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PIVOT MECHANISMS IN PROBABILITY REVELATION

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SOCIAL SCIENCE WORKING PAPER 596

March 1987

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ABSTRACT

The Groves mechanism and k^{th} price auctions are well-known examples of pivot mechanisms. In this paper an analogous pivot mechanism is defined for probability relevation and then the Bayesian equilibria are characterized for the three pivot mechanisms. The main result is that in Bayesian games with these pivot mechanisms, equilibria must satisfy a simple fixed point condition. The result does not require signal ordering properties and thus generalizes and simplifies results by Milgrom and others. When the fixed point is unique there is "no regret." The result also holds for games less structured than Bayesian games (where the common knowledge and consistency assumptions are relaxed).

The pivot mechanism in probability revelation is shown to generalize and characterize proper scoring rules. The characterization yields an optimization of research incentives for proper scoring rules and suggests that under some conditions the new mechanisms, which are pivot mechanisms but not proper scoring rules, outperform proper scoring rules.

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"Rational actors" make two types of judgment: one of belief (probability) and the other of preference (utility or willingness to pay). Although these two judgments are brought symmetrically together in the formation of an expected utility, applications using revealed probabilities tend to differ from those using revealed valuations. Mechanisms revealing preferences arise in the theory of auctions (Milgrom and Weber [1982]) and public goods (Green and Laffont [1979]). Mechanisms for revealing probabilities arise in attempts to establish incentive compatible rules of liability, forecasting, and risk assessment (Savage [1971]).

A chemical firm, facing a rule of strict liability, may want its toxicologists to assess the probability that a given chemical is toxic. Weather forecasters routinely make probabilistic predictions of precipitation, cloud cover, and other conditions. In a firm the director of R and D may ask experts to assess the probability that various projects will meet performance targets at given costs and dates. In the regulation of nuclear power, decisions on investments in safety are based on quantitative assessments of the probabilities of accidents.

Concern with incentive compatibility in probability revelation has a long history, going back from Bayes to Ramsey, de Finetti, and Savage. Savage [1971] devoted his last paper to *proper scoring rules*, a class of mechanisms for revealing probability. (Proper scoring rules are virtually the only probability revealing mechanisms to have received formal study.) In experimental economics, proper scoring rules have been used by Grether [1981] and McKelvey and Page [1985]. Eleven years before Vickery's seminal paper on incentive compatibility and second price auctions, Brier [1950] proposed a mechanism for eliciting probabilistic weather forecasting which would be immune to manipulation, or as he put it, "playing the system." Since 1965 his mechanism, a proper scoring rule, has been widely adopted in weather forecasting (Murphy and Winkler [1985]).

Interestingly, Savage traced his idea for characterizing proper scoring rules to Marschak's seller's price auction, and Green and Laffont traced the "essence" of the Groves mechanism to the same source. The identification of a common source suggests a parallel between the two theories of relevation mechanism (probability and preference). However, by and large the two theories have grown separately, and the parallel has not been developed. Pivot mechanisms play a central role in the Groves mechanism, but Savage, and others, defined proper scoring rules as non-pivot mechanisms.

The purpose of this paper is to explore the role of pivot mechanisms in probability revelation and to bring out the parallel between probability and preference revelation. I begin by defining a Bayesian game for pivot mechanisms in probability revelation. The main result (Theorem 1) is that the Bayesian equilibria for this game satisfy, as a necessary condition, a fixed point property. The fixed point property is also a necessary condition for the Bayesian equilibria of other Bayesian games with pivot mechanisms. In particular the property characterizes the Groves mechanism for public goods (Theorem 3) and k^{th} price auctions (immediate consequence of Theorem 3).

Further, the fixed point property obtains for games less structured then Bayesian games. So, the theorems are derived with weaker assumptions, compared to those of a Bayesian game, but apply to the equilibria of a Bayesian game as a specialization. In the more general setting, following the parallel between probability and preference revelation in the latter direction, I use the idea of Theorem 1 to extend the basic theorem of the Groves mechanism (that truthful reporting is a dominant strategy) to the case where there can be conditional valuations. For the k^{th} price auction I derive a fixed point characterization of equilibria similar to Milgrom's theorem 3.1 [1981], but with weaker assumptions (I do not require signal ordering assumptions and the consistency assumptions of Bayesian games.)

Following the parallel in the direction of probability revelation, I show (Theorem 5) that pivot mechanisms are equivalent, in expectation, to a generalization of proper scoring rules. I use this characterization of proper scoring rules to obtain the proper scoring rule with optimal research incentives (Theorem 6). And by Monte Carlo methods I show that at least for some specific cases more general pivot mechanisms outperform proper scoring rules.

Pivot mechanisms in Bayesians games, both for probability revelation and for preference revelation, provide a simple context which may help clarify the relationship between Bayesian games and rational expectations equilibria. I show (Theorem 2 and 4) that when the fixed point is unique, equilibria of the Bayesian (and less structured) games are "regret free," a property Green and Laffont [1985] call "posterior implementability." It appears that a rational expectations equilibrium is close to, but not quite, a Bayesian equilibrium with the "regret free" property.

The paper is organized as follows. In section 1, I define pivot mechanisms as a special class of probability revelation mechanisms, define Bayesian strategies for the less structured game, derive the fixed point characterization for Bayesian strategies in response to pivot mechanisms, and find sufficient conditions for the regret-free property. In section 2, I derive the corresponding fixed point and regret-free properties for the well-known pivot mechanisms in preference revelation—Groves mechanisms and k^{th} price auctions. In this section, I discuss how the fixed point and regret-free properties can be interpreted to relate the concepts of Bayesian and rational expectations equilibria. In section 3, I show that pivot mechanisms can be used to characterize proper scoring rules. In section 4, I relax the assumption of risk neutrality. And finally, in section 5, I discuss some applications of pivot mechanisms in probability revelation.

1. PIVOT MECHANISMS IN PROBABILITY REVELATION

We begin by defining a Bayesian game of probability assessment. Denote:

$N = \{1, \ldots, n\}$ *group of assessors*

$X \, \varepsilon \, \{0,1\}$ *uncertain event be assessed*

$Pr(X = 1) = p$ *probability of event* $X = 1$

$y_i \, \varepsilon \, Y_i$ *i's private information*

$r_i(\cdot): Y_i \to [0,1]$ *a strategy function* for i

$R_i = \{r_i(\cdot)\}$ *i's strategy space*

$r_i = r_i(y_i)$ *a strategy choice or report* for i

$q_i(\cdot):[0,1]^{n-1} \to [0,1]$ *aggregation function*

$q_i = q_i(r_{-i})$ *a pivot* for i, *where*

$r_{-i} = (r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n)$

$t_i = t_i(r_i, q_i, X)$ *i's transfer or reward*

We use the notational conventions $y = (y_1, \ldots, y_n)$, $y_{-i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$, $Y = \prod_i Y_i$, $Y_{-i} = \prod_{j \neq i} Y_j$, and $r_{-i}(y_{-i}) = (r_1(y_1), \ldots, r_{i-1}(y_{i-1}), r_{i+1}(y_{i+1}), \ldots, r_n(y_n))$.

