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GAME FORMS FOR NASH IMPLEMENTATION OF GENERAL SOCIAL CHOICE CORRESPONDENCES

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SOCIAL SCIENCE WORKING PAPER 579

August 1985
Revised March 1986

ABSTRACT

Several game forms are given for implementing general social choice correspondences (SCC's) which satisfy Maskin's conditions of monotonicity and No Veto Power. The game forms have smaller strategy spaces than those used in previously discovered mechanisms: the strategy for an individual consists of an alternative, two subsets (of alternatives), and a player number. For certain types of economic and political SCC's, including α-majority rule, the Walrasian, and Lindahl correspondence, the strategy space reduces to an alternative and a vector, where the number of components of the vector is at most twice the dimension of the alternative space.

GAME FORMS FOR NASH IMPLEMENTATION

OF GENERAL SOCIAL CHOICE CORRESPONDENCES

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1. Introduction

This paper provides some game forms for Nash implementation of general social choice correspondences. We give three different mechanisms, which are based on similar ideas as those used by Saijo [1985]. The mechanisms can be used to implement any social choice correspondence (SCC) involving three or more individuals which satisfies Maskin's conditions of monotonicity and no veto power, but we require smaller strategy spaces than are used in previous methods.

We briefly review the existing literature in this area. Maskin [1977] first proved that a special type of monotonicity, referred to in the subsequent literature as "Maskin monotonicity", was a necessary condition for Nash implementability of a social choice function. Maskin further proposed a game form which could be used to implement an arbitrary social choice function which satisfied the additional condition of "No Veto Power". He also required at least three individuals, and a finite number of alternatives. His proof was incomplete, but a complete proof was subsequently provided by Williams [1984], who also relaxed the restriction of finiteness of the

alternative space to allow for infinite sets satisfying a certain condition. The Maskin and Williams theorems, however, require that the strategy spaces be quite large. Letting N be the set of individuals, A be the set of alternatives, and $\mathcal{R} = \Pi_{i \in N} \mathcal{R}_i$ be the space of preference profiles, they require that individual strategy spaces be $A \times \mathcal{R}$. Thus, each individual must report an alternative, together with a complete preference profile for the society.

Subsequent research by Saijo [1985] investigated the question of whether the strategy space can be significantly reduced. Saijo provided a mechanism in which the strategy space for each individual is $A \times \mathcal{R}_i \times \mathcal{R}_{i+1} \times N$. So, instead of reporting a complete profile, it suffices under Saijo's mechanism to report two preference orders and a player number. Saijo's mechanism also allows for arbitrary alternative sets (finite or infinite). The mechanism works on the principle of what we call a "Tweed ring"[1]: the individuals stand in a circle. The individual strategy spaces are designed so that each individual reports information that affects the feasible outcomes that can be achieved by his neighbor to the right. The payoff function for the game is then designed to make all reports consistent in equilibrium. In Saijo's mechanism, the information reported is a preference order. Other authors have previously used versions of the Tweed ring in economic contexts (see e.g., Hurwicz [1979] and Walker [1981].)

This paper is closely related to the work of Saijo, as we use a version of the Tweed ring that is similar in structure to his. We propose three mechanisms. For all three mechanisms presented here, the strategy space for individual i is a subset of $A \times 2^A \times 2^A \times N$, where 2^A is the set of all subsets of A. Thus, instead of reporting two preference orders (which are each elements of $2^{A \times A}$), the individual reports subsets of A (which are each elements of 2^A). In all of the mechanisms, the sets that are reported can be restricted to be valid lower contour sets for the individual and his neighbor under some preference profile in $\mathcal{R}$, so the strategy space proposed here is always at least as small, and usually much smaller than that of Saijo.

Depending on which of the mechanisms is used, and on what is known about the SCC, degenerate subsets of the basic strategy space can be used. If the set of feasible preference profiles is large enough, then N can be eliminated, so that the individual strategy space is a subset of $A \times 2^A \times 2^A$ (Mechanism I'). If there is an alternative that is worst for everyone, regardless of the true preference profile, then one can use strategy sets of the form $A \times 2^A \times N$, or $A \times 2^A$ (Mechanisms II and II'), depending on the size of the set of feasible profiles. For SCCs satisfying a continuity requirement, the strategy space can be parameterized by the minimal preference profiles that give rise to specific alternatives (Mechanism III). In several common political and economic mechanisms (such as α-majority rule, and the constrained Walrasian and Lindahl correspondences), which are defined over continuous alternative spaces with concave preferences, these minimal profiles can be described by vectors. So, in this case the strategy space reduces to a set of the form $A \times E^k$, where E^k is Euclidian space of dimension k, and k is less than or equal to twice the dimension of A.

Finally, for the case where the alternative space is finite, and the space of preference profiles is large enough, we provide a mechanism which implements any implementable SCC that satisfies unanimity (Mechanism III). As a corollary, it follows that in these circumstances, necessary and sufficient conditions for a unanimous SCC to be implementable are that it satisfy monotonicity and a weak version of No Veto Power.

The results seem to suggest a reason why certain social choice correspondences can be implemented with much more economy in the size of the message space than others. The reason may be related to the number of parameters needed to describe the minimal profiles for a given SCC.

2. Notation and Definitions

Let $N = \{1,2,\ldots,n\}$ be a set of ***individuals***, and A be a set of ***alternatives***. For each $i \in N$, let $\mathcal{R}_i$ be a set of reflexive binary relations on A, and let $\mathcal{R} \subseteq \Pi_{i \in N} \mathcal{R}_i$. A ***Social Choice C****orrespondence* ***(SCC)*** is a correspondence $F: \mathcal{R} \twoheadrightarrow A$. Elements of $\mathcal{R}$ are denoted R $(R_1, R_2, \ldots, R_n)$, and are referred to as ***preference profiles***, with R_i denoting the preference relation of individual $i \in N$. For any $R_i \in \mathcal{R}_i$, and $a \in A$, we write $L(a,R_i) = \{x \in A: aR_i x\}$ for the lower contour set of R_i about a. Note that we do not necessarily assume that R_i is transitive or complete.

A ***game form*** is a pair (S,g), where $S = \Pi_{i \in N} S_i$, and $g: S \rightarrow A$. Elements of S are written $s = (s_1, \ldots, s_n)$. For any $s \in S$ and $j \in N$, write $s_{-j} = (s_1, \ldots, s_{j-1}, \circ, s_{j+1}, \ldots, s_n)$, and $s = (s_j, s_{-j})$, and write $g(S_j, s_{-j}) = \{a' \in A: a' = g(s_j', s_{-j})$ for some $s_j' \in S_j\}$. Given any preference profile $R \in \mathcal{R}$, a strategy n-tuple, $s \in S$ is said to be a ***Nash Equilibrium*** for g if, for all $i \in N$, $g(S_i, s_{-i}) \subseteq L(g(s), R_i)$. We write $N_g(R)$ for the set of Nash equilibria for g under the preference profile R.

The social choice correspondence F, is said to be ***Nash implementable*** if there exists a game form (S,g), such that $F = g \circ N_g$. I.e., for all $R \in \mathcal{R}$, $F(R) = g \circ N_g(R)$. In this case, we say that g ***implements*** F.

We now define several conditions on social choice functions that are used in the proofs.

