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THE ECONOMIC VALUE OF RESOURCE FLEXIBILITY

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SOCIAL SCIENCE WORKING PAPER 570

May 1985

ABSTRACT

The economic value of resource flexibility is discussed in the context of a model of a two-stage production process with stochastic variability. A unique optimal barrier policy is characterized with any level of capacity flexibility. The more flexible the capacities, the lower the optimal inventory limit and the higher the profit of the producer. A diffusion approximation of the model is also discussed.

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1. FORMULATION OF THE MODEL

This paper studies the issues concerning the economics of multiple-use resources, or the economics of resource flexibility. The economic value of flexibility can only be discussed in the context of a model that has uncertainty, seasonality, or in general terms, some sort of variability. It is also necessary that one's model explicitly recognize the existence of more than one type of task or processing requirement within the firm. Thus, to consider any issue in the economics of resource flexibility, we must consider a multi-stage process (some sort of network structure). A simple model is considered as follows.

Imagine a production process that consists of two work stations, 1 and 2, producing an intermediate good 1 and a final good 2, and an inventory storing intermediate goods. Both production processes are independent Poisson if at their full capacities. Worker 1 can produce good 1 with rate α, or he can produce good 2 with rate $\delta_1\alpha (0 \leq \delta_1 \leq 1)$. Similarly, worker 2 can produce good 2 using good 1 with rate β, or produce good 1 with rate $\delta_2\beta (0 \leq \delta_2 \leq 1)$. Suppose the management monitoring the production can change the production rates within the range of their maximum rates, and can also switch either

* I gratefully acknowledge the insightful suggestions of Erhan Cinlar and Michael J. Harrison.

worker from one type of production to the other. The mathematical formulation of this problem is as follows. Let α_t be the actual production rate of intermediate good 1 at time t, and A(t) its cumulative output up to time t. Similarly, β_t and B(t) denote the actual production rate and the cumulative output of final good 2 at time t respectively. Mathematically, the control mode stated above can be expressed as

$$0 \leq \alpha_t \leq (\alpha + \delta_2\beta) \cdot 1_{\{0\}}(\beta_t) + \alpha \cdot 1_{(0,\beta]}(\beta_t), \tag{1}$$

$$0 \leq \beta_t \leq (\beta + \delta_1\alpha) \cdot 1_{\{0\}}(\alpha_t) + \beta \cdot 1_{(0,\alpha]}(\alpha_t). \tag{2}$$

And then

$$A(t) = \int_0^t\int_0^{\alpha_s} N(ds,sy), \text{ and} \tag{3}$$

$$B(t) = \int_0^t\int_{-\beta_s}^0 N(ds,dy), \tag{4}$$

where N is a Poisson random measure with unit intensity. Production of intermediate goods flows into inventory, and Z(t) is the inventory level at time t. Then

$$Z(t) = x + A(t) + B(t), \tag{5}$$

with x the initial inventory level. Note that this definition of inventory content process Z implies that the intermediate good 2 which is in process of being converted to final product and not yet finished

is still counted as inventory. In other words, one may view the inventory as work-in-process. Assume that the production of good 2 has no choice other than waiting if its demand of intermediate products can not be met from stock on hand. This means the inventory level is not allowed to go negative.

Given the capacity α and β, and the indicators of resource flexibility δ_1 and δ_2, a feasible policy is defined as a pair of stochastic processes (α_t,β_t) that jointly satisfy the following:

(α_t) and (β_t) are left-continuous with right-hand limits, (6)

(α_t) and (β_t) are adapted to Z, (7)

$$0 \leq \alpha_t \leq (\alpha + \delta_2\beta) \cdot 1_{\{0\}}(\beta_t) + \alpha \cdot 1_{(0,\beta]}(\beta_t), \text{ and}$$
$$0 \leq \beta_t \leq (\beta + \delta_1\alpha) \cdot 1_{\{0\}}(\alpha_t) + \beta \cdot 1_{(0,\alpha]}(\alpha_t),$$
$$\text{for all } t \geq 0, \tag{8}$$

Z(t) is non-negative for all t. (9)