Before i chooses r_i (to be interpreted as his reported assessment of p), i observes y_i but not X, p, r_j $(j \neq i)$, or q_i. To define a Bayesian game, we also assume:

A1 (CONSISTENCY IN BELIEF). There is a prior $\rho(p, y)$ such that i's belief after observing y_i can be written $\rho(p, y_{-i} \mid y_i)$;

A2 (COMMON KNOWLEDGE). The game form, including the functional forms $q_i(\cdot)$, $t_i(\cdot,\cdot,\cdot)$, and $\rho(\cdot,\cdot)$ is common knowledge;

we specify preferences by assuming that i's utility is t_i (and hence i is risk neutral); and we adopt the following solution concept: *Bayesian Equilibrium*. By definition, the n-tuple $(r_1(\cdot), \ldots, r_n(\cdot))$ is a *Bayesian equilibrium* if for each $i \, \varepsilon \, N$ and $y_i \, \varepsilon \, Y_i$, $r_i(y_i) = r_i$ maximizes

$$\int_{[0,1]} \int_{Y_{-i}} E[t_i(r_i', q_i(r_{-i}(y_{-i})), X) \mid p, y_{-i}] \rho(p, y_{-1} \mid y_i) dp dy_{-i} \qquad (1.0)$$

$$over\ r_i' \, \varepsilon \, [0,1]$$

The above specification of game form, preferences and solution concept defines a Bayesian game. In a Bayesian equilibrium there are three consistency conditions, the first two being A1 and A2. The third follows from the solution concept. In a Bayesian equilibrium i's belief about the others' equilibrium strategies must be coordinated with the others' beliefs about the others' strategies. That is, the $(n-1)$-tuple $(r_i(\cdot), \ldots, r_{i-1}(\cdot), r_{i+1}(\cdot), \ldots, r_n(\cdot))$ against which i maximizes in (1.0) is drawn from the same n-tuple $(r_1(\cdot), \ldots, r_n(\cdot))$ as the $(n-1)$-tuple against which j maximizes in (1.0). We will call this third consistency condition C3.

When the transfer rule is pivotal (defined below), Bayesian equilibria satisfy a fixed point property, stated as follows:

Suppose $(r_1(\cdot), \ldots, r_n(\cdot))$ is a Bayesian equilibrium. Denote $\rho(p, q_i \mid y_i)$ as the measure induced by $q_i = q_i(r_{-i}(y_{-i}))$ and $\rho(p, y_{-i} \mid y_i)$ and write $E[p \mid q_i, y_i] = \int_0^1 p \, \rho_{p \mid q_i, y_i}(p \mid q_i, y_i) dp$.

Fixed Point Property. Let $(r_1(\cdot), \ldots, r_n(\cdot))$ be a Bayesian equilibrium. Then for any $y \in Y$, $q_i = q_i(r_{-i}(y_{-i}))$ is a fixed point of $E[p \mid \cdot, y_i]$.

This property is sometimes enough to characterize symmetric Bayesian equilibria for invertible strategies (see Examples 1 and 2). We derive the property in Theorems 1 and 3 for games of less structure than Bayesian games. For Theorem 1 we modify the above described Bayesian game as follows:

Drop A1 and A2 and in their place assume:

A1′. Each i, after observing y_i and before reporting r_i, forms a belief on (p, q_i), written as the measure $\rho_i(p, q_i \mid y_i)$.

Define $r_i(\cdot)$ to be *Bayesian strategy for i* if, for each $y_i \in Y_i$, $r_i(y_i) = r_i$ maximizes

$$\int_0^1 \int_0^1 E[t_i(r_i', q_i, X) \mid p, q_i] \rho_i(p, q_i \mid y_i) dp dq_i \qquad (1.1)$$

$$\text{over } r_i' \in [0,1]$$

And in place of a Bayesian equilibrium for the solution concept, adopt the following solution concept:

The n-tuple $(r_i(\cdot), \ldots, r_n(\cdot))$ is an *equilibrium of Bayesian strategies*, if for each $i \in N$, $r_i(\cdot)$ is a Bayesian strategy for i.

Note that when we have a Bayesian game with A1, A2 and a Bayesian equilibrium $(r_1(\cdot), \ldots, r_n(\cdot))$ we also have $A1'$ and $(r_1(\cdot), \ldots, r_n(\cdot))$ is an equilibrium of Bayesian strategies. However, when we have the weaker game, assuming $A1'$ but not A1 and A2, and $(r_1(\cdot), \ldots, r_n(\cdot))$ is an equilibrium of Bayesian strategies, A1 and A2 are not implied, and $(r_1(\cdot), \ldots, r_n(\cdot))$ may not be a Bayesian equilibrium (Example 3 is a counterexample). In the weaker game, none of the three consistency conditions A1, A2 and C3 are required to hold.

In what follows we can suppress the conditioning y_i without confusion, by writing $h_i(p, q_i) = \rho_i(p, q_i \mid y_i)$. Write:

$$f_i(p) = \int_0^1 h_i(p, q_i) dq_i$$

$$g_i(q_i) = \int_0^1 h_i(p, q_i)dp$$

$$\bar{p}_i = E_i(p) = \int_0^1 p f_i(p)dp = E_i(X)$$

$$\bar{p}_i(q_i) = E(p \mid q_i)$$

Example (1). $n = 2$ and assume A1 and A2. Each assessor knows (as common knowledge) that X is a trial from a Bernoulli process with unknown parameter p, which is drawn from a uniform distribution over [0,1]. It is also common knowledge that each assessor will observe M independent trials from the same process. The number of successes that i observes, y_i, is i's private information.

Assessor i's prior for p is uniform and his posterior, after observing y_i, is the beta distribution $f_i(p) \sim beta(1 + y_i, 1 + M - y_i)$ and where $\bar{p}_i = \dfrac{1 + y_i}{2 + M}$.

We define a ***pivot mechanism*** t as a transfer function $t = (t_i, \ldots, t_n)$ where

$$t_i(r_i, q_i, X) = \begin{cases} (1-q_i)A_i(q_i) + B_{i1} & \text{if } X = 1 \text{ and } r_i \ge q_i \\ B_{i2} & \text{if } X = 0 \text{ and } r_i \ge q_i \\ q_i A_i(q_i) + B_{i2} & \text{if } X = 0 \text{ and } r_i < q_i \\ B_{i1} & \text{if } X = 1 \text{ and } r_i < q_i \end{cases}$$

where $A_i : [0,1] \to R^+$ is a continuous function and B_{ik} is a constant. It turns out that even though i does not observe q_i before reporting r_i, and hence does not observe the function value $\bar{p}_i(q_i)$, the function $\bar{p}_i(\cdot)$ plays a central role in characterizing i's Bayesian strategy for pivot mechanisms (and thus for characterizing Bayesian equilibria).

Theorem 1. For any pivot mechanism, if $\bar{p}_i(\cdot)$ is continuous and $g_i(\cdot) > 0$, then i's Bayesian strategy exists and is a fixed point of $\bar{p}_i(\cdot)$.