Monotonicity: Let $a \in A$ and $R, R' \in \mathcal{R}$, satisfy $a \in F(R)$ and $L(a,R_i) \subseteq L(a,R_i')$ for all $i \in N$, then $a \in F(R')$.

No Veto Power (NVP): Let $a \in A$ and $R \in \mathcal{R}$ satisfy, for some $j \in N$, $L(a,R_i) = A$ for all $i \neq j$, then $a \in F(R)$.

Unanimity (U): Let $a \in A$ and $R \in \mathcal{R}$ satisfy $L(a,R_i) = A$ for all $i \in N$, then $a \in F(R)$.

For any $R \in \mathcal{R}$, and $a \in A$, we say R is ***F-minimal*** for a if $a \in F(R)$, and for every $R' \in \mathcal{R}$ with $L(a,R_j') \subseteq L(a,R_j)$ for all $j \in N$ and $L(a,R_j') \neq L(a,R_j)$ for some $j \in N$, we have $a \notin F(R')$.

Weak NVP (NVP'): Let $a \in A$ and $R \in \mathcal{R}$ satisfy, for some $j \in N$, $L(a,R_i) = A$ for all $i \neq j$. Assume $\exists\ a' \in A$ and $R' \in \mathcal{R}$ with

i) R' F-minimal for a'

ii) $a \in L(a',R_j') \subseteq L(a,R_j)$.

Then, $a \in F(R)$.

Clearly, NVP implies both NVP' and U.

Maskin [1977] first proved that any Nash implementable social choice correspondence is monotonic. The proof is short, so we include it for completeness.

<u>Lemma 2.1</u>: If $F:\mathcal{R} \twoheadrightarrow A$ is Nash implementable, then F is monotonic.

<u>Proof</u>: Since F is implementable, there is a game form, (S,g), with $F = g \circ N_g$. Let $a \in A$, and $R, R' \in \mathcal{R}$ satisfy $a \in F(R)$ and $L(a,R_i) \subseteq L(a,R_i')$ for all $i \in N$. Then, $\exists\ s \in S$ with $g(s) = a$ and $s \in N_g(R)$. I.e., $g(S_i, s_{-i}) \subseteq L(a,R_i')$ for all $i \in N$. But, since $L(a,R_i) \subseteq L(a,R_i')$, we have $g(S_i, s_{-i}) \subseteq L(a,R_i')$ for all $i \in N$. I.e., $s \in N_g(R')$, and hence $a \in g \circ N_g(R) = F(R)$.

Q.E.D.

Under an additional assumption on the domain of preference profiles, we show that any implementable social choice correspondence satisfies NVP'. We say that $\mathcal{R}$ is *complete* if, for all $a \in A$, all

$j \in N$, all $R \in \mathcal{R}$, and all $B \subseteq A$ with $a \in B$, $\exists R' \in \mathcal{R}$ with $L(a,R_j') = B$ and $L(a,R_i) = L(a,R_i')$ for all $i \neq j$. Examples of complete preference domains would be the set of all profiles of weak orders on A, the set of all profiles of linear orders on A, or the set of all profiles of binary relations on A.

Lemma 2.2: If $F:\mathcal{R} \twoheadrightarrow A$ is Nash implementable, and $\mathcal{R}$ is complete, then F satisfies NVP'.

Proof: Since F is implementable, $\exists g:S \to A$ with $F(R) = g \circ N_g(R)$ for all $R \in \mathcal{R}$. So, let $R \in \mathcal{R}$ and $a \in A$ be such that, for some $j \in N$, $L(a,R_i) = A$ for all $i \neq j$. Assume $\exists a' \in A$ and $R' \in \mathcal{R}$ with (i) R' minimal for a', and (ii) $a \in L(a',R_j') \subseteq L(a,R_j)$. Then, by (i), $a' \in F(R')$, so since g implements F, $\exists s' \in S$ with $g(s') = a'$ and $s' \in N_g(R')$. I.e., $g(S_j,s_{-j}') \subseteq L(a',R_j')$. But, since $\mathcal{R}$ is complete, and R' is minimal for a', it follows also that $L(a',R_j') \subseteq g(S_j,s_{-j}')$. If not, then find $R'' \in \mathcal{R}$ with $L(a',R_j'') = g(S_j,s_{-j})$, and $L(a',R_i'') = L(a',R_i')$ for all $i \neq j$. Then $s' \in N_g(R'')$, since $g(S_i,s_{-i}') \subseteq L(a',R_i'')$ for all $i \in N$. Thus, since g implements F, $a' \in F(R'')$. But this contradicts the fact that R' is minimal for a'. Thus, we have $g(S_j,s_{-j}') = L(a',R_j')$. But now, by (ii), we have $a \in g(S_j,s_{-j}')$. So, $\exists$ $s_j \in S_j$ with $g(s_j,s_{-j}') = a$. Let $s = (s_j,s_{-j}')$. Then clearly, $g(S_j,s_{-j}) = g(S_j,s_{-j}')$, since $s_{-j} = s_{-j}'$. Hence, $g(S_j,s_{-j}) = L(a',R_j') \subseteq L(a,R_j)$. But now, since for $i \neq j$, $L(a,R_i) = A$, we have, for all i, $g(S_i,s_{-i}) \subseteq L(a,R_i)$. Hence, $s \in N_g(R)$. So, since g implements F, we have $a \in$

F(R). This proves NVP'.

Q.E.D.

3. A General Game Form

In this section, a general game form is presented, which implements any social choice function satisfying Maskin's conditions of monotonicity and No Veto Power. The strategy spaces required for each individual are to report an alternative, two subsets of the alternative space, and a player number. The game form is a modified version of Saijo's [1985] mechanism. However, whereas Saijo requires individual strategy spaces to include the report of two preference orders, the strategy space used here replaces these with two subsets of the alternatives. Otherwise, the basic structure of the mechanism is very similar to that of Saijo. In particular, we use the trick of Rule 2 to eliminate the possiblity of undesireable equilibrium when there is not enough agreement. Saijo attributes Rule 2 to an unidentified author. Similar ideas have also appeared in Postlewaite and Shmeidler [1984] and been used by Palfrey and Srivastava [1985].

Throughout the remainder of the paper, we use the convention that any non negative integer appearing in a subscript is computed modulo n, and if the result is 0, it is set to be n. As before, negative integers appearing in subscripts refer to the n-1 tuple excluding the indicated individual.

We begin by defining the strategy spaces of the game, (S,g): For any $i \in N$,

$S_i \subseteq A \times 2^A \times 2^A \times N$ is any set satisfying:

$$S_i \supseteq \{(a_i, A_i, B_i, n_i) : a_i \in A,\ n_i \in N, \text{ and } \exists\ R \in \mathcal{R} \text{ with either } A_i = L(a_i, R_i) \text{ and } B_i = L(a_i, R_{i+1}), \text{ or } A_i = L(a_i, R_i) \text{ and } B_i = \phi\} \qquad (3.1)$$

$$S = \Pi_{i \in N} S_i \qquad (3.2)$$

Typical elements of S_i and S are written $s_i = (a_i, A_i, B_i, n_i)$ and $s = (s_1, \ldots, s_n)$, respectively. As before, for any $s \in S$, we write $s_{-i} = (s_1, \ldots, s_{i-1}, \circ, s_{i+1}, \ldots, s_n)$ and $s = (s_i, s_{-i})$. Finally, for any $s \in S$, define

$$k(s) = \textstyle\sum_{i \in N} n_i \pmod n. \qquad (3.3)$$

Definition 3.1: For any $s \in S$, and $j \in N$, define s_{-j} to be ***F-consistent*** if $\exists\ a^* \in A$ and $R^* \in \mathcal{R}$ with

(i) $a^* \in F(R^*)$

(ii) $a_i = a^*$ for all $i \in N - \{j\}$

(iii) $\forall\ i \neq j$, $A_i = L(a^*, R_i^*)$, and $B_i = L(a^*, R_{i+1}^*)$.

