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ENDOGENOUS AGENDA FORMATION IN THREE-PERSON COMMITTEES

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ABSTRACT

This paper analyzes a 3-person voting game in which two or three players have the ability to choose alternatives to be considered. Once the set of possible alternatives and the structure of the voting procedure are known, the players can solve for the outcome. Thus, the actual choice over outcomes takes place in the choice of alternatives to be voted on, i.e., the agenda. An equilibrium to this agenda-formation game is shown to exist under different assumptions about the information relative to the order of the players in the voting game. Further, this equilibrium is computed and found to possess certain features which are attractive from a normative point of view.

ENDOGENOUS AGENDA FORMATION IN THREE-PERSON COMMITTEES*

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1. INTRODUCTION

In recent years a branch of political science and social choice theory has examined the outcomes of voting games where the set of alternatives is exogenously determined [Miller (1980), Shepsle and Weingast (1984), Banks (1985)]. The actual process by which outcomes are arrived at can be seen as encompassing numerous stages prior to this final (voting) stage; however, in the spirit of dynamic programming, to solve for this larger game one must solve these stages recursively. For one common rule (amendment procedure) and one behavioral assumption (sophisticated voting), the solution to this final stage for any finite set of alternatives is well established [cf. the citations above]. Given this, a next logical step would be to study the determination of the set of alternatives under these assumptions. This paper makes a modest contribution to such a task.

We analyze a 3-person voting game where two or three players have the ability to choose alternatives to be considered. Given the set of alternatives and the order in which they will be voted on, the players can solve for the outcome. Thus, the game reduces to the choice of the alternatives given their place in the voting procedure.

The paper is organized as follows: Section 2 introduces the structure of preferences in the voting game; Section 3 presents the different methods for choosing alternatives as well as the different assumptions on the information possessed in regard to the order of the alternatives to be considered. Section 4 characterizes the solutions to the games along with certain interesting geometric properties of the solutions. An explicit characterization of the main solution concept is contained in the Appendix as are the proofs of two of the theorems. Finally, some concluding remarks are contained in section 5.

2. NOTATION, DEFINITIONS, ASSUMPTIONS

Consider a voting game consisting of a set I of players, $|I| = 3$, an alternative set $X \subset \mathbb{R}^n$ which includes the Pareto optimal points, and assume that each player has euclidean preferences over X:

$$u_i(x) = -d(x, x^i),$$

where x^i constitutes i's ideal point. Note that with three players the set of Pareto optimal points is simply a 2-dimensional surface. We say that x is <u>majority-preferred</u> to y, xPy, <u>iff</u>

$$|\{i \in I: \ u_i(x) > u_i(y)\}| \geq 2,$$

and define P: $X \to\to X$ as $P(y) = \{x \in X: \ xPy\}$. Thus, P(y) is the set of points majority-preferred to y.

We assume that a status quo point either does not exist, or is sufficiently distant from the Pareto surface.

Given a finite set of points $B = \{X_1,\ldots,X_t\} \subseteq X$ with $|B| = t$ and $T = \{1,2,\ldots,t\}$, define an agenda as an element of

$$\overline{A} = \{(x_{\sigma(1)},\ldots,x_{\sigma(t)}) \in B^t : \sigma : T \rightarrow T \text{ and } \sigma \text{ is } 1-1\};$$

thus $\overline{A}$ is the set of all permutations of B. Letting $y_i = x_{\sigma(i)}$, we assume that voting follows an amendment procedure, where for a given $A \subset \overline{A}$ an aggregate decision rule is arrived at by: i) comparing y_t and y_{t-1} via the majority preference relation; ii) comparing the preferred alternative to y_{t-2}, etc. After the $t - 1$ pairwise comparisons the remaining alternative is declared the voting outcome. We further assume that voters adopt sophisticated voting strategies; cf. Farquharson (1960). [A complete description of sophisticated voting under an amendment procedure can be found in Shepsle and Weingast (1984).]

Given that the outcome of this voting game is well defined for any finite set of alternatives [Banks (1985)], we focus our attention on the game describing the choice of alternatives to be voted on, with the corresponding payoffs from the subsequent voting game.

3. GAME FORMS AND SOLUTION CONCEPTS

3.1 1-Amendment Games

In the 1-amendment game, two players choose alternatives to be considered, a bill and an amendment (though these terms lose much of their content without restrictions on the set of alternatives labeled "amendments" to a particular "bill"). Two different assumptions on

the information available to the bill proposer will be examined. In the certainty case, the player proposing the bill knows the identity of the player proposing the amendment; in the uncertainty case, this identity is unknown. Let player i propose a bill x_i and, after x_i is announced, j proposes an amendment x_j. The strategy space for both i and j is simply X; however, j knows that if x_j does not beat x_i, x_i will be the outcome. Hence we define j's effective strategy space as $P(x_i)$. Note that there is no need to incorporate sophisticated voting in the 1-amendment game; players simply vote for their preferred alternative from the set $\{x_i,x_j\}$. A (pure) strategy for i is an element of X, while a (pure) strategy for j is a function $x_j : X \rightarrow X$.

3.2 2-Amendment Games

Here all three players choose alternatives, with i choosing the bill, j the first amendment, and k the second amendment. Again we examine the certainty case, where i knows the order of the players proposing amendments, and the uncertainty case, where this order is not known by i. In the voting game, x_j is paired with x_k, with the majority-preferred alternative paired against x_i; hence the assumption of sophisticated voting has a measurable effect. Suppose x_j is offered prior to x_k; the effective strategy space for j is still $P(x_i)$, while for k, the effective strategy space is $P(x_i) \cap P(x_j)$, since for x_k to be the voting outcome it must be that $x_k P x_i$ and $x_k P x_j$. In the 2-amendment game, (pure) strategies are defined as above for i and j, while for k a strategy is a function $x_k : X \times X \rightarrow X$.

3.3 Solution Concepts

In the spirit of sophisticated behavior and, more generally, dynamic programming, the equilibrium concept we employ in the certainty case is a modification of the Stackelberg equilibrium. Given that the player putting forward the last amendment chooses from his effective strategy space, we define a Stackelberg equilibrium (in the 2-amendment game with certainty) as $(x_i^*, x_j^*(\cdot), x_k^*(\cdot,\cdot))$, where

$$x_k^*(\cdot,\cdot) = \underset{x_k \varepsilon P(x_i) \cap P(x_j)}{\text{argmax}} u_k(x_k)$$

$$x_j^*(\cdot) = \underset{x_j \varepsilon P(x_i)}{\text{argmax}} u_j(x_k^*(x_i,x_j))$$

$$x_i^* = \underset{x_i \varepsilon X}{\text{argmax}} u_i(x_k^*(x_i,x_j^*(x_i))).$$

Since $P(x_i)$ and $P(x_j)$ are open sets, however, this concept might not be well defined. Thus we define an ε-Stackelberg equilibrium as a triple $(x_i^*,x_j^*(\cdot),x_k^*(\cdot,\cdot))$ such that, given $\varepsilon > 0$,

$$u_k(x_k^*) \geq \sup_{x_k \varepsilon P(x_i) \cap P(x_j)} u_k(x_k) - \varepsilon$$

$$u_j(x_k^*(x_i,x_j^*(x_i)) \geq \sup_{x_j \varepsilon P(x_i)} u_j(x_k^*(x_i,x_j)) - \varepsilon$$

$$u_i(x_k^*(x_i^*,x_j^*(x_i^*)) \geq \sup_{x_i \varepsilon X} u_i(x_k^*(x_i,x_j^*(x_i)) - \varepsilon.$$

Thus, if the strategies imply payoffs within ε of the highest (supremum) payoff, for small (fixed) ε, we will call the strategies an ε-Stackelberg equilibrium.

In the uncertainty case, we assume that player i (the bill proposer) adopts a modification of a minimax strategy where (for the 2-amendment game) x_i^* is i's minimax strategy if

$$x_i^* = \underset{x_i \varepsilon X}{\text{argmax}} \{\min\{u_i(x_k^*(x_i,x_j^*(x_i)),u_i(x_j^*(x_i,x_k^*(x_i))\}\},$$

where $x_j^*(x_i,x_k^*(x_i))$ is the outcome when k chooses the first amendment and j the second amendment. Again we define an ε-approximation to the minimax strategy; we say that $(x_i^*,x_j^*(\cdot),x_k^*(\cdot,\cdot)$ is an ε-minimax-Stackelberg equilibrium if j and k adopt ε-Stackelberg strategies and i adopts a minimax strategy such that the payoff to i from x_i^* is within ε of the highest (supremum) possible.

4. SOLUTIONS TO THE UNDERLYING GAMES

In section 3 we have described four different games; we seek now to characterize their solutions. We propose a point that we denote by M^* in the paper, which will turn out to be either the limiting outcome itself in the sense defined above or else closely related to it, for each of the four games. In this latter case, it follows that the limiting outcome can easily be found starting from M^*.

This section contains two parts. The first part (4.1) introduces the point M^*. The aim there is to make the reader familiar with the geometric properties of this point. We describe the procedure by which we construct the point M^* and show that in fact it is the intersection of three well-defined curves. In section 1 of the Appendix we give the mathematical equations of these curves.

The second part of this section (4.2) contains the main results. The proofs of Theorem 1.2 and Theorem 3.1 are given respectively in section 2.1 and section 2.2 of the Appendix.