Proof. Assessor i's expected transfer, for a given p and q_i and as a function of his report r_i is, omitting subscript i,

$$E(t \mid p, q, r) = \begin{cases} (1-q)A(q)p + (B_1 - B_2)p + B_2 & \text{if } r \ge q \\ qA(q)(1-p) + (B_1 - B_2)p + B_2 & \text{if } r < q \end{cases} \tag{1.2}$$

Taking the expectation over p and q, i's expected transfer as a function of r is

$$T(r) = \int_0^1\int_0^1 E(t \mid p, q, r)h(p, q)dpdq$$

$$= \int_0^1 dp \int_0^r ((1-q)A(q) + B_1 - B_2)ph(p,q)dq$$

$$+ \int_0^1 dp \int_r^1 (qA(q)(1-p) + (B_1 - B_2)p)h(p,q)dq + B_2$$

$$= \int_0^r dq \int_0^1 ((1-q)A(q) + B_1 - B_2)ph_{p|q}(p\,|\,q)g(q)dp$$

$$+ \int_r^1 dq \int_0^1 (qA(q)(1-p) + (B_1 - B_2)p)h_{p|q}(p\,|\,q)g(q)dp + B_2$$

$$= \int_0^r (1-q)A(q)\bar{p}(q)g(q)dq + \int_r^1 qA(q)(1-\bar{p}(q))g(q)dq$$

$$+ (B_1 - B_2)\int_0^1 \bar{p}(q)g(q)dq + B_2 \tag{1.3}$$

$$T(r) = \int_0^r (\bar{p}(q) - q)A(q)g(q)dq + K \tag{1.4}$$

for a constant K. Since $A(q)g(q) > 0$, if the maximum of T is attained at r^* inside the unit interval we must have $\bar{p}(r^*) = r^*$. If the maximum is attained at $r^* = 0$, we must have $\bar{p}(0) \leq 0$; but $p(\cdot) \geq 0$ (since $p(\cdot)$ is the expectation of a probability), so $\bar{p}(0) = 0$. If the maximum is attained at $r^* = 1$, we must have $\bar{p}(1) \geq 1$; but $p(\cdot) \leq 1$, so $\bar{p}(1) = 1$. Thus in each case the maximum occurs at a fixed point of $\bar{p}(\cdot)$. Existence of a Bayesian strategy (i.e., existence of a global maximum for each y_i) is guaranteed since $T(\cdot)$ is a continuous function over a compact set.

Q.E.D.

Figure 1 illustrates the theorem. $T(r)$ is made up of little slices like C, weighted by the positive weights $A(r)g(r)$. As long as $\bar{p}(r) > r$, $T(r)$ is increasing with r. Thus as r increases T increases over the region $(0,r_1)$, decreases over (r_1,r_2), increases over (r_2,r_3) and decreases over $(r_3,1)$. $T(r_2)$ is a local minimum, and one of the two local maxima, $T(r_1)$ and $T(r_3)$, is the global maximum.

The conditions on $g_i(\cdot)$ and $\bar{p}_i(\cdot)$ can be weakened while retaining the fixed point characterization, or something similar to it. The assumption that $g_i(\cdot) > 0$ (which can be interpreted as a perfectness requirement on i's beliefs) is necessary for $\bar{p}_i(\cdot)$ to be defined everywhere on $[0,1]$. But we can allow g_i to vanish over some regions. In Figure 2, such a region bridges the diagonal and any r between r_4 and r_5 is a Bayesian strategy. In Figure 3, we allow $\bar{p}_i(\cdot)$ to be defined over $[0,1]$ but be discontinuous; and the Bayesian strategy is r_6.[1]

Example (1) (Continued). Define $q_i = r_j$ $(i \neq j)$. Suppose that for the information structure of

Example (1), both assessors believe that the Bayesian strategies $r_i(\cdot)$ and $r_2(\cdot)$ are symmetric and invertible; and write $r_1(\cdot) = r_2(\cdot) = r(\cdot)$ $(i = 1,2)$. Then, when i observes y_i^*, $\bar{p}_i(r_j) = \frac{1 + y_i^* + r^{-1}(r_j)}{2+2M}$. Write $r_i^* = r(y_i^*)$ for i's Bayesian report. At $r_i = r_i^* = r(y_i^*)$, $\bar{p}_i(r_i^*) = \frac{1+2y_i^*}{2+2M}$. But by Theorem 1 at the Bayesian strategy r_i^*, $\bar{p}_i(r_i^*) = r_i^*$ and so $r_i^* = \frac{1+2y_i^*}{2+2M}$. Since this holds for any y_i^*, i's Bayesian strategy is

$$r_i(y_i) = \frac{1+2y_i}{2+2M}. \tag{1.5}$$

Assessor i's Bayesian strategy is unique, given his beliefs, and $(r_1(\cdot), r_2(\cdot))$ is a Bayesian equilibrium.

Example (2). $n = 2$ and assume A1 and A2. Each assessor knows that X is a Bernoulli random variable with known probability α. If $X = 1$, i observes M Bernoulli trials with parameter $\beta > 0.5$. If $X = 0$, i observes M Bernoulli trials with parameter $(1 - \beta)$. (The M trials i observes are independent, conditional on β or $(1-\beta)$, of the M trials j observes.) This structure of information is common knowledge. The number of successes i observes is y_i, which is i's private information.

Then by Bayes Theorem

$$\bar{p}_i = Pr(X = 1 \mid y_i) = 1/(1 + \underline{\alpha}\underline{\beta}^{M-2y_i}) \tag{1.6}$$

where $\underline{\alpha} = (1-\alpha)/\alpha$ and $\underline{\beta} = \beta/(1-\beta)$. Define $q_i = r_j$ $(j \neq i)$. We assume that each i believes that $r_1(\cdot) = r_2(\cdot) = r(\cdot)$ and $r(\cdot)$ is invertible. Then

$$\bar{p}_i(q_i) = 1/(1 + \underline{\alpha}\underline{\beta}^{2M-2y_i-2r^{-1}(q_i)})$$

By Theorem 1, when i observes y_i^*, at the Bayesian strategy $r_i^* = r(y_i^*)$ we have $\bar{p}_i(r_i^*) = r_i^*$ so

$$r_i(y_i) = 1/(1 + \underline{\alpha}\underline{\beta}^{2M-4y_i}). \tag{1.7}$$

Again $(r_1(\cdot), r_2(\cdot))$ is a Bayesian equilibrium.

Example (3). $n = 2$ and $q_i = r_j$ $(i \neq j)$ but A1 and A2 are not assumed. Assessor 1 believes that the information structure is that of Example (1) except 1 believes that 2's signal is uncorrelated with X, and he does not believe $r_1(\cdot) = r_2(\cdot)$. Assessor 2 believes that the information structure is that of Example (2) except 2 believes that 1's signal is uncorrelated with X and he does not believe $r_1(\cdot) = r_2(\cdot)$. Clearly neither A1 nor A2 hold. Here, $\bar{p}_1(q_1) = \bar{p}_1$ and 1's unique Bayesian strategy is $r_1(y_1) = \frac{1+y_1}{2+M}$. Similarly $\bar{p}_2(q_2) = \bar{p}_2$ and 2's unique Bayesian strategy is $r_2(y_2) = 1/(1 + \underline{\alpha}\underline{\beta}^{M-2y_2})$. Thus $(r_1(\cdot), r_2(\cdot))$ is an equilibrium of Bayesian strategies for the more general game but not a

Bayesian equilibrium in a Bayesian game.

The fixed point property of Theorem 1 does not depend on signal ordering assumptions such as the "monotone ratio property," "good news," or "affinity." Signal ordering assumptions play an important role, not in the fixed point characterization of Bayesian strategies, but in limiting the number of fixed points and obtaining the "regret free" property. This property is defined as follows.