The cost structure is assumed as follows. Each final product is worth p dollars. In one way, it can be thought that the demand for final goods is infinite, and then the firm can sell its final good right away at price p whenever it is available. Or p can be viewed as the value of each final product coming off the production line, net of later processing costs such as storage, transportation, etc. We simplify the market side where the firm serves as a supplier for the purpose of focusing our attention on the issues of resource flexibility. The plant incurs linear variable costs, c_1 dollars per unit of good 1 actually produced and c_2 dollars per unit of good 2 actually produced. The variable cost may comprise material cost and also labor cost if workers are paid piece-rate. The variable cost c_2 can be interpreted as the material cost and perhaps labor cost as well, in addition to c_1 to produce a final product. The selling price p is reasonably assumed to cover the total variable cost of a final good, that is,

$$p > c_1 + c_2. \tag{10}$$

A physical holding cost of h dollars per unit time is incurred for each unit of intermediate goods held in inventory. Assume that the firm earns interest at rate $r > 0$, compounded continuously, on the funds which are not required for production operations and the production is planned over an infinite time horizon. Therefore, given that the initial inventory is x, the expected revenue is

$$TR(x) \equiv E_x\{\int_0^\infty e^{-rt} p dB(t)\} \tag{11}$$

and the expected cost is

$$TC(x) \equiv E_x\{\int_0^\infty e^{-rt}[c_1 dA(t) + c_2 dB(t) + hZ(t)dt]\} \tag{12}$$

The firm's problem is to choose a pair of control processes (α_t, β_t) to maximize the expected profit

$$\Pi(x) = TR(x) - TC(x) \tag{13}$$

such that equations (3)-(5) and feasibility constraints (6)-(9) are satisfied.

Applying integration by parts theorem to (13), we have

$$\Pi(x) = V(x) - \frac{h}{r} \tag{14}$$

where

$$V(x) = E_x\{\int_0^\infty e^{-rt}[(p - c_2 + \frac{h}{r})dB(t) - (c_1 + \frac{h}{r})dA(t)]\}. \tag{15}$$

Since $V(x)$ is the only part in $\Pi(x)$ which the operating control processes (α_t) and (β_t) may affect, maximizing $\Pi(x)$ is equivalent to maximizing $V(x)$. For notation simplification, let

$$q \equiv p - c_2 + \frac{h}{r}, \text{ and} \tag{16}$$

$$w \equiv c_1 + \frac{h}{r}. \tag{17}$$

Then

$$V(x) = E_x\{\int_0^\infty e^{-rt}[qdB(t) - wdA(t)]\} \tag{18}$$

with $q > w$ by assumption (10).

2. THE BARRIER POLICIES

In this section we investigate a class of feasible policies, namely, barrier policies, and show the existence of an optimal barrier policy for given α, β, δ_1 and δ_2.

A barrier policy is such that, for some parameter $b > 0$,

$$\alpha_t = \alpha \cdot 1_{(0,b)}(Z_{t-}) + (\alpha + \delta_2\beta) \cdot 1_{\{0\}}(Z_{t-}), \text{ and} \tag{1}$$

$$\beta_t = \beta \cdot 1_{(0,b)}(Z_{t-}) + (\beta + \delta_1\alpha) \cdot 1_{\{b\}}(Z_{t-}), \text{ for } t \geq 0. \tag{2}$$

This policy requires that worker 1 works at its full capacity on his regular task until inventory reaches b, at this point he switches to assist worker 2 with full effort until the inventory has been depleted by one; similarly, worker 2 works at full capacity on his regular task until inventory is zero, then switches to assist worker 1.

Under a barrier policy with parameter $b > 0$, the inventory content process Z is a Markov process with generator

$$G = \begin{bmatrix} -(\alpha+\delta_2\beta) & \alpha+\delta_2\beta & & & & \\ \beta & -(\alpha+\beta) & \alpha & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & 0 & & \beta & -(\alpha+\beta) & \alpha \\ & & & & \beta+\delta_1\alpha & -(\beta+\delta_1\alpha) \end{bmatrix} \tag{3}$$

For ease of analysis, we may change the form of the value function V(x) once more by noting the following important lemma (see Cinlar (1982) for the proof).

<u>Lemma</u>: Suppose A and B are the counting processes with intensity

processes (α_t) and (β_t) respectively and G is a left-continuous and right-limited stochastic process adapted to Z. Then,

$$E[\int_0^t G(s)dA(s)] = E[\int_0^t G(s)\alpha_s ds] \text{ and}$$

$$E[\int_0^t G(s)dB(s)] = E[\int_0^t G(s)\beta_s ds].$$

Then,

$$V(x) = E_x\{\int_0^\infty e^{-rt}(q \cdot \beta_t - w \cdot \alpha_t)dt\}. \tag{4}$$

