^2 (1-u^2) - \frac{\mu^2}{1-u^2} \right) \varphi = 0, \qquad (36)$$

where we have made the substitution $\lambda = h(h+1) + 2m^2$, $\gamma^2 = -m^2/4$ and $\mu^2 = m^2$. When $\gamma = 0$, Eq. (36) reduces to the Legendre differential equation and the solutions are Legendre polynomials. When $\gamma^2 < 0$, the general solution of Eq. (36) is the same as the radial solution of spheroidal equation in the oblate spheroidal coordinates ($u = i\xi$),

$$\varphi(\xi) = a_1 S_n^{\mu(1)}(i\xi, \gamma) + b_1 S_n^{\mu(2)}(i\xi, \gamma), \qquad (37)$$

where $S_n^{\mu(1)}(i\xi, \gamma)$ and $S_n^{\mu(2)}(i\xi, \gamma)$ are radial spheroidal harmonics. These solutions only exist for eigenvalues $\lambda = \lambda_n^\mu(\gamma^2)$, where $\mu = 0, 1, 2, \ldots$, and $n = \mu, \mu+1, \mu+2, \ldots$. Thus, there are only discrete values of the irrep label h which satisfy regularity at the poles $u = \pm 1$.

C. Maxwell system

Let's look at another system of physical importance, the Maxwell system, and verify that we can separate variables in Maxwell's equations. The inhomogeneous Maxwell equations in presence of a source vector field $\mathcal{J}$ are

$$\nabla_a \mathcal{F}^{ab} = \mathcal{J}^b, \qquad (38)$$

where the electromagnetic tensor $\mathcal{F}$ is built from the vector potential A according to

$$\mathcal{F}_{ab} = \nabla_a A_b - \nabla_b A_a. \qquad (39)$$

We again assume that we can expand the vector potential in the scalar and vector bases. Define a one-form $n_a = \mathrm{d}u$, this expansion is given by

$$A^a = \sum_{mhk} \left(C_u(u) n^a F^{(m\,h\,k)} + \sum_B C_B(u) V_B^{a\,(m\,h\,k)} \right), \qquad (40)$$

where $B \in \{T, \Phi, R\}$, $C_B(u)$ and $C_u(u)$ are unknown functions of u. Notice that B is *not* a tensor index. It is the label of a specific choice of vector bases and their corresponding unknown C-functions. The expression of $F^{(m\,h\,k)}$ and the projection of $V_B^{a\,(m\,h\,k)}$ onto Σ_u, i.e. $V_B^{i\,(m\,h\,k)}$ are both given in App. A 1. Then at the highest weight $k = 0$, the left hand side of Maxwell's equation can be rewritten as

$$\nabla_a \mathcal{F}^{ab}|_{k=0} = \mathcal{D}_u^{(m,h)}[\mathbf{C}(u)] n^b F^{(m\,h\,0)} \qquad (41)$$
$$+ \sum_B \mathcal{D}_B^{(m,h)}[\mathbf{C}(u)] V_B^{b\,(m\,h\,0)}\,,$$

where we have collected the four C-functions into the vector $\mathbf{C}(u)$, and defined the general differentiation as $\mathcal{D}^{(m,h)}[\mathbf{C}(u)]$, whose expressions are given in App. B. As long as the source field can also be decomposed using the scalar and vector bases, the inhomogeneous Maxwell equations in NHEK will reduce to four ordinary differential equations with four unknown C-functions. Although we only show this is true for the highest weight case, this conclusion holds for any k. This is due to the commutation of the lowering operator and the covariant differentiation. For explicit calculations of Maxwell's system using the highest weight vector basis we refer our readers to [22, 23].

D. Linearized Einstein system

In this subsection we show that we can separate variables on the left hand side of linearized Einstein equation, using our scalar, vector, and tensor bases for NHEK. Consider the metric perturbation $g'_{ab} = g_{ab} + \epsilon h_{ab} + \mathcal{O}(\epsilon^2)$, where g_{ab} is the NHEK metric and h_{ab} is a perturbation. The linearized Einstein equations (i.e. at order ϵ^1) are

$$G_{ab}^{(1)}[h] = 8\pi T_{ab}\,, \qquad (42)$$

where T_{ab} is the stress-energy tensor of a source term. The linearized Einstein operator $G^{(1)}[h]$ can be written in terms of the background covariant derivative ∇ as

$$-2G^{(1)}_{ab}[h] = \Box\overline{h}_{ab} + g_{ab}\nabla^c\nabla^d\overline{h}_{cd} - 2\nabla^c\nabla_{(a}\overline{h}_{b)c} - g_{ab}R^{cd}\,\overline{h}_{cd} + R\,\overline{h}_{ab}\,, \tag{43}$$

where $\overline{h}_{ab} = h_{ab} - \frac{1}{2}g_{ab}g^{cd}h_{cd}$ is the trace-reverse of h_{ab}, R_{ab} is the background Ricci curvature, R is the background Ricci scalar, and parentheses around n indices means symmetrizing with a factor of $1/n!$. This operator, again, is $SL(2,\mathbb{R})\times U(1)$-equivariant.

We assume that we can expand the metric perturbation in our scalar, vector, and tensor bases, according to

$$h_{ab} = \sum_{mhk} h^{(m\,h\,k)}_{ab} = \sum_{mhk}\Bigg(n_a n_b F^{(m\,h\,k)} C_{uu}(u) \tag{44}$$
$$+ \sum_B 2n_{(a}V^{B(m\,h\,k)}_{b)} C_{uB}(u) + \sum_{A,B} W^{AB(m\,h\,k)}_{ab} C_{AB}(u)\Bigg),$$

where $A, B \in \{T, \Phi, R\}$, C_{uu}, C_{uB}, C_{AB} are unknown functions of u. Notice that A and B are *not* tensor indexes but only labels of a specific choice of the vector and tensor bases (introduced in App. A 1 b and A 1 c) and their corresponding unknown C-functions. Thus there are no differences between a subscript and a superscript A or B. We choose the three highest weight vector bases $V^{B(m\,h\,0)}_b$ and the six highest weight tensor bases $W^{AB(m\,h\,0)}_{ab}$ such that the metric perturbation with $k = 0$ can be written as Eq. (45). We substitute the highest weight metric perturbation into the left hand side of the linearized Einstein equation and the result is given by Eq. (46).

$$h^{(m\,h\,0)}_{ab} = R^h e^{im\Phi}\begin{bmatrix} R^{+2}C_{TT}(u) & R^{+1}C_{T\Phi}(u) & R^{+0}C_{TR}(u) & R^{+1}C_{uT}(u)\\ * & R^{+0}C_{\Phi\Phi}(u) & R^{-1}C_{R\Phi}(u) & R^{+0}C_{u\Phi}(u)\\ * & * & R^{-2}C_{RR}(u) & R^{-1}C_{uR}(u)\\ * & * & * & R^{+0}C_{uu}(u)\end{bmatrix} \tag{45}$$

$$G^{(1)}_{ab}[h^{(m\,h\,0)}] = R^h e^{im\Phi}\begin{bmatrix} R^{+2}\mathcal{D}^{(m,h)}_{TT}[\mathbf{C}(u)] & R^{+1}\mathcal{D}^{(m,h)}_{T\Phi}[\mathbf{C}(u)] & R^{+0}\mathcal{D}^{(m,h)}_{TR}[\mathbf{C}(u)] & R^{+1}\mathcal{D}^{(m,h)}_{uT}[\mathbf{C}(u)]\\ * & R^{+0}\mathcal{D}^{(m,h)}_{\Phi\Phi}[\mathbf{C}(u)] & R^{-1}\mathcal{D}^{(m,h)}_{R\Phi}[\mathbf{C}(u)] & R^{+0}\mathcal{D}^{(m,h)}_{u\Phi}[\mathbf{C}(u)]\\ * & * & R^{-2}\mathcal{D}^{(m,h)}_{RR}[\mathbf{C}(u)] & R^{-1}\mathcal{D}^{(m,h)}_{uR}[\mathbf{C}(u)]\\ * & * & * & R^{+0}\mathcal{D}^{(m,h)}_{uu}[\mathbf{C}(u)]\end{bmatrix} \tag{46}$$

