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ELECTIONS WITH LIMITED INFORMATION: A MULTIDIMENSIONAL MODEL

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Abstract

We develop a game theoretic model of 2 candidate competition over a multidimensional policy space, where the participants have incomplete information about the preferences and strategy choices of other participants. The players consist of the voters and the candidates. Voters are partitioned into two classes, depending on the information they observe. Informed voters observe candidate strategy choices while uninformed voters do not. All players (voters and candidates alike) observe contemporaneous poll data broken down by various subgroups of the population.

The main results of the paper give conditions on the number and distribution of the informed and uninformed voters which are sufficient to guarantee that any equilibrium (or voter equilibrium) extracts all information.

1. Introduction

We develop a game theoretic model of 2 candidate competition over a multidimensional policy space, where the participants have incomplete information about the preferences and strategy choices of other participants. The players consist of the voters and the candidates. Voters are partitioned into two classes, depending on the information they observe. Informed voters observe candidate strategy choices while uninformed voters do not. All players, voters and candidates alike, observe contemporaneous poll data broken down by various subgroups of the population. Also, all players have some basic knowledge about the structure of the electorate.

Each participant has beliefs about the parameters he does not observe. I.e. uninformed voters have beliefs about the candidate strategy choices, and candidates have beliefs about which voters are "informed." They then each choose a strategy conditional on their beliefs. A voter strategy is a choice of a candidate to vote for, and a candidate strategy is a choice of a policy position in the multidimensional policy space.

We define a set of strategies together with a set of beliefs to be in equilibrium if it satisfies two conditions: First, all participants are maximizing their payoff subject to their beliefs. Second, all participants must have beliefs which are consistent with the information they observe. A situation in which the voters are in equilibrium, but the candidates are not is referred to as a voter equilibrium. An equilibrium (or voter equilibrium) is said to extract all information if all players behave as if they have complete information.

The main results of the paper give conditions on the number and distribution of the informed and uninformed voters which are sufficient to guarantee that any equilibrium (or voter equilibrium) extracts all information.

This paper is related to but makes somewhat different assumptions than a previous paper of ours [1982] which develops a similar model in one dimension. In our previous paper, we assumed that voters observed both endorsement information as well as poll data. Here the voter only sees poll data, but the poll data must be broken down by subgroups in order to provide the voter with enough information to draw inferences about candidate positions. Also, our previous paper did not require the voters or candidates to have as much structural information about the electorate as we require here. Here, we require the voters to have some knowledge of the distribution of preferences in each subgroup of the population. An assumption of this sort seems to be necessary for the multidimensional extension.

2. The Formal Development

We are given a set, N, of voters, a set $X \subseteq \mathbb{R}^m$ of alternatives, and for each voter, $\alpha \in N$, a utility function, $u_\alpha : X \to \mathbb{R}$ representing voter α's preferences. We assume that the population, N, of voters can be partitioned into two subgroups, I and U, representing the informed and uninformed voters, respectively. We further assume that t subpopulations, $N_1, N_2, \ldots, N_t$ of N can be identified. These can be thought of as ethnic, or other socio-economic subdivisions of N. Note that the N_i need not necessarily be a partition of N, nor need any two N_i necessarily be disjoint. We let μ be a measure on the measurable subsets, $\underline{N}$ of N, and, for each i, let μ_i be the probability measure induced on the measurable subsets of N, conditional on being in N_i. Thus for any $C \in \underline{N}, \mu_i(C)$ $= \mu(N_i \cap C)/\mu(N_i)$.

In addition to the voters, we assume there are two candidates, labeled 1 and 2 , and we let $K = \{1,2\}$ be the set of candidates. If $k \in K$ is a candidate, we use the notation $\bar{k}$ to represent the other candidate, i.e., $\{\bar{k}\} = K - \{k\}$.

We now define a game, in which the players are the voters, N, together with the candidates, K. The strategy spaces for a voter $\alpha \in N$ and candidate $k \in K$ are denoted B_α and S_k respectively. The strategy spaces are defined as follows

Voter Strategy Space: $B_\alpha = K \cup \{0\}$.

$$(2.1)$$

Candidate Strategy Space: $S_k = X$.

We let $\underline{B}$ denote the set of functions from N into $K \cup \{0\}$, and $\underline{S}$ denote the set of functions from K to X. Elements of $\underline{B}$ are denoted b, with $b(\alpha) \varepsilon B_\alpha$ representing the choice of strategy by voter α. Elements of $\underline{S}$ are denoted s, with $s(k) \varepsilon S_k$ representing the choice of strategy. Alternatively, we also write b_α for $b(\alpha)$ and s_k for $s(k)$. We call b_α voter α's ballot, and s_k candidate k's policy position. We let $\underline{\Omega} = \underline{S} \times \underline{B}$, and an element $\omega = (s,b) \varepsilon \underline{\Omega}$ represents a choice of strategies by all players.