Using the same notation V(x) for the value function under a barrier policy, we have,

$$\begin{aligned} V(x) &= E_x\{\int_0^\infty e^{-rt}[q(\beta \cdot 1_{(0,b)}(Z_{t-}) + (\beta + \delta_1\alpha)1_{\{b\}}(Z_{t-})) \\ &\quad - w(\alpha \cdot 1_{(0,b)}(Z_{t-}) + (\alpha + \delta_2\beta)1_{\{0\}}(Z_{t-}))]dt\} \\ &= q \cdot \frac{\beta}{r} - w \cdot \frac{\alpha}{r} + E_x\{\int_0^\infty e^{-rt}[(w + \delta_1 q)\alpha \cdot 1_{\{b\}}(Z_t) \\ &\quad - (q + \delta_2 w)\beta \cdot 1_{\{0\}}(Z_t)]dt\}. \end{aligned} \tag{5}$$

Let

$$\bar{V}(x) = E_x\{\int_0^\infty e^{-rt}[(w + \delta_1 q)\alpha \cdot 1_{\{b\}}(Z_t) - (q + \delta_2 w)\beta \cdot 1_{\{0\}}(Z_t)]dt\}$$

$$= E_x\{\int_0^\infty e^{-rt}[w\alpha \cdot 1_{\{b\}}(Z_t) - q\beta \cdot 1_{\{0\}}(Z_t)]dt\}$$

$$+ E_x\{\int_0^\infty e^{-rt}[\delta_1 q\alpha \cdot 1_{\{b\}}(Z_t) - \delta_2 w\beta \cdot 1_{\{0\}}(Z_t)]dt\}. \qquad (6)$$

Then,

$$V(x) = q \cdot \frac{\beta}{r} - w \cdot \frac{\alpha}{r} + \bar{V}(x). \qquad (7)$$

From (6), $\bar{V}(x)$ consists of two parts. The first term in the second equality of (6) is the cost saving of production 1 due to limited inventory capacity, minus the revenue loss due to stockout of the intermediate goods, while the second term indicates the profit gain from the resource flexibility. In the case that each work station can be used for only one purpose, i.e., $\delta_1 = \delta_2 = 0$, the second term in (6) becomes zero. In view of equation (6) and (7), the explicit form of $V(x)$ can be obtained by the calculation of $E_x\{\int_0^\infty e^{-rt}1_{\{0\}}(Z_t)dt\}$ and $E_x\{\int_0^\infty e^{-rt}1_{\{b\}}(Z_t)dt\}$, and the barrier policy which maximizes $\bar{V}$ will maximize V, and hence, Π as well.

For simplicity, we adopt the following notations:

$$T(y) \equiv \inf\{t \geq 0 : Z(t) = y\}, \qquad (8)$$

$$\theta(x,y) \equiv E_x[e^{-rT(y)}], \qquad (9)$$

$$\bar{a}^* \equiv (\rho_1 - 1)(1 + \delta_1\rho_1^{-1}), \quad \bar{a}_* \equiv (\rho_2 - 1)(1 + \delta_1\rho_2^{-1})$$

$$\bar{\theta}^* \equiv (\rho_1^{-1} - 1)(1 + \delta_2\rho_1), \quad \bar{\theta}_* \equiv (\rho_2^{-1} - 1)(1 + \delta_2\rho_2), \qquad (10)$$

$$\bar{g}(x) \equiv \bar{a}_*\rho_1^{-x} - \bar{a}^*\rho_2^{-x}, \text{ and}$$

$$\bar{e}(x) \equiv \bar{\theta}_*\rho_1^{x} - \bar{\theta}^*\rho_2^{x}, \qquad (11)$$

where ρ_1 and ρ_2 ($\rho_1 < \rho_2$) are the two roots of the quadratic equation $\rho = \frac{\alpha}{\alpha + \beta + r}\rho^2 + \frac{\beta}{\alpha + \beta + r}$. Noting that $\bar{a}^*$, $\bar{a}_*$, $\bar{\theta}^*$, $\bar{\theta}_*$, $\bar{g}$ and $\bar{e}$ are functions of δ_1 and δ_2, we denote by a^*, a_*, θ^*, θ_*, g and e respectively their values when $\delta_1 = \delta_2 = 0$.