4.1 The point M^*

4.1.1 Construction and Existence of M^*

First we construct the locus of points M_1, denoted by H_1^- as follows: we draw an arbitrary indifference curve for voter 1, which intersects the two contract curves x^1x^2 and x^2x^3 in, respectively I_{12} and I_{13}, say. We then draw voter 2's and voter 3's indifference curves which contain respectively I_{12} and I_{13}. Finally, we find the intersection M_1 of these two curves inside the Pareto set, if it exists. See Figures 1.1 and 1.2. Note that for any such point M_1 we have:

$$d(x^1,x^2) - d(x^2,M_1) = d(x^1,x^3) - d(x^3,M_1),$$

or equivalently:

$$d(x^3,M_1) - d(x^2,M_1) = d(x^1,x^3) - d(x^1,x^2).$$

Let $d(x^1,x^3) - d(x^1,x^2) \equiv K_1$. The locus of points H_1^- is then characterized by the equation:

$$d(x^3,M_1) - d(x^2,M_1) = K_1. \qquad (1)$$

This merely says that H_1^- is a subset of the locus of points, the difference of whose numerical distances from two fixed points (namely x^2 and x^3) is a constant ($+K_1$ or $-K_1$). This is by definition the hyperbola H_1 say, whose foci are x^2 and x^3. Note that the point x^1 (voter 1's ideal point) satisfies equation (1) and hence belongs to H_1^-. Consequently H_1^- is the piece of the hyperbola H_1 contained in the Pareto set and containing x^1. Indeed, the intersection of H_1^- and the contract curve x^2x^3 can easily be found. Such a point, I, satisfies (1). So,

$$d(x^3,I) - d(x^2,I) = K_1.$$

On the other hand since I lies on the contract curve x^2x^3 we have: $d(x^3,I) = d(x^2,x^3) - d(x^2,I)$. Thus,

$$d(x^2,I) = (d(x^2,x^3) - K_1)/2. \qquad (2)$$

In a similar fashion, starting from voter 2's and voter 3's ideal points, x^2 and x^3, respectively, we construct the loci of points M_2 and M_3 denoted respectively by H_2^- and H_3^-. The locus H_2^- is characterized by the equation:

$$d(x^3,M_2) - d(x^1,M_2) = K_2. \qquad (3)$$

where, $K_2 = d(x^2,x^3) - d(x^1,x^2)$, whereas H_3^- is characterized by the equation:

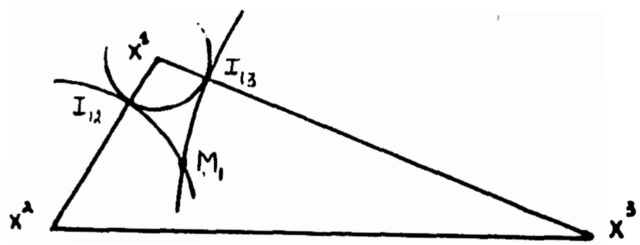

FIGURE 1.1 A POINT M_1 OF THE LOCUS H_1^-

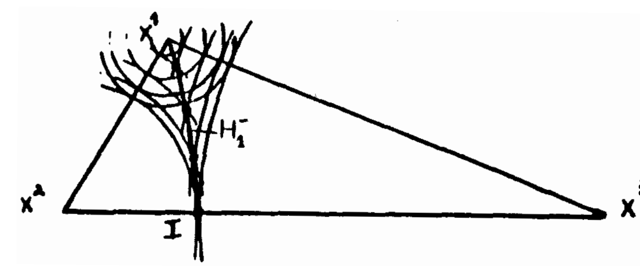

FIGURE 1.2 CONSTRUCTION OF THE LOCUS OF POINTS H_1^-

$$d(x^2,M_3) - d(x^1,M_G) = K_3 \quad (4)$$

where, $K_3 = d(x^2,x^3) - d(x^1,x^3)$.

For similar reasons as for H_1^-, the locus H_2^- is the piece of the hyperbola whose foci are x^1 and x^3, contained in the Pareto set and containing x^2. The point of H_2^- on the contract curve x^1x^3(H) is characterized by:

$$d(x^1,H) = (d(x^1,x^3) - K_2)/2. \quad (5)$$

The locus H_3^- is the piece of the hyperbola whose foci are x^1 and x^2, contained in the Pareto set and containing the point x^3. The intersection of H_3^- and the contract curve x^1x^3 (the point G) is characterized by:

$$d(x^1,G) = (d(x^1,x^2) - K_3)/2. \quad (6)$$

<u>Proposition 1</u>: There exists a unique point M^* in the Pareto set such that:

$$M^* = H_1^- \cap H_2^- \cap H_3^-.$$

<u>Proof</u>:

As already seen H_1^- and H_2^- are pieces of hyperbolas, contained in the Pareto set which is the area of the triangle $x^1x^2x^3$. The locus H_1^- contains the vertex x^1 and the point I on the opposite side x^2x^3 whereas H_2^- contains the vertex x^2 and the point H on the side x^1x^3. It follows that these two loci intersect in a unique point. Call M^* this point. Now, since H_3^- is also a piece of a hyperbola, in order to

prove the proposition, it suffices to show that $M^* \varepsilon H_3^-$. Since $M^* \varepsilon H_1^-$ we have by (1):

$$d(x^3,M^*) - d(x^2,M^*) = K_1 = d(x^1,x^3) - d(x^1,x^2). \quad (7)$$

Similarly $M^* \varepsilon H_2^-$, so that by (3):

$$d(x^3,M^*) - d(x^1,M^*) = K_2 = d(x^2,x^3) - d(x^1,x^2). \quad (8)$$

Subtracting (7) from (8) we obtain:

$$d(x^2,M^*) - d(x^1M^*) = K_2 - K_1 = d(x^2,x^3) - d(x^1,x^3)$$

or

$$d(x^2,M^*) - d(x^1,M^*) = K_3$$

which says that $M^* \varepsilon H_2^-$ (see (4)).

Q.E.D.

To summarize, we have constructed three loci H_1^-, H_2^- and H_3^- which as seen are pieces of hyperbolas, and have a unique common point inside the Pareto set, M^*. Figure 2 visualizes this point. The reader is referred to section 1 of the Appendix for a complete mathematical characterization of M^*. There, the equations of the complete hyperbolas, of which H_1^-, H_2^- and H_3^- are subsets, are given with respect to a simple system of coordinates. Therefore, given the three voters' ideal points x^1, x^2, and x^3, the point M^* of interest can easily be found.

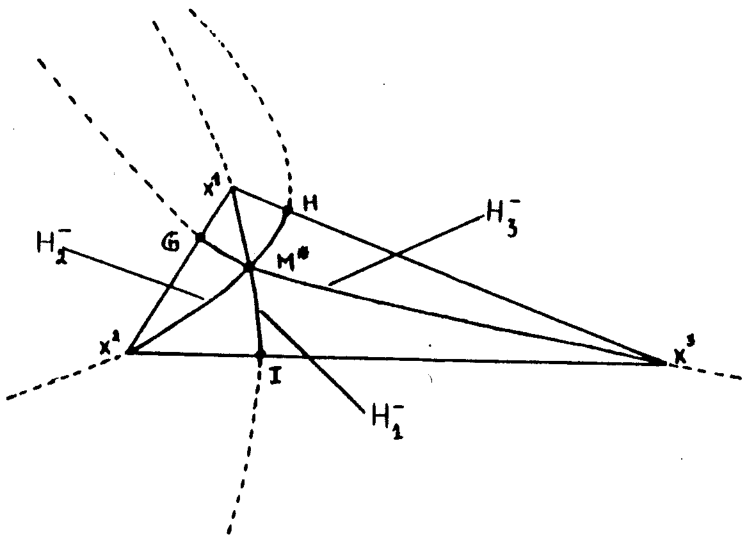

FIGURE 2: THE POINT $M^* = H_1^- \cap H_2^- \cap H_3^-$

Next, we observe that M^* displays a nice geometric property which as we shall see is the basis of the proofs of our main results.

4.1.2 Geometric Properties of M^*:

Draw the indifference curves of the three voters going through the point M^* and find the well-known area constituted by the three leaves (1,2), (1,3) and (2,3) of points that get a simple majority against M^*, i.e., $P(M^*)$. These indifference curves intersect the three contract curves x^1x^2, x^1x^3 and x^2x^3 in the set of points {A,B,C,D,E,F}. See Figure 3.

Proposition 2:

$$d(x^1,B) = d(x^1,C). \tag{9}$$

$$d(x^2,A) = d(x^2,F). \tag{10}$$

$$d(x^3,E) = d(x^3,D). \tag{11}$$

Proof: Since $M^* \varepsilon H_1^-$ we have by (1):

$$d(x^3,M^*) - d(x^2,M^*) = K_1 \equiv d(x^1,x^3) - d(x^1,x^2) \Leftrightarrow$$

$$d(x^1,x^2) - d(x^2,M^*) = d(x^1,x^3) - d(x^3,M^*) \text{ or}$$

$$d(x^1,x^2) - d(x^2,B) = d(x^1,x^3) - d(x^3,C), \text{ i.e.,}$$

$$d(x^1,B) = d(x^1,C).$$

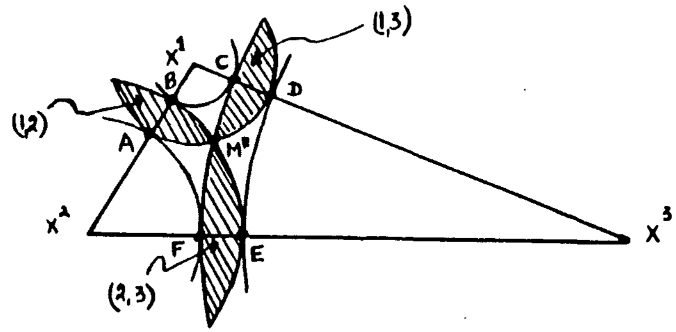

FIGURE 3: A GEOMETRIC PROPERTY OF THE POINT M^*

In a similar fashion,

since $M^* \varepsilon H_2^-$, by (3): $d(x^2,A) = d(x^2,F)$ and

since $M^* \varepsilon H_3^-$, by (4): $d(x^3,E) = d(x^3,D)$.