Again notice that the (T,Φ,R) dependence has factored straight through the differential operator, resulting in ten coupled ODEs for the ten C-functions, which we have collected together as $\mathbf{C}(u)$. The expressions for all these differential operators are given in App. C.

We can easily verify that $G^{(1)}$ commutes with $\mathcal{L}_{H_-}$, therefore the linearized Einstein operator acting on a basis function with arbitrary weight can be obtained easily by repeatedly applying the lowering operator $\mathcal{L}_{H_-}$, k times, on Eq. (46). While applying the lowering operator, in general different components of $G^{(1)}_{ab}[h^{(m\,h\,k)}]$ will get mixed up, but the separation of variables still holds. Therefore we conclude that with these scalar, vector and tensor bases, we can separate variables in the linearized Einstein system in NHEK.

Given some source terms, these bases can be directly applied to solving for the corresponding metric perturbations. For instance, we have obtained the highest weight metric deformations in NHEK sourced by the decoupling limits of dynamical Chern-Simons and Einstein-dilaton-Gauss-Bonnet gravity [24].

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed an isometry-inspired method to study metric perturbations in the near-horizon extremal Kerr spacetime. That is, we separated variables in the metric perturbation equations in the NHEK spacetime, by expanding the perturbation in terms of basis functions adapted to the isometry group. With the separable linearized Einstein equation, one obtains the perturbed metric directly, without the complication of metric reconstruction. Further, our formalism does not depend on gauge choice. Within our formalism, partial differential equations built from $SL(2,\mathbb{R})\times U(1)$-equivariant operators can be converted into ordinary differential equations in the polar angle, which are simpler to solve.

We accomplished three things: (i) we used the highest weight method to obtain the scalar, vector, and symmetric tensor bases for the isometry group of NHEK; (ii) in global coordinates, we showed that these bases form orthogonal basis sets when the labels of irreps satisfy $h < -1$; and (iii) with these basis functions, we separated variables in many physical equations like the scalar wave equation, Maxwell's equations, and the linearized Einstein equations.

Future work. Although we have shown that bases in global coordinates are orthogonal, we did not mention completeness. There are clues that, in global coordinates, combining the highest- and lowest-weight modules will give a complete set of states. We leave a rigorous treatment of completeness to future work. However, many problems can already be attacked without worrying about completeness—for example, if the source term lives in exactly one irrep.

Since the near-horizon near-extremal black hole exhibits the same isometry as NHEK, we expect all discussions in this paper can be applied to understanding metric perturbations in near-NHEK, which is more astrophysically relevant. With the knowledge of isometry-adapted bases in NHEK, we hope to enhance our understanding of the Kerr/CFT conjecture [25] from the gravity side.

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Appendix A: Scalar, vector and symmetric tensor bases

In this section we present the expressions of scalar, vector and symmetric tensor bases both in Poincaré coordinates and global coordinates, up to constant factors. All the basis functions are defined on the three dimensional hypersurface Σ_u. To promote these basis functions to the full four dimensional manifold $\mathcal{M}$, one promotes all constant coefficients c_β to become unknown functions of the (cosine) polar angle, $c_\beta(u)$. The basis functions given here are (mostly) obtained using the highest weight method introduced in Sec. III, i.e. they form the highest weight modules for $SL(2,\mathbb{R})\times U(1) \circlearrowleft \mathcal{M}$. Such a highest weight module is infinite dimensional, the length of this paper, however, is supposed to be finite. Therefore, we give the highest three weights for scalar bases, the highest two weights for vector bases, and only the highest weight for tensor bases. Note all other basis functions can be generated by applying the lowering operator on the highest weight basis order by order. In order to compare the basis functions in different modules, in global coordinates, we also give the expressions of the scalar bases obtained using the lowest weight method.

1. Basis functions in Poincaré coordinates

a. Scalar bases

The scalar bases in Poincaré coordinates are given by

$$F^{(m\,h\,k)} \propto R^{h-k} e^{im\Phi} \times f^{(m\,h\,k)}\,, \tag{A1}$$

where

$$\begin{aligned} f^{(m\,h\,0)} =& 1\,, \\ f^{(m\,h\,1)} =& -2(hRT+im)\,, \\ f^{(m\,h\,2)} =& -2[-2i(2h-1)mRT+ \\ & + h(1-2h)R^2T^2 + h + 2m^2]\,. \end{aligned} \tag{A2}$$

b. Vector bases

The vector bases in Poincaré coordinates can be decomposed using the dual basis one-forms $\{\mathrm{d}T, \mathrm{d}\Phi, \mathrm{d}R\}$ via

$$\mathbf{V}^{(m\,h\,k)} = V_i^{(m\,h\,k)} \mathrm{d}x^i, \quad x \in \{T,\Phi,R\}\,. \tag{A3}$$

The vector components are given by

$$V_i^{(m\,h\,k)} \propto \begin{bmatrix} v_T^{(m\,h\,k)} R^{+1} \\ v_\Phi^{(m\,h\,k)} R^{+0} \\ v_R^{(m\,h\,k)} R^{-1} \end{bmatrix} R^{h-k} e^{im\Phi}\,, \tag{A4}$$

where

$$v_T^{(m\,h\,0)} = c_1\,, \quad v_\Phi^{(m\,h\,0)} = c_2\,, \quad v_R^{(m\,h\,0)} = c_3\,, \tag{A5}$$

and

$$\begin{aligned} v_T^{(m\,h\,1)} &= -2[c_3 + c_1(hRT+im)]\,, \\ v_\Phi^{(m\,h\,1)} &= -2c_2(hRT+im)\,, \\ v_R^{(m\,h\,1)} &= -2[c_3(hRT+im) + c_1 - c_2]\,. \end{aligned} \tag{A6}$$

Notice that there are three unknown coefficients c_1, c_2 and c_3. They endow us the freedom of choosing the three vector bases. In particular, we introduce a specific set of vector bases labeled by B where $B \in \{T,\Phi,R\}$. They are defined by

$$\begin{aligned} \mathbf{V}_T^{(m\,h\,k)} &= \mathbf{V}^{(m\,h\,k)}|_{c_2=c_3=0}\,, \\ \mathbf{V}_\Phi^{(m\,h\,k)} &= \mathbf{V}^{(m\,h\,k)}|_{c_1=c_3=0}\,, \\ \mathbf{V}_R^{(m\,h\,k)} &= \mathbf{V}^{(m\,h\,k)}|_{c_1=c_2=0}\,. \end{aligned} \tag{A7}$$

c. Symmetric tensor bases

The symmetric tensor bases in Poincaré coordinates can be decomposed using the dual basis one-forms $\{\mathrm{d}T, \mathrm{d}\Phi, \mathrm{d}R\}$ via

$$\mathbf{W}^{(m\,h\,k)} = W_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad x \in \{T,\Phi,R\}\,. \tag{A8}$$