Given a choice of strategies, say $\omega = (s,b)$ by all players, we can compute a poll outcome and the outcome function. For each $1 \leq i \leq t$, and $k \varepsilon K \cup \{0\}$, we define

$$V_k(s,b) = \{\alpha\varepsilon N | b_\alpha = k\} \tag{2.2}$$

$$v_k(s,b) = \mu(V_k(s,b)) \tag{2.3}$$

and

$$p_{ik}(s,b) = \mu_i(V_k(s,b)) \tag{2.4}$$

So $V_k(s,b)$ represents the set of voters voting for candidate k (or abstaining if $k = 0$), $v_k(s,b)$ is the total vote for candidate k and $p_{ik}(s,b)$ represents the poll result in group i (i.e., the proportion of group i voting for candidate k). For each i, we use the notation

$$v(s,b) = (v_0(s,b),v_1(s,b),v_2(s,b)) \tag{2.5}$$

and

$$p_i(s,b) = (p_{i0}(s,b),p_{i1}(s,b),p_{i2}(s,b)) \tag{2.6}$$

We also write

$$p(s,b) = (p_1(s,b),\ldots,p_t(s,b)) \tag{2.7}$$

We let $\Delta = \{q = (q_0,q_1,q_2) \varepsilon \mathbb{R}^3 | \sum_{i=0}^{2} q_i = 1, q_i \geq 0 \text{ all } i\}$ to be the unit simplex in $\mathbb{R}^3$. Clearly, for all $(s,b) \varepsilon \Omega$, $p_i(s,b) \varepsilon \Delta$, and $p(s,b) \varepsilon \Delta^t$. We let $r = (r_1,\ldots,r_t) \varepsilon \mathbb{R}^t$, with $r_i > 0$ for all i. (For example, we could define $r_i = \mu(N_i)$ for each i.) Then for any $p^1,p^2 \varepsilon \Delta$, we define $||p^1 - p^2||_r = \sum_{i=1}^{t} r_i|p_i^1 - p_i^2|$, where $|p_i^1 - p_i^2| = \sum_{k=1} |p_{ik}^1 - p_{ik}^2|$. We next define the outcome function by

$$k(s,b) = \begin{cases} 1 \text{ if } v_1(s,b) > v_2(s,b) \\ 2 \text{ if } v_2(s,b) > v_1(s,b) \\ 0 \text{ otherwise.} \end{cases} \tag{2.9}$$

So k(s,b) represents the winning candidate, given the choice of strategies (s,b) by all players.

With these definitions, we can now define the payoff function to the game. We write $M_\alpha(s,b)$ and $M_k(s,b)$ for the payoff function for a voter $\alpha \varepsilon N$ and a candidate, $k \varepsilon K$, and they are defined by; for all $\alpha \varepsilon N$,

$$M_\alpha(s,b) = u_\alpha(s_{k(s,b)}), \tag{2.10}$$

where we define $u_\alpha(s_0) = \frac{1}{2}u_\alpha(s_1) + \frac{1}{2}u_\alpha(s_2)$. For $k \varepsilon K$,

$$M_k(s,b) = \begin{cases} 1 \text{ if } k(s,b) = k \\ -1 \text{ if } k(s,b) = \bar{k} \\ 0 \text{ otherwise} \end{cases} \tag{2.11}$$

In addition to the above more or less standard structure, we assume that each actor has <u>beliefs</u> about certain parameters of the game. The <u>belief space</u> for voter α is denoted $\tilde{S}^\alpha$, and that for candidate k is denoted C^k. We assume the belief spaces are given by:

$$\begin{aligned} &\text{Voter Belief Space:} \quad \tilde{S}^\alpha = \tilde{X}^2 \\ &\text{Candidate Belief Space:} \quad C^k = \underline{N} \end{aligned} \tag{2.12}$$

Here $\tilde{X}^2$ is the set of probability measures over $X^2 = X \times X$. We let $\underline{\tilde{S}}$ denote the set of functions from N into $\tilde{X}^2$ and $\underline{C}$ the set of functions from K into $\underline{N}$. Elements of $\underline{\tilde{S}}$ are denoted $\tilde{s}$, with $\tilde{s}^\alpha$ being used to represent $\tilde{s}(\alpha)$, for $\alpha \varepsilon N$. Similarly, elements of $\underline{C}$ are denoted C, with C^k representing C(k), for $k \varepsilon N$. So $\tilde{s}^\alpha \varepsilon \tilde{X}^2$, and $C^k \varepsilon \underline{N}$. We can think of $\tilde{s}^\alpha$ as being a probability measure representing voter α's belief of the probable location of the candidate positions, $s = (s_1, s_2)$. We use the notation $\text{supp}(\tilde{s}^\alpha)$ to represent the support

set of the measure $\tilde{s}^\alpha$. On the other hand, C^k represents candidate k's belief about the subset $C^k \subseteq N$ of voters who are "concerned" – i.e., who know the candidate positions. We let $\Lambda = \underline{C} \times \underline{\tilde{S}}$.