MECHANISM I: Let S satisfy (3.1) and (3.2), $|N| \geq 3$, and for any $s \in S$, let $k(s)$ be defined by (3.3). Define $g : s \to A$ by

Rule 1: If $\exists\ j \in N$ with $a_j \neq a_{j-1}$, and if s_{-j} is F-consistent, then

$$g(s) = \begin{cases} a_j & \text{if } a_j \in B_{j-1} \\ a_{j-1} & \text{otherwise} \end{cases}$$

Rule 2: Otherwise

$$g(s) = a_{k(s)}$$

Note that since $|N| \geq 3$, there can be at most one $j \in N$ which satisfies the conditions of Rule 1, so $g(s)$ is well defined.

Theorem 1: Let $|N| \geq 3$ and let F be any monotonic SCC satisfying NVP. Then, the game form g, defined in Mechanism I, implements F.

Proof: The proof is provided by Lemmas 3.1 and 3.2.

Lemma 3.1: For all $R \in \mathcal{R}$, $F(R) \subseteq g \circ N_g(R)$.

Proof: Pick $R \in \mathcal{R}$, and let $a \in F(R)$. We must show $a \in g \circ N_g(R)$. For all $i \in N$, define $s_i = (a, L(a,R_i), L(a,R_{i+1}), 1)$. Then, $s = (s_1, \ldots, s_n)$ falls in Rule 2, and $g(s) = a_n = a$. Further, for all $j \in N$, s_{-j} is F-consistent. Thus, given any $s_j' = (a_j', A_j', B_j', n_j') \in S_j$, either $a_j' = a$, in which case Rule 2 applies, and $g(s_j', s_{-j}) = a \in L(a,R_j)$, or $a_j' \neq a$, in which case Rule 1 applies, so $g(s_j', s_{-j}) \in \{a, a_j'\} \subseteq B_{j-1} = L(a,R_j)$. Thus, $g(S_j, s_{-j}) \subseteq L(a,R_j)$, which shows that $s \in N_g(R)$. So, $a \in g \circ N_g(R)$.

Q.E.D.

Lemma 3.2: Let $|N| \geq 3$ and let F be any monotonic social choice function satisfying NVP. For all $R \in \mathcal{R}$, $g \circ N_g(R) \subseteq F(R)$.

Proof: Pick $R \in \mathcal{R}$, and assume $a \in g \circ N_g(R)$. Then $\exists\ s \in S$ with

$s \in N_g(R)$ and $a = g(s)$. We must show $a \in F(R)$. We first establish two properties of s:

(P1) For all $j \in N$, if s_{-j} is not F-consistent, then $L(a,R_j) = A$.

(P2) For all $j \in N$, $B_{j-1} \subseteq L(a,R_j)$.

To prove (P1), note that if s_{-j} is not F-consistent, then $g(S_j, s_{-j}) = A$. In particular, to achieve $a' \in A$, set $a_j' = a'$, $B_j' = \phi$, and pick n_j' so that $\sum_{i \neq j} n_i + n_j' \pmod{n} = j$. Now, pick A_j' so $s_j' = (a_j', A_j, B_j', n_j') \in S_j$. It follows that $s' = (s_j', s_{-j})$ falls under Rule 2, since s_{-i}' is F-consistent for no $i \in N$. But $k(s') = j$, hence $g(s') = a_j = a'$. Since $s \in N_g(R)$, it follows that $g(S_j, s_{-j}) \subseteq L(a,R_j)$. Thus, $L(a,R_j) = A$.

To prove (P2), first note that if s_{-j} is not F-consistent, we can apply (P1) to get the result. If s_{-j} is F-consistent, then we show $g(S_j, s_{-j}) = B_{j-1}$. To achieve $a' \in B_{j-1}$, set $a_j' = a'$, and pick A_j', B_j' so $s_j' = (a_j', A_j', B_j', n_j) \in S_j$. If $a' = a_{j-1}$, Rule 2 applies, since $a_i = a'$ for all i, and $g(s_j, s_{-j}) = a'$. If $a' \neq a_{j-1}$, then Rule 1 applies, and $g(s_j', s_{-j}) = a_j' = a'$. Since $s \in N_g(R)$, it follows $B_{j-1} \subseteq L(a,R_j)$.

To prove the lemma, we consider three cases:

<u>Case 1</u>: For all $j \in N$, s_{-j} is not F-consistent.

By (P1), $A = L(a,R_j)$ for all j. It follows by NVP that $a \in F(R)$.

<u>Case 2</u>: For some $j \in N$, s_{-j} is F-consistent, and $s_{-(j+1)}$ is not.

Note that in this case $a_i = a_{j-1}$ for all $i \neq j$. Now if $a \neq a_{j-1}$, then we must have $a = a_j \neq a_{j-1}$, and for all $i \neq j$, s_{-i} is not F-consistent. Hence, by (P1), $L(a,R_i) = A$ for all $i \in N - \{j\}$. By

NVP, $a \in F(R)$. So, we may assume $a = a_{j-1}$. But then, since s_{-j} is F-consistent, $\exists R^* \in \mathcal{R}$ with $a \in F(R^*)$, and $B_i = L(a, R_{i+1}^*)$ for all $i \neq j$. By (P2), $L(a,R_i^*) \subset L(a,R_i)$ for all $i \neq j+1$. But, by assumption, $s_{-(j+1)}$ is not consistent. So, by (P1), $L(a,R_{i+1}) = A$. Hence, we get $L(a,R_i^*) \subseteq L(a,R_i)$ for all $i \in N$. By monotonicity, $a \in F(R)$.

<u>Case 3</u>: For all $j \in N$, s_{-j} is F-consistent.

Here, $a_i = a$ for all i, and $B_i = A_{i+1}$ for all $i \in N$. Consequently, since s_{-j} is F-consistent for some $j \in N$, $\exists R^* \in \mathcal{R}$ with $a \in F(R^*)$, and with $B_i = L(a,R_{i+1}^*)$ for all $i \neq j$, and with $A_{j+1} \neq L(a,R_{j+1}^*)$. But, since $B_j = A_{j+1}$, we have $B_i = L(a,R_{i+1}^*)$ for all i. By (P2), we get $L(a,R_i^*) \subseteq L(a,R_i)$ for all i. By monotonicity, $a \in F(R)$.

Q.E.D.

We make a few final comments on the differences between Theorem 1 and Saijo's results. Saijo requires two additional conditions in his proofs. Specifically, he requires that the domain of preference profiles satisfy the coordinate property (i.e., $\mathcal{R} = \Pi_{i \in N} \mathcal{R}_i$), and also that the domain of preferences consist of weak orders. Neither of those conditions are required in Theorem 1.