Suppose $r_i(\cdot)$ is a Bayesian strategy (by definition for each initial private information y_i, $r_i(y_i)$ maximizes the full expectation $E[t_i(r_i,q_i,X)]$ given by (1.1). Suppose that after reporting $r_i(y_i)$, i observes q_i and conditions on the new information. His expected transfer is

$$R_i(r_i,q_i) = \int_0^1 E[t_i(r_i,q_i,X) \mid p,q_i] h_i(p \mid q_i) dp \tag{1.8}$$

Then we say i's Bayesian strategy $r_i(\cdot)$ is *regret free* under the mechanism t if

$$\forall y_i \in Y_i, \forall q_i \in [0,1] \;\; r_i(y_i) \; maximizes \; R_i(r_i,q_i) \quad over \; r_i \in [0,1] \tag{1.9}$$

The intuitive idea is that for given initial information y_i^*, the Bayesian strategy r_i^* (satisfying (1.1)) may not remain optimal once the additional information q_i is observed (may not satisfy (1.9)). Green and Laffont suggest that regret free[2] is a useful property because it without individuals may anticipate future regret and "vitiate" their initially optimal Bayesian strategies (perhaps leading to the "intertemporal tussle" described by Strotz [1955-6] in another context).

However, for any probability revealing mechanism which provides an incentive for i to reveal information, eventual regret is unavoidable, as long as i eventually observes X. (To see this note that if, for every fixed q_i, r_i^* maximizes both $t_i(r_i,q_i,1)$ and $t_i(r_i,q_i,0)$, then i has an incentive to report $r_i = r_i^*$ no matter what his initial information y_i and later information q_i.) In contrast, as we shall see, Groves mechanisms and k^{th} price auctions can be permanently regret free.

Theorem 2. For any pivot mechanism, if $\bar{p}_i(\cdot)$ has a unique fixed point then i's Bayesian strategy is regret free.

Proof. Under a pivot mechanism i's expected transfer, after observing q_i and conditioned on q_i, is by (1.2) and (1.8), omitting subscript i,

$$R(r,q) = \begin{cases} \dot{\bar{p}}(q)A(q) - \bar{p}(q)qA(q) + (B_1 - B_2)\bar{p}(q) + B_2 & \text{if } r \geq q \\ qA(q) - \bar{p}(q)qA(q) + (B_1 + B_2)\bar{p}(q) + B_2 & \text{if } r < q \end{cases}$$

Assume r^* is the unique fixed point of $\bar{p}(\cdot)$. If $r^* = 0$, then $\bar{p}(q) < q$ for all $q > 0$, otherwise by continuity there would be at least one additional fixed point. If $r^* > 0$, then by uniqueness $\bar{p}(0) > 0$. Hence for $q < r^*$, $\bar{p}(q) > q$, for otherwise would require at least one more fixed point. Similarly if $r^* = 1$, then $\bar{p}(q) > q$ for all $q < 1$ and if $r^* < 1$ then $\bar{p}(q) < q$ for $q > r^*$. Thus

$$\bar{p}(q) \gtreqless q \quad as \quad r^* \gtreqless q$$

We must show, for all r and q that $R(r^*,q) \geq R(r,q)$. If $r^* \geq q$ then $\bar{p}(q) \geq q$ and $R(r^*,q) \geq R(r,q)$. If $r^* < q$ then $\bar{p}(q) < q$ and $R(r^*,q) \geq R(r,q)$.

Q.E.D.

2. FIXED POINTS IN PREFERENCE REVELATION

In deriving the corresponding fixed point and regret-free properties for the Groves mechanism and for the k^{th} price auction, I follow the development of Green and Laffont (pp. 39-43), modifying the notation a little to bring out the parallel. In the public good model write $K = 1$ if the public good is produced and $K = 0$ if not. Write w_i for i's reported willingness to pay and $q_i = -\sum_{j \neq i} w_j$ for the negative of the sum of others' reported willingnesses to pay. The Groves *decision function* is to produce the good ($K = 1$) if $\sum_j w_j \geq 0$, or equivalently if $w_i \geq q_i$, and not to produce the good ($K = 0$) if $w_i < q_i$.

The Groves *transfer mechanism* is the function $t = (t_1, \ldots, t_n)$ where

$$t_i = \begin{cases} -q_i + H_i & \text{if } w_i \geq q_i \\ H_i & \text{if } w_i < q_i \end{cases}$$

where H_i is an arbitrary function of others' reported willingnesses to pay.

When $K = 1$, i gets v_i utility from the public good; when $K = 0$, i gets 0 utility from the unproduced public good. With additive, separable utility, i's utility is

$$U^i = \begin{cases} v_i - q_i + H_i & \text{if } w_i \geq q_i \\ H_i & \text{if } w_i < q_i \end{cases}$$

Typically, although not often explicitly, it is assumed that i knows his true valuation v_i with certainty. We relax this assumption. Suppose for example, that the proposed public good is a dam and i's valuation depends upon the unknown future state, which describes among other things whether it will be dry or rainy in the next ten years. In this more general case i expresses his uncertainty as to his own valuation, and the sum of the others' willingness to pay by a pdf $h_i(v_i, q_i)$.

Corresponding to Section 1, we define the for the public good model:

$$f_i(v_i) = \int_{-\infty}^{\infty} h_i(v_i, q_i) dq_i$$

$$g_i(q_i) = \int_{-\infty}^{\infty} h_i(v_i, q_i) dv_i$$

$$\bar{v}_i = E_i(v_i) = \int_{-\infty}^{\infty} v_i f_i(v_i) dv_i$$

$$\bar{v}_i(q_i) = E_i(v_i \mid q_i)$$

Theorem 3. For the Groves mechanism, if $\bar{v}_i(\cdot)$ is continuous and $g_i(\cdot) > 0$, then i's Bayesian strategy exists and is a fixed point of $\bar{v}_i(\cdot)$.

Proof. Write $V_i = U^i - H_i$. Since H_i is not a function of w_i, i's maximizing of his expected utility is the same as maximizing V_i. Omitting the subscript i,

$$E(V) = T(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Vh(v,q) dv dq$$

$$= \int_{-\infty}^{\infty} dv \int_{-\infty}^{w} (v - q) h(v,q) dq + \int_{-\infty}^{\infty} dv \int_{w}^{\infty} (0) h(v,q) dq$$

$$= \int_{-\infty}^{w} dq \int_{-\infty}^{\infty} (v - q) h(v,q) dv$$

$$= \int_{-\infty}^{w} g(q) dq \int_{-\infty}^{\infty} (v - q) h_{v \mid q}(v \mid q) dv$$

$$= \int_{-\infty}^{w} (\bar{v}(q) - q) g(q) dq \qquad (2.1)$$

Note (2.1) is in the same form as (1.4). If max $T(w)$ is attained for finite w^* we must have $\bar{v}(w^*) = w^*$. If there is no finite fixed point either T monotonically increases as $w \to \infty$, with $\bar{v}(w) > w$, in which case we write $w^* = \infty$ maximizes T and $\bar{v}(\infty) = \infty$; or T monotonically decreases, with $\bar{v}(w) < w$, in which case we write $w^* = -\infty$ and $\bar{v}(-\infty) = -\infty$.

Q.E.D.

And in parallel to Theorem 2 we have:

Theorem 4. For the Groves mechanism, if $\bar{v}_i(\cdot)$ has a unique fixed point, then i's Bayesian strategy is regret free.

Proof. Omitting subscript i, i's expected utility after observing q is

$$R(w,q) = \begin{cases} \bar{v}(q) - q + H & \text{if } w \geq q \text{ (and } K = 1) \\ H & \text{if } w < q \text{ (and } K = 0) \end{cases}$$

and the proof is similar to Theorem 2.

. Q.E.D.