Here we only list the results under a barrier policy and the detailed calculation can be found in Li (1984):

$$\theta(x,0) = \frac{\bar{g}(b-x)}{\bar{g}(b)}, \qquad (12)$$

$$\theta(x,b) = \frac{\bar{e}(x)}{\bar{e}(b)}, \qquad (13)$$

$$E_x\{\int_0^\infty e^{-rt}1_{\{0\}}(Z_t)dt\} = \frac{\theta(x,0)}{(\alpha + \delta_2\beta + r) - (\alpha + \delta_2\beta)\theta(1,0)}, \text{ and} \qquad (14)$$

$$E_x\{\int_0^\infty e^{-rt}1_{\{b\}}(Z_t)dt\} = \frac{\theta(x,b)}{(\beta + \delta_1\alpha + r) - (\beta + \delta_1\alpha)\theta(b-1,b)}. \qquad (15)$$

By substituting (14) and (15) into (6), we have,

<u>Proposition 1</u>: The value function under a barrier policy with parameter $b > 0$ is of the form

$$\bar{V}^b(x) = (w + \delta_1 q) \cdot \frac{\beta}{r} \cdot \bar{e}(x) \cdot [(1 + \delta_1\rho_1^{-1})(1 + \delta_2\rho_2)\rho_1^{b+1}$$

$$- (1 + \delta_1\rho_2^{-1})(1 + \delta_2\rho_1)\rho_2^{b+1}]^{-1}$$

$$+ (q + \delta_2 w) \cdot \frac{\alpha}{r} \cdot \bar{g}(b-x) \cdot [(1 + \delta_1\rho_2^{-1})(1 + \delta_2\rho_1)\rho_1^{-(b+1)}$$

$$- (1 + \delta_1\rho_1^{-1})(1 + \delta_2\rho_2)\rho_2^{-(b+1)}]^{-1} \tag{16}$$

The existence and uniqueness of an optimal barrier policy is proved in Proposition 3 by using the result in Proposition 2.

<u>Proposition 2</u>: $\bar{g}(\cdot)$ is strictly increasing and $e(\cdot)$ is strictly decreasing. Hence $\theta(\cdot,0)$ is strictly decreasing and $\theta(\infty,0) = 0$.

<u>Proof</u>: Note that

$$\bar{g}(x) = g(x) + \delta_1\frac{\alpha}{\beta}g(x-1), \text{ and} \tag{17}$$

$$\Delta g(x) = g(x) - g(x-1)$$

$$= (\rho_2 - 1)(1 - \rho_1)\rho_1^{-x}(1 - (\rho_1/\rho_2)^x)>0, \text{ for } x \geq 1, \tag{18}$$

by the fact that $0 < \rho_1 < 1 < \rho_2$. So, $\bar{g}(x)$ is strictly increasing for $x \geq 0$. Also note that $\bar{g}(\infty) = \infty$ and $\bar{g}(0) = g(0)(1 + \delta_1\frac{\alpha}{\beta}) > 0$. It follows that $\theta(b,0)$ decreases to zero as b increases to infinity by (12). The monotonicity of e is subject to a similar argument.

Q.E.D.

<u>Proposition 3</u>: There exists an optimal barrier with one critical number b (inventory limit), which is uniquely determined by the condition:

$$\theta(b+1,0)^{-1} + \delta_2\theta(b,0)^{-1} \geq k \cdot [1 + \delta_1\theta(1,0)^{-1}], \text{ and} \tag{19}$$

$$\theta(b,0)^{-1} + \delta_2\theta(b-1,0)^{-1} \leq k \cdot [1 + \delta_1\theta(1,0)^{-1}]; \tag{20}$$

or equivalently

$$\bar{g}(b+1) + \delta_2\bar{g}(b) \geq k \cdot [\bar{g}(0) + \delta_1\bar{g}(1)], \text{ and} \tag{21}$$

$$\bar{g}(b) + \delta_2\bar{g}(b-1) \leq k \cdot [\bar{g}(0) + \delta_1\bar{g}(1)], \tag{22}$$

where $k = \frac{q + \delta_2 w}{w + \delta_1 q}$.

<u>Proof</u>: Denote by V^b the value function under the barrier policy with parameter b. For fixed $x \in [0,b]$, it can be calculated that

$$V^b(x) - V^{b-1}(x)$$

$$= \bar{V}^b(x) - \bar{V}^{b-1}(x)$$

$$= H \cdot [k(1 + \delta_1\theta(1,0)^{-1}) - (\theta(b,0)^{-1} + \delta_2\theta(b-1,0)^{-1})] \tag{23}$$

where

$$H \equiv \frac{\beta\rho_1^b\rho_2^b}{r\Delta}(w + \delta_1 q)(\bar{a}_* - \bar{a}^*)(\bar{\theta}^*\rho_2^x - \bar{\theta}_*\rho_1^x) > 0, \text{ and}$$

$$\Delta = [(1 + \delta_1\rho_1^{-1})(1 + \delta_2\rho_2)\rho_1^{b+1} - (1 + \delta_1\rho_2^{-1})(1 + \delta_2\rho_1)\rho_2^{b+1}]$$

$$\times [(1 + \delta_1\rho_1^{-1})(1 + \delta_2\rho_2)\rho_1^b - (1 + \delta_1\rho_2^{-1})(1 + \delta_2\rho_1)\rho_2^b].$$