The inclusion of the possibility that $A_i = B_i = \emptyset$ in the definition of the strategy spaces for Mechanism I is simply to make the proof easier. This can be eliminated, and the theorem is still true under the conditions assumed by Saijo, but then a different argument is needed to deal with the case of $|N| \leq 4$ and $|A| \leq 3$. With this proviso, it should be noted that the strategy space for Mechanism I is always at least as small as that of Saijo [1985], and usually much smaller.

If the set of profiles which can give rise to any given alternative is always large enough, then we can eliminate the set N from the strategy space: For any $a \epsilon A$ and $i \epsilon N$, define $L_i(a) = \{L(a,R_i)\}_{R \epsilon \mathcal{R}}$. Now, if $|L_i(a)| \geq n$ for all $a \epsilon A$, and $i \epsilon N$, then we can define a function $n_i : A \times 2^A \to N$ with $n_i(a,L_i(a)) = N$ for all $a \epsilon A$, $i \epsilon N$. Now, for each $i \epsilon N$, set

$S_i \supseteq A \times 2^A \times 2^A$ to be any set satisfying

$$S_i \subseteq \{(a_i,A_i,B_i): a_i \epsilon A, \text{ and } \exists R \epsilon \mathcal{R} \text{ with } A_i = L(a_i,R_i) \text{ and } B_i = L(a_i,R_{i+1})\}. \quad (3.1)'$$

$$S = \Pi_{i \epsilon N} S_i. \quad (3.2)'$$

For any $s \epsilon S$, define

$$k(s) = \textstyle\sum_{i \epsilon N} n_i(a_i,A_i) \pmod n. \quad (3.3)'$$

Then, define Mechanism I' in exactly the same manner as in Mechanism I, except replace (3.1)-(3.3) by (3.1)'-(3.3)', respectively.

Theorem 1': Let $|N| \geq 3$, and let F be any monotonic SCC satisfying NVP. Assume $|\{L(a,R_i)\}_{R \epsilon \mathcal{R}}| \geq n$ for all $i \epsilon N$ and $a \epsilon A$. Then, Mechanism I' implements F.

Proof: Deleting the n_i's from the strategies, the proof follows exactly the lines of Theorem 1, except in the proof of property P1.

Proof of (P1): We prove that $g(S_j,s_{-j}) = A$. From the assumption that $|\{L(a,R_i)\}_{R \epsilon \mathcal{R}}| \geq n$ for all $i \epsilon N$, it follows that $|a| \geq 3$. Write $A(s) = \{a_i : i \neq j\}$. To achieve $a' \epsilon A$, pick $a'' \epsilon A$ so that $a' \epsilon A(s) \cup \{a''\}$ and so that $|A(s) \cup \{a''\}| \geq 3$ unless $|A(s)| = 1$. Set $a_j' = a''$, and pick A_j' so that $\sum_{i \neq j} n_i(a_i,A_i) + n_j(a'',A_j') = k \pmod n$, where k is chosen with either $a_k = a'$ or $a_k' = a'$. Pick B_j' so $s_j' = (a_j',A_j',B_j') \epsilon S_j$. Then, $s' = (s_j',s_{-j})$ falls under Rule 2, since s_{-i} is F-consistent for no $i \epsilon N$. But, $k(s') = k$, so $g(s') = a'$.

Q.E.D.

4. The Holocaust Mechanism

It is possible to achieve some strategy space reduction if there are restrictions on the feasible preference profiles. In this section we assume that there is some alternative which is unconditionally worst for everybody, and give a mechanism which can implement any SCC satisfying monotonicity and NVP. The strategy space of an individual consists of an alternative, a subset of the alternatives, and a player number. It is smaller than that in Mechanism I in that it requires only one, instead of two sets be reported.

We assume there exists an alternative, $a_H \epsilon A$, with $a_H \notin F(\mathcal{R})$, such that for all $R \epsilon \mathcal{R}$, and all $a \epsilon A-\{a_H\}$, $a \notin L(a_H,R_i)$ and $a_H \epsilon L(a,R_i)$ for all $i \epsilon N$. The alternative a_H is called the holocaust. Note that the existence of a_H implies that $|A| \geq 2$.

We define the strategy spaces and payoff function of the game (S,g):

$S_i \subseteq A \times 2^A \times N$ is defined as:

$S_i \supseteq \{(a_i, A_i, n_i): a_i \in A,\ n_i \in N, \text{ and } \exists\ R \in \mathcal{R} \text{ with } A_i = L(a_i, R_{i+1})\}$ (4.1)

$S = \Pi_{i \in N} S_i$ (4.2)

Typical elements of S_i and S are denoted $s_i = (a_i, A_i, n_i)$ and $s = (s_1, \ldots, s_n)$, respectively. For any $s \in S$, define

$k(s) = \sum_{i \in N} n_i \pmod{n}$ (4.3)

MECHANISM II: Let S satisfy (4.1) and (4.2), $N \geq 3$, and for any $s \in S$, let $k(s)$ be defined by (4.3). Define $g: S \to A$ by:

Rule 1: $\exists\ a^* \in A$ such that $\forall\ i \in N$, $a_i = a^*$. Then,

(i) If $\exists\ R^* \in \mathcal{R}$ s.t. $a^* \in F(R^*)$, and $\forall\ i \in N$, $A_i = L(a^*, R^*_{i+1})$,

$g(s) = a^*$

(ii) Otherwise,

$g(s) = a_H$

Rule 2: $\exists\ a^* \in A$ and $\exists\ j \in N$ such that $\forall\ i \in N-\{j\}$, $a_i = a^* \neq a_j$. Then,

$$g(s) = \begin{cases} a_j & \text{if } a_j \in A_{j-1} - \{a_H\} \text{ or } a^* = a_H \\ a^* & \text{otherwise} \end{cases}$$

Rule 3: Otherwise, set

$g(s) = a_{k(s)}$

Theorem 2: Let $|N| \geq 3$, let A contain a holocaust, and let F be any monotonic SCC satisfying NVP. Then, the game form g, defined in Mechanism II, implements F.

Proof: The proof follows from Lemmas 4.1 and 4.2.

Lemma 4.1: For all $R \in \mathcal{R}$, $F(R) \subseteq g \circ N_g(R)$.

Proof: Let $a \in F(R)$. Then, for all $i \in N$, define $s_i = (a, L(a, R_{i+1}), i)$. Clearly, s falls under Rule 1 (i), and hence $g(s) = a$. But also, s is a Nash Equilibrium, because for any $i \in N$, consider $s_i' \neq s_i$. Then, if $a_i' \neq a$, $s' = (s_i', s_{-i})$ falls in Rule 2, and hence $g(s') \in A_{i+1} = L(a, R_i)$. On the other hand, if $a_i' = a$, then s' falls in Rule 1, in which case $g(s') \in \{a, a_H\} \subset L(a, R_i)$. Hence, for all $s_i' \in S_i$, $g(s_i', s_{-i}) \in L(a, R_i) = L(g(s), R_i)$, so $s \in N_g(R)$. Thus, $a \in g \circ N_g(R)$, and we have shown $F(R) \subset g \circ N_g(R)$

Q.E.D.

Lemma 4.2: Let $|N| \geq 3$, and let F be monotonic and satisfy NVP, then for all $R \in \mathcal{R}$, $g \circ N_g(R) \subseteq F(R)$.

Proof: Pick $R \in \mathcal{R}$, and let $a \in g \circ N_g(R)$. Then, since $a \in g \circ N_g(R)$, it follows that $\exists s \in S$ with $g(s) = a$ and $s \in N_g(R)$. We must show $a \in F(R)$. Now, s could fall under any of the three rules. We consider each in turn.