Theorems 3 and 4 can be reinterpreted for k^{th} price auctions as follows. Define q_i as the k^{th} highest bid of the bidders other than i and w_i as i's bid. Write $K_i = 1$ if i receives one of the k identical goods and $K_i = 0$ if not. If $w_i > q_i$, then $K_i = 1$; if $w_i < q_i$, then $K_i = 0$. If $w_i = q_i$, then the tie is broken by lottery; if i wins the lottery $K_i = 1$, if not $K_i = 0$. Bidder i's transfer is $-q_i$ if $K_i = 1$, and 0 if $K_i = 0$. Assuming linear separable utility, i's utility is $U^i = v_i - q_i$ if $K_i = 1$, and $U^i = 0$ if $K_i = 0$. Then, Theorems 3 and 4 go through as before, but for k^{th} price auctions.[3]

For a Groves mechanism (or a k^{th} price auction), if there is a unique fixed point to $\bar{v}_i(\cdot)$, i not only is regret free after observing q_i, but also after observing K (or K_i). Once i reports w_i and observes q_i he infers K (or K_i). Thus he obtains no new information by observing K (or K_i) and remains regret free. This is in contrast with pivot mechanism for probability revelation. Here X is not a direct function of r_i and q_i and i learns additional information upon observing X. Thus we see that although there is a strong parallel between pivot mechanisms for probability revelation and pivot mechanisms for preference revelation, the parallel is not complete.

An important special case for the public good model is when i's valuation is known by i with certainty and does not depend on q_i (v_i is a constant known to i). Then $\bar{v}_i(q_i) = v_i$ (for all q_i), $\bar{v}_i = v_i$ and the fixed point of $\bar{v}_i(\cdot)$ is v_i. For this case i's Bayesian strategy is to report $w_i = v_i = \bar{v}_i$, i.e., for i to reveal his true willingness to pay. This is the standard result which says that when v_i is constant and known to i, i has a truthful dominant strategy.[4]

The preceding analysis suggests a possible interpretation of how Bayesian and rational expectations equilibria are related. (I hasten to add that the interpretation to follow is not the standard one of a rational expectations equilibrium.) Typically in the definition of a rational expectations equilibrium it is assumed that each i can condition on a public signal *at the same time* he contributes to it. There are several problems with this idea. As is well known, there can be a "paradox" of research incentives. As a practical problem there are few institutions for which an individual can simultaneously contribute to a public signal and condition on it.[5]

To develop an alternative concept of a rational expectations equilibrium, we keep the idea that each i both contributes to and conditions on a public signal, but we give up the idea that each i does so simultaneously. Let $q = -\sum_j w_j$ in the public good model, q be the $(k+1)^{th}$ price in the auction model, and $q = \sum_j r_j/n$ and $q_i = \sum_{j \neq i} r_j/(n-1)$ in the probability model.

Assume that q is made public *after* i reports w_i (or r_i in the probability revelation model). In each model once i observes the public signal q he can infer q_i. Now consider the following two step process, which we describe for the public good model, omitting the corresponding discussion for k^{th} price auctions and probability revelation.

In the first step, i observes y_i^*, forms his Bayesian strategy $r_i(\cdot)$, and takes the action of reporting $w_i^* = r_i(y_i^*)$. After taking this action the signal q^* becomes publicly available. In the second step i conditions on y_i^* and q_i^* and he may or may not experience regret. If all i are regret free we interpret the n-tuple $(w_1^*, \ldots, w_n^*)$ as a rational expectations equilibrium. The idea is that if i could modify his action after learning q^* he would have no incentive to do so.[6]

Under this interpretation $(w_1^*, \ldots, w_n^*)$ could be a rational expectations equilibrium without being a Bayesian equilibrium (one or more of A1, A2, or C3 might not hold yet i is regret free, as in Example 3). Or $(w_1^*, \ldots, w_n^*)$ could be a Bayesian equilibrium with no regret and hence a rational expectations equilibrium (as in Examples 1 and 2). Or $(w_1^*, \ldots, w_n^*)$ could be a Bayesian equilibrium without being a rational expectations equilibrium, or neither.

A possible drawback from the above interpretation is that it seems to exclude learning. However, we can think of learning as an iterative process. In the first step of the first iteration i observes his private information and takes an action. In the second step of the first iteration, i observes an aggregate public signal and updates his current posterior probabilities. In the first step of the second iteration i takes a second action. In the second step he observes another aggregate signal and updates his information. And so iterative process goes. Compared with later information an earlier action may no longer be optimal (it may be regretted). The process may eventually reach a stage where the previous action remains optimal under the new information. The process may then be in an equilibrium, with no new information being generated, and learning stops. The interesting thing about the Groves mechanism is that even though it is defined for only one iteration, it achieves this regret free property at the first step, if the $\bar{v}_i(\cdot)$ have unique fixed points.

3. CHARACTERIZING PROPER SCORING RULES AS PIVOT MECHANISMS

We define a *scoring rule for X* as a transfer mechanism

$$t_i = \begin{cases} F(r_i) & \text{if } X = 1 \\ G(r_i) & \text{if } X = 0 \end{cases} \tag{3.1}$$

Note that a scoring rule, applied to i, depends only on i's report and X, and not on the others' reports. Write i's expected transfer $E[t_i \mid r_i] = S(r_i) = \bar{p}_i F(r_i) + (1-\bar{p}_i)G(r_i)$. For convenience we denote a scoring rule by (F, G). A (strictly) *proper scoring rule for X* is a scoring rule for which i (uniquely) maximizes his expected transfer by reporting $\bar{p}_i$ his unconditioned expectation of the probability of X. As part of the definition we assume that proper scoring rules are at least minimally responsive in the sense that $F(1) \neq F(0)$ and $G(0) \neq G(1)$. Minimal responsiveness insures that when $\bar{p}_i = 1$, i is not indifferent between reporting 0 and 1, and similarly when $\bar{p}_i = 0$.

For a risk neutral assessor i, strictly proper scoring rules elicit $\bar{p}_i$; while pivot mechanisms elicit a fixed point of $\bar{p}_i(\cdot)$. We know that when $f_i(p)g_i(q_i) = h_i(p, q_i)$, the fixed point of $\bar{p}_i(\cdot)$ is $\bar{p}_i$. Thus it might be possible to view proper scoring rules as special cases of pivot mechanisms. The specialization would correspond the specialization in the public good model where there are fixed private valuations (v_i is constant) and i's Bayesian strategy specializes to $w_i = \bar{v}_i$. Theorem 5 confirms the parallel (at least where differentiability is not a problem).

Theorem 5. Let (F, G) be a twice differentiable, proper scoring rule. Then for any information y_i and any report r_i, (F, G) is equivalent in expectation to the pivot mechanism where

$$A(q_i) = A = 1/(F(1) - F(0) + G(0) - G(1))$$

$$B_{i1} = F(0) \text{ and } B_{i2} = G(1)$$

$$g_i(q_i) = (F'(q_i) - G'(q_i))/A \text{ and } g_i(q_i)f(p) = h_i(p, q_i)$$

Proof. (Omit subscript i.) By the definition of a proper scoring rule $F(1) > F(0)$ and $G(0) > G(1)$ so $0 < A < \infty$. To check that $g(\cdot)$ is well defined as a pdf, note $\int_0^1 g(q)dq = \frac{1}{A}\int_0^1 (F'(q) - G'(q)dq = 1$. Also the FOC for max $S(r)$ is $\bar{p}(F'(r) - G'(r)) = -G'(r)$. By the definition of a proper scoring rule, for each $\bar{p}$ the max is attained at r = $\bar{p}$, so $G'(r) = -rg(r)A$, and $S'(r) = \bar{p}g(r)A - rg(r)A$. So $S''(r) = (\bar{p} - r)g'(r)A - g(r)A$. For S to be a maximum at $r = \bar{p}$, we must have $S''(\bar{p}) \leq 0$; hence $g(r) \geq 0$.