Q.E.D.

An immediate consequence of this proposition is the following.

Corollary 1:

$$d(A,B) = d(C,D) = d(E,F). \tag{12}$$

Proof: from the previous proposition, $d(x^1,B) = d(x^1,C)$. On the other hand, $d(x^1,A) = d(x^1,D)$. Thus $d(A,B) = d(C,D)$. From $d(x^2,A) = d(x^2,F)$ and $d(x^2,B) = d(x^2,E)$ we have, $d(A,B) = d(E,F)$.

Q.E.D.

Next we see that the three points I, H, G which are respectively $H_1^- \cap x^2x^3$, $H_2^- \cap x^1x^3$ and $H_3^- \cap x^1x^2$ exhibit also a nice geometric property. See Figure 4.

Proposition 3:

$$d(x^1,G) = d(x^1,H). \tag{13}$$

$$d(x^2,G) = d(x^2,I). \tag{14}$$

$$d(x^3,I) = d(x^3,H). \tag{15}$$

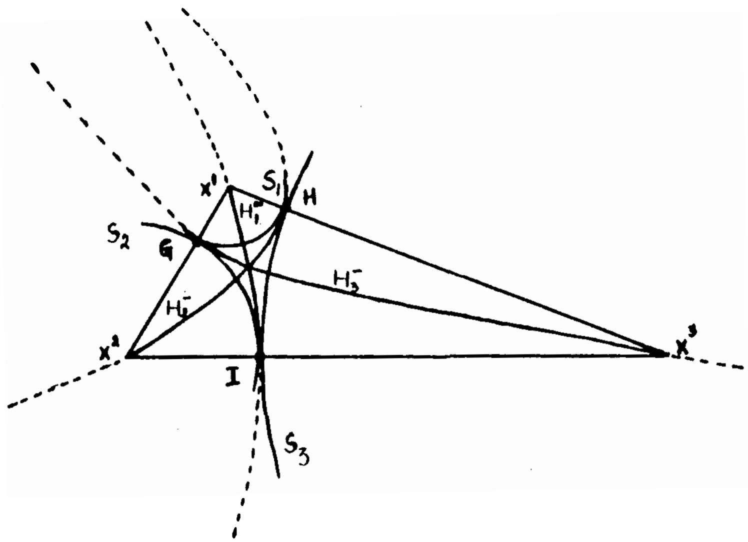

FIGURE 4: A GEOMETRIC PROPERTY OF THE POINTS $G = H_3^- \cap x^1x^2$, $H = H_2^- \cap x^1x^3$ AND $I = H_1^- \cap x^2x^3$

<u>Proof</u>: we have seen in (5) that $d(x^1,H) = (d(x^1,x^3) - K_2)/2$. Substituting for the expression of K_2 this can be rewritten as:

$$d(x^1,H) = [d(x^1,x^3 - d(x^2,x^3) + d(x^1,x^2)]/2$$

$$= [d(x^1,x^2) - (d(x^2,x^3) - d(x^1,x^3))]/2$$

$$= (d(x^1,x^2) - K_3)/2 = d(x^1,G).$$

$$d(x^2,G) = d(x^1,x^2) - d(x^1,G)$$

$$= d(x^1,x^2) - \frac{(d(x^1,x^2) - K_3)}{2} \text{ (by (6))}$$

$$= \frac{d(x^1,x^2) + K_3}{2} = \frac{d(x^1,x^2) + d(x^2,x^3) - d(x^1,x^3)}{2}$$

$$= \frac{d(x^2,x^3) - (d(x^1,x^3) - d(x^1,x^2))}{2} = \frac{d(x^2,x^3) - K_1}{2} = d(x^2,I).$$

$$d(x^3,I) = d(x^2,x^3) - d(x^2,I) = d(x^2,x^3) - \frac{(d(x^2,x^3) - K_1)}{2}$$

$$= \frac{d(x^2,x^3) + d(x^1,x^3) - d(x^1,x^2)}{2}.$$

$$d(x^3,H) = d(x^1,x^3) - d(x^1,H) = d(x^1,x^3) - \frac{(d(x^1,x^3) - K_2)}{2}$$

$$= \frac{d(x^1,x^3) + d(x^2,x^3) - d(x^1,x^2)}{2}.$$

Q.E.D.

From proposition 3, it follows that there exists a triple of indifference curves of respectively voter 1,2 and 3, say (S_1,S_2,S_3) such that: S_1 and S_2 are tangent at G, S_1 and S_3 at H, and S_2 and S_3 at I [Note that S_1, S_2, S_3 are each VNM solutions, as well as the set of points {G,H,I}]. Next we show that the point M^* is in the centered region delimited by S_1, S_2 and S_3. See Figure 4.

Proposition 4:

M^* belongs to the centered region delimited by S_1, S_2 and S_3.

Proof: It suffices to show that

$$d(x^1,G) \leq d(x^1,M^*).$$

$$d(x^2,I) \leq d(x^2,M^*).$$

$$d(x^3,H) \leq d(x^3,M^*).$$

$$M^* \in H_1^- \Rightarrow d(x^3,M^*) - d(x^2,M^*) = d(x^1,x^3) - d(x^1,x^2).$$

$$M^* \in H_2^- \Rightarrow d(x^3,M^*) - d(x^1,M^*) = d(x^2,x^3) - d(x^1,x^2).$$

By adding these two equations we get:

$$d(x^3,M^*) = \frac{d(x^1,M^*) + d(x,M^*) + d(x^1,x^3) + d(x^2,x^3) - 2d(x^1,x^2)}{2}$$

$$= \frac{d(x^1,x^3) + d(x^2,x^3) + d(x^1,A) + d(x^2,B) - 2d(x^1,x^2)}{2}$$

$$= \frac{d(x^1,x^3) + d(x^2,x^3) + d(x^1,x^2) + d(A,B) - 2d(x^1,x^2)}{2}$$

$$= \frac{d(x^1,x^3) + d(x^2,x^3) - d(x^1,x^2)}{2} + \frac{d(A,B)}{2}.$$

From the proof of proposition 3 recall that

$$d(x^3,H) = \frac{d(x^1,x^3) + d(x^2,x^3) - d(x^1,x^2)}{2}.$$

Thus $d(x^3,H) \leq d(x^3,M^*)$.

Similarly, using the fact that $M^* \in H_1^- \cap H_3^-$, we obtain $d(x^2,I) \leq d(x^2,M^*)$, and that $M^* \in H_2^- \cap H_3^-$, we obtain $d(x^1,G) \leq d(x^1,M^*)$.

Q.E.D.

4.2 Solutions

4.2.1 Solution to the 1-amendment game with certainty

The following theorem gives the solution to the game described in section 3.1 where the bill proposer, or the leader, has the information as to who will be the amendment proposer or the follower.

Theorem 1.1

(i) The point G is the unique limiting outcome as $\varepsilon \to 0$ of the ε-Stackelberg equilibrium of the 1-amendment game with certainty where player 1 is the leader and player 2 the follower. See Figure 5.

(ii) If player 3 is the follower, then the point H is the unique limiting outcome.

Proof:

As we will see, the limiting equilibrium strategy for player 1 is, in (i) the point H and in (ii) the point G. For convenience and simplicity of our demonstration we will divide the Pareto set (the area of the triangle $x^1x^2x^3$) into several regions. A natural way to do so is to consider the regions delimited by the loci H_2^- and H_3^-. Indeed, we notice that by the way the locus H_2^- has been constructed, for any point to the left of it proposed by player 1, player 2 will put forward an amendment on the contract curve x^2x^3. On the other hand, for any proposed point to the right of H_2^-, player 2 will put forward an amendment on the contract curve x^1x^2. See Figure 6. As to the locus H_3^-, for any point proposed by player 1 above it, player 3 will put forward an amendment on x^2x^3 and for any proposed point below it, player 3 will go on x^1x^3.

We will then look at the Pareto set as the union of four subsets or regions, the areas I_1, II_1, III_1, and IV_1, delimited by the loci H_2^- and H_3^-. See Figure 7.

Suppose now that player 2 is the follower. By the previous remark for any proposal made by player 1 in regions I_1 or IV_1, the optimizing behavior of player 2 will lead to an outcome on the contract curve x^2x^3.

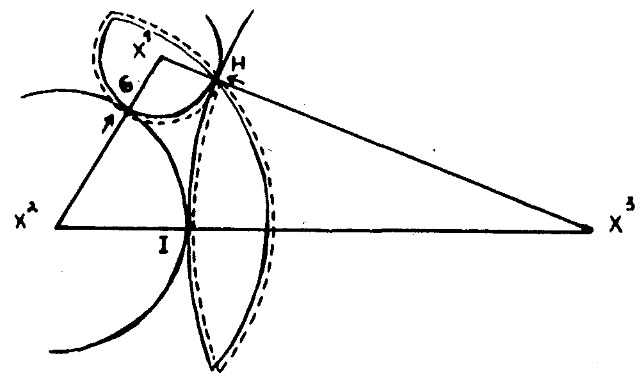

FIGURE 5: THE POINT G UNIQUE LIMITING OUTCOME OF THE ONE AMENDMENT GAME WITH CERTAINTY WHEN PLAYER 1 IS THE LEADER AND PLAYER 2 THE FOLLOWER.