Rule 1: For some $a^* \in A$, $a_i = a^*$ for all $i \in N$.

Consider first, subcase (i). Then, $a = a^*$, and $\exists R^* \in \mathcal{R}$ with $F(R^*) = a$ and with $A_{i-1} = L(a, R_i^*)$ for all $i \in N$. But then, we show for all i, $A_{i-1} - \{a_H\} \subseteq g(S_i, s_{-i})$. In particular, to achieve $a' \in A_{i-1} - \{a_H\}$, set $a_i' = a'$, and pick A_i' so $s_i' = (a_i', A_i', n_i) \in S_i$. Then, $g(s_i', s_{-i}) = a'$. Now, using $s \in N_g(R)$, we must have $L(a, R_i^*) - \{a_H\} = A_{i-1} - \{a_H\} \subseteq g(S_i, s_{-i}) \subseteq L(a, R_i)$ for all $i \in N$. But, since $a_H \in L(a, R_i)$, it follows $L(a, R_i^*) \subseteq L(a, R_i)$ for all $i \in N$. But then, by monotonicity, since $a \in F(R^*)$, we must have $a \in F(R)$.

Next, consider subcase (ii). Here, $a = g(s) = a_H$. Pick any $a' \in A$ with $a' \neq a^*$, and pick any $i \in N$. Pick A_i' with $s_i' = (a', A'_i, n_i) \in S_i$. Then, $s' = (s_i', s_{-i})$ falls in Rule 2. So, $g(s') \neq a_H$. Hence, $g(s_i', s_{-i}) \notin L(a, R_i) = \{a_H\}$. So, $s \notin N_g(R)$, a contradiction.

<u>Rule 2</u>: Here, $\exists j \in N$ and $a^* \in A$ with $a_i = a^* \neq a_j$ for all $i \neq j$. So, $|A| \geq 2$. By Rule 2, $a = g(s) \in \{a_j, a^*\}$, and $g(s) \neq a_H$. We show that for any $i \neq j$, $A - \{a_H\} \subseteq g(S_i, s_{-i})$. If $|A| = 2$, then since $g(s) \neq a_H$, we get $A - \{a_H\} = \{g(s)\} \subseteq g(S_i, s_{-i})$. If $|A| > 2$, then to achieve any $a'' \in A - \{a_H\}$, pick $a' \in A$ so that $a' \notin \{a_j, a^*\}$ and so that $a'' \in \{a_j, a^*, a'\}$. Then, pick A_i' so $s_i' = (a', A_i', n_i') \in S_i$. Then, Rule 3 applies, and n_i' can be chosen so that $g(s_i', s_{-i})$ is any element in $\{a_j, a^*, a'\}$, in particular, a''. Thus, for all $i \neq j$, since $s \in N_g(R)$, we have $A - \{a_H\} \subseteq g(S_i, s_{-i}) \subseteq L(a, R_i)$. But, since $a_H \in L(a, R_i)$ for all $i \neq j$, it follows that $L(a, R_i) = A$ for all $i \neq j$. By NVP, we have $a \in F(R)$.

<u>Rule 3</u>: Here, by the same argument as the latter part of Rule 2, we have $L(g(s), R_i) = A$ for all $i \in N$, implying $a \in F(R)$.

Q.E.D.

Just as in Mechanism I, we can delete N from the strategy space if, for all $a \in A$ and $i \in N$, $\{L(a, R_i)\}_{R \in \mathcal{R}}$ is large enough. Since the method is very similar to that used for Mechanism I, we just state the result, leaving the proof for the reader.

Define $n_i: A \times 2^A \rightarrow N$ exactly as in Section 3. Then, for each $i \in N$, set

$S_i \subseteq A \times 2^A$ to be any set satisfying

$$S_i \supseteq \{(a_i, A_i): a_i \in A \text{ and } \exists R \in \mathcal{R} \text{ with } A_i = L(a_i, R_i)\} \quad (4.1)'$$

$$S = \Pi_{i \in N} S_i. \quad (4.2)'$$

For any $s \in S$, define

$$k(s) = \textstyle\sum_{i \in N} n_i(a_i, A_i) \pmod{n}. \quad (4.3)'$$

Then, define Mechanism II' identically to Mechanism I, except replace (4.1)-(4.3) by (4.1)'-(4.3)', respectively.

<u>Theorem 2'</u>: Let $|N| \geq 3$, let A contain a holocaust, and let F be any monotonic SCC satisfying NVP. Assume $|\{L(a, R_i)\}_{R \in \mathcal{R}}| \geq n$ for all $i \in N$ and $a \in A$. Then, Mechanism II' implements F.

5. <u>Strategy Space Reduction</u>

We return to the case where there is no holocaust alternative, and give a mechanism which is useful in reducing the size of the strategy space for certain SCCs. The mechanism is a modified version of Mechanism I, which has some advantages and disadvantages over that game form.

The primary advantage of the modified mechanism is that the strategy spaces are smaller than in Mechanism I. The reduction is quite dramatic for certain common social choice functions such as Walrasian, Lindahl, and α-majority rules defined over spaces of quasi concave preferences. In all these cases, the strategy spaces can be reduced to an alternative, a player number, and one or two (price) vectors. This yields mechanisms that are very similar to mechanisms that have been constructed by Hurwicz [1979], and others for use in economic contexts. The mechanism presented here has the advantage that it always produces feasible outcomes, even out of equilibrium, although it is not completely "decentralized", as it requires individuals to know the feasible set for the whole society.

A second advantage of the modified mechanism is that it implements a wider class of SCCs than Mechanism I. The SCC need only satisfy U and NVP′ rather than NVP. For the case of complete preferences over finite sets, the mechanism implements all implementable SCCs satisfying unanimity (U). As a corollary, in this case we find that necessary and sufficient conditions for a unanimous SCC to be implementable are that it satisfy NVP′ and monotonicity.

One disadvantage of the revised mechanism is that it requires an additional restriction on the SCC. The condition is a type of "closedness", requiring that for every alternative and every profile yielding that alternative as a social choice, there must be an F-minimal profile yielding the same alternative, with lower contour sets contained in those of the given profile. Any SCC defined on a

finite set of alternatives, as well as many common examples of economic and political SCCs, satisfy this condition.

A second disadvantage of the revised mechanism is that it is not "direct". In Mechanism I, there is an equilibrium in which individuals reveal truthfully attributes of themselves and their neighbors. Namely, they reveal their own and their neighbor's correct lower contour sets. In the revised mechanism, the equilibrium strategies are not quite so natural.

We first need to define the additional condition on the SCC that is needed. We say that the SCC, $F: \mathcal{R} \twoheadrightarrow A$, satisfies ***F-closedness*** if, for all $a \in A$ and $R \in \mathcal{R}$ with $a \in F(R)$, there is an $R' \in \mathcal{R}$, with $L(a,R_i') \subsetneq L(a,R_i)$ for all $i \in N$, such that R' is F-minimal for a.