The tensor components are given by

$$W_{ij}^{(m\,h\,k)} \propto \begin{bmatrix} R^{+2}w_{TT} & R^{+1}w_{T\Phi} & R^{+0}w_{TR} \\ * & R^{+0}w_{\Phi\Phi} & R^{-1}w_{R\Phi} \\ * & * & R^{-2}w_{RR} \end{bmatrix} R^{h-k}e^{im\Phi}\,, \tag{A9}$$

where

$$\begin{aligned} w_{TT}^{(m\,h\,0)} &= c_1\,, \quad w_{\Phi\Phi}^{(m\,h\,0)} = c_2\,, \quad w_{RR}^{(m\,h\,0)} = c_3\,, \\ w_{T\Phi}^{(m\,h\,0)} &= c_4\,, \quad w_{\Phi R}^{(m\,h\,0)} = c_5\,, \quad w_{RT}^{(m\,h\,0)} = c_6\,. \end{aligned} \tag{A10}$$

Notice that there are six unknown c-coefficients. They endow us the freedom of choosing the six tensor bases. In particular, we introduce a specific set of highest weight tensor bases labeled by A, B where $A, B \in \{T, \Phi, R\}$. They are defined by

$$\begin{aligned} \mathbf{W}_{TT}^{(m\,h\,k)} &= \mathbf{W}^{(m\,h\,k)}\big|_{c_{\beta\neq 1}=0}\,, \\ \mathbf{W}_{\Phi\Phi}^{(m\,h\,k)} &= \mathbf{W}^{(m\,h\,k)}\big|_{c_{\beta\neq 2}=0}\,, \\ \mathbf{W}_{RR}^{(m\,h\,k)} &= \mathbf{W}^{(m\,h\,k)}\big|_{c_{\beta\neq 3}=0}\,, \\ \mathbf{W}_{T\Phi}^{(m\,h\,k)} &= \mathbf{W}^{(m\,h\,k)}\big|_{c_{\beta\neq 4}=0}\,, \\ \mathbf{W}_{\Phi R}^{(m\,h\,k)} &= \mathbf{W}^{(m\,h\,k)}\big|_{c_{\beta\neq 5}=0}\,, \\ \mathbf{W}_{RT}^{(m\,h\,k)} &= \mathbf{W}^{(m\,h\,k)}\big|_{c_{\beta\neq 6}=0}\,. \end{aligned} \tag{A11}$$

This specific choice of tensor bases will be utilized to write the metric perturbation as in Eq. (45).

2. Basis functions in global coordinates

a. Scalar bases (highest weight module)

The scalar bases from the highest weight module in global coordinates are given by

$$F^{(m\,h\,k)} \propto (\sin\psi)^{-h} e^{i[(h-k)\tau + m\varphi] + m\psi} \times f^{(m\,h\,k)}\,, \tag{A12}$$

where

$$\begin{aligned} f^{(m\,h\,0)} &= 1\,, \\ f^{(m\,h\,1)} &= -2(m\sin\psi - h\cos\psi)\,, \\ f^{(m\,h\,2)} &= 2\left[h^2 + m^2 + \left(h^2 - h - m^2\right)\cos 2\psi + (m - 2hm)\sin 2\psi\right]\,. \end{aligned} \tag{A13}$$

b. Scalar bases (lowest weight module)

The scalar bases from the lowest weight module in global coordinates are given by

$$F_L^{(m\,h\,k)} \propto (\sin\psi)^{+h} e^{i[(h+k)\tau + m\varphi] - m\psi} \times f_L^{(m\,h\,k)}\,, \tag{A14}$$

where

$$\begin{aligned} f_L^{(m\,h\,0)} &= 1\,, \\ f_L^{(m\,h\,1)} &= -2(m\sin\psi - h\cos\psi), \\ f_L^{(m\,h\,2)} &= 2\left[h^2 + m^2 + \left(h^2 + h - m^2\right)\cos 2\psi - (m + 2hm)\sin 2\psi\right]\,. \end{aligned} \tag{A15}$$

c. Vector bases

The vector bases in global coordinates can be decomposed using the dual basis one-forms $\{\mathrm{d}\tau, \mathrm{d}\varphi, \mathrm{d}\psi\}$ via

$$\mathbf{V}^{(m\,h\,k)} = V_i^{(m\,h\,k)}\mathrm{d}x^i, \quad x \in \{\tau, \varphi, \psi\}\,. \tag{A16}$$

The vector components are given by

$$V_j^{(m\,h\,k)} \propto \begin{bmatrix} v_\tau^{m\,h\,k}(\sin\psi)^{-1} \\ v_\varphi^{m\,h\,k}(\sin\psi)^{+0} \\ v_\psi^{m\,h\,k}(\sin\psi)^{-1} \end{bmatrix} (\sin\psi)^{-h} e^{i[(h-k)\tau + m\varphi] + m\psi}\,, \tag{A17}$$

where

$$v_\tau^{(m\,h\,0)} = -\frac{1}{4}\left(c_1 e^{-i\psi} + 2c_1 e^{i\psi} - 2c_2 e^{-i\psi} + 4c_3 e^{i\psi}\right) , \tag{A18}$$
$$v_\varphi^{(m\,h\,0)} = c_1 ,$$
$$v_\psi^{(m\,h\,0)} = +\frac{1}{4}\left(c_1 e^{-i\psi} + 2c_2 e^{-i\psi} + 4c_3 e^{i\psi}\right) ,$$

and

$$\begin{aligned} v_\tau^{(m\,h\,1)} =& -\frac{1}{4}\Big\{c_1[2(h+im)e^{2i\psi} + (3h-im-1) + (h-im+1)e^{-2i\psi}]- \\ & -2c_2[(h+im+1) + (h-im-1)e^{-2i\psi}] + 4c_3[(h+im-1)e^{2i\psi} + (h-im+1)]\Big\} , \end{aligned} \tag{A19}$$
$$v_\varphi^{(m\,h\,1)} = -2c_1(m\sin\psi - h\cos\psi) ,$$
$$\begin{aligned} v_\psi^{(m\,h\,1)} =& +\frac{1}{4}\Big\{c_1[(h+im+1) + (h-im-1)e^{-2i\psi}] + +2c_2[(h+im+1) + (h-im-1)e^{-2i\psi}] \\ & + 4c_3[(h+im-1)e^{2i\psi} + (h-im+1)]\Big\} . \end{aligned}$$

d. *Symmetric tensor bases*

The symmetric tensor bases in global coordinates can be decomposed using the dual basis one-forms $\{\mathrm{d}\tau, \mathrm{d}\varphi, \mathrm{d}\psi\}$ via

$$\mathbf{W}^{(m\,h\,k)} = W_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j , \quad x \in \{\tau, \varphi, \psi\} . \tag{A20}$$