Next we check the equivalence in expectation. Since q and p are assumed independent, $\bar{p}(q) = \bar{p}$ and (1.3) specializes to $T(r) = \bar{p}(A\int_0^r (1 - q)g(q)dq + B_1) + (1 - \bar{p})(A\int_r^1 qg(q)dq + B_2)$ and $T'(r) = (\bar{p} - r)Ag(r)$. So for each $\bar{p}$ and r, T(r) and $S(r)$ differ at most by a constant k. But when $\bar{p} = 1$, $T(0) = B_1$ and $S(0) = F(0) = B_1$, so $k = 0$.

Q.E.D.

Write

$$W(\bar{p}) = \bar{p}(A\int_0^{\bar{p}} (1 - q)g(q)dq + B_1) + (1 - \bar{p})(A\int_{\bar{p}}^1 qg(q)dq + B_2) \tag{3.2}$$

for i's expected transfer, as a function of $\bar{p}$, when i reports his maximizing $r = \bar{p}$. Then it is easy to check that $W''(\bar{p}) = Ag(\bar{p})$.

Define a *normalized scoring rule* as a scoring rule where $F(0) = G(1) = 0$ and $F(1) + G(0) = 1$. To illustrate a normalized proper scoring rule as in Figure 4 note that the pivot mechanism representation of a normalized proper scoring rule has $A = 1$, and $B_1 = B_2 = 0$; further $W(0) = -W'(0) = G(0)$ and $W(1) = W'(1) = F(1)$.

Theorem 5 is a representation theorem showing the close connection between pivot mechanisms and proper scoring rules. A principal can choose among two types of pivot mechanisms. If the principal defines q_i to be a function if the others' reports, he has an interactive pivot mechanism, with Bayesian strategy as analyzed in Section 1. If the principal defines q_i to be a random variable, drawn independently of p, he has a non-interactive pivot mechanism, equivalent in expectation to a proper scoring rule.[7]

4. BAYESIAN STRATEGIES WITHOUT RISK NEUTRALITY

In this section we weaken the assumption of risk neutrality for normalized pivot mechanisms to a weaker assumption of monotonicity, defined as follows. Suppose i faces two lotteries. In the first lottery he has probability p of winning a valued prize and probability $1-p$ of winning nothing. In the second lottery he has probability p' of winning the same prize, and $1-p'$ of winning nothing. Then we can say i has *monotonic* preferences if he prefers the first lottery to the second when $p > p'$.[8]

For normalized pivot mechanisms where $A_i(q_i) = 1, B_{1i} = B_{2i} = 0$, we have $0 \le t_i \le 1$. So we can define a lottery version of a normalized pivot by

$$t_i' = \begin{cases} 1 & \textit{with probability} \quad t_i \\ 0 & \textit{with probability} \quad 1 - t_i \end{cases}$$

Then all the theorems go through as before but with the assumption of risk neutrality replaced by monotonicity, and with what was previously an expected transfer now the probability of winning a zero-one lottery.[9]

5. APPLICATIONS OF PIVOT MECHANISMS IN REVEALING PROBABILITY

Applications of pivot mechanisms for revealing preferences have been intensively studied. In this section we look at a few applications of pivot mechanisms for revealing probability.

The first application arises when a principal wants the revelation mechanism to pool information. Suppose that the principal knows that the information structure is that of Example 1, except he does not know M. The principal knows from (1.5) that if he uses an interactive pivot mechanism, where $q_i = r_j$ $(j \ne i)$, and the assessors have symmetric Bayesian strategies, the principal will elicit

$$r_i = \frac{1 + 2y_i}{2 + 2M}$$

The principal also knows that by taking the average $(r_1 + r_2)/2$, he will obtain the pooled information expectation of p which is $E[p \mid y_1, y_2] = \frac{1 + y_1 + y_2}{2 + 2M}$.

In contrast, if the principal uses a proper scoring rule, he will elicit

$$\bar{p}_i = r_i = \frac{1 + y_i}{2 + M}$$

But there is no aggregation of r_1 and r_2 which yields the full information expectation.[10]

Similarly, suppose the principal knows that the information structure is that of Example 2, but he does not know M. Then if he uses an interactive pivot mechanism where $q_i = r_j$ and the assessors have symmetric Bayesian strategies, then the principal knows he can obtain the pooled information expectation by taking the geometric mean of the reported odds $(1 - r_1)/r_1$ and $(1 - r_2)/r_2$ (this can be verified from (1.6) and (1.7)). But if the principal uses a proper scoring rule he cannot find an aggregation of the elicited r_1 and r_2 (from (1.6)) which yields the full information expectation. These two examples show that there are at least some cases where an interactive pivot mechanism aggregates information better than a proper scoring rule (i.e., a non-interactive pivot mechanism).

A second application arises when a principal wants to choose a proper scoring rule to maximize research incentives for an assessor, per unit expected cost to the principal. We derive the optimal rule by two lemmas. By Theorem 5 and (3.2), the expected transfer for a normalized proper scoring rule can be written, for the Bayesian strategy $r = \bar{p}$

$$W(\bar{p}) = \bar{p}\int_0^{\bar{p}} (1 - q)g(q)dq + (1 - \bar{p})\int_{\bar{p}}^1 qg(q)dq \tag{5.1}$$

Assessor i can simply reveal his current $\bar{p}$, or undertake research to update $\bar{p}$. Suppose, if he undertakes research, the research is positive, and his new expectation of p is $p_b > \bar{p}$. And suppose, if the new research is negative, his new expectation of p is $p_a < \bar{p}$. Write the probability of a positive research finding as α. (Then of course $\alpha p_b + (1 - \alpha)p_a = \bar{p}$). Then the expected value of information (VOI) is

$$VOI = \alpha W(p_b) + (1 - \alpha)W(p_a) - W(\bar{p}) \tag{5.2}$$

Lemma 1. If i's expectation of p is currently $\bar{p}$, the normalized proper scoring rule which maximizes i's expected VOI is characterized by a $g(\cdot)$ fully concentrated at $\bar{p}$.

Proof. Applying (5.1) to (5.2)

$$VOI = \alpha(p_b\int_0^{p_b} g(q)dq + \int_{p_b}^1 qg(q)dq - p_b\int_0^1 qg(q)dq)$$

$$+ (1 - \alpha)(p_a\int_0^{p_a} g(q)dq + \int_{p_a}^1 qg(q)dq - p_a\int_0^1 qg(q)dq)$$

$$-\bar{p}\int_0^{\bar{p}} g(q)dq - \int_{\bar{p}}^1 qg(q)dq + \bar{p}\int_0^1 qg(q)dq$$

$$= \alpha\int_{\bar{p}}^{p_b}(p_b - q)g(q)dq + (1-\alpha)\int_{p_a}^{\bar{p}}(q - p_a)g(q)dq \tag{5.3}$$

We are looking for a function $g(\cdot)$ which maximizes (5.3) subject to $\int_0^1 g(q)dq = 1$ and $g(\cdot) \geq 0$. This is an isoperimetric control problem and its solution is a $g(\cdot)$ fully concentrated at $\bar{p}$. Q.E.D.