Before we define the notion of equilibrium used here, we develop some further notation. For any $s \varepsilon \underline{S}$, and $k \varepsilon K$, we write

$$\hat{V}_k(s) = \{\alpha \varepsilon N | u_\alpha(s_k) > u_\alpha(s_{\bar{k}})\}$$

and $\hfill (2.13)$

$$\hat{V}_0(s) = \{\alpha \varepsilon N | u_\alpha(s_k) = u_\alpha(s_{\bar{k}})\}.$$

So $\hat{V}_k(s)$ is the set of voters who prefer the policy position of candidate k over that of $\bar{k}$, while $\hat{V}_0(s)$ is the set of voters who are indifferent between the two candidates.

Next, given any measurable $C \subseteq N$, we define, for all $k \varepsilon K \cup \{0\}$,

$$\hat{b}_\alpha(s|C) = \begin{cases} k \text{ if } \alpha \varepsilon \hat{V}_k(s) \text{ and if } \alpha \varepsilon C \\ 0 \text{ if } \alpha \notin C \end{cases} \tag{2.14}$$

and

$$\hat{p}_{ik}(s|C) = p_{ik}(s, \hat{b}(s|C)) = \begin{cases} \mu_i(\hat{V}_k(s) \cap C) \text{ if } k \varepsilon K \\ \mu_i(\hat{V}_k(s) \cap C) + \mu_i(N - C) \text{ if } k = 0 \end{cases} \tag{2.15}$$

We write $\hat{p}_i(s|C) = (\hat{p}_{i0}(s|C), \hat{p}_{i1}(s|C), \hat{p}_{i2}(s|C))$, and $\hat{p}(s|C) = (\hat{p}_1(s|C), \dots, \hat{p}_t(s|C))$. Given any set $C \subseteq N$, $\hat{b}_\alpha(s|C)$ is the ballot that would result if all voters in C voted correctly, and all those in N − C abstained. We call the set C the set of <u>concerned voters</u>. So, $\hat{p}_i(s|C)$ is the predicted poll in group i when the voters behave according to $\hat{b}(s|C)$.

Now, for any $s, s' \varepsilon \underline{S}$ we define an equivalence relation $\approx$ on $\underline{S}$ as follows:

$$s \approx s' \iff \hat{V}_k(s) = \hat{V}_k(s') \text{ for all } k \varepsilon K \cup \{0\}. \qquad (2.16)$$

For any subset $A \subseteq \underline{S}$, we write $A \approx s' \iff s \approx s'$ for all $s \varepsilon A$. We let $\underline{S}^0 \subseteq \underline{S}$ be a subspace of $\underline{S}$ which contains exactly one representative of each equivalence class. And finally, for any $p \varepsilon \Delta^t$, we define $\underline{S}^0(p) \subseteq \underline{S}^0$ by

$$\underline{S}^0(p) = \arg \min_{s^0 \varepsilon \underline{S}^0} \| p - \hat{p}(s^0|N) \|_r \qquad (2.17)$$

So $\underline{S}^0(p)$ is the set of $s^0 \varepsilon \underline{S}$ which give the best fit of the actual to the predicted poll based on s^0.

<u>Definition 2.1</u>: An <u>equilibrium</u> is a pair (ω,η), where $\omega = (s,b) \varepsilon \underline{\Omega}$ and $\eta = (C,\tilde{s}) \varepsilon \Lambda$ satisfy

<u>Voters</u>: For all $\alpha \varepsilon N$,

V1: $b_\alpha \varepsilon \arg \max_{b_\alpha \varepsilon B_\alpha} E[u_\alpha(s_{b_\alpha})]$,

where the expectation is with respect to $\tilde{s}^\alpha$,

V2: $\alpha \varepsilon I \Rightarrow \text{supp}(\tilde{s}^\alpha) = \{s\}$,

$\alpha \varepsilon U \Rightarrow \text{supp}(\tilde{s}^\alpha) \approx s^\alpha$ for some $s^\alpha \varepsilon \underline{S}^0(p(s,b))$

<u>Candidates</u>: For all $k \varepsilon K$,

C1: $s_k \varepsilon \arg \max_{s_k \varepsilon S_k} M_k(s, \hat{b}(s|C^k))$

C2: $C^k \varepsilon \arg \min_{C \subseteq N} [\| p(s,b) - \hat{p}(s|C) \|_r]$

The equilibrium conditions require that all players maximize their expected payoff subject to their beliefs (Conditions V1 and C1). Further, the beliefs which the players hold must be as consistent as possible with the information they observe (Conditions V2 and C2). We also define a "partial equilibrium," or "voter equilibrium," in which the voters are in equilibrium, but the candidates are not. This type of equilibrium is useful in describing what might occur if there are exogenous constraints on candidate positions.