The tensor components are given by

$$W_{ij}^{(mhk)} \propto \begin{bmatrix} w_{\tau\tau}^{mhk}(\sin\psi)^{-2} & w_{\tau\varphi}^{mhk}(\sin\psi)^{-1} & w_{\tau\psi}^{mhk}(\sin\psi)^{-2} \\ * & w_{\varphi\varphi}^{mhk}(\sin\psi)^{+0} & w_{\varphi\psi}^{mhk}(\sin\psi)^{-1} \\ * & * & w_{\psi\psi}^{mhk}(\sin\psi)^{-2} \end{bmatrix} (\sin\psi)^{-h} e^{i[(h-k)\tau + m\varphi] + m\psi} , \tag{A21}$$

where

$$w_{\tau\tau}^{(m\,h\,0)} = +\frac{1}{16}(c_1 e^{-2i\psi} + 4c_1 e^{2i\psi} - 6c_2 e^{-2i\psi} + 16c_3 e^{2i\psi} + 8c_5 e^{-2i\psi} + 16c_6 e^{2i\psi} + 4c_1 - 8c_2 + 16c_3 + 8c_4) , \tag{A22}$$
$$w_{\varphi\varphi}^{(m\,h\,0)} = c_1 ,$$
$$w_{\psi\psi}^{(m\,h\,0)} = +\frac{1}{16}(-8c_4 + 16c_6 e^{2i\psi} + c_1 e^{-2i\psi} + 2c_2 e^{-2i\psi} + 8c_5 e^{-2i\psi}) ,$$
$$w_{\tau\varphi}^{(m\,h\,0)} = -\frac{1}{4}\left(2c_1 e^{i\psi} + 4c_3 e^{i\psi} + c_1 e^{-i\psi} - 2c_2 e^{-i\psi}\right) ,$$
$$w_{\varphi\psi}^{(m\,h\,0)} = +\frac{1}{4}\left(4c_3 e^{i\psi} + c_1 e^{-i\psi} + 2c_2 e^{-i\psi}\right) ,$$
$$w_{\psi\tau}^{(m\,h\,0)} = -\frac{1}{16}\left(2c_1 + 4c_2 + 8c_3 + 8c_3 e^{2i\psi} + 16c_6 e^{2i\psi} + c_1 e^{-2i\psi} + 2c_2 e^{-2i\psi} - 8c_5 e^{-2i\psi}\right) .$$

Appendix B: Expressions of $\mathcal{D}_A^{(m,h)}[\mathbf{C}(u)]$ in Maxwell systems

$\mathcal{D}_A$	$C_T''(u)$	$C_\Phi''(u)$	$C_R''(u)$	$C_u''(u)$
$\mathcal{D}_T$	$\frac{u^2-1}{\left(u^2+1\right)^2}$	$\frac{1-u^2}{\left(u^2+1\right)^2}$	0	0
$\mathcal{D}_\Phi$	$\frac{1-u^2}{\left(u^2+1\right)^2}$	$\frac{u^4+6u^2-3}{4\left(u^2+1\right)^2}$	0	0
$\mathcal{D}_R$	0	0	$\frac{1-u^2}{\left(u^2+1\right)^2}$	0
$\mathcal{D}_u$	0	0	0	0

	$C_T'(u)$	$C_\Phi'(u)$	$C_R'(u)$	$C_u'(u)$
$\mathcal{D}_T$	$\frac{4u}{\left(u^2+1\right)^3}$	$-\frac{4u}{\left(u^2+1\right)^3}$	0	$\frac{im\left(u^2-1\right)}{\left(u^2+1\right)^2}$
$\mathcal{D}_\Phi$	$-\frac{4u}{\left(u^2+1\right)^3}$	$\frac{u\left(u^4+2u^2+9\right)}{2\left(u^2+1\right)^3}$	0	$-\frac{im\left(u^4+6u^2-3\right)}{4\left(u^2+1\right)^2}$
$\mathcal{D}_R$	0	0	$-\frac{4u}{\left(u^2+1\right)^3}$	$\frac{h\left(u^2-1\right)}{\left(u^2+1\right)^2}$
$\mathcal{D}_u$	$\frac{im\left(u^2-1\right)}{\left(u^2+1\right)^2}$	$-\frac{im\left(u^4+6u^2-3\right)}{4\left(u^2+1\right)^2}$	$\frac{(h+1)\left(u^2-1\right)}{\left(u^2+1\right)^2}$	0

	$C_T(u)$	$C_\Phi(u)$	$C_R(u)$	$C_u(u)$
$\mathcal{D}_T$	$-\frac{4\left(u^2-1\right)h^2+4\left(u^2-1\right)h+m^2\left(u^2+1\right)^2}{4\left(u^2-1\right)\left(u^2+1\right)^2}$	$\frac{h^2}{\left(u^2+1\right)^2}$	$-\frac{ihm}{\left(u^2+1\right)^2}$	$\frac{4imu}{\left(u^2+1\right)^3}$
$\mathcal{D}_\Phi$	$\frac{(h+1)^2}{\left(u^2+1\right)^2}$	$-\frac{h(h+1)\left(u^4+6u^2-3\right)}{4\left(u^2-1\right)\left(u^2+1\right)^2}$	$\frac{i(h+1)m\left(u^4+6u^2-3\right)}{4\left(u^2-1\right)\left(u^2+1\right)^2}$	$-\frac{imu\left(u^4+2u^2+9\right)}{2\left(u^2+1\right)^3}$
$\mathcal{D}_R$	$-\frac{i(h+1)m}{\left(u^2+1\right)^2}$	$\frac{ihm\left(u^4+6u^2-3\right)}{4\left(u^2-1\right)\left(u^2+1\right)^2}$	$\frac{m^2\left(u^4+6u^2-3\right)}{4\left(u^2-1\right)\left(u^2+1\right)^2}$	$\frac{4hu}{\left(u^2+1\right)^3}$
$\mathcal{D}_u$	0	0	0	$-\frac{4\left(u^2-1\right)h^2+4\left(u^2-1\right)h+m^2\left(u^4+6u^2-3\right)}{4\left(u^2+1\right)^2}$

TABLE I. The coefficient table that gives the expressions of $\mathcal{D}_A^{(m,h)}[\mathbf{C}(u)]$, $A \in \{T, \Phi, R, u\}$ in Maxwell systems. Each row is labeled by $\mathcal{D}_A^{(m,h)}$, while each column is labeled by a C-function or its derivative. Each table component is the coefficient in front of the (derivative of) corresponding C-function in $\mathcal{D}_A^{(m,h)}[\mathbf{C}(u)]$. To recover $\mathcal{D}_A^{(m,h)}[\mathbf{C}(u)]$, one just multiplies each table component with its column label and then add up all those with the same row label $\mathcal{D}_A$.