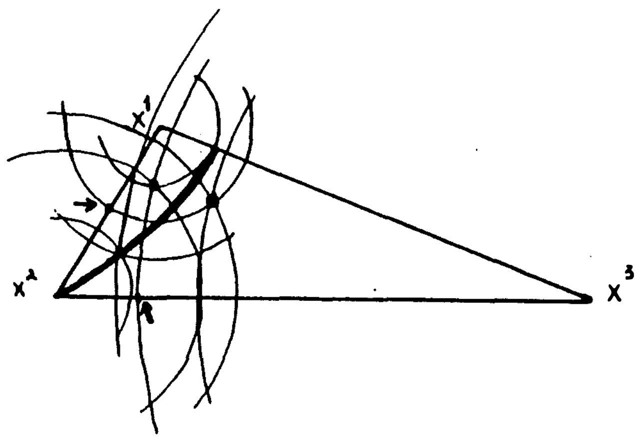

FIGURE 6: FOR POINTS TO THE LEFT OF H_2^- PROPOSED BY PLAYER 1, PLAYER 2 WOULD PUT AN AMENDMENT ON x^2x^3. FOR POINTS TO THE RIGHT, PLAYER 2 WOULD GO ON x^1x^2.

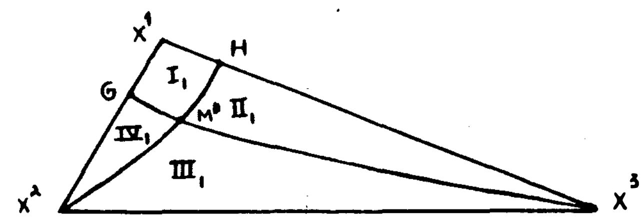

FIGURE 7: REGIONS I_1, II_1, III_1, AND IV_1, DELIMITED BY THE LOCI OF POINTS H_2^- AND H_3^- USED IN THE PROOF OF THEOREMS 1.1 AND 2.

For any point in region II_1 or III_1 (i.e., situated to the right of H_2^-) player 2 will go on x^1x^2 but at a point farther away than the limiting outcome (G) from player 1's ideal point. Similarly, if player 3 is the follower, for any proposal made by player 1 in regions I_1 or II_1, the outcome will be on x^2x^3. For any point in regions III_1 or IV_1, the resulting outcome is on x^1x^3, however less desired by player 1 than the limiting outcome (H).

Q.E.D.

Next, we examine the cases where either player 2 or player 3 is the leader.

<u>Theorem 1.2</u>

(i) The point G is the unique limiting outcome as $\varepsilon \to 0$ of the ε-Stackelberg equilibrium of the 1-amendment game with certainty where player 2 is the leader and player 1 the follower. If player 3 is the follower then the point I is the unique limiting outcome.

(ii) The point I is the unique limiting outcome as $\varepsilon \to 0$ of the ε-Stackelberg equilibrium of the 1-amendment game with certainty where player 3 is the leader and player 1 the follower. If player 2 is the follower, then the point H is the unique limiting outcome.

The proof of this theorem utilizes a similar argument as the proof of Theorem 1.1. The reader is referred to section 2.1 of the Appendix.

4.2.2 <u>Solution to the 1-Amendment Game with Uncertainty</u>

The next theorem gives the solution to the 1-amendment game with uncertainty. A nice feature of this solution is that it is independent of who the leader is.

<u>Theorem 2</u>: The point M^* is the unique limit as $\varepsilon \to 0$ ε-minimax strategy for the leader $i \in \{1,2,3\}$ in the 1-amendment game with uncertainty.

<u>Proof</u>: We will demonstrate this theorem for the case where $i = 1$, namely when player 1 is the leader. The other oases ($i = 2,3$) are briefly indicated and make use of a similar argument.

Note, first that by making a bill proposal in the neighborhood of M^*, the leader (player 1) induces an outcome in the neighborhood of A (if player 2 is the follower) or in the neighborhood of D (if player 3 is the follower). In either cases player 1 gets the same level of utility. See Figure 8.

Next, we show that if he makes a proposal different from the ε-minimax "proposal," then he will be worse off. More specifically, it is the case that when at least one of players 2 or 3 puts forward an amendment, the resulting outcome from the voting game is less desired by the original bill proposer. As seen in the proof of Theorem 1.1 the geometric characteristics of the loci H_2^- and H_3^- allow us to observe that any proposal by player 1 in regions I_1, II_1 or IV_1 is not advantageous for him. Indeed, if he proposes a bill in region II_1, he runs the risk that player 3 is the one who puts forward an

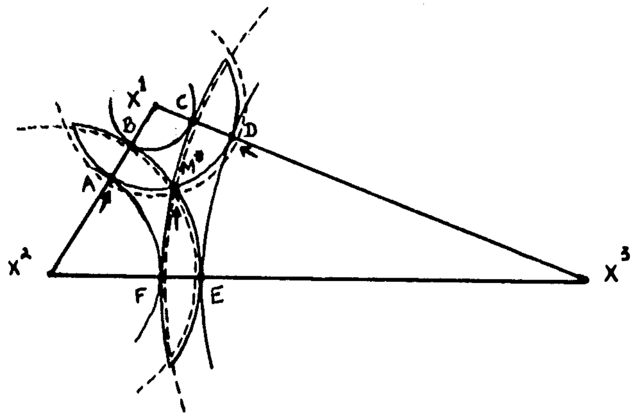

FIGURE 8: M^*, ε-MINIMAX STRATEGY FOR PLAYER 1 IN THE ONE AMENDMENT GAME WITH UNCERTAINTY AS TO WHO IS THE FOLLOWER.

amendment resulting in an outcome on the contract curve x^2x^3. For points in regions IV_1, he runs the risk that player 2 is the follower which will lead to an outcome on x^2x^3. Finally for points in region I_1, the outcome will be on x^2x^3 when either of the two players is the follower. Now, if player 1 makes a proposal in region III_1, the outcome will be either on x^1x^2 (when player 2 is the follower) or on x^1x^3 (when player 3 is the follower). However, in both cases the outcome is farther away from x^1 than the outcome resulting from the ε-minimax strategy.

By using regions $\{I_2, II_2, III_2, IV_2\}$ and $\{I_3, II_3, III_3, IV_3\}$ (see the proof of theorem 1.2 in section 2.1 of the Appendix) and an analogous argument, the remainder of the theorem follows (cases where $i = 2,3$).

Q.E.D.

4.2.3. <u>Solution to the 2-Amendment Game with Certainty</u>

When all players make proposals and the order of the players proposing amendments is known to the leader, the following result holds.

<u>Theorem 3.1</u>

(i) The point M^* is the unique limiting outcome as $\varepsilon \to 0$ of the ε-Stackelberg equilibria of the 2-amendment game with certainty where player 1 is the leader and the order of the followers is 2,3.

(ii) If the order of the followers is 3,2 then, the point M^* is still the unique limiting outcome.

The proof of this theorem is in section 2.2 of the Appendix.

Theorem 3.2

The point M^* is the unique limiting outcome as $\varepsilon \to 0$ of the ε Stackelberg equilibria of the 2-amendment game with certainty where either player 2 or 3 is the leader and for any ordering of the followers.

The proof of this theorem is analogous to that of theorem 3.1 and is omitted.

4.2.4 Solution to the Two Amendment Game with Uncertainty

Notice from the proof of theorem 3.1 (see section 2.2 of the Appendix) that player 1 is indifferent between two alternatives in the limit, namely A and D, regardless of the order of the followers. Similar observations are true for the cases where either player 2 or player 3 is the leader. Therefore, the following result holds.

Theorem 4

(i) The point M^* is the limiting outcome as $\varepsilon \to 0$ of the ε-minimax-Stackelberg equilibrium of the 2-amendment game with uncertainty as to the order of the followers, where player 1 is the leader.

(ii) when player 2 or player 3 is the leader, the point M^* is still the limiting outcome.

5. CONCLUSION

This paper characterizes the solutions to a number of endogenous agenda formation games with three players. Our main result concerns the importance of the point M^* in arriving at such solutions. In relation to other solution concepts for voting games with alternative spaces in $\mathbb{R}^m$, it is easily shown that:

(i) M^* is in the uncovered set of X;

(ii) M^* is not in general the center of the yolk [McKelvey (1983)];

(iii) M^* is not in general the strong point [Grofman, et al. (1985)].

From a normative point of view, M^* has certain attractive features. If the ideal points of the three players are equidistant from each other, M^* is then the barycenter of the Pareto set, giving all players equal utility; see Figure 9. As the ideal points become less symmetrical, M^* tends to the closer pair of ideal points, leaving the player whose preferences are less similar in a relatively disadvantageous position. Thus, one can argue that, irrespective of the institution involved, the point M^* provides an equitable solution to policy-type games.