<u>Definition 2.2</u> A <u>voter equilibrium</u>, conditional on $s \varepsilon \underline{S}$, is a pair $(b,\tilde{s})$, where $b \varepsilon \underline{B}$ and $\tilde{s} \varepsilon \tilde{S}$ satisfy

V1: $b_\alpha \varepsilon \arg \max_{b_\alpha \varepsilon B_\alpha} E[u_\alpha(s_{b_\alpha})]$, where the expectation is with respect to $\tilde{s}^a$.

$$V2: \quad a \in I \Rightarrow \text{supp}(\tilde{s}^a) = \{s\}$$

$$a \in U \Rightarrow \text{supp}(\tilde{s}^a) \gtrsim s^a \text{ for some } s^a \in \underline{S}(p(s,b))$$

3. Interpretation

The formulation of the previous section makes certain implicit assumptions about what information each participant observes, and about what each participant knows about the underlying structure of the game. We discuss these assumptions in more detail before proceeding.

The Information Assumptions

Our definition of equilibrium assumes that each participant observes certain contemporaneous data on the strategy choices of other participants. Candidates and informed voters observe the candidate positions, s, as well as the poll results, p(s,b). However, uninformed voters do not observe candidate positions. They only observe poll results.

In addition to this contemporaneous information about player strategies which they directly observe, all actors are assumed to have some basic knowledge about the preferences and likely behavior of other participants. This underlying structural information is captured in their knowledge of the function $\hat{p}(s|C)$, which is a reconstruction of the likely voting behavior that will result when the

candidate positions are given by s, and the set of concerned voters is C. Note, however, that the voters need only know $\hat{p}(s|N)$, while the candidates have the more particularistic knowledge of $\hat{p}(s|C)$ for any measurable $C \in \underline{N}$.

The information which is assumed of each participant is summarized in the following table. For the case when preferences are in the class of "intermediate preferences," we show later that the structural information which is generated by $\hat{p}$ is equivalent to the players having certain knowledge about the distribution of voter characteristics in each group N_i. This equivalent structural information is given in the last column of the table (the measure μ_i^C will be defined later).

	Contemporaneous Information	Structural Information	Equivalent Structural Information for Intermediate Preferences
Candidates	s,p(s,b)	$\hat{p}(\cdot\|\cdot)$	$\mu_i, \mu_i^C,\ 1 \leq i \leq t$, $C \in \underline{N}$
Informed Voters*	s,p(s,b)	$\hat{p}(\cdot\|N)$	$\mu_i, 1 \leq i \leq t$
Uninformed Voters	p(s,b)	$\hat{p}(\cdot\|N)$	$\mu_i, 1 \leq i \leq t$

*Actually the only information which is required of the informed voters by the equilibrium definition is s. Since it does not make sense to assume that they have less information available to them than the uninformed voters, we assume that informed voters also have information on p(s,b), $\hat{p}(s|N)$, and μ_i, but chose to ignore it because of the precedence of s.

Equilibrium Conditions

We next justify each of the four Equilibrium conditions. We consider first the voters, then the candidates.

Voters

For a standard Bayesian equilibrium, each voter would try to maximize his expected payoff, given his beliefs about the strategies of the other players. Thus, applying (2.10), voter $\alpha \varepsilon N$ should solve

$$\max_{b_\alpha \varepsilon B_\alpha} E[u_\alpha(s_{k(s,b)})] \tag{3.1}$$

where the expectation is taken with respect to α's belief of (s,b). However, since the ballot aggregation procedure (i.e., majority rule) is positively responsive, given any beliefs $\tilde{s}^\alpha$ of the candidate positions, voter α has a dominant strategy regardless of the value of b. Namely, voting for the candidate with the highest expected utility can never hurt that candidate and might sometimes help. In this analysis, we assume that voters adopt this dominant strategy. Thus, we can dispense with voter beliefs about b, and assume that voter α will choose b_α to

$$\max_{b_\alpha \varepsilon B_\alpha} E[u_\alpha(s_{b_\alpha})] \tag{3.2}$$

where the expectation is now with respect to the voter's belief

$\tilde{s}^\alpha$ of s. This is exactly statement (V1) of Definition 2.1.