Appendix C: Expressions of $\mathcal{D}_{AB}^{(m,h)}[\mathbf{C}(u)]$ in linearized Einstein equations

The general second order differentiation $\mathcal{D}^{(m,h)}$ on the ten unknown C-functions, denoted as $\mathcal{D}_{AB}^{(m,h)}[\mathbf{C}(u)]$, can be written compactly by putting all C-functions together to form a vector $\mathbf{C}(u)$,

$$\mathcal{D}_{AB}^{(m,h)}[\mathbf{C}(u)] = (\mathcal{A}_{AB}\partial_u^2 + \mathcal{B}_{AB}\partial_u + \mathcal{C}_{AB}) \cdot \Big(C_{TT}(u), \ldots, C_{\Phi u}(u)\Big)^{\mathrm{T}}. \tag{C1}$$

Here $\mathcal{A}_{AB}$, $\mathcal{B}_{AB}$ and $\mathcal{C}_{AB}$ are covectors whose components are obtained by collecting coefficients in front of C-functions. We further stack all the covectors $\mathcal{A}_{AB}$ to form a matrix, and similarly do for $\mathcal{B}_{AB}$ and $\mathcal{C}_{AB}$. We label the resulting coefficient matrices as $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ respectively. They are given in Tables II, III, IV, V, and VI.

$\mathcal{D}_{AB}$	$C''_{TT}(u)$	$C''_{T\Phi}(u)$	$C''_{\Phi\Phi}(u)$	$C''_{RR}(u)$	$C''_{Ru}(u)$	$C''_{uu}(u)$	$C''_{TR}(u)$	$C''_{Tu}(u)$	$C''_{\Phi R}(u)$	$C''_{\Phi u}(u)$
$\mathcal{D}_{TT}$	$-\frac{2(u^2-1)^2}{(u^2+1)^3}$	$\frac{u^6+5u^4-9u^2+3}{(u^2+1)^3}$	$-\frac{(u^4+6u^2-3)^2}{8(u^2+1)^3}$	$\frac{u^6+5u^4-9u^2+3}{2(u^2+1)^3}$	0	0	0	0	0	0
$\mathcal{D}_{T\Phi}$	$-\frac{2(u^2-1)^2}{(u^2+1)^3}$	$\frac{u^6+9u^4-17u^2+7}{2(u^2+1)^3}$	$-\frac{u^6+5u^4-9u^2+3}{2(u^2+1)^3}$	$\frac{2(u^2-1)^2}{(u^2+1)^3}$	0	0	0	0	0	0
$\mathcal{D}_{\Phi\Phi}$	$-\frac{2(u^2-1)^2}{(u^2+1)^3}$	$\frac{4(u^2-1)^2}{(u^2+1)^3}$	$-\frac{2(u^2-1)^2}{(u^2+1)^3}$	$\frac{2(u^2-1)^2}{(u^2+1)^3}$	0	0	0	0	0	0
$\mathcal{D}_{RR}$	$\frac{u^2-1}{2(u^2+1)}$	$\frac{1-u^2}{u^2+1}$	$\frac{u^4+6u^2-3}{8(u^2+1)}$	0	0	0	0	0	0	0
$\mathcal{D}_{Ru}$	0	0	0	0	0	0	0	0	0	0
$\mathcal{D}_{uu}$	0	0	0	0	0	0	0	0	0	0
$\mathcal{D}_{TR}$	0	0	0	0	0	0	$\frac{u^2-1}{2(u^2+1)}$	0	0	0
$\mathcal{D}_{Tu}$	0	0	0	0	0	0	0	0	0	0
$\mathcal{D}_{\Phi R}$	0	0	0	0	0	0	0	0	$\frac{u^2-1}{2(u^2+1)}$	0
$\mathcal{D}_{\Phi u}$	0	0	0	0	0	0	0	0	0	0

TABLE II. $\mathcal{A}$ matrix.

$\mathcal{D}_{AB}$	$C'_{TT}(u)$	$C'_{T\Phi}(u)$	$C'_{\Phi\Phi}(u)$	$C'_{RR}(u)$	$C'_{Ru}(u)$
$\mathcal{D}_{TT}$	$\frac{2u(u^4-4u^2+3)}{(u^2+1)^4}$	$-\frac{4u(u^2-3)(u^2-1)}{(u^2+1)^4}$	$-\frac{u(u^{10}+u^8-22u^6+66u^4-123u^2+45)}{8(u^2-1)(u^2+1)^4}$	$-\frac{u(u^6+u^4-13u^2+3)}{(u^2+1)^4}$	$-\frac{h(u^2-1)(u^4+6u^2-3)}{(u^2+1)^3}$
$\mathcal{D}_{T\Phi}$	$\frac{2u(u^4-4u^2+3)}{(u^2+1)^4}$	$-\frac{4u(u^4-4u^2+3)}{(u^2+1)^4}$	$\frac{2u(u^4-4u^2+3)}{(u^2+1)^4}$	$-\frac{2u(u^4-4u^2+3)}{(u^2+1)^4}$	$-\frac{2(2h+1)(u^2-1)^2}{(u^2+1)^3}$
$\mathcal{D}_{\Phi\Phi}$	$\frac{2u(u^4-4u^2+3)}{(u^2+1)^4}$	$-\frac{4u(u^2-3)(u^2-1)}{(u^2+1)^4}$	$\frac{2u(u^4-4u^2+3)}{(u^2+1)^4}$	$-\frac{2u(u^4-4u^2+3)}{(u^2+1)^4}$	$-\frac{4(h+1)(u^2-1)^2}{(u^2+1)^3}$
$\mathcal{D}_{RR}$	$-\frac{u(u^2-3)}{(u^2+1)^2}$	$\frac{2u(u^2-3)}{(u^2+1)^2}$	$\frac{u(u^2-3)^3}{8(u^2-1)(u^2+1)^2}$	0	$\frac{u^2-1}{u^2+1}$
$\mathcal{D}_{Ru}$	$\frac{h+1}{2(u^2+1)}$	$-\frac{2h+1}{2(u^2+1)}$	$\frac{h(u^4+6u^2-3)}{8(u^4-1)}$	$\frac{1}{2(u^2+1)}$	0
$\mathcal{D}_{uu}$	$-\frac{u}{2(u^4-1)}$	$\frac{u}{u^4-1}$	$-\frac{u(u^2+3)}{4(u^4-1)}$	$\frac{u}{2(u^4-1)}$	0
$\mathcal{D}_{TR}$	0	0	0	0	0
$\mathcal{D}_{Tu}$	$\frac{im}{2(u^2+1)}$	$-\frac{im(u^4+6u^2-3)}{8(u^4-1)}$	0	0	0
$\mathcal{D}_{\Phi R}$	0	0	0	0	$-\frac{im(u^2-1)}{2(u^2+1)}$
$\mathcal{D}_{\Phi u}$	$\frac{im}{2(u^2+1)}$	$-\frac{im}{2(u^2+1)}$	0	$-\frac{im}{2(u^2+1)}$	0

