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MORE ON HARSANYI'S UTILITARIAN CARDINAL WELFARE THEOREM*

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ABSTRACT

If individuals and society both obey the expected utility hypothesis and social alternatives are uncertain, then the social utility must be a linear combination of the individual utilities, provided the society is indifferent when all its members are. This result was first proven by Harsanyi [4] who made implicit assumptions in the proof not actually needed for the result (see [5]). This note presents a straightforward proof of Harsanyi's theorem based on a separating hyperplane argument.

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INTRODUCTION

Harsanyi [4, Theorem V] states his well-known utilitarian cardinal welfare theorem which provides conditions under which the social utility is a weighted sum of individual utilities and which has often been cited as a justification of classical utilitarianism. The validity of this result has recently called into question by Resnick [5]. Resnick rightfully points out that Harsanyi's proof uses implicit assumptions which may not be satisfied in many of the situations where one would like to apply the theorem. Fishburn [3] has pointed out that nonetheless the result is true and that Resnick's counterexample is actually a counterexample to a claim which is stronger than Harsanyi's actual claim. In this paper we present an alternative proof of Harsanyi's theorem which is considerably different from Fishburn's. While the details of the proof are somewhat technical, the structure of the proof is straightforward and interesting in its own right.

There is a set X of basic social alternatives, but rankings must be made over lotteries on X, that is, countably additive probability measures defined on a σ-field Ω of subsets of X. Individuals and society are assumed to rank lotteries according to the expected utility hypothesis. That is, each individual i has a Ω-measurable von Neumann-Morgenstein utility function $u_i : X \to \mathbb{R}$ so that lottery μ is weakly preferred to lottery π if and only if $\int u_i d\mu \geq \int u_i d\pi$. In order to guarantee the finiteness of these integrals we assume that all utilities are bounded. See Fishburn [2] for conditions sufficient to gaurantee this. Likewise there is a bounded Ω-measurable social utility $w : X \to \mathbb{R}$ used for ranking lotteries. Harsanyi's main result is that if the social utility is indifferent whenever all individuals are indifferent, then the social utility is an affine combination of the individual utilities. Formally we have the following result.

<u>Theorem</u>: Let (X,Ω) be a measurable space and let $u_i, i = 1,\ldots,n$, and w be bounded real-valued Ω-measurable functions on X. Suppose that

$$\int w d\mu = \int w d\pi \text{ whenever } \int u_i d\mu = \int u_i d\pi \text{ for all } i = 1,\ldots,n \qquad (1)$$

for any countably additive probability measures μ, π on Ω. Then there are real numbers $\lambda_0,\lambda_1,\ldots,\lambda_n$ such that

$$w = \lambda_0 + \sum_{i=1}^{n} \lambda_i u_i. \qquad (2)$$

<u>Proof</u>: Let B denote the set of all bounded real-valued Ω-measurable functions on X and let $S = \text{span}\{\underline{1},u_1,\ldots,u_n\}$, where $\underline{1}$ is the constant function taking on the value 1. Let M be the set of countably additive signed measures on Ω. Each measure $\mu \in M$ defines a linear functional on B by $u \mapsto \int u d\mu$. Furthermore, since M contains all countably additive measures of the form μ_x where $\mu_x(E) = \begin{cases} 0 \text{ if } x \notin E \\ 1 \text{ if } x \in E, \end{cases}$

M separates the points of B. Thus the weak topology on B generated by M makes B a locally convex Hausdorff topological vector space ([1, V.3.3]). Since S is a finite dimensional subspace of B it is closed in any vector space topology ([6, 1.3.3]). Suppose condition (2) fails. Then $w \notin S$ and so we have by a separating hyperplane theorem ([1, V.3.12]) that there is some nonzero $\mu \in M$ satisfying

$$\int w d\mu = 1 \text{ and } \int u d\mu = 0 \text{ for all } u \in S. \tag{3}$$

Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ [1, III.1.8]. By (2) and the fact that $\underline{1} \in S$ we have

$$\int \underline{1}\, d\mu = \int \underline{1}\, d\mu^+ - \int \underline{1} d\mu^-(X) = 0.$$

Since μ is nonzero, $\mu^+(X) = \mu^-(X) \neq 0$, so we can normalize μ^+ and μ^- to be probabilities. Set $\tilde{\mu}^+ = \frac{1}{\mu^+(X)} \mu^+$ and $\tilde{\mu}^- = \frac{1}{\mu^+(X)} \mu^-$. Then from (3) it follows that

$$\int u_i d\tilde{\mu}^+ = \int u_i d\tilde{\mu}^- \quad i = 1,\ldots,n. \tag{4}$$

Thus by (1) we must have that

$$\int w d\tilde{\mu}^+ = \int w d\tilde{\mu}^-. \tag{5}$$

So that $1 = \int w d\mu = \mu^+(X)[\int w d\tilde{\mu}^+ - \int w d\tilde{\mu}^-] = 0$, a contradiction.

Q.E.D.

Remark: This proof offers no proof of the nonnegativity of the coefficients. This requires additional hypotheses such as may be found in Fishburn [3].

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