We define the strategy spaces and the payoff function of the modified game form, (S,g), as follows:

$S_i \subseteq A \times 2^A \times 2^A \times N$ is any set satisfying

$$S_i \supseteq \{(a_i,A_i,B_i,n_i): a_i \in A,\ n_i \in N, \text{ and } \exists\ R \in \mathcal{R} \text{ which is F-minimal for } a_i \text{ with either } A_i = L(a_i,R_i) \text{ and } B_i = L(a_i,R_{i+1}) \text{ or } A_i = L(a_i,R_i) \text{ and } B_i = \phi\} \quad (5.1)$$

$$S = \Pi_{i \in N} S_i \quad (5.2)$$

Typical elements of S_i and S are denoted $s_i = (a_i,A_i,B_i,n_i)$ and $s = (s_1,\ldots,s_n)$, respectively. Again, for any $s \in S$, set

$$k(s) = \textstyle\sum_{i \in N} n_i \pmod{n} \quad (5.3)$$

<u>Definition 5.1</u>: For any $s \in S$, and $j \in N$, define s_{-j} to be ***minimally F-consistent*** if $\exists\ a^* \in A$ and $R^* \in \mathcal{R}$ satisfying:

(i) R^* is F-minimal for a^*

(ii) $\forall\, i \neq j$, $a_i = a^*$

(iii) $\forall\, i \neq j$, $A_i = L(a^*, R_i^*)$, and $B_i = L(a^*, R_{i+1}^*)$.

MECHANISM III: Let S satisfy (5.1) and (5.2), $|N| \geq 3$, and for any $s \in S$, let k(s) be defined by (5.3). Define $g: S \to A$ by:

<u>Rule 1</u>: If $\exists\, j \in N$ with $a_j \neq a_{j-1}$, and if s_{-j} is minimally F-consistent, then

$$g(s) = \begin{cases} a_j & \text{if } a_j \in B_{j-1} \\ a_{j-1} & \text{otherwise} \end{cases}$$

<u>Rule 2</u>: Otherwise,

$$g(s) = a_{k(s)}$$

Note that there can be at most one $j \in N$ which satisfies the conditions of Rule 1, so g(s) is well defined.

<u>Theorem 3</u>: Let $|N| \geq 3$ and let F be any F-closed monotonic SCC satisfying NVP′ and U. Then, the game form g, defined in Mechanism III, implements F.

<u>Proof</u>: The proof follows from Lemmas 5.1 and 5.2.

<u>Lemma 5.1</u>: Let F satisfy F-closedness and monotonicity. Then for all $R \in \mathcal{R}$, $F(R) \subseteq g \circ N_g(R)$.

<u>Proof:</u> Pick $R \in \mathcal{R}$, and let $a \in F(R)$. We must show $a \in g \circ N_g(R)$. Pick $R^* \in \mathcal{R}$ with R^* F-minimal for a, and $L(a,R^*) \subseteq L(a,R_i)$ for all $i \in N$. For all $i \in N$, define $s_i = (a, L(a,R_i^*), L(a,R_{i+1}^*), 1)$. Then $s = (s_1, \ldots, s_n)$ falls in Rule 2, and $g(s) = a_n = a$. Further, for all $j \in N$, s_{-j} is F-consistent. Thus, given any $s_j' = (a_j', A_j', B_j', n_j') \in S_j$, either $a_j' = a$, in which case Rule 2 applies, and $g(s_j', s_{-j}) = a \in L(a,R_j)$, or $a_j' \neq a$, in which case Rule 1 applies, so $g(s_j', s_{-j}) \in \{a, a_j'\} \subseteq B_{j-1} = L(a,R_j^*) \subseteq L(a,R_j)$. Thus, $g(S_j, s_{-j}) \subseteq L(a,R_j)$, which shows that $s \in N_g(R)$. So, $a \in g \circ N_g(R)$.

Q.E.D.

<u>Lemma 5.2</u>: Let $|N| \geq 3$ and let F be any F-closed, monotonic social choice function satisfying U and NVP′. For all $R \in \mathcal{R}$, $g \circ N_g(R) \subseteq F(R)$.

<u>Proof</u>: Replacing "minimal F-consistency" for "F-consistency", the proof is exactly the same as the proof of Lemma 3.2, except for case 1 and case 2. So we just present that portion of the proof.

<u>Case 1</u>: For all $j \in N$, s_{-j} is not minimally F-consistent.

By (P1), $A = L(a,R_j)$ for all j. It follows by unanimity that $a \in F(R)$.

<u>Case 2</u>: For some $j \in N$, s_{-j} is minimally F-consistent, and $s_{-(j+1)}$ is not.

In this case, $a_i = a_{j-1}$ for all $i \neq j$. If $a \neq a_{j-1}$, then we must have $a = a_j \neq a_{j-1}$, and for all $i \neq j$, s_{-i} is not minimally

F-consistent. Hence, by (P1), $L(a,R_i) = A$ for all $i \neq j$. Further, since s_{-j} is minimally F-consistent, $\exists R^* \in \mathcal{R}$ and $a^* = a_{j-1} \in A$ with R^* minimal for for a^*. By P2, $B_{j-1} = L(a^*,R^*_j) \subseteq L(a,R_j)$, and by Rule 1 $a \in B_{j-1} = L(a^*,R^*_j)$. The conditions of NVP′ are satisfied, so $a \in F(R)$. So we may assume $a = a_{j-1}$. The remainder of the proof of this case is exactly as before.

Q.E.D.

In Mechanism III, as in Mechanisms I and II, we can delete N from the strategy space when the set of profiles is large enough. Define $\mathcal{R}(a)$ to be the set of profiles in $\mathcal{R}$ which are F-minimal for a. Then, let $L_i(a) = \{L(a,R_i)\}_{R \in \mathcal{R}(a)}$. If $|L_i(a)| \geq n$ for all $a \in A$ and $i \in N$, then, define $n_i: A \times 2^A \to N$ with $n_i(a,L_i(a)) = N$ for all $a \in A$, $i \in N$. Now, for each $i \in N$, set

$S_i \subseteq A \times 2^A \times 2^A$ is any set satisfying

$$S_i \supseteq \{(a_i,A_i,B_i): a_i \in A, \text{ and } \exists R \in \mathcal{R}(a) \text{ with } A_i = L(a_i,R_i) \text{ and } B_i = L(a_i,R_{i+1}), \text{ or } A_i = L(a_i,R_i) \text{ and } B_i = \phi\} \qquad (5.1)'$$

$$S = \Pi_{i \in N} S_i. \qquad (5.2)'$$

For any $s \in S$, define

$$k(s) = \textstyle\sum_{i \in N} n_i(a_i,A_i) \pmod n. \qquad (5.3)'$$

Then, define Mechanism III′ in exactly the same manner as Mechanism III, except replace (5.1)-(5.3) by (5.1)′-(5.3)′, respectively.

Theorem 3′: Let $|N| \geq 3$, and let F be any F-closed, monotomic SCC satisfying NVP′ and U. Assume $|\{L(a,R_i)\}_{R \in \mathcal{R}(a)}| \geq n$ for all $i \in N$ and $a \in A$. Then, Mechanism III′ implements F.

Proof: The changes to the proof of Theorem 3 are exactly the same as those made in Theorem 1′ to Theorem 1.

Q.E.D.