By Proposition 2, $\theta(b,0)^{-1}$ is strictly increasing in b. So, b is optimal if and only if

$$V^b(x) - V^{b-1}(x) \geq 0 \text{ and } V^{b+1}(x) - V^b(x) \leq 0, \text{ for } 0 \leq x \leq b. \tag{24}$$

And condition (24) is equivalent to condition (19) and (20) on account of (23).

Q.E.D.

Several remarks can be drawn from the above results:

1. Suppose $\delta_1 = \delta_2 = 0$, that is, each work station can be used for a single purpose only. The inventory process z degenerates to a M/M/1/b queue and the conditions (19) and (20) become

$$\theta(b+1,0) \leq \frac{W}{q}, \text{ and } \theta(b,0) \geq \frac{W}{q}. \tag{25}$$

The barrier policy with parater b determined by (25) is optimal among all the feasible policies as shown in Li (1984, chapter 3).

2. From condition (20), in order to abtain a finite optimal upper barrier b which is at least 1, it is necessary to have

$$(q + \delta_2 w)[1 + \delta_1\frac{\bar{g}(1)}{\bar{g}(0)}] - (w + \delta_1 q)[\frac{\bar{g}(1)}{\bar{g}(0)} + \delta_2]$$

$$= (1 - \delta_1\delta_2)(q - w \cdot \frac{\delta_1\alpha + \beta + r}{\delta_1\alpha + \beta}) \geq 0 \tag{26}$$

Whence there exists a finite optimal buffer size which is at least one if either

$$\frac{W}{q} \leq \frac{\delta_1\alpha + \beta}{\delta_1\alpha + \beta + r}, \text{ or} \tag{27}$$

$$\delta_1 = \delta_2 = 1. \tag{28}$$

Observe that condition (27) is equivalent to

$$\frac{W}{q} \leq \theta(1,0). \tag{29}$$

3. Above calculation also shows that if $\delta_1 = \delta_2 = 1$, then the optimal inventory limit is exactly 1 no matter how other parameters are chosen. That is, if the workers can help each other at their regular task rates, then the optimal policy is to let them always work together in a fashion that they work on product 1 when product 1 is not available, and they switch to work on product 2 as soon as they finish product 1, and back and forth.

The following propositon constitutes the main result of this chapter showing the properties of the optimal upper barrier b as a function of δ_1 and δ_2.

<u>Proposition 4</u>: Suppose b is the optimal limit of the WIP inventory determined by the conditions in Proposition 3. Then b decreases as δ_1 or δ_2 increases.

<u>Proof</u>: Define a function

$$h(b,\delta_1,\delta_2) \equiv (w + \delta_1 q)[\bar{g}(b) + \delta_2\bar{g}(b-1)] - (q + \delta_2 w)[\bar{g}(0) + \delta_1\bar{g}(1)]. \tag{30}$$

Condition (21) and (22) determining b is equivalent to

$$h(b+1,\delta_1,\delta_2) \geq 0, \text{ and } h(b,\delta_1,\delta_2) \leq 0. \tag{31}$$

First note that h is a strictly increasing function of b. If it can be shown that an increase in δ_1 or δ_2 will cause an increase in h, then b is required to decrease to maintain conditon (31). Since $\bar{g}(x) = g(x) + \delta_1\frac{\alpha}{\beta}g(x-1)$ does not depend on δ_2,

$$\frac{\partial h}{\partial \delta_2} = (w + \delta_1 q)\bar{g}(b-1) - w \cdot [\bar{g}(0) + \bar{g}(1)]. \tag{32}$$

This derivatives is again a strictly increasing function of b for $b \geq 1$. Let b = 1,

$$\left.\frac{\partial h}{\partial \delta_2}\right|_{b=1} = \delta_1[q \cdot [\bar{g}(0) - w \cdot \bar{g}(1)]$$

$$= \delta_1 q \cdot \bar{g}(1)[\theta(1,0) - \frac{w}{q}] \geq 0, \text{ by}(29). \tag{33}$$

Therefore, $\frac{\partial h}{\partial \delta_2}(b,\delta_1,\delta_2) \geq 0$ for $b \geq 1$ and $0 \leq \delta_1 \leq 1$, and strict inequality holds for $b > 1$. It follows that b decreases as δ_2 increases.