	$C'_{uu}(u)$	$C'_{TR}(u)$	$C'_{Tu}(u)$	$C'_{\Phi R}(u)$	$C'_{\Phi u}(u)$
$\mathcal{D}_{TT}$	$\frac{u(u^2-1)(u^6+11u^4-13u^2+9)}{2(u^2+1)^4}$	0	$-\frac{im(u^2-1)(u^4+6u^2-3)}{(u^2+1)^3}$	0	$\frac{im(u^4+6u^2-3)^2}{4(u^2+1)^3}$
$\mathcal{D}_{T\Phi}$	$\frac{4u(u^2-1)^3}{(u^2+1)^4}$	0	$-\frac{im(u^6+9u^4-17u^2+7)}{2(u^2+1)^3}$	0	$\frac{im(u^6+5u^4-9u^2+3)}{(u^2+1)^3}$
$\mathcal{D}_{\Phi\Phi}$	$\frac{4u(u^2-1)^3}{(u^2+1)^4}$	0	$-\frac{4im(u^2-1)^2}{(u^2+1)^3}$	0	$\frac{4im(u^2-1)^2}{(u^2+1)^3}$
$\mathcal{D}_{RR}$	$-\frac{u(u^2-1)}{2(u^2+1)}$	0	$\frac{im(u^2-1)}{u^2+1}$	0	$-\frac{im(u^4+6u^2-3)}{4(u^2+1)}$
$\mathcal{D}_{Ru}$	0	$\frac{im}{2(u^2+1)}$	0	$-\frac{im(u^4+6u^2-3)}{8(u^4-1)}$	0
$\mathcal{D}_{uu}$	0	0	0	0	0
$\mathcal{D}_{TR}$	0	$-\frac{u(u^2-3)}{(u^2+1)^2}$	$-\frac{(u^2-1)(-u^4-6u^2+h(u^2+1)^2+3)}{2(u^2+1)^3}$	$\frac{u(u^2-3)}{(u^2+1)^2}$	$-\frac{(u^2-1)(u^4+6u^2-3)}{2(u^2+1)^3}$
$\mathcal{D}_{Tu}$	0	$\frac{h+2}{2(u^2+1)}$	0	0	0
$\mathcal{D}_{\Phi R}$	0	0	$\frac{2(u^2-1)^2}{(u^2+1)^3}$	0	$-\frac{(u^2-1)(h(u^2+1)^2+4(u^2-1))}{2(u^2+1)^3}$
$\mathcal{D}_{\Phi u}$	0	0	0	$\frac{h+1}{2(u^2+1)}$	0

TABLE III. $\mathcal{B}$ matrix.

$\mathcal{D}_{AB}$	$C_{TT}(u)$	$C_{T\Phi}(u)$
$\mathcal{D}_{TT}$	$\frac{\left(u^2-1\right)\left(u^4+2u^2+2h^2\left(u^2+1\right)^2+6h\left(u^2+1\right)^2+9\right)}{\left(u^2+1\right)^5}$	$-\frac{u^8-28u^6-42u^4+36u^2+2h^2\left(u^8+8u^6+10u^4-3\right)+3h\left(u^8+8u^6+10u^4-3\right)-15}{2\left(u^2+1\right)^5}$
$\mathcal{D}_{T\Phi}$	$\frac{\left(u^2-1\right)\left(2h^2\left(u^2+1\right)^2+5h\left(u^2+1\right)^2+8\right)}{\left(u^2+1\right)^5}$	$-\frac{h^2\left(u^4+10u^2-7\right)\left(u^2+1\right)^2+h\left(u^4+10u^2-7\right)\left(u^2+1\right)^2-8\left(3u^6+4u^4-5u^2+2\right)}{2\left(u^2+1\right)^5}$
$\mathcal{D}_{\Phi\Phi}$	$\frac{2\left(u^2-1\right)\left(h^2\left(u^2+1\right)^2+2h\left(u^2+1\right)^2+4\right)}{\left(u^2+1\right)^5}$	$-\frac{2\left(u^2-1\right)\left(-3u^4-6u^2+2h^2\left(u^2+1\right)^2+h\left(u^2+1\right)^2+5\right)}{\left(u^2+1\right)^5}$
$\mathcal{D}_{RR}$	$\frac{8\left(u^6-8u^4+9u^2-2\right)-m^2\left(u^2+1\right)^4}{8\left(u^2-1\right)\left(u^2+1\right)^3}$	$\frac{-3u^4+30u^2+h\left(u^2+1\right)^2-7}{2\left(u^2+1\right)^3}$
$\mathcal{D}_{Ru}$	$-\frac{(h+1)u}{\left(u^2+1\right)^2}$	$\frac{2u\left(u^2+h\left(u^2-1\right)-2\right)}{\left(u^2-1\right)\left(u^2+1\right)^2}$
$\mathcal{D}_{uu}$	$\frac{m^2\left(u^2+1\right)^4+4h^2\left(u^2-1\right)\left(u^2+1\right)^2+8h\left(u^2-1\right)\left(u^2+1\right)^2+8\left(u^6-u^4+u^2-1\right)}{8\left(u^2-1\right)^2\left(u^2+1\right)^3}$	$-\frac{3u^4-2u^2+2h^2\left(u^2+1\right)^2+3h\left(u^2+1\right)^2+3}{2\left(u^2-1\right)\left(u^2+1\right)^3}$
$\mathcal{D}_{TR}$	$\frac{im\left(u^4-2u^2+2h\left(u^2+1\right)^2+5\right)}{4\left(u^2+1\right)^3}$	$-\frac{im\left(u^4+6u^2-3\right)\left(-u^4-6u^2+h\left(u^2+1\right)^2+3\right)}{8\left(u^2-1\right)\left(u^2+1\right)^3}$
$\mathcal{D}_{Tu}$	$\frac{imu}{2-2u^4}$	$\frac{imu\left(u^4+6u^2-3\right)}{4\left(u^2-1\right)\left(u^2+1\right)^2}$
$\mathcal{D}_{\Phi R}$	$\frac{im\left(u^4+h\left(u^2+1\right)^2+3\right)}{2\left(u^2+1\right)^3}$	$-\frac{im\left(-u^4-6u^2+h\left(u^2+1\right)^2+3\right)}{2\left(u^2+1\right)^3}$
$\mathcal{D}_{\Phi u}$	$\frac{imu}{2-2u^4}$	$\frac{imu}{\left(u^2+1\right)^2}$

	$C_{\Phi R}(u)$	$C_{\Phi u}(u)$
$\mathcal{D}_{TT}$	$-\frac{ihm\left(u^4+6u^2-3\right)^2}{4\left(u^2-1\right)\left(u^2+1\right)^3}$	$\frac{imu\left(u^4+6u^2-3\right)^2}{4\left(u^2-1\right)\left(u^2+1\right)^3}$
$\mathcal{D}_{T\Phi}$	$-\frac{ihm\left(u^4+6u^2-3\right)}{\left(u^2+1\right)^3}$	$\frac{imu\left(u^4+6u^2-3\right)}{\left(u^2+1\right)^3}$
$\mathcal{D}_{\Phi\Phi}$	$-\frac{4ihm\left(u^2-1\right)}{\left(u^2+1\right)^3}$	$\frac{4imu\left(u^2-1\right)}{\left(u^2+1\right)^3}$
$\mathcal{D}_{RR}$	$\frac{im\left(u^4+6u^2-3\right)}{4\left(u^4-1\right)}$	$-\frac{imu\left(u^6+3u^4+19u^2-15\right)}{4\left(u^2-1\right)\left(u^2+1\right)^2}$
$\mathcal{D}_{Ru}$	$\frac{imu\left(u^4+6u^2-3\right)}{4\left(u^2-1\right)\left(u^2+1\right)^2}$	$-\frac{ihm\left(u^4+6u^2-3\right)}{8\left(u^4-1\right)}$
$\mathcal{D}_{uu}$	$-\frac{i(h+1)m\left(u^4+6u^2-3\right)}{4\left(u^2-1\right)^2\left(u^2+1\right)}$	$\frac{imu\left(u^2+3\right)}{2\left(u^4-1\right)}$
$\mathcal{D}_{TR}$	$-\frac{u^4-12u^2+3}{\left(u^2+1\right)^3}$	$\frac{2u\left(u^4-14u^2+9\right)}{\left(u^2+1\right)^4}$
$\mathcal{D}_{Tu}$	$\frac{(h+2)u\left(u^2-3\right)}{\left(u^2-1\right)\left(u^2+1\right)^2}$	$-\frac{(h+2)\left(u^4+6u^2-3\right)}{2\left(u^2+1\right)^3}$
$\mathcal{D}_{\Phi R}$	$\frac{6u^2-2}{\left(u^2+1\right)^3}$	$-\frac{2u\left(h\left(u^2+1\right)^2-2\left(u^4-6u^2+5\right)\right)}{\left(u^2+1\right)^4}$
$\mathcal{D}_{\Phi u}$	$-\frac{2(h+1)u}{\left(u^2-1\right)\left(u^2+1\right)^2}$	$-\frac{(h+1)\left(h\left(u^2+1\right)^2+4\left(u^2-1\right)\right)}{2\left(u^2+1\right)^3}$

TABLE IV. Part I of $\mathcal{C}$ matrix.