Lemma 2. If i's expectation of p is currently $\bar{p}$, the normalized proper scoring rule which minimizes the principal's expected cost, without research when i reports $r = \bar{p}$, is characterized by a $g(\cdot)$ fully concentrated at $\bar{p}$.

Proof. The principal's expected cost is

$$W = \bar{p}\int_0^{\bar{p}}(1-q)g(q)dq + (1-\bar{p})\int_{\bar{p}}^1 qg(q)dq \tag{5.4}$$

We are looking for a $g(\cdot)$ which minimizes (4.5) subject to $\int_0^1 g(q)dq = 1$ and $g(\cdot) \geq 0$. This is another isoperimetric control problem and its solution is a $g(\cdot)$ fully concentrated at $\bar{p}$. Q.E.D.

Putting the two lemmas together, we have:

Theorem 6. If i's current expectation of p is $\bar{p}$, the proper scoring rule which maximizes i's VOI per unit expected cost to the principal is characterized by a $g(\cdot)$ fully concentrated at $\bar{p}$.

The proof is immediate since the constants A, B_1 and B_2 simply act as scaling factors.

For Theorem 6 to be useful, the principal must know i's current $\bar{p}_i$, the pivot. The principal might have acquired knowledge of $\bar{p}_i$ from a previous use of a proper scoring rule. In other cases the principal might not know the current $\bar{p}_i$. An attractive feature of the interactive pivot mechanism is that it may allow the principal to achieve some of the benefits of Theorem 6, in stronger research incentives, without the principal himself knowing the current $\bar{p}$'s of the assessors. If i believes that the consensus of others is likely to be close to his own current $\bar{p}_i$—a plausible assumption—he will feel some of the incentives described in Theorem 6. From (1.4), with $A(q) \equiv 1$

$$T''(r) = (\bar{p}'(r) - 1)g(r) + (\bar{p}(r) - r)g'(r).$$

Thus i's expected transfer function at the point $r = \bar{p}(r)$ (his Bayesian report) has convexity $(\bar{p}'(r) - 1)g(r)$. The more concentrated $g(r)$ at the point of i's current Bayesian report, the greater the convexity and greater i's incentive to undertake research which might revise his expectation of

p. Conversely the more sensitive i's conditional expectation $\bar{p}_i(q_i)$ to q_i (the closer $\bar{p}_i(\cdot)$ to the diagonal in Figure 1, the smaller the convexity and the smaller i's research incentives.

We can see how these two factors trade off in Examples (1) and (2). Table 1 compares the interactive pivot mechanism with $q_i = r_j$ to four of the most well known proper scoring rules. In each case in Table 1 assessor 1's information is specified by $M_1 = 5$. Assessor 2's information is better, with $M_2 = 10$ or $M_2 = 15$. For Example (2), the case is taken for $\alpha = 0.5$ and $\beta = 0.6$. The differences in expected transfers, as a percent of assessor 1's expected transfer are shown in Table 1. (Comparing the expected transfers in relative terms normalizes differences in expected transfers among the various rules and provides a measure of the relative incentives to undertake research.) As can be seen for the two examples, the interactive pivot mechanism provides a considerably higher relative expected transfer, compared with the other mechanisms, and thus relatively stronger research incentives.

As a final observation, when $A_i(q_i) = 2/q_i(1 - q_i)$ and $B_{ik} = -2$ $(i = 1, \ldots, n;\ k = 1,2)$, the pivot mechanism specializes to

$$t_i = \begin{cases} 2/q_i - 2 & \text{if } X = 1 \text{ and } r_i \geq q_i \\ -2 & \text{if } X = 0 \text{ and } r_i \geq q_i \\ 2/(1 - q_i) - 2 & \text{if } X = 0 \text{ and } r_i < q_i \\ -2 & \text{if } X = 1 \text{ and } r_i < q_i \end{cases}$$

This payoff structure is the same as for a parimutuel betting rule, for a race of two horses, at the \$2.00 betting window, with the following interpretation and modification. Write $r_i = 1$ for "i^{th} bet on horse 1," $r_i = 0$ for "i^{th} bet on horse 2," and $X = 1$ for "horse 1 wins." Write q_i for the fraction of the total betting pool, aside from bet i, on horse 1, and q for the fraction of the total pool (including i) on horse 1. (A single bettor can be associated with many \$2.00 bets). In the parimutuel bet the i^{th} strategy space is restricted to $\{0,1\}$, and $A_i(q_i)$ is replaced with $A_i(q) = 2/q(1 - q)$—this last difference between q_i and q is lost in the round-off error when there are thousands of bets. More importantly, q is a signal, made public *during* the betting period (q is the current odds on the totalizer). Thus to a limited extent a bettor can do what is often assumed in a rational expectations model—he can condition on the public signal at the same time he contributes to it. In reaction to the odds on the totalizer a bettor is free to add new bets. But his ability to recontract is limited because he cannot cancel the old ones.[11]

In a similar way it may be possible to model other markets, where individuals take actions based on private information and public prices, part of the private information becomes incorporated into new prices, and then individuals use the new public information to augment their private information and take new actions, and so on.

TABLE 1

RELATIVE VALUE OF INFORMATION: THE DIFFERENCE BETWEEN ASSESSOR 2's EXPECTED TRANSFER AND ASSESSOR 1's EXPECTED TRANSFER AS A PERCENT OF ASSESSOR 1's EXPECTED TRANSFER

	M_2	Interactive Pivot	Non-interactive Proper Scoring Rules: Sphere	Brier	Log	Marschak
Information Structure of Example (1)	10	12.4	5.2	1.2	.8	.8
	15	18.7	7.3	1.7	1.1	1.1
Information Structure of Example (2)	10	28.9	5.1	4.1	2.3	2.5
	15	57.1	9.1	7.3	4.3	4.5

Notes

$M_1 = 5$

The proper scoring rules are defined by:

	F(r)	G(r)
Sphere	$r/\sqrt{r^2+(1-r)^2}$	$(1-r)/\sqrt{r^2+(1-r)^2}$
Brier	$2r - r^2$	$1 - r^2$
Log	$log(r)$	$log(1-r)$
Marschak	1 *w.p.* r z *w.p.* $1-r$	O *w.p.* r z *w.p.* $1-r$

For the Marschak rule, z is a uniform random variable on [0,1] (see Becker et al. [1964]). Grether's [1981] proper scoring rule can be shown to be a lottery version of the Marschak rule, with the same expectation. McKelvey and Page [1985] used a lottery version of the Brier rule. To bound the range of the Logarithmic rule, the rule was truncated by setting $r = .02$ for $r < .02$. The truncated rule is not proper for $r < .02$ but there is little effect of the truncation in Table 1 since $r < .02$ arises infrequently for Example (2) and not at all for Example (1).

For $M_1 \neq M_2$ I modified the assumption of symmetric strategy functions. I assumed that the strategies are "symmetrically asymmetric," in the sense that $M_1 r_2^{-1}(\cdot) = M_2 r_1^{-1}(\cdot)$. This reduces to the symmetric case when $M_1 = M_2$ and provides $r_1^{-1}(\cdot)$ and $r_2^{-1}(\cdot)$ having full ranges compared with the expectations on the full information $y_1 + y_2$. This assumption yields Bayesian strategies of $r_1 = (1 + (1 + M_1/M_2)y_1)/(2 + M_1 + M_2)$ and $r_2 = (1 + (1 + M_2/M_1)y_2)/(2 + M_1 + M_2)$ for Example (1) and $r_1 = 1/[1 + \underline{\alpha}^{M_1+M_2-2y_1(1+M_2/M_1)}]$ and $r_2 = 1/[1 + \underline{\beta}^{M_1+M_2-2y_2(1+M_1/M_2)}]$ for Example (2).