Note that if F is onto (i.e., $F(\mathcal{R}) = A$) and $|A| \geq 3$, then the requirement that $B_i = \phi$ be admissible in the definition of S_i can be dropped. In this case, we can replace (5.1) by

$$S_i \supseteq \{(a_i,A_i,B_i,n_i): a_i \in A,\ n_i \in N, \text{ and } \exists R \in \mathcal{R}(a) \text{ with } A_i = L(a_i,R_i) \text{ and } B_i = L(a_i,R_{i+1})\} \qquad (5.4)$$

and Theorem 3 is still true. Similarly, in Mechanism III′, if F is onto, we can replace (5.1) by

$$S_i \supseteq \{(a_i,A_i,B_i): a_i \in A, \text{ and } \exists R \in \mathcal{R}(a) \text{ with } A_i = L(a_i,R_i) \text{ and } B_i = L(a_i,R_{i+1})\} \qquad (5.4)'$$

and Theorem 3′ is still true. The proofs just involve minor changes to the proof of property (P1) in Theorems 3 and 3′, so we leave them for the reader.

Finally, if $|A|$ is finite, Theorem 3 yields the following characterization of implementable SCCs.

Corollary 5.1: If $\mathcal{R}$ is complete, $|A|$ is finite, and $|N| \geq 3$, necessary and sufficient conditions for a social choice correspondence satisfying U to be implementable are that it satisfy monotonicity and NVP′.

Proof: Since any SCC defined on a finite set always satisfies F-closedness, the result is a direct consequence of Theorem 2.

Q.E.D

6. Some Examples

In this section, we give some examples showing how the theorems of the previous section apply to give game forms for implementation of well known SCCs in political and economic settings. Sepcifically, we consider α-majority rule, the Walrasian correspondence and the Lindahl correspondence. In each case, we identify the minimal profiles giving rise to a given alternative, and then use these minimal profiles to construct strategy spaces satisfying the conditions of Mechanism III and III′. Since in all three cases, the minimal profiles can be described by vectors, it follows also that the strategy spaces can be parameterized by these vectors. Thus, all these SCCs can be implemented using message spaces consisting of an alternative and a vector (of dimension no more than twice the dimension of A).

These results are not new. At least for the Walrasian and Lindahl cases, other researchers have already constructed mechanisms whose message spaces are on the order of magnitude of those given here (see, e.g., Hurwicz [1979]). Nevertheless, the results are of interest for two reasons. First, since the methods used here can be applied to any SCC, they provide a technique for constructing a game form, with "small" strategy spaces, for an arbitrary SCC. Secondly, the results do suggest a reason why certain SCCs can be implemented with smaller

message spaces than others. We conjecture that the minimum size of the message space required to implement a given SCC may be related to this number of parameters needed to pararmeterize the minimal profiles of the SCC.

Example 1: α-majority rule.

Let $A \subseteq E^n$ be compact and convex, $\mathcal{R}_i = \{R_i \subseteq A \times A: R_i$ has a pseudoconcave utility representation$\}$, and $\mathcal{R} = \Pi_{i \in N} \mathcal{R}_i$. For any $R \in \mathcal{R}$, and $a, a' \in A$, define

$$v(a,a',R) = |\{i \in N: a' \notin L(a,R_i)\}|,$$

$$v(a,R) = \max_{a' \in A} v(a,a',R).$$

Then, for any $0 \le \alpha \le 1$, define the SCC, $F_\alpha: \mathcal{R} \twoheadrightarrow A$ by

$$F_\alpha(R) = \{a \in A: v(a,R) \le \alpha n\}.$$

Greenberg [1979] has shown, if $\alpha > n\cdot[m/(m+1)]$, that F_α is always non-empty. It also follows easily that F_α satisfies monotonicity and NVP. So Mechanism I can be used to implement F_α. However, Mechanism III can yield substantial strategy space reduction. For any $a \in A$, and $q \in E^m$, write $H(a,q) = \{a' \in A: a' \cdot q \le a \cdot q\}$. For any $p = (p_1,\ldots,p_n) \in (E^m)^n$, write $|p| = \max_{a' \in A} |\{i \in N: a' \notin H(a,p_i)\}|$. Then, it can be verified that F_α is F-closed, and that for any $a \in A$, $R \in \mathcal{R}$ is F-minimal for a only if $\exists\, p \in (E^m)^n$, with $|p| \le \alpha$, satisfying $L(a,R_i) = H(a,p_i)$ for all $i \in N$.

Since the F-minimal profiles for F_α can be parameterized by $p \in (E^m)^n$, and each individual need only specify lower contour sets for himself and his neighbor, we can define a strategy set satisfying the conditions of Mechanism III as follows: For $i \in N$, define

$$T_i = A \times E^m \times E^m \times N$$

$$T = \Pi_{i \in N} T_i ,$$

We write $t_i = (a_i, p_i, q_i, n_i)$ and $t = (t_1, \ldots, t_n)$ for typical elements of T_i and T, respectively. For any $t \in T$ define $s(t) = (s_1(t), \ldots, s_n(t))$ by setting $s_i(t) = (a_i, A_i, B_i, n_i)$, where $A_i = H(a_i, p_i)$ and $B_i = H(a_i, q_i)$. The set $S = s(T)$ so constructed satisfies (5.4) and (5.2). Hence, since F_α is onto, S satisfies the conditions required for Mechanism III (see comment after Theorem 3). Thus, $g: S(T) \to A$ implements F_α. The game form can be defined directly on T instead of on S by defining $h: T \to A$ by $h(t) = g(s(t))$. It is immediate from Theorem 2 that h implements F_α.

In this example, as explained in Section 5, above, the strategy space can be further reduced to $A \times E^m \times E^m$ by partitioning E^m into n non empty regions, say $\{D_i\}_{i \in N}$, and then setting $n_i(a_i, A_i) = j \leftrightarrow p_i \in D_i$.

Example 2: The constrained Walrasian correspondence.

An example in Hurwicz, Maskin and Postlewaite [1984] shows that the Walrasian correspondence does not satisfy monotonicity when the final allocation is on the boundary. Hence, the usual Walrasian correspondence is not Nash implementable. However, they connstruct a "constrained Walrasian correspondence" which is implementable. This correspondence is a superset of the Walrasian correspondence, and the two are equal on non boundary points.

For each $i \in N$, let $Z_i \subseteq E^L$ be a closed convex set, bounded from below, containing the origin. Thus, Z_i represents the consumption set (in net trades) for agent i. Let $Z = \Pi_{i \in N} Z_i$, and let Δ^L be the L dimensional price simplex. Then, set

$$A = \{z \in Z: \Sigma_{i \in N} z_i = 0\}$$

$$\mathcal{R}_i = \{R_i \subseteq A \times A: R_i = R_i' \cap (A \times A) \text{ for some } R_i' \subseteq Z \times Z, \text{ where } R_i' \text{ is continuous, convex, monotonic, and selfish}\}$$

$$\mathcal{R} = \Pi_{i \in N} \mathcal{R}_i .$$

We then define the constrained Walrasian correspondence,[2] $F_W: \mathcal{R} \twoheadrightarrow A$ by

$$F_W(R) = \{z \in A: \exists\, p \in \Delta^L \text{ such that } \forall\, z' \in A \text{ and } \forall\, i \in N, \; p \cdot z_i' \le 0 \text{ implies } z' \in L(z, R_i)\}$$

It follows from standard results (e.g., Debreu [1959]) that F_W is always non-empty. It also follows easily that F_W satisfies monotonicity and NVP. So Mechanism I can be used to implement F_W. However, again, Mechanism III can yield substantial strategy space reduction. For $p \in \Delta^L$, write $H_i(p) = \{z' \in A : p \cdot z_i' \le 0\}$. Then, it can be verified that F_W is F-closed, and that for any $z \in A$, $R \in \mathcal{R}$ is F-minimal for z only if $\exists\, p \in \Delta^L$, with $p \cdot z_i = 0$ for all i, satisfying $L(z, R_i) = H_i(p)$ for all $i \in N$.