	$C_{RR}(u)$	$C_{Ru}(u)$
$\mathcal{D}_{TT}$	$\frac{8\left(u^{10}-2u^8-6u^6-8u^4+21u^2-6\right)-m^2\left(u^6+7u^4+3u^2-3\right)^2}{8\left(u^2-1\right)\left(u^2+1\right)^5}$	$-\frac{4u\left((2h+3)u^4+2(h-6)u^2+9\right)}{\left(u^2+1\right)^4}$
$\mathcal{D}_{T\Phi}$	$\frac{-\left(u^8+8u^6+10u^4-3\right)m^2+2h\left(u^2-1\right)\left(u^2+1\right)^2+8\left(u^6+u^4-3u^2+1\right)}{2\left(u^2+1\right)^5}$	$-\frac{4u\left(u^2-1\right)\left(hu^2+2u^2+h-4\right)}{\left(u^2+1\right)^4}$
$\mathcal{D}_{\Phi\Phi}$	$\frac{2\left(u^2-1\right)\left(-m^2\left(u^2+1\right)^2+h\left(u^2+1\right)^2+2\left(u^4+2u^2-1\right)\right)}{\left(u^2+1\right)^5}$	$-\frac{4(h+1)u\left(u^2-1\right)}{\left(u^2+1\right)^3}$
$\mathcal{D}_{RR}$	$\frac{u^2-1}{\left(u^2+1\right)^3}$	$\frac{4u}{\left(u^2+1\right)^2}$
$\mathcal{D}_{Ru}$	$-\frac{u}{\left(u^2+1\right)^2}$	$\frac{8\left(u^6+3u^4-5u^2+1\right)-m^2\left(u^8+8u^6+10u^4-3\right)}{8\left(u^2-1\right)\left(u^2+1\right)^3}$
$\mathcal{D}_{uu}$	$\frac{-\left(u^8+8u^6+10u^4-3\right)m^2+4h\left(u^2-1\right)\left(u^2+1\right)^2+16u^2\left(u^2-1\right)}{8\left(u^2-1\right)^2\left(u^2+1\right)^3}$	$-\frac{(h+1)u}{u^4-1}$
$\mathcal{D}_{TR}$	$\frac{im\left(u^4+6u^2-3\right)}{4\left(u^2+1\right)^3}$	$-\frac{imu\left(u^2-3\right)}{\left(u^2+1\right)^2}$
$\mathcal{D}_{Tu}$	$-\frac{imu\left(u^2-3\right)}{2\left(u^2-1\right)\left(u^2+1\right)^2}$	$\frac{im\left(u^4+6u^2-3\right)}{2\left(u^2+1\right)^3}$
$\mathcal{D}_{\Phi R}$	$\frac{im\left(u^4+4u^2-1\right)}{2\left(u^2+1\right)^3}$	$-\frac{imu\left(u^2-1\right)}{\left(u^2+1\right)^2}$
$\mathcal{D}_{\Phi u}$	$\frac{imu}{2\left(u^4-1\right)}$	$\frac{im\left(u^4+6u^2+h\left(u^2+1\right)^2-3\right)}{2\left(u^2+1\right)^3}$

	$C_{uu}(u)$	$C_{TR}(u)$
$\mathcal{D}_{TT}$	$\frac{4h^2\left(u^6+5u^4-9u^2+3\right)\left(u^2+1\right)^2+m^2\left(u^6+7u^4+3u^2-3\right)^2+8\left(5u^8+34u^6-68u^4+54u^2-9\right)}{8\left(u^2+1\right)^5}$	$\frac{i(2h+3)m\left(u^4+6u^2-3\right)}{2\left(u^2+1\right)^3}$
$\mathcal{D}_{T\Phi}$	$\frac{\left(u^2-1\right)\left(\left(u^8+8u^6+10u^4-3\right)m^2+4h^2\left(u^2-1\right)\left(u^2+1\right)^2+2h\left(u^2-1\right)\left(u^2+1\right)^2+8\left(u^6+8u^4-11u^2+2\right)\right)}{2\left(u^2+1\right)^5}$	$\frac{im\left(2\left(u^4+8u^2-5\right)+h\left(u^4+10u^2-7\right)\right)}{2\left(u^2+1\right)^3}$
$\mathcal{D}_{\Phi\Phi}$	$\frac{2\left(u^2-1\right)^2\left(h^2\left(u^2+1\right)^2+m^2\left(u^2+1\right)^2+h\left(u^2+1\right)^2+2\left(u^4+9u^2-2\right)\right)}{\left(u^2+1\right)^5}$	$\frac{2i(2h+3)m\left(u^2-1\right)}{\left(u^2+1\right)^3}$
$\mathcal{D}_{RR}$	$-\frac{\left(u^8+8u^6+10u^4-3\right)m^2+4h\left(u^2-1\right)\left(u^2+1\right)^2+8\left(u^4+4u^2-1\right)}{8\left(u^2+1\right)^3}$	$-\frac{im}{2\left(u^2+1\right)}$
$\mathcal{D}_{Ru}$	$-\frac{hu}{2\left(u^2+1\right)}$	$-\frac{imu}{\left(u^2+1\right)^2}$
$\mathcal{D}_{uu}$	$\frac{u^2\left(u^2+3\right)}{\left(u^2+1\right)^3}$	$\frac{i(2h+3)m}{2\left(u^4-1\right)}$
$\mathcal{D}_{TR}$	$\frac{im\left(u^2-1\right)\left(u^4+6u^2-3\right)}{4\left(u^2+1\right)^3}$	$\frac{8\left(u^6-7u^4+7u^2-1\right)-m^2\left(u^8+8u^6+10u^4-3\right)}{8\left(u^2-1\right)\left(u^2+1\right)^3}$
$\mathcal{D}_{Tu}$	$-\frac{imu\left(u^2-3\right)}{2\left(u^2+1\right)^2}$	$-\frac{(h+2)u}{\left(u^2+1\right)^2}$
$\mathcal{D}_{\Phi R}$	$\frac{im\left(u^2-1\right)\left(h\left(u^2+1\right)^2+2\left(u^2-1\right)\right)}{2\left(u^2+1\right)^3}$	$-\frac{m^2}{2\left(u^2+1\right)}$
$\mathcal{D}_{\Phi u}$	$-\frac{imu\left(u^2-1\right)}{\left(u^2+1\right)^2}$	0

TABLE V. Part II of $\mathcal{C}$ matrix.