Similarly calculate

$$\frac{\partial h}{\partial \delta_1} = q[\bar{g}(b) + \delta_2\bar{g}(b-1)] + (w + \delta_1 q)[g(b-1) + \delta_2 g(b-2)]\frac{\alpha}{\beta}$$

$$- (q + \delta_2 w)[\bar{g}(1) + (1 + \delta_1) \cdot \frac{\alpha}{\beta} \cdot g(0)]. \tag{34}$$

Note that

$$\frac{\partial^2}{\partial\delta_2\partial\delta_1}h(b,\delta_1,\delta_2) = q \cdot \bar{g}(b-1) + (w + \delta_1 q)g(b-2) \cdot \frac{\alpha}{\beta}$$

$$- w \cdot [\bar{g}(1) + (1 + \delta_1) \cdot \frac{\alpha}{\beta} \cdot g(0)]$$

$$> w[\bar{g}(b-1) - \bar{g}(1)] + (w + \delta_1 q)\frac{\alpha}{\beta}[g(b-2) - g(0)] \geq 0. \tag{35}$$

So, it is sufficient to consider the case $\delta_2 = 0$ since if $\frac{\partial h}{\partial \delta_1} \geq 0$ holds for $\delta_2 = 0$, then it holds for $\delta_2 > 0$ as well. Also,

$$\frac{\partial^2 h}{\partial \delta_1^2}(b,\delta_1,0) = 2q \cdot \frac{\alpha}{\beta} \cdot [g(b-1) - g(0)] \geq 0. \tag{36}$$

Hence it is sufficient to consider the case $\delta_1 = \delta_2 = 0$. By (34)

$$\frac{\partial h}{\partial \delta_1}(b,0,0) = q[g(b) - g(1)] + \frac{\alpha}{\beta}[w \cdot g(b-1) - q \cdot g(0)]. \tag{37}$$

And

$$h(b,0,0) = wg(b) - qg(0). \tag{38}$$

If $h(b,0,0) = 0$ (at which b jumps), then

$$\frac{\partial h}{\partial \delta_1}(b+1,0,0) = q[g(b+1) - g(1)] \geq 0. \tag{39}$$

By (36), this implies $h(b+1,\delta_1,0) \geq h(b+1,0,0) \geq 0$ for $0 \leq \delta_1 \leq 1$. That is, an increase in δ_1 will never cause an increase in b since b is so determined that $h(b) \leq 0$, and $h(b+1) \geq 0$.

Q.E.D.

One of the most fundamental effect of stochastic variability in a production system is that it creates a requirement for buffer stocks to decouple operations. In fact, the result in the paper shows that the optimal inventory size chosen by the firm is strictly positive as long as it is in business. What we have shown here is that introducing multi-purpose work stations reduces the buffer stocks relative to the case where work stations can be used for a single purpose only. Moreover parameter δ_1, δ_2 can be viewed as indicators of the resource flexibility. Then, the more the resource flexibility is the lower the optimal inventory limit will be. This is a special example, and there is the important question of what general or qualitative wisdom can be extracted from this.

3. A DIFFUSION LIMIT

Harrison-Taylor (1977) and Harrison (1982) study a diffusion model of inventory and production control where the difference of cumulative potential input and cumulative demand is modeled by a Brownian motion (with general drift and variance parameters) , and the optimal policy (involving a single critical number b*) is very simple. We shall show that this diffusion model may represent the limit of the model described earlier as certain parameters approach critical values. This helps one to better understand conditions under which the diffusion model applies, and justifies a very trachtable approximation for the general additive process formulation under such conditions.

Let us first restate Harrison-Taylor's model. Consider a controller who continuously monitors the content of a storage system. In the absence of control, the content process $Z = \{Z_t, t > 0\}$ fluctuates as a (μ,σ) Brownian motion. The controller can at any time increase or decrease the content of the system by any amount desired, but he is obliged to keep $Z_t \geq 0$.