Thus, the F-minimal profiles for F_W can be parameterized by $p \in \Delta^L$. As in Example 1, we can use this parameterization to define a strategy set satisfying the conditions for Mechanism III. Set

$$T_i = A \times \Delta^L \times N$$

$$T = \Pi_{i \in N} T_i ,$$

and write $t_i = (a_i, p_i, n_i)$ and $t = (t_1, \ldots, t_n)$ for typical elements of T_i and T, respectively. For any $t \in T$, define $s(t) = (s_1(t), \ldots, s_n(t))$ by setting $s_i(t) = (a_i, A_i, B_i, n_i)$, where $A_i = H_i(p_i)$ and $B_i = H_{i+1}(p_i)$ if $p_i \cdot z_i = 0$ and otherwise $A_i = H_i(p)$ and $B_i = \phi$. The set $S = s(T)$ so constructed satisfies conditions (5.1) and (5.2), so S satisfies the conditions for Mechanism III. Hence, $g:S(T) \twoheadrightarrow A$ implements F_W. Again, the game form can be defined directly on T instead of on S by defining $h:T \rightarrow A$ by $h(t) = g(s(t))$. It is immediate from Theorem 2 that h implements F_W.

As in Example 1 noted above, the strategy space can be further reduced to $A \times \Delta^L$ by partitioning Δ^L into n regions, say $\{D_i\}_{i \in N}$, and then setting $n_i(a_i, A_i) = j \leftrightarrow p_i \in D_j$.

Example 3: The constrained Lindahl correspondence.

For each $i \in N$, let $Z_i \subseteq E^m \times E^L$ be a closed convex set, bounded from below, with non empty interior, and containing the origin. Let $Z = \Pi_{i \in N} Z_i$, and let Δ^m, Δ^L, and Δ^{m+L} be the m, L, and m+L dimensional price simplices. Set $\Delta_i = \Delta^{m+L}$ for each i, and $\Delta^n = \Pi_{i \in N} \Delta_i$. Elements of Z_i and Δ_i are denoted $z_i = (z_i^1, z_i^2)$ and $p_i = (p_i^1, p_i^2)$ respectively, where $z_i^1 \in E^m$, $z_i^2 \in E^L$, $p_i^1 \in \Delta^m$, and $p_i^2 \in \Delta^L$. Elements of Z and Δ are denoted $z = (z_1, \ldots, z_n)$ and $p = (p_1, \ldots, p_n)$, respectively. Write $\bar{z} = \Sigma_{i \in N} z_i$, and $\bar{p} = \Sigma_{i \in N} p_i/n$. Let $Y \subseteq E^m \times E^L$ be a closed, convex cone, with typical elements (x,y), satisfying conditions B.1 - B.5 of Foley [1970]. Define

$$A = \{z \in Z: z_i^1 = z_j^1 \text{ for all } i,j \in N, \text{ and } \bar{z} \in Y\}$$

$$\Delta = \{p \in \Delta^n: p_i^2 = p_j^2 \text{ for all } i,j \in N\}$$

$$\mathcal{R}_i = \{R_i \subseteq A \times A: R_i = R_i' \cap (A \times A) \text{ for some } R_i' \subseteq Z \times Z, \text{ where } R_i' \text{ is continuous, convex, monotonic, and selfish}\}$$

$$\mathcal{R} = \Pi_{i \in N} \mathcal{R}_i.$$

We then define the constrained Lindahl correspondence, $F_L:\mathcal{R} \twoheadrightarrow A$ by

$$F_L(R) = \{z \in A: \exists\, p \in \Delta \text{ such that } \forall\, z' \in A,\ \bar{p} \cdot \bar{z}' \leq \bar{p} \cdot \bar{z}, \text{ and } \forall\, i \in N,\ p_i \cdot z_i' \leq 0 \text{ implies } z' \in L(z, R_i)\}$$

It follows from Foley [1970] that F_L is always non-empty.[3] It also follows easily that F_L satisfies monotonicity and NVP. So Mechanism I can be used to implement F_L. However, again, Mechanism III can yield substantial strategy space reduction. For $p \in \Delta^{m+L}$, and $i \in N$, write $H_i(p) = \{z' \in A: p \cdot z_i' \leq 0\}$. It can be verified that F_L is F-closed, and that for any $z \in A$, $R \in \mathcal{R}$ is F-minimal for z only if $\exists\, p \in \Delta^{m+L}$, with $p_i \cdot z_i = 0$ for all i, satisfying $L(z, R_i) = H_i(p_i)$ for all $i \in N$. Thus, the F-minimal profiles for F_L can be parameterized by $p \in \Delta$. As before, we can use this to define a strategy space satisfying the conditions for Mechanism III. Define

$$T_i = A \times \Delta^L \times \Delta^m \times \Delta^m \times N$$

$$T = \Pi_{i \in N} T_i,$$

and write $t_i = (a_i, p_i, q_i, r_i, n_i)$ and $t = (t_1, \ldots, t_n)$ for typical elements of T_i and T, respectively. For any $t \in T$ define $s(t) = (s_1(t), \ldots, s_n(t))$ by setting $s_i(t) = (a_i, A_i, B_i, n_i)$, where $A_i = H_i(p_i, q_i)$ and $B_i = H_{i+1}(p_i, r_i)$. If $(p_i, q_i) \cdot z_i = 0$ and $A_i = H_i(p_i, q_i)$

and $B_i = \phi$ otherwise. It can be checked that the set $S = s(T)$ so defined satisfies (5.1) and (5.2). Hence, the conditions for Mechanism III are met, and $g:S(T) \twoheadrightarrow A$ implements F_L. The game form can then be defined directly on T instead of on S by defining $h:T \to A$ by $h(t) = g(s(t))$. It is immediate from Theorem 2 that h implements F_L.

As before, the strategy space can be further reduced to $A \times \Delta^L \times \Delta^m \times \Delta^m$ by partitioning Δ^L into n regions, say $\{D_i\}_{i \in N}$, and then setting $n_i(a_i, A_i) = j \leftrightarrow p_i \in D_j$.

FOOTNOTES

* This paper was inspired by Tatsuyoshi Saijo's dissertation. I am greatly indebted to Kim Border, Leo Hurwicz, and Tatsuyoshi Saijo for some very helpful discussions. I acknowledge support of NSF grant No. SES 8208184.

1. The basic idea of a Tweed ring is illustrated by Thomas Nast, Harpers Weekly, August 19, 1871, and reprinted in Morton Keller [1968], p 118.

2. Hurwicz, Maskin and Postlewaite actually use a different definition of a constrained Walrasian equilibrium, in which individuals maximize over their own consumption sets subject to a constraint on total endowment. However, with selfish preferences, that definition is equivalent to the one given here.

3. Foley makes an additional assumption outlawing allocations that are on the boundary of the constraint set (his assumption (C.1). However, this condition is only required for the first half of his paper, and is not required in his existence proof. (In fact, the existence theorem is incorrect with this assumption.) Hence, we do not make it here.

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