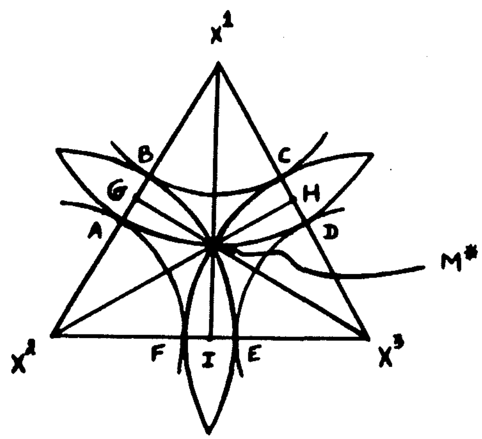

FIGURE 9: THE CASE WHERE THE 3 PLAYERS' IDEAL POINTS FORM AN EQUILATERAL TRIANGLE.

APPENDIX

<u>Section 1</u>:

In this section we give a complete mathematical characterization of the point M^*. Namely, we give the equations of the hyperbolas H_1, H_2 and H_3 which contain respectively the loci H_1^-, H_2^- and H_3^-.[1]

We will elaborate in detail the equation of the hyperbola H_1. The equations of H_2 and H_3 are derived in an analogous manner. Given the three voters' ideal points x^1, x^2, and x^3 recall that H_1^- was defined as the locus of points M_1 such that:

$$d(x^3,M_1) - d(x^2,M_1) = K_1 \text{ where } K_1 = d(x^1,x^3) - d(x^1,x^2).$$

Furthermore, H_1^- contains x^1 and is contained in the area of the triangle $x^1x^2x^3$ (the Pareto set). This locus H_1^- is a piece of the complete hyperbola H_1 which foci are x^2 and x^3. The point midway between x^2 and x^3 is the center of the hyperbola. We denote the distance between the foci by $2c_1$ and the difference of the distances of any point on H_1 from the foci by $2k_1$. In order to derive the equation of H_1 we let the line through the foci be the x-axis and the center be the origin. Thus, the coordinates of the foci are $x^2(-c_1,0)$ and $x^3(c_1,0)$. Any point $M_1(x,y)$ on the curve H_1 satisfies:

$$d(x^3,M_1) - d(x^2,M_1) = \pm K_1 \equiv \pm 2k_1 \qquad \text{(A1)}$$

where in our case the positive sign holds for points to the left of the y-axis and the negative sign for the points to the right. Since $k_1 < c_1$ (as the difference of two sides of a triangle is less that the third side) define:

$$b_1^2 = c_1^2 - k_1^2. \tag{A2}$$

Substituting for $d(x^3,M_1)$ and $d(x^2,M_1)$ their values from the distance formula in (A1) and using (A2), we obtain the equation of H_1 as:

$$\frac{x^2}{k_1^2} - \frac{y^2}{b_1^2} = 1 \tag{A3}$$

the lines of equation:

$$y = \frac{b_1}{k_1} x \text{ and } y = -\frac{b_1}{k_1} x \tag{A4}$$

are the asymptotes of the hyperbola. These equations can be combined into a single equation:

$$\frac{x^2}{k_1^2} - \frac{y^2}{b_1^2} = 0. \tag{A5}$$

The points I and I' which are the intersections of the hyperbola with the x-axis (called the vertices) have coordinates $I(-k_1,0)$ and $I'(k_2,0)$. See Figure A1.

The equations of H_2 and H_3 are derived in a similar way and are given by:

$$\frac{x^2}{k_2^2} - \frac{y^2}{b_2^2} = 1 \tag{A6}$$

where

$$2k_2 = K_2 = d(x^2,x^3) - d(x^1,x^2)$$

$$b_2^2 = c_2^2 - k_2^2. \tag{A7}$$

$$d(x^1,x^3) = 2c_2$$

and

$$\frac{x^2}{k_3^2} - \frac{y^2}{b_3^2} = 1 \tag{A8}$$

where

$$2k_3 = K_3 = d(x^2,x^3) - d(x^1,x^3)$$

$$b_3^2 = c_3^2 - k_3^2. \tag{A9}$$

$$d(x^1,x^2) = 2c_3.$$

The equations of the asymptotes are respectively:

$$\frac{x^2}{k_2^2} - \frac{y^2}{b_2^2} = 0 \tag{A10}$$

and

$$\frac{x^2}{k_3^2} - \frac{y^2}{b_3^2} = 0. \tag{A11}$$

See Figures A2 and A3.