A policy is defined as a pair of processes L and U such that

L and U are adapted, (1)

L and U are right-continuous, increasing, and positive almost surely. (2)

Interpret L_t as the cumulative increase in the system content effected by the controller up to time t, and U_t as the corresponding cumulative decrease effected. Associated with policy (L,U) is the controlled process $Z \equiv X + L - U$, where X is a (μ,σ) Brownian motion with starting state x. And (L,U) is said to be feasible if

$$P_x\{Z_t \geq 0 \text{ for all } t \geq 0\} = 1 \quad \text{for all } x \geq 0, \tag{3}$$

$$E_x[\int_0^\infty e^{-rt} dL_t] \leq \infty \text{ for all } x \geq 0, \tag{4}$$

$$E_x[\int_0^\infty e^{-rt} dL_t] \leq \infty \text{ for all } x \geq 0, \tag{5}$$

We associate with a feasible policy (L,U) the cost function

$$v(x) \equiv E_x\{\int_0^\infty e^{-rt}[\gamma dU_t - \xi dL_t]\}.$$

with $\xi > \gamma > 0$.

It is shown that the optimal policy (L^*,U^*) enforces a lower reflecting barrier at zero and an upper reflecting barrier at zero and an upper reflecting barrier at b^*, where b^* is the unique solution of equation

$$\theta(b,0) = \frac{w}{q}, \tag{6}$$

with the notations

$$\theta(b,0) \equiv E_b[e^{-rT(0)}], \text{ and} \tag{7}$$

$$T(y) \equiv \inf\{t \geq 0: Z_t = y\}, 0 \leq y \leq b. \tag{8}$$

This means that L^* increases in the minimal amounts necessary to ensure $Z^* \geq 0$, while U^* increases in the minimal amount necessary to ensure $Z^* \leq b^*$.

Under a barrier policy, the following results may be calculated explicitly. First,

$$\theta(X,0) = \frac{g(b-x)}{g(b)} \text{ for } 0 \leq x \leq b, \tag{9}$$

where

$$g(y) = a_*(r)e^{-a^*(r)y} + a(r)e^{a_*(r)y}, \tag{10}$$

$$a_*(r) \equiv \sigma^{-2}[(\mu^2 + 2\sigma^2 r)^{1/2} + \mu] > 0, \text{ and} \tag{11}$$

$$a^*(r) \equiv \sigma^{-2}[(\mu^2 + 2\sigma^2 r)^{1/2} + \mu] > 0. \tag{12}$$

And $g(\cdot)$ is strictly increasing on $[0,\infty)$. Secondly, let v be the value function for the barrier policy (L,U) with parameter $b > 0$. Then

$$v(x) = Ae^{-a_*(r)x} + Be^{a^*(r)x}, 0 \leq x \leq b, \tag{13}$$

where A and B are (uniquely) chosen so that

$$v'(0) = q, \text{ and } v'(b) = w. \tag{14}$$

Note that suppose

$$X_n(t) \equiv x + \int_0^t\int_0^{\alpha_n} N(ds,dy) - \int_0^t\int_{-\beta n}^0 N(ds,dy), \text{ and} \tag{15}$$

$$X_n^*(t) \equiv \sigma(\alpha_n + \beta_n)^{-1/2} X_n(t). \tag{16}$$

Then

$$X_n^* \text{ converges weakly to a } (\mu,\sigma) \text{ Brownian motion as } n \uparrow \infty \tag{17}$$

provided that as $n \uparrow \infty$,

$$\alpha_n + \beta_n \uparrow \infty, \text{ and} \tag{18}$$

$$\sigma(\alpha_n + \beta_n)^{-1/2}(\alpha_n - \beta_n) \rightarrow \mu, \tag{19}$$

where μ is an arbitrary constant.

We consider a sequence of Poisson-Poisson problems indexed by n with the objectives of type (1.18) with the same parameters except the capacities α_n and β_n. The value function indexed by n is

$$V_n(x_n) = E_{x_n}\{\int_0^\infty e^{-rt}[qdB_n(t) - wdA_n(t)]\}. \tag{20}$$

Multiply $\alpha(\alpha_n + \beta_n)^{1/2}$ to both sides of (20), we have

$$\sigma(\alpha_n + \beta_n)^{-1/2} V_n(x_n) = E_{x_n^*}\{\int_0^\infty e^{-rt}[qdB_n^*(t) - wdA_n^*(t)]\} \equiv V_n^*(x_n^*) \tag{21}$$

where the following transformations are made:

$$A_n^*(t) \equiv \sigma(\alpha_n + \beta_n)^{-1/2} A_n(t), \tag{22}$$

$$B_n^*(t) \equiv \sigma(\alpha_n + \beta_n)^{-1/2} B_n(t), \tag{23}$$

$$Z_n^*(t) \equiv \sigma(\alpha_n + \beta_n)^{-1/2} Z_n(t), \tag{25}$$

and σ is a positive constant.