Note that by assuming (3.2) directly instead of (3.1), we avoid one difficulty for the infinite voter case: If N is infinite, no one voter has any impact on the outcome, so <u>any</u> strategy is equally good if we assume (3.1). By assuming (3.2) instead, we insure that even in the infinite voter case, voters will vote for the candidate whose policy position gives them the highest utility.

Our definition of equilibrium requires not only that voters maximize expected utility (V1) with respect to their beliefs, but also that their beliefs be consistent with the information they observe (V2). Here, we must differentiate between the informed and uninformed voters.

Informed Voters

Informed voters have perfect information. I.e., given $s \varepsilon \underline{S}$ and $\alpha \varepsilon I$, for $\tilde{s}^\alpha$ to be in equilibrium, V2 must hold:

$$\operatorname{supp}(\tilde{s}^\alpha) = \{s\}. \tag{3.3}$$

So an informed voter's beliefs of candidate positions must coincide with what the candidates actually decide to do.

Uninformed Voters

An uninformed voter also has a belief, $\tilde{s}^\alpha$ of s.

However, the uninformed voter does not observe the candidate positions, rather he only observes aggregate data, namely the poll data p(s,b). Requirement (V2) for the uninformed voter $\alpha \in U$ states that

$$\operatorname{supp}(\tilde{s}^{\alpha}) \gtrsim s^0 \text{ for some } s^0 \in \underline{S}^0(p(s,b)) \tag{3.4}$$

where

$$\underline{S}^0(p(s,b)) = \arg \min_{s^0 \in \underline{S}^0} \| p(s,b) - \hat{p}(s^0|N) \| \tag{3.5}$$

Thus the uninformed voter uses the poll data to inform his belief, $\tilde{s}^{\alpha}$, of candidate positions in such a way as to make his predicted poll outcome correspond as closely as possible with the observed poll outcomes. According to (3.4) and (3.5), the uninformed voter uses his structural information of the rest of the electorate to infer that the poll result which will occur given a choice $s \in \underline{S}$ of candidate strategies is $\hat{p}(s|N)$. I.e., he assumes that all other voters who are voting are perfectly informed and vote rationally. Using (2.15), we can write, for $k \in K \cup \{0\}$, $1 \leq i \leq t$,

$$\hat{p}_{ik}(s|N) = \mu_i(\hat{V}_k(s)). \tag{3.6}$$

There are several things to note about the above expressions for (V2). First, note that the form of the objective function is consistent with the view that each voter believes that as few other voters are making errors as possible. Second, in light of (3.6), the structural information necessary for the uninformed voter to be able to solve (3.5), is simply that he know μ_i for $1 \leq i \leq t$. Next, note that the model used by the uninformed voter for predicting poll outcomes is quite simple. Namely, given any candidate positions $s' \in \underline{S}$, the voter assumes that the supporting coalitions for candidates k and $\bar{k} \in K$ are described by the sets of voters who, under full information, would prefer s_k' or $s_{\bar{k}}'$, respectively.

Candidates

The candidates will choose policy positions to maximize their expectation of winning the election, subject to the beliefs they have about the voter utility functions, and hence about the voting behavior of the electorate. These beliefs, summarized by their belief, C^k, of the "concerned electorate," must be consistent with the information they have about p(s,b). Now for a standard Bayesian equilibrium, candidate $k \in K$ should choose $s_k \in X$ to solve

$$\max_{s_k \in X} E[M_k(s,b)] \tag{3.7}$$

where the expectation is taken over his belief of b and of $s_{\bar{k}}$ for $\bar{k} \in K - \{k\}$. However, here we assume that candidate k knows $s_{\bar{k}}$, and we require that for any $s \in \underline{S}$, and belief $C^k \subseteq N$, candidate k assumes that b is generated according to $\hat{b}(s|C^k)$. It follows that M_k can be written as a function only of $s_k, s_{\bar{k}}$ and C^k.

4. Results: Voter Equilibria

We now consider the existence and properties of equilibria in the model developed in the previous two sections. Specifically, we are concerned with conditions under which the equilibrium to the model corresponds to the behavior which would occur under full information. In this situation, we will say that the equilibrium extracts all available information. We consider first only voter equilibria, given fixed positions of the candidates:

<u>Definition 4.1</u> A voter equilibrium $(b,\tilde{s}) \varepsilon \underline{B} \times \tilde{\underline{S}}$, conditional on $s \varepsilon \underline{S}$ is said to <u>extract all available information</u> iff, for all $\alpha \varepsilon N$, $k \varepsilon K$,

$$\alpha \varepsilon \hat{V}_k(s) \Rightarrow b_\alpha = k \tag{4.1}$$

Thus, a voter equilibrium extracts all available information iff all voters, informed and uninformed alike, vote for the candidate they would prefer if they had full information.