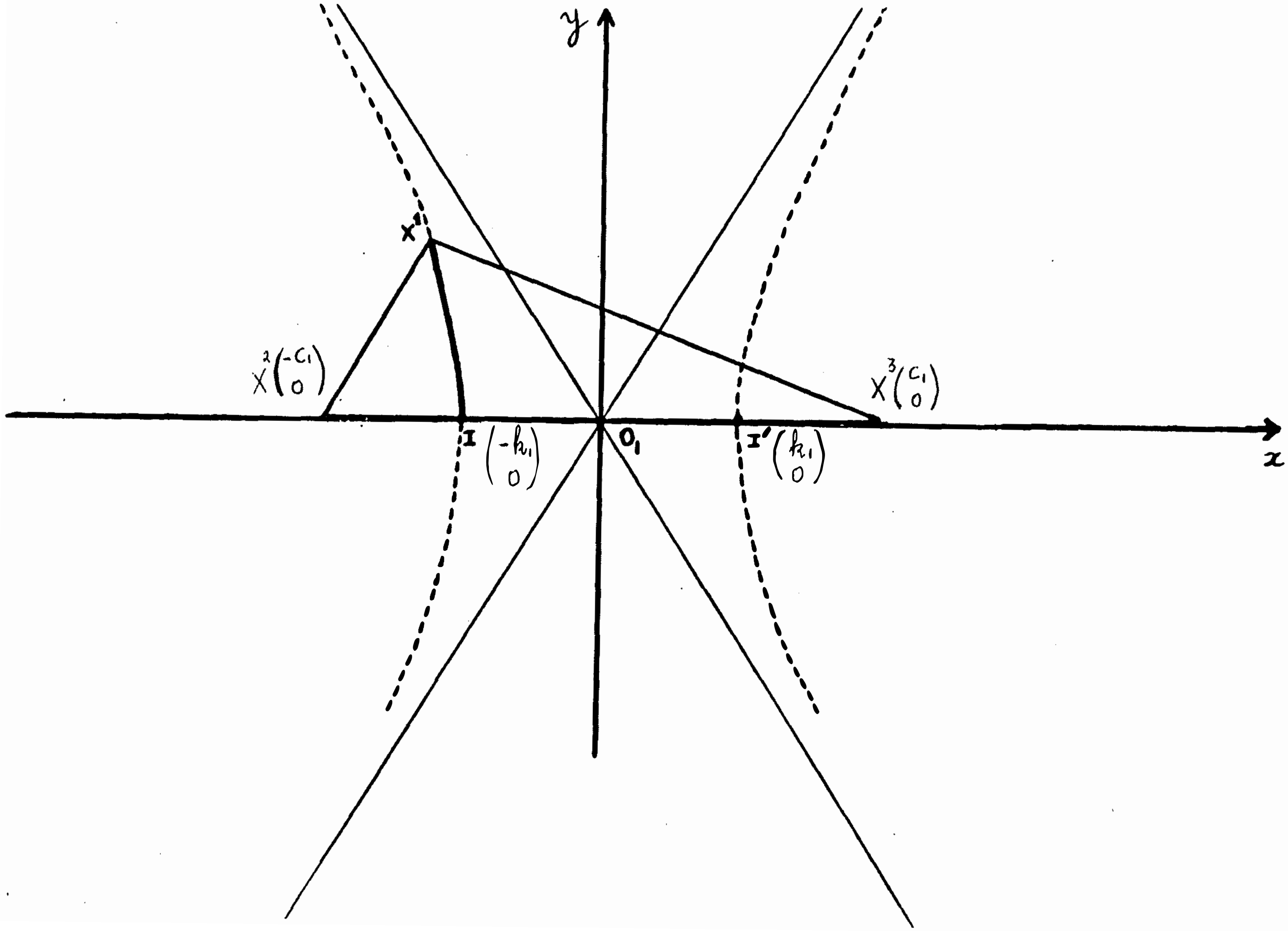

FIGURE A1: THE COMPLETE HYPERBOLA H_1 CONTAINING H_1^-

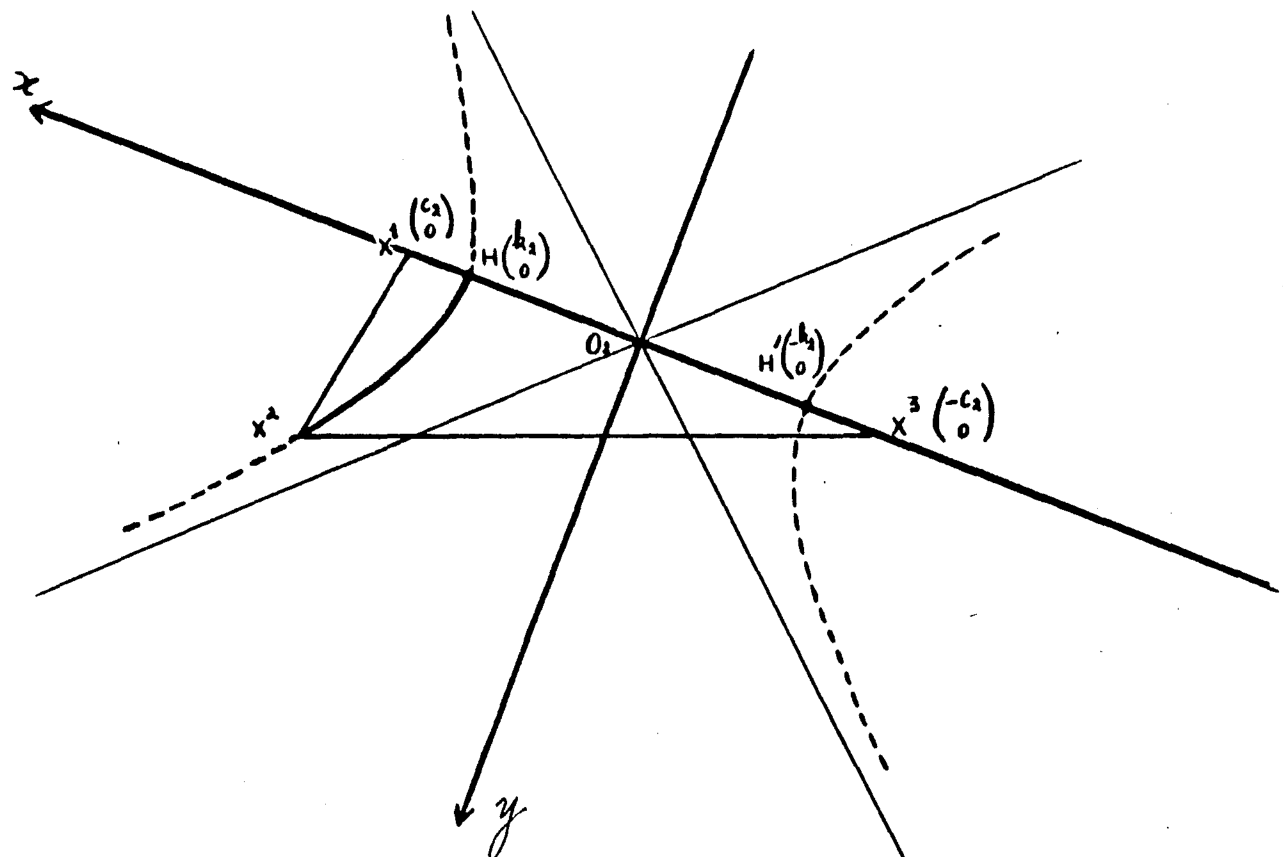

FIGURE A2: THE COMPLETE HYPERBOLA H_2 CONTAINING H_2^-

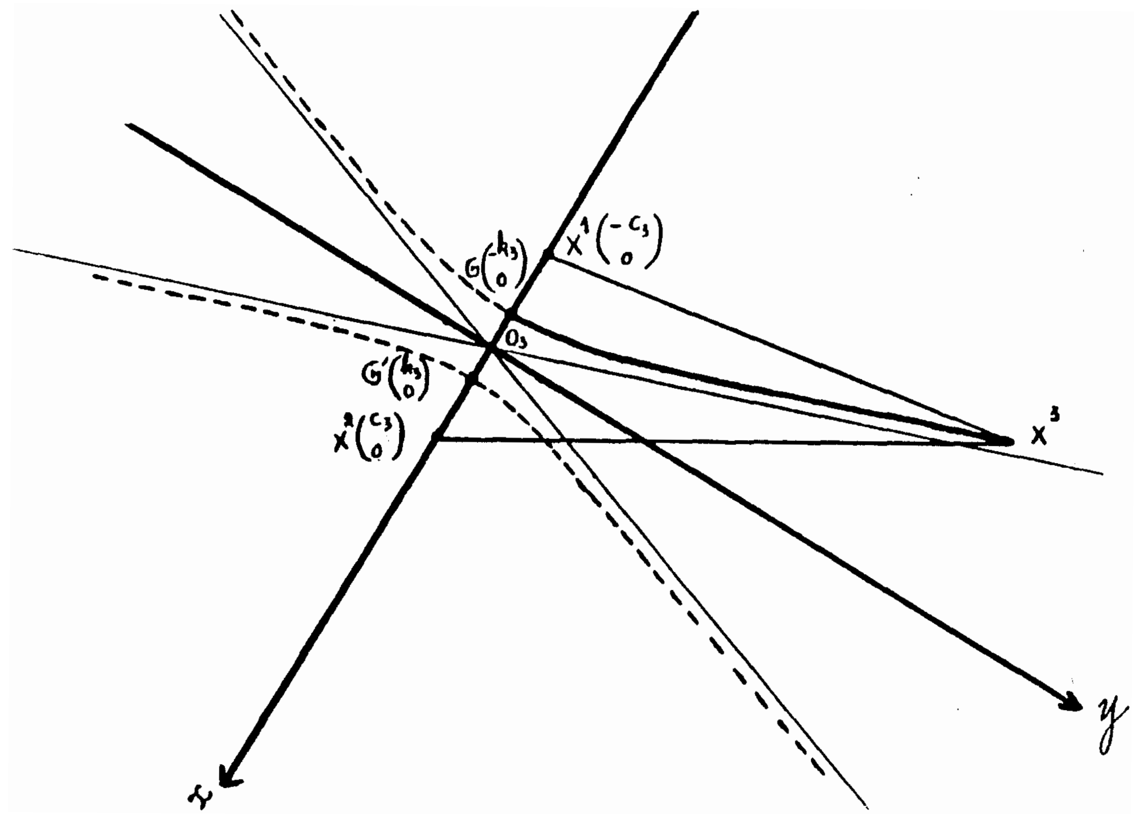

FIGURE A3: THE COMPLETE HYPERBOLA H_3 CONTAINING H_3^-

<u>Section 2.1</u>

<u>Proof of Theorem 1.2</u>

(i) Here player 2 is the bill proposer and when player 1 is the follower, the limiting equilibrium strategy for player 2 is the point I. Consider the loci H_1^- and H_3^-. By construction of H_1^- for any point proposed by player 2 to the left of H_1^- player 1 would put forward an amendment of x^1x^3 (or at x^1 as there will be cases where x^1 is in his effective strategy space).[2] For any point to the right he would go on x^1x^2. As to H_3^-, player 3 will behave in the same way as when player 1 is the leader (see theorem 1.1). Consider the regions I_2, II_2, III_2 and IV_2 delimited by these two loci. See Figure A4. Now, when player 1 is the follower, for any proposed point in regions I_2 and IV_2 player 2 is worse off than at the limiting outcome. Now, notice that in regions II_2 and III_2 the limiting equilibrium "proposal" is his best alternative. When player 3 is the follower, the limiting equilibrium strategy for player 2 is the point G. The regions I_2 and II_2 are not favorable to player 2 as any point proposed by him will lead to an outcome on x^1x^3. Again, from the remaining regions the limiting equilibrium proposal is player 2's best strategy.

(ii) Player 3 is now the bill proposer. By examining the region I_3, II_3, III_3 and IV_3 delimited by the loci H_1^- and H_2^- (see Figure A5) and applying a similar argument as above the results follow.

<u>Section 2.2</u>

In what follows, we shall expose in detail the proof of the first part of theorem 3.1 (i), that is when player 1 is the bill proposer and the order of the players proposing amendments is known to be 2,3. Again, the essence of this result, lies in the geometric properties of the point M^*. The second part of the theorem follows from a similar argument. Incidentally, it is quite useful in order to understand the results, to notice the symmetry of the problem.

<u>Proof of Theorem 3.1(i)</u>

As we shall see, in this case $x_1^* = A$ or $x_1^* = D$ and $x_2^*(x_1^*) = B$. The strategy used to demonstrate this theorem is as follows. Notice first that the behavioral assumption of sophistication of the player (see section 3.2) implies that the amendment proposed by player 3 will be the outcome. Indeed, recall that for any proposal x_1 made by player 1, the effective strategy space for player 2 is $P(x_1)$ while for player 3 it is $P(x_1) \cap P(x_2)$, i.e., the set of alternatives that beat x_1 and x_2. Thus, the third player's amendment will be the voting outcome. Given the assumption of euclidean preferences, player 3's equilibrium strategy is simply $x_3^*(x_1,x_2) = \underset{x \in P(x_1)\cap P(x_2)}{\operatorname{argmin}} d(x,x^3)$. Consequently, for our two proposed equilibria, it is the case that the voting

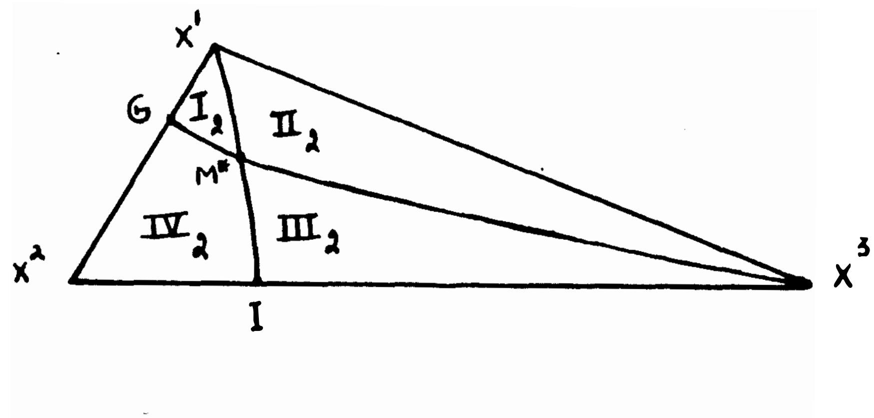

FIGURE A4: REGIONS I_2, II_2, III_2, AND IV_2 DELIMITED BY THE LOCI H_1^- AND H_3^- USED IN THE PROOF OF THEOREMS 1.2. (i) and 2.