Clearly, our original objective is equivalent to maximizing V_n^*. Specifically, suppose the optimal inventory limit for problem (20) is b_n, then the optimal inventory limit for the transformed problem (21) is $b_n^* = \sigma(\alpha_n + \beta_n)^{-1/2} b_n$. In light of the fact (17), the natural diffusion approximation for our original problem should be the instantaneous control problem solved by Harrison and Taylor. Note that condition (19) implies

$$\alpha_n - \beta_n \rightarrow 0, \text{ as } n \uparrow \infty. \tag{25}$$

Therefore, conditions (18), (19) under which the diffusion model applies actually says that the capacity are sufficiently large and the firm is approaching a balanced production situation.

<u>Proposition 5</u>: Suppose there is a sequence of transformed Poisson-Poisson problems indexed by n, and condition (18) and (19) hold as $n \uparrow \infty$. Then as $n \uparrow \infty$

$$b_n^* \to b^*, \tag{26}$$

and with the further condition

$$x_n^* \to x, \tag{27}$$

$$\overline{V}_n^*(x_n^*) \to v(x) \tag{28}$$

Where b^* is the optimal upper barrier, x is the initial inventory level, and v(x) is the optimal value function in Harrison-Taylor's (μ,σ) Brownian motion setting with $\gamma = \frac{w + \delta_1 q}{1 + \delta_1}$ and $\xi = \frac{q + \delta_2 w}{1 + \delta_2}$ where q amd w are defined in (1.16), (1.17) and $\frac{q + \delta_2 w}{1 + \delta_2} > \frac{w + \delta_1 q}{1 + \delta_1}$ if $0 \leq \delta_1, \delta_2 < 1$.

In the limiting case, the optimal upper barrier b is uniquely determined by equation

$$\theta(b,0) = \frac{(w + \delta_1 q)(1 + \delta_2)}{(q + \delta_2 w)(1 + \delta_1)}, \tag{29}$$

and it is very easy to check that b decreases as δ_1 or δ_2 increases.

Define

$$f(b,\delta_1,\delta_2) \equiv \theta(b,0)^{-1} - \frac{(q + \delta_2 w)(1 + \delta_1)}{w + \delta_1 q)(1 + \delta_2)}. \tag{30}$$

Then condition (29) is equivalent to $f(b,\delta_1,\delta_2) = 0$. Since $\frac{\partial f}{\partial b} > 0$, it is sufficient to check,

$$\frac{\partial f}{\partial \delta_1} = \frac{(q + \delta_2 w)(q - w)}{(1 + \delta_2)(w + \delta_1 q)^2} > 0, \text{ and} \tag{31}$$

$$\frac{\partial f}{\partial \delta_1} = \frac{(q + \delta_2 w)(q - w)}{(w + \delta_1 q)(1 + \delta_2)^2} > 0. \tag{32}$$

4. CONCLUSION

This paper attempts to discuss the economic value of flexible resources in a production system and show that introducing multi-purpose work stations reduces the fundamental effects of stochastic variability, i.e., the level of buffer stocks. Moreover, the higher flexibility of the resources results in a higher profit of the producer because increasing δ_1 or δ_2 simply implies a larger feasible set of the value of control processes (α_t, β_t). However, the cost of building a product-flexible capacity is higher than that of a dedicated (single-purpose) capacity since it involves more investment both in physical and human productive assets. We can incorporate these design decisions into our simple model. Suppose that $\Pi(x)$ is the expected gross profit resulting from the optimal operating policy given α, β, δ_1 and δ_2. Assume the firm incurs a fixed cost $C(\alpha,\beta,\delta_1,\delta_2)$ of building capacities α, β and flexibility levels δ_1, δ_2, and C is increasing in α, β, δ_1, and δ_2. Then the net profit becomes

$$\Pi(\alpha,\beta,\delta_1,\delta_2) = \overline{\Pi}(0) - C(\alpha,\beta,\delta_1,\delta_2). \tag{1}$$

It can be shown that Π and, hence, $\overline{\Pi}$ is continuous and almost everywhere differentiable given that the optimal operating policy follows. Therefore, the usual calculus techniques can be applied to determine the optimal α^*, β^*, δ_1^*, and δ_2^*.

Certainly, this is a very preliminary study on the topic of flexible capacity in a production system. There are many potentially fruitful directions for future research along this line.

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