We start with a couple of simple Lemmas characterizing individual voting behavior in any voter equilibrium (independent of whether it extracts information).

<u>Lemma 4.1</u> Given fixed candidate strategies $s^* \varepsilon \underline{S}$, with $s^*_k \neq s^*_{\underline{k}}$, if $(b,\tilde{s}) \varepsilon \underline{B} \times \tilde{\underline{S}}$ is a voter equilibrium, conditional on s^*, and $p^* = p(s^*,b)$, Then for $k \varepsilon K$,

a) for all $\alpha \varepsilon I$

$$\alpha \varepsilon \hat{V}_k(s^*) \Rightarrow b_\alpha = k$$

b) for all $\alpha \varepsilon U$, $\exists\, s^\alpha \varepsilon \underline{S}^0(p^*)$ such that

$$\alpha \varepsilon \hat{V}_k(s^\alpha) \Rightarrow b_\alpha = k.$$

<u>Proof</u> For all voters, we have, from (V1) that

$$b_\alpha \varepsilon \arg \max_{b_\alpha \varepsilon B_\alpha} E[u_\alpha(s_{b_\alpha})] \tag{4.2}$$

where the expectation is with respect to $\tilde{s}^\alpha$. But by (V2), for $\alpha \varepsilon I$, supp $(\tilde{s}^\alpha) = \{s^*\}$. Hence, we must have

$$b_\alpha \varepsilon \arg \max_{b_\alpha \varepsilon B_\alpha} u_\alpha(s^*_{b_\alpha}). \tag{4.3}$$

So, $\alpha \varepsilon \hat{V}_k(s^*) \Rightarrow u_\alpha(s^*_k) > u_\alpha(s^*_{\underline{k}}) \Rightarrow b_\alpha = k$, which proves part (a).

For $\alpha \varepsilon U$, on the other hand, we have that $\text{supp}(\tilde{s}^\alpha) \underset{\sim}{\sim} s^\alpha$ for some $s^\alpha \varepsilon \underline{S}^0(p^*)$. But then, for all $s \varepsilon$ supp $(\tilde{s}^\alpha)$, we have $s \underset{\sim}{\sim} s^\alpha$, or $\hat{V}_k(s) = \hat{V}_k(s^\alpha)$. Hence, $u_\alpha(s^\alpha_k) > u_\alpha(s^\alpha_{\underline{k}}) \Leftrightarrow \alpha \varepsilon \hat{V}_k(s^\alpha)$ $\Leftrightarrow \alpha \varepsilon \hat{V}_k(s) \Leftrightarrow u_\alpha(s_k) > u_\alpha(s_{\underline{k}})$ for all $s \varepsilon \text{supp}(\tilde{s}^\alpha)$. So

$$\arg \max_{b_\alpha \varepsilon B_\alpha} E[u_\alpha(s_{b_\alpha})] = \arg \max_{b_\alpha \varepsilon B_\alpha} [u_\alpha(s^\alpha_{b_\alpha})], \tag{4.4}$$

and hence $\alpha \in \hat{V}_k(s^\alpha) \Rightarrow u_\alpha(s^\alpha_k) > u_\alpha(s^\alpha_{\underline{k}}) \Rightarrow b_\alpha = k.$

Q.E.D.

So, in short, in equilibrium, the informed voters will always vote correctly, while the uninformed voters will each vote according to an idiosyncratic, normalized, nonstochastic representative of their private beliefs. Each of these idiosyncratic private beliefs, of course, must be as consistent as possible with the observed poll. When $\underline{S}^0(p^*)$ is single valued, then it follows that all uninformed voters will vote according to the <u>same</u> (although possibly incorrect) belief s^α of s^*.

Next, for any measureable $E \subseteq N$, we define the conditional probability measure μ_i^E by

$$\mu_i^E(C) = \frac{\mu_i(E \cap C)}{\mu_i(E)} = \frac{\mu(N_i \cap E \cap C)}{\mu(N_i \cap E)} \tag{4.5}$$