$\mathcal{D}_{AB}$	$C_{\Phi\Phi}(u)$
$\mathcal{D}_{TT}$	$\frac{h^2\left(u^2-1\right)\left(u^6+7u^4+3u^2-3\right)^2-2\left(3u^{12}+68u^{10}-5u^8-128u^6+153u^4-36u^2+9\right)}{8\left(u^2-1\right)^2\left(u^2+1\right)^5}$
$\mathcal{D}_{T\Phi}$	$-\frac{-2\left(u^8+8u^6+10u^4-3\right)h^2+\left(u^8+8u^6+10u^4-3\right)h+4\left(9u^6+13u^4-9u^2+3\right)}{4\left(u^2+1\right)^5}$
$\mathcal{D}_{\Phi\Phi}$	$\frac{\left(u^2-1\right)\left(-3u^4-6u^2+2h^2\left(u^2+1\right)^2-2h\left(u^2+1\right)^2+5\right)}{\left(u^2+1\right)^5}$
$\mathcal{D}_{RR}$	$\frac{2\left(7u^8-30u^6+72u^4-42u^2+9\right)-h\left(u^2+1\right)^2\left(u^6+5u^4-9u^2+3\right)}{8\left(u^2-1\right)^2\left(u^2+1\right)^3}$
$\mathcal{D}_{Ru}$	$-\frac{u\left(8\left(u^4-4u^2+3\right)+h\left(u^6+11u^4-13u^2+9\right)\right)}{8\left(u^4-1\right)^2}$
$\mathcal{D}_{uu}$	$\frac{\left(u^8+8u^6+10u^4-3\right)h^2+\left(u^8+8u^6+10u^4-3\right)h+2\left(7u^6+3u^4+9u^2-3\right)}{8\left(u^2-1\right)^2\left(u^2+1\right)^3}$
$\mathcal{D}_{TR}$	$-\frac{im\left(u^4+6u^2-3\right)^2}{16\left(u^2-1\right)\left(u^2+1\right)^3}$
$\mathcal{D}_{Tu}$	$-\frac{imu\left(u^6+3u^4-21u^2+9\right)}{8\left(u^4-1\right)^2}$
$\mathcal{D}_{\Phi R}$	$-\frac{im\left(u^4+6u^2-3\right)}{4\left(u^2+1\right)^3}$
$\mathcal{D}_{\Phi u}$	$-\frac{imu\left(u^2-3\right)}{2\left(u^2-1\right)\left(u^2+1\right)^2}$

	$C_{Tu}(u)$
$\mathcal{D}_{TT}$	$-\frac{2imu\left(u^2-1\right)\left(u^2+3\right)}{\left(u^2+1\right)^3}$
$\mathcal{D}_{T\Phi}$	$-\frac{imu\left(u^4+4u^2-5\right)}{\left(u^2+1\right)^3}$
$\mathcal{D}_{\Phi\Phi}$	$-\frac{4imu\left(u^2-1\right)}{\left(u^2+1\right)^3}$
$\mathcal{D}_{RR}$	$\frac{4imu}{\left(u^2+1\right)^2}$
$\mathcal{D}_{Ru}$	$\frac{i(h+1)m}{2\left(u^2+1\right)}$
$\mathcal{D}_{uu}$	$-\frac{imu}{u^4-1}$
$\mathcal{D}_{TR}$	$-\frac{2u\left(u^4-14u^2+h\left(u^2+1\right)^2+9\right)}{\left(u^2+1\right)^4}$
$\mathcal{D}_{Tu}$	$-\frac{4h^2\left(u^2-1\right)\left(u^2+1\right)^2+4h\left(u^6-3u^4+7u^2-5\right)+\left(u^4+6u^2-3\right)\left(-8u^2+m^2\left(u^2+1\right)^2+8\right)}{8\left(u^2-1\right)\left(u^2+1\right)^3}$
$\mathcal{D}_{\Phi R}$	$-\frac{4u\left(u^4-6u^2+5\right)}{\left(u^2+1\right)^4}$
$\mathcal{D}_{\Phi u}$	$\frac{-m^2\left(u^2+1\right)^2+4h\left(u^2-1\right)+4\left(u^2-1\right)}{2\left(u^2+1\right)^3}$

TABLE VI. Part III of $\mathcal{C}$ matrix.

[1] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago and London, 1984).
[2] S. A. Teukolsky, Class. Quant. Grav. **32**, 124006 (2015), arXiv:1410.2130 [gr-qc].
[3] S. A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972).
[4] S. A. Teukolsky, Astrophys. J. **185**, 635 (1973).
[5] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).
[6] P. L. Chrzanowski, Phys. Rev. **D11**, 2042 (1975).
[7] L. S. Kegeles and J. M. Cohen, Phys. Rev. **D19**, 1641 (1979).
[8] B. F. Whiting and L. R. Price, Class. Quant. Grav. **22**, S589 (2005).
[9] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[10] C. V. Vishveshwara, Phys. Rev. **D1**, 2870 (1970).
[11] F. J. Zerilli, Phys. Rev. Lett. **24**, 737 (1970).
[12] F. J. Zerilli, Phys. Rev. **D2**, 2141 (1970).
[13] O. Sarbach and M. Tiglio, Phys. Rev. **D64**, 084016 (2001), arXiv:gr-qc/0104061 [gr-qc].
[14] K. Martel and E. Poisson, Phys. Rev. **D71**, 104003 (2005), arXiv:gr-qc/0502028 [gr-qc].
[15] J. M. Bardeen and G. T. Horowitz, Phys. Rev. **D60**, 104030 (1999), arXiv:hep-th/9905099 [hep-th].
[16] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
[17] R. H. Boyer and R. W. Lindquist, J. Math. Phys. **8**, 265 (1967).
[18] J. M. Lee, *Introduction to smooth manifolds*, 2nd ed., Graduate Texts in Mathematics, Vol. 218 (Springer, New York, 2013) pp. xvi+708.
[19] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (Singapore: World Scientific, 1986).
[20] I. Bredberg, T. Hartman, W. Song, and A. Strominger, JHEP **04**, 019 (2010), arXiv:0907.3477 [hep-th].
[21] DLMF, "*NIST Digital Library of Mathematical Functions*," http://dlmf.nist.gov/, Release 1.0.15 of 2017-06-01, f. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.
[22] A. Lupsasca, M. J. Rodriguez, and A. Strominger, JHEP **12**, 185 (2014), arXiv:1406.4133 [hep-th].
[23] G. Compère and R. Oliveri, Phys. Rev. **D93**, 024035 (2016), [Erratum: Phys. Rev.D93,no.6,069906(2016)], arXiv:1509.07637 [hep-th].
[24] B. Chen and L. C. Stein, (2017), forthcoming.
[25] M. Guica, T. Hartman, W. Song, and A. Strominger, Phys. Rev. **D80**, 124008 (2009), arXiv:0809.4266 [hep-th].
[26] "xAct: Efficient tensor computer algebra for the Wolfram Language," `http://www.xact.es/`.
[27] J. M. Martín-García, Comp. Phys. Commun. **179**, 597 (2008), arXiv:0803.0862 [cs.SC].