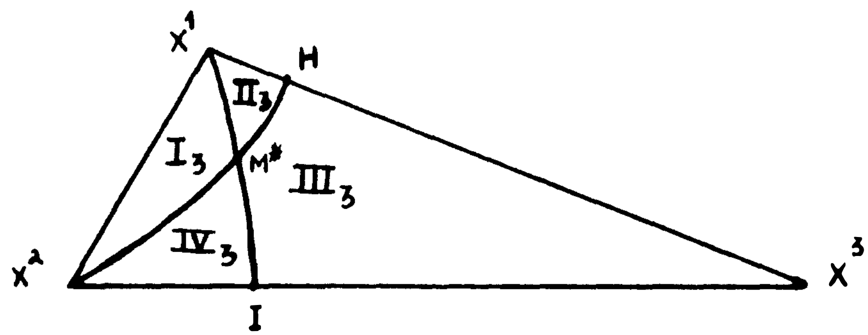

FIGURE A5: REGIONS I_3, II_3, III_3, AND IV_3 DELIMITED BY THE LOCI H_1^- AND H_2^- USED IN THE PROOF OF THEOREMS 1.2. (ii) and 2.

outcome is within ε of M^*. Now, player 1 as a leader, makes his decision taking into account the optimizing behavior of player 2, his immediate follower, which in turn does the same thing as to the "rational" behavior of player 3. We will therefore investigate the best "reaction" function of player 2. Namely, for any point proposed by player 1, we will search for the best strategy of player 2 and the subsequent voting outcome. Finally, we will show that by proposing a point different from the equilibrium strategies (within ε of A or D) player 1 will induce an outcome less desired by him.

Figure A6 shows the Pareto set divided into six subsets (or regions). For any point in region I proposed by player 1, player 2 will make an amendment at a point within ε of the point G and player 3 will make an amendment within ε of the point I. See Figure A7 for a sample of such points. For any proposal made by player 1 in region V, the induced outcome will be on the contract curve x^2x^3 to the right of the point I. For any point in region IV picked by player 1 the resulting outcome will be on x^2x^3 again to the right of I. See Figure A8. For any point in region III, the induced outcome will be on player 1's indifference curve going through that point and below M^* or else on x^2x^3 to the right of I. See Figure A9. Now, for any point proposed by player 1 in regions II or IV the induced outcome will be on player 1's indifference curve passing through that point and a player 2's indifference curve which goes through an appropriate point in the (1,3) leaf. See Figure A10. Notice that when player 1 picks precisely the equilibrium strategies, namely points within ε of A or

D, the intersection of these two indifference curves will be very close to M^*. See Figure A11. This completes the proof.

Q.E.D.

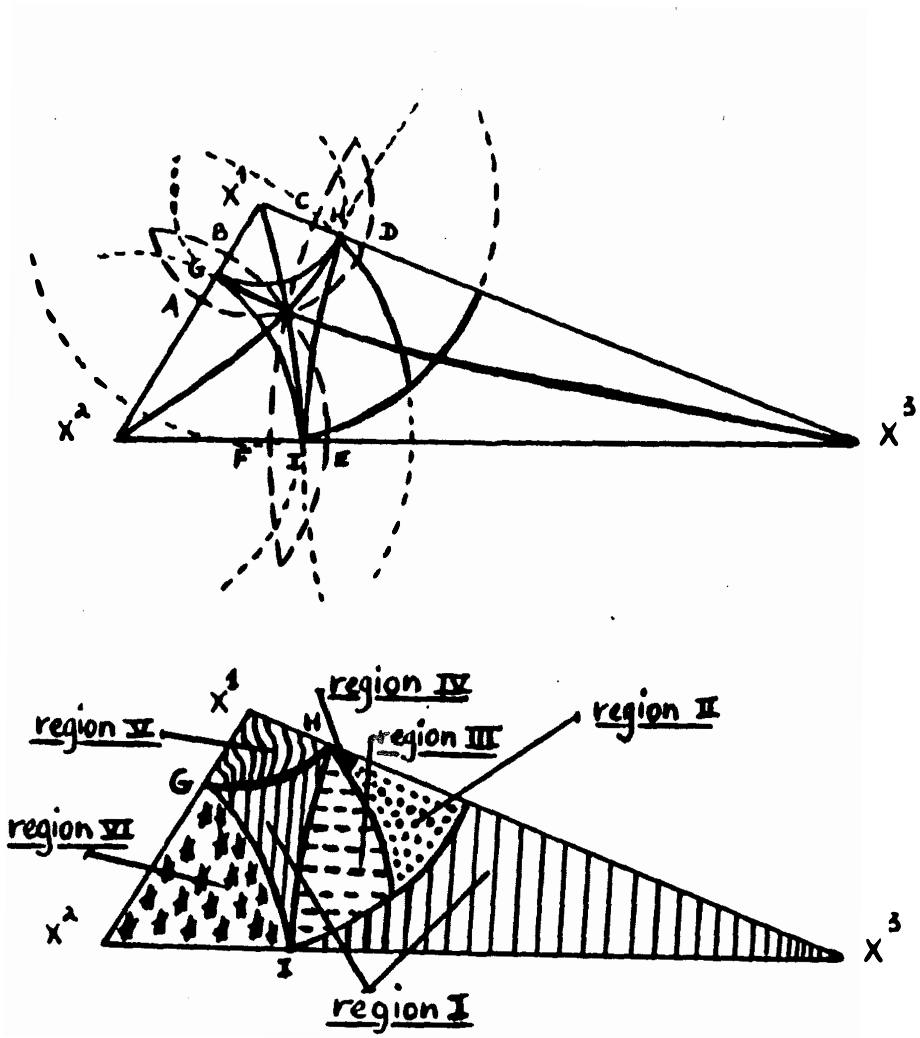

FIGURE A6: THE CURVES DELIMITING REGIONS I, II, III, IV, V AND VI USED IN THE PROOF OF THEOREMS 3.1. (i) AND THE REGIONS THEMSELVES.

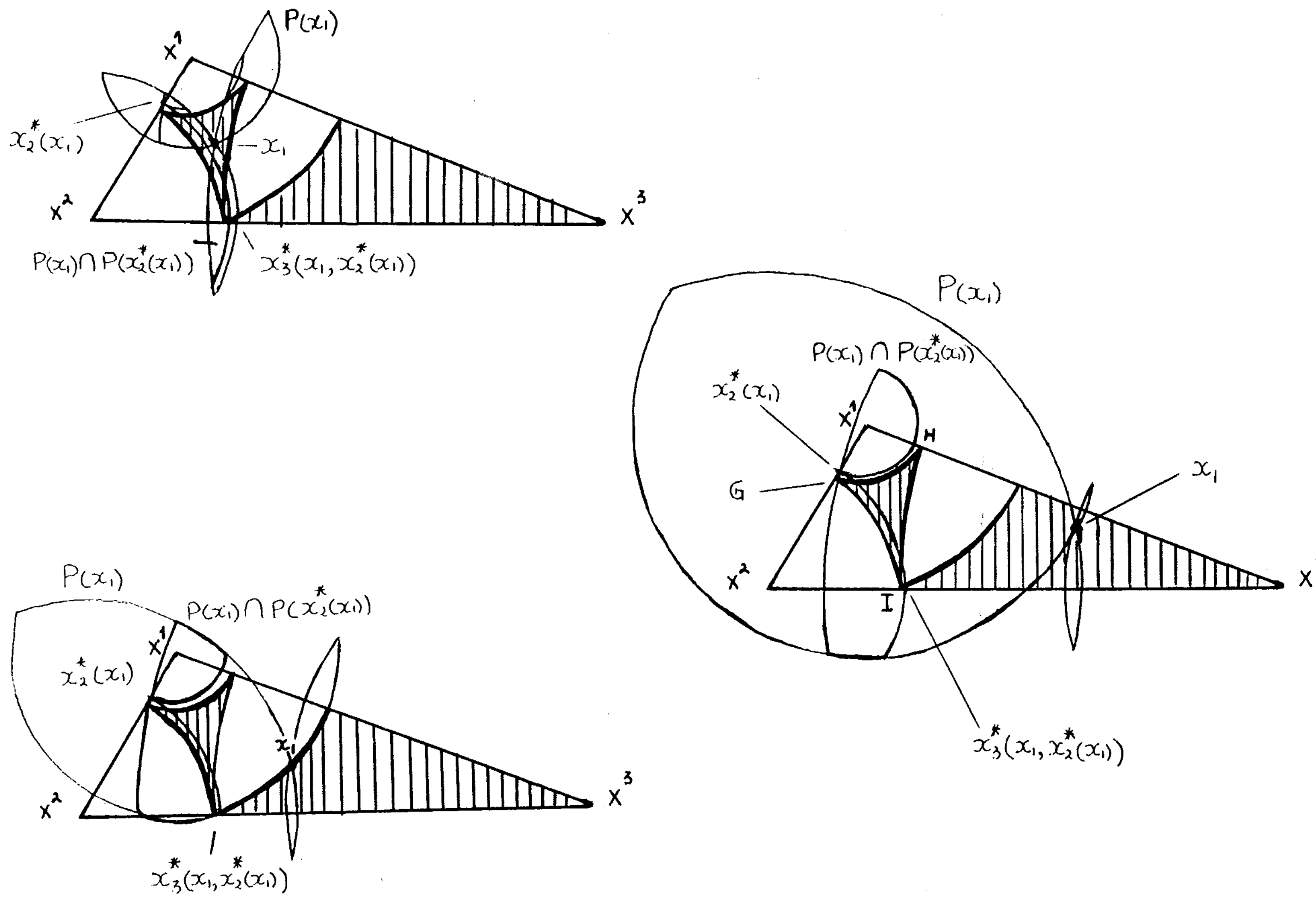
P(x1)
X1
x2*(x1)
x1
X2
X3
P(x1) ∩ P(x2*(x1))
x3*(x1, x2*(x1))
P(x1)
P(x1) ∩ P(x2*(x1))
x2*(x1)
X1
H
G
x1
X2
I
X3
x3*(x1, x2*(x1))
P(x1)
P(x1) ∩ P(x2*(x1))
X1
x2*(x1)
x1
X2
X3
x3*(x1, x2*(x1))