Setting $t_i^I = \mu_i(I)$ and $t_i^U = \mu_i(U)$, it follows that (since I and U partition N) for all $1 \leq i \leq t$

$$\mu_i = t_i^I \mu_i^I + t_i^U \mu_i^U \tag{4.6}$$

where $t_i^I + t_i^U = 1$. (In other words, for all measureable $C \subseteq X$, $\mu_i(C) = t_i^I \mu_i^I(C) + t_i^U \mu_i^U(C)$. For any $s \in \underline{S}$, $1 \leq i \leq t$, and $k \in K^0$, we define

$$\hat{q}^E_{ik}(s) = \mu_i^E(\hat{V}_k(s)) \tag{4.7}$$

As usual, we write $\hat{q}^E_i(s) = (\hat{q}^E_{i0}(s), \hat{q}^E_{i1}(s), \hat{q}^E_{i2}(s))$, and $\hat{q}^E(s) = (\hat{q}^E_1(s), \ldots, \hat{q}^E_t(s))$. We also write $\hat{q}_{ik}(s) = \hat{q}^N_{ik}(s)$, $\hat{q}_i(s) = \hat{q}^N_i(s)$ and

$$\hat{q}(s) = \hat{q}^N(s) \tag{4.8}$$

Clearly, with the above notation, we have

$$\hat{q}(s) = \hat{p}(s|N) \tag{4.9}$$

Given any fixed candidate position $s^* \in \underline{S}$, we can define a correspondence, $T: \Delta^t \rightarrow\!\!\!\rightarrow \Delta^t$ by setting

$$T(p) = Co\{p' \in \Delta^t |\text{ for some } s^\alpha \in \underline{S}^0(p),\ p_i' = t_i^I \hat{q}_i^I(s^*) + t_i^U \hat{q}_i^U(s^\alpha) \text{ for all } 1 \leq i \leq t\}. \tag{4.10}$$

So T(p) is the set of polls that could result if all voters vote optimally according to their beliefs, generated by $\tilde{s}$, and their beliefs are consistent with the information p. (see Lemma 4.1).

<u>Lemma</u> <u>4.2</u> Given fixed candidate strategies $s^* \in \underline{S}$, with $s^*_k \neq s^*_{\underline{k}}$, if $(b, \tilde{s}) \in \underline{B} \times \tilde{\underline{S}}$ is a voter equilibrium, conditional on s^*, and $p^* = p(s^*, b)$, then we must have $p^* = T(p^*)$.

<u>Proof</u>: Follows immediately from Lemma 4.1 together with the definition (4.10).

Q.E.D.

I.e., for any equilibrium $(b,\tilde{s})$, $p = p(s^*,b)$ must be a fixed point for the correspondence, T. The correspondence T, can also be thought of as a dynamic describing the convergence of the model to equilbrium. This will be elaborated on later.

Next, we define a notion of consistency of poll outcomes:

<u>Definition 4.2</u> A poll $p \in \Delta^t$ is said to be <u>consistent for C</u> $\subseteq N$ if $\exists$ $s \in S$ such that

$$p = \hat{q}^C(s) \tag{4.11}$$

If p is consistent for N, we say it is <u>consistent</u>.

Thus, the poll is consistent for C if the poll results restricted to C could have been generated by some pair of candidate positions with all voters in C voting as if they had complete information. For all of our results, we need an assumption on consistent polls which requires that each consistent poll be generated by a unique $s \in \underline{S}^0$.

<u>Assumption 4.1</u> If $p \in \Delta^t$ is consistent for C, where C is either N, I, or U, and $s, s' \in \underline{S}$ satisfy $p = \hat{q}^C(s) = \hat{q}^C(s')$, then $s \approx s'$.

Our next Lemma proves the existence of equilibria that extract all information and shows that if the poll resulting from a voter equilibrium is consistent, then the equilibrium must extract all available information. Thus, the only equilibria of the model occur either when all voters vote correctly or when the resulting poll is inconsistent.

<u>Lemma 4.3</u> Given fixed candidate strategies $s^* \in \underline{S}$, with $s^*_k \neq s^*_{\bar{k}}$, there exists a voter equilibrium that extracts all information. Further, under Assumption 4.1, any voter equilibrium $(b,\tilde{s})$ based on s^* for which $p(b,s^*)$ is consistent extracts all information.

<u>Proof</u>: For existence, define $(b,\tilde{s})$ by, for $\alpha \in N$, $k \in K \cup \{0\}$,

$$b_\alpha = k \text{ if } \alpha \in \hat{V}_k(s) \tag{4.12}$$

$$\operatorname{supp}(\tilde{s}^\alpha) = \{s^*\} \tag{4.13}$$

From (2.4), $p_{ik}(s^*,b) = \mu_i(\hat{V}_k(s^*)) = \hat{q}_{ik}(s)$, so $p(s^*,b)$ is consistent. But then from (4.9), it follows that $\hat{p}_i(s^*|N) = p_i(s^*,b)$ for all i. So pick $s^0 \in \underline{S}^0$ with $s^0 \approx s^*$. Then $s^0 \in \underline{S}^0(p(s^*,b))$. Hence (V2) is satisfied for all $\alpha \in N$ (since it is satisfied trivally for $\alpha \in I$,) so $(b,\tilde{s})$ is a voter equilibrium. Clearly, by (4.12), $(b,\tilde{s})$ extracts all information.