The expectations in Table 1 are calculated by a Monte Carlo method.

APPENDIX

In the following example there is an asymmetric Bayesian equilibrium, which satisfies the fixed point property of Theorem 3, but not Milgrom's and Milgrom and Weber's signal ordering assumptions. The example is for a second price auction; analogous examples can be constructed for Theorem 1.

Let $n = 2$ and $Y_1 = Y_2 = Y = \{1,2,3\}$. Suppose $Pr(y_1, y_2)$ is given by Table A1; from Table A1 $Pr(y_2 \mid y_1)$ are easily calculated in Table A2.

TABLE A1

$Pr(y_1, y_2)$

	y_2		
y_1	.01	.01	.01
	.10	.26	.16
	.01	.43	.01

TABLE A2

$Pr(y_2 \mid y_1)$

	y_2		
y_1	.33	.33	.33
	.19	.50	.31
	.02	.96	.02

Define v_1 and v_2 by

$$v_1 = \begin{cases} 1 & \text{if } y_1 = 1 \text{ or } y_2 = 1 \\ 4 & \text{if } (y_1, y_2) = (2,3) \text{ or } (3,2) \\ 7 & \text{if } (y_1, y_2) = (3,3) \\ 10 & \text{if } (y_1, y_2) = (2,2) \end{cases}$$

$$v_2 = \begin{cases} 1 & \text{if } y_2 = 1 \\ 7 & \text{if } y_2 = 3 \\ 8 & \text{if } y_2 = 2 \end{cases}$$

To check that the signals are not affiliated consider the two pairs of signals $(y_1,y_2) = (2,2)$ and $(y_1,y_2) = (1,3)$. By Milgrom [p. 1098, 1982] $(2,2) \vee (1,3) = (2,3)$ and $(2,2) \wedge (1,3) = (1,2)$, and for affiliation we must have $Pr(2,3) \cdot Pr(1,2) \geq Pr(2,2) \cdot Pr(1,3)$. But $(.16)(.01) < (.26)(.01)$ so the signals are not affiliated.

To check that the signal ordering property does not obtain, we compute $Pr(v_1 = 1 \mid y_1 = 2) = 0.19$, $Pr(v_1 = 4 \mid y_1 = 2) = 0.31$, $Pr(v_1 = 1 \mid y_1 = 3) = 0.02$ and $Pr(v_1 = 4 \mid y_1 = 3) = 0.96$. Clearly the conditional distribution $Pr(v_1 \mid y_1 = 2)$ does not stochastically dominate $Pr(v_1 \mid y_1 = 3)$, or vice versa. So $y_1 = 2$ is not more "favorable" to assessor 1 then $y_1 = 3$, or vice versa, and there is no signal ordering property.

To check that the monotone likelihood ratio property does not hold note that $Pr(y_1 = 2 \mid v_1 = 7)/Pr(y_1 = 2 \mid v_1 = 10) = 0/1 = 0$ and $Pr(y_1 = 3 \mid v_1 = 7)/Pr(y_1 = 3 \mid v_1 = 10) = 1/0 = \infty$. So for increasing y_1 the likelihood ratio increases and the monotone likelihood ratio property does not obtain.

Nonetheless we can still use the discrete version of Theorem 3 to find a Bayesian equilibrium. First note that when Mr. 2 observes y_2 he knows his valuation with certainty. Thus he can do no better than report $w_2(1) = 1$, $w_2(2) = 8$, and $w_2(3) = 7$. Write $w_2^*(\cdot)$ for this reporting function, which is Mr. 2's dominant strategy.

Next, we plot $\bar{v}_1(\cdot)$ for each y_1 in Figure A1:

[Figure A1 here]

By Theorem 3, candidates for Mr. 1's best response to $w_2^*(\cdot)$ are shown by the circled dots. When $y_1 = 2$ calculations of 1's expected utility shows that $w_1(2) = 10$ is a better response than $w_1(2) = 1$. So define the function $w_1^*(\cdot)$ by $w_1^*(1) = 1$, $w_1^*(2) = 10$, and $w_1^*(3) = 7$. Direct calculation confirms that $(w_1^*(\cdot), w_2^*(\cdot))$ is a Bayesian equilibrium.

FOOTNOTES

* This research was supported by the National Science Foundation and by the Mellon Foundation. I would like to thank John Ferejohn, Leonid Hurwicz, Lode Li, Richard McKelvey, Roger Noll, and Jennifer Reinganum for many helpful comments.

1. A problem can arise if $\bar{p}_i(\cdot)$ is not upper semi-continuous, and $\bar{p}_i(\cdot)$ jumps across the diagonal at a mass point of $g_i(\cdot)$. In this case there may be no maximum.

2. The condition of "regret free" is the same as the condition of "posterior optimal" in Green and Laffont [1985] under the proviso that q_i is the only additional information i receives. "Regret free" is also the same as Milgrom's "no regret" under the proviso that the $(k + 1)$ price is the only additional information each bidder receives.

3. Theorem 3 is similar to Milgrom's [1981] Theorem 3.1 or Milgrom and Weber's [1982] Theorem 6. The main difference is that the fixed point property established in this paper does not depend on signal ordering assumptions.

4. See for example, Theorem 3.1 of Green and Laffont [1979].

5. In some economic experiments individuals are allowed to submit strategies which are conditional on a later-to-be revealed public signal, which in turn depends on everyone's conditional report. For example in the Groves mechanism, individuals sometimes submit a reported demand curve. With such an institutional arrangement the simultaneity condition can be met. In Section 5, I note that the parimutuel mechanism meets the simultaneity condition in a limited way.

6. Recall that for the probability model i will not in general remain regret free after observing X, but in the Groves mechanism and the k^{th} price auction, i is regret free after observing K, if he is regret free after observing q.

7. As a scoring rule depends only on an agent's report and a random variable, it corresponds to an "independent contract" in Green and Stokey [1983], (see pp. 353-4). In an interactive pivot mechanism i wins or loses, depending on others' actions as in a "tournament," but i's winning does not solely depend on rank order; hence an interactive pivot mechanism does not closely correspond to a "tournament."

8. The monotonicity condition is the same as Assumption 6 in Luce and Raiffa [1957].

9. The idea of "paying off" in probabilities can be traced back to Savage [1971], Smith [1961], and Marschak [1975]. In an experiment Grether [1981] used a mechanism which is a lottery version of Marschak's proper scoring rule, and is an application of the idea. McKelvey and Ordeshook [1984] also "paid in probabilities" in an experimental setting.

10. To see this consider case 1, where $M = 1$ and $y_1 = y_2 = 0$. Then under a proper scoring rule $r_1 = r_2 = 1/3$. In case 2, $M = 4$ and $y_1 = y_2 = 1$. Then under a proper scoring rule $r_1 = r_2 = 1/3$. But in case 1 the full information expectation is 1/4, but in case 2 it is 3/10.

11. For an analysis of how the parimutuel rule aggregates information see McKelvey and Page [1985].

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