FIGURE A7

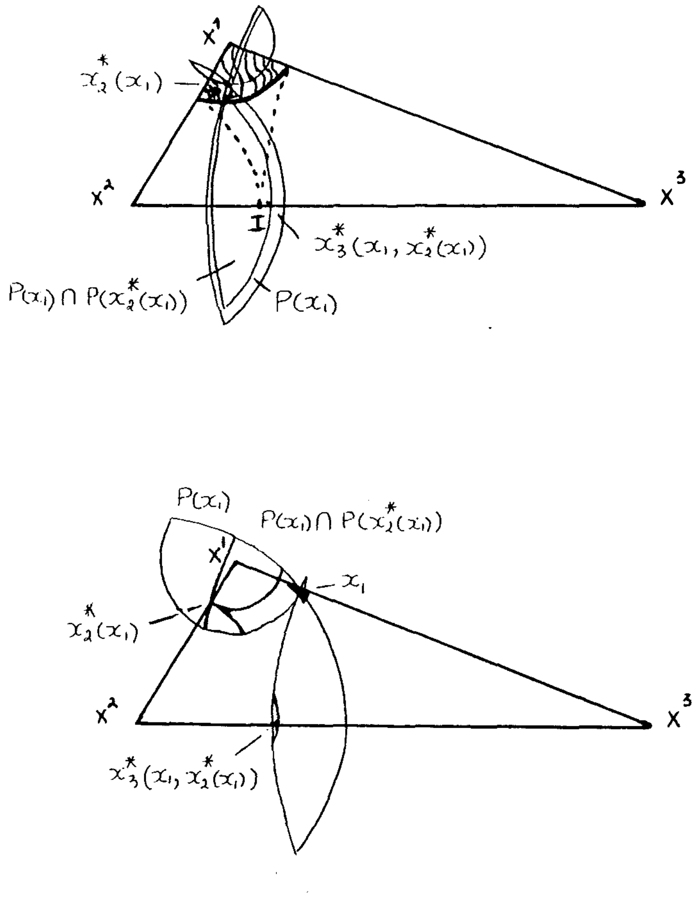

FIGURE A8

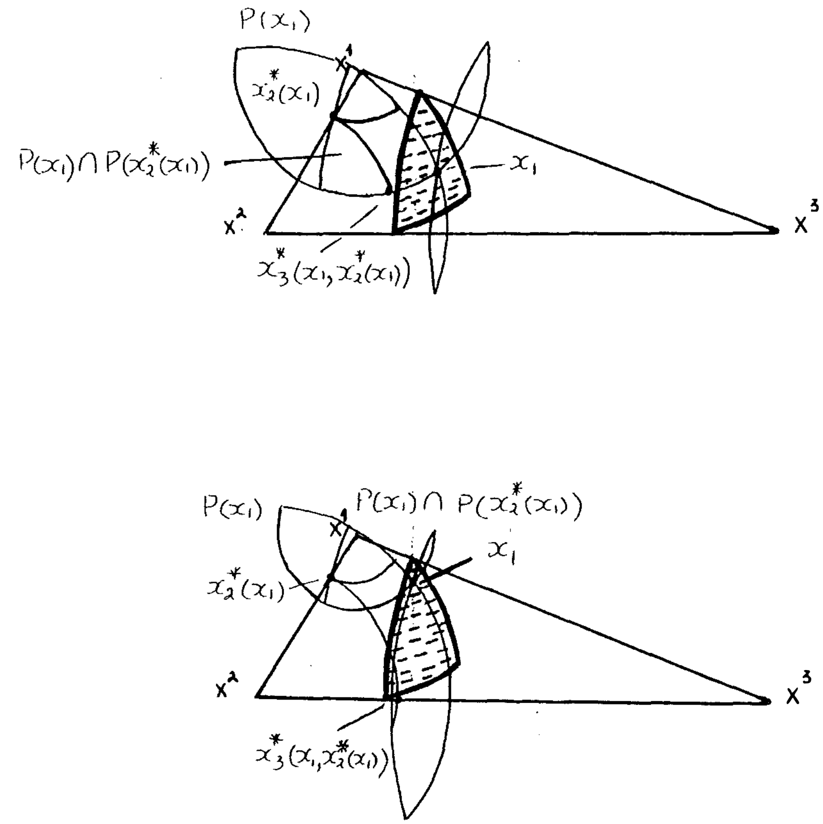

FIGURE A9

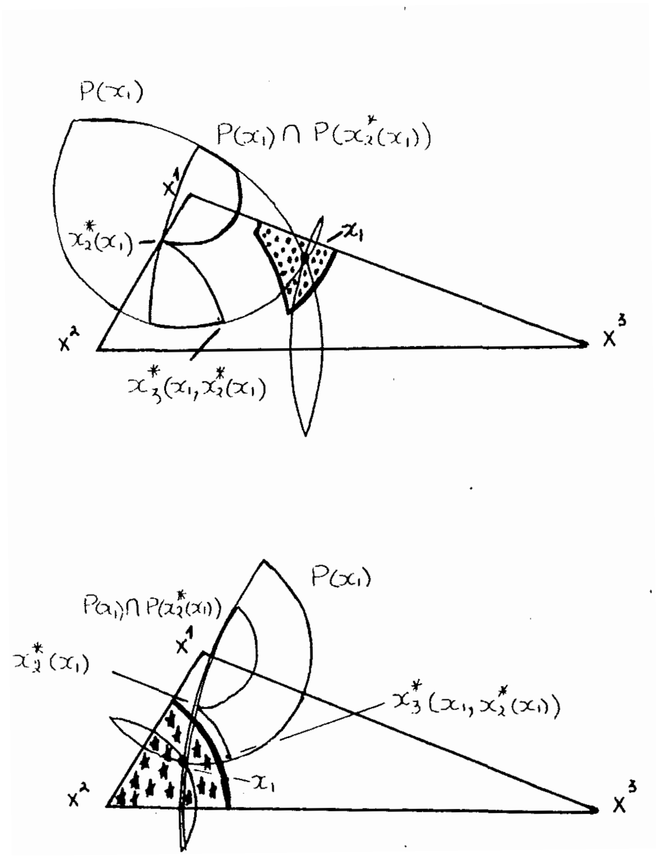

FIGURE A10

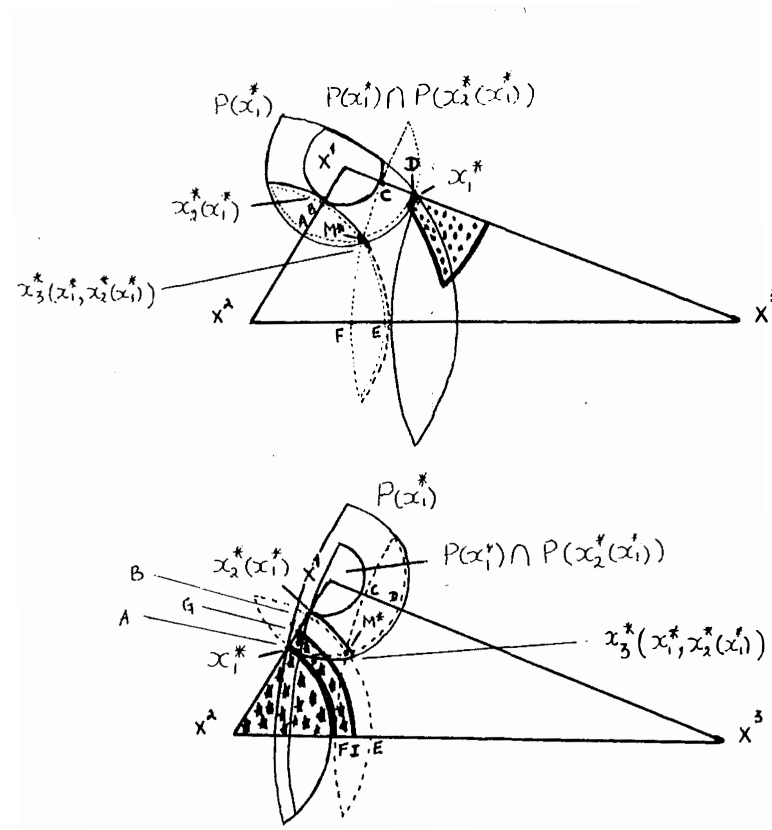

FIGURE A11

FOOTNOTES

* Prepared for delivery at the 1985 Annual Meeting of the American Political Science Association, the New Orleans Hilton, August 29–September 1, 1985. Copyright by the American Political Science Association. We would like to thank Richard McKelvey and Norman Schofield for valuable comments and suggestions.

1. Recall that $M^* = H_1^- \cap H_2^- \cap H_3^-$ (see proposition 1).

2. This is so whenever the angle $x^2x^1x^3$ is obtuse.

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