Now let $(b,\tilde{s})$ be a voter equilibrium based on s^*, and write $p = p(s^*,b)$. Then if p is consistent, $\exists$ $s' \in \underline{S}^0$ with $p = \hat{q}(s')$. So clearly $s' \varepsilon \underline{S}^0(p)$. Further, by assumption 4.1, it follows that any other element of $\underline{S}^0(p)$ must have $s \approx s'$. Hence $\underline{S}^0(p)$ is single valued, and we let $s' \varepsilon \underline{S}^0(p)$. Then we have $p = \hat{q}(s')$. Further by Lemma 4.2, we must have $p \varepsilon T(p)$. But since $\underline{S}^0(p)$ is single valued, this means that

$$\begin{aligned} p_i = p_i(s^*,b) &= t_i^I\hat{q}_i^I(s^*) + t_i^U\hat{q}_i^U(s') \\ &= [t_i^I\hat{q}_i^I(s^*) - t_i^I\hat{q}_i^I(s')] \\ &+ [t_i^I q_i^I(s') + t_i^U\hat{q}_i^U(s')] \\ &= t_i^I[\hat{q}_i^I(s^*) - \hat{q}_i^I(s')] + \hat{q}_i(s') \end{aligned} \tag{4.14}$$

But, since p is consistent, it follows that $p = \hat{q}(s')$, or equivalently, for all $1 \leq i \leq t$, that $p_i = p_i(s^*,b) = \hat{q}_i(s')$. Therefore, equation (4.14) implies that for all $1 \leq i \leq t$, $q_i^I(s^*) = q_i^I(s')$. Now, by Assumption 4.1, we have $s^* \approx s'$, so $\hat{V}_k(s^*) = \hat{V}_k(s')$ for all $k \varepsilon K \cup \{0\}$. Hence, for $\alpha \varepsilon U$, we have $b_\alpha = k$ if $\alpha \varepsilon \hat{V}_k(s^*)$, and the result is proven.

Q.E.D.

The above lemma does not rule out the possibility of equilibria which do not extract all information, since it is quite possible for inconsistent polls to be in equilibrium. We introduce a further assumption on the distribution of the voters.

<u>Assumption 4.2</u> (Identical Distributions) For all $s \varepsilon \underline{S}$,

$$\hat{q}^I(s) = \hat{q}^U(s) = \hat{q}(s)$$

We prove that under Assumption A1, if there are more informed than uninformed voters, there is always an equilibrium.

<u>Theorem 1</u> If Assumptions 4.1 and 4.2 are met and $\mu_i(U) < 1/2$ for all i, then for any fixed candidate strategies $s^* \varepsilon \underline{S}$ with $s_k \neq s_{\bar{k}}$, then if $(b,\tilde{s})$ is a voter equilibrium, it extracts all information.

<u>Proof</u>: Let $(b,\tilde{s})$ be a voter equilibrium, with corresponding poll $p(s^*,b) = p = (p_1,\ldots,p_t)$. Since $(b,\tilde{s})$ is an equilibrium we must have $p \varepsilon T(p)$. So we can write, for $1 \leq i \leq t$

$$p_i = t_i^I\hat{q}_i(s^*) + t_i^U\sum_{j=1}^{t^*} w_j\hat{q}_i(s^j)$$

for some $w_1,\ldots,w_{t^*} \varepsilon \mathbb{R}^t$ with $\sum_{j=1}^{t^*} w_j = 1$ and $s^j \varepsilon \underline{S}^0(p)$. We let $t^* = \max\limits_{1 \leq i \leq t} t_i^U$. Then, if $||p - \hat{q}(s^*)||_r \neq 0$, we get

$$\begin{aligned} ||p - \hat{q}(s^*)||_r &= \sum_{i=1}^{t} r_i|p_i - \hat{q}_i(s^*)|. \\ &= \sum_{i=1}^{t} r_i|t_i^I\hat{q}_i(s^*) + t_i^U[\sum_{j=1}^{t} w_j\hat{q}_i(s^j)] - \hat{q}_i(s^*)| \\ &= \sum_{i=1}^{t} r_i t_i^U|[\sum_{j=1}^{t} w_j\hat{q}_i(s^j)] - \hat{q}_i(s^*)| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{t} r_i t_i^U \left| \left[\sum_{j=1}^{t} w_j (\hat{q}_i(s^j) - p_i) \right] + (p_i - \hat{q}_i(s^*)) \right| \\
&\le \sum_{i=1}^{t} r_i t_i^U \left[\sum_{j=1}^{t} w_j |\hat{q}_i(s^j) - p_i| + |p_i - \hat{q}_i(s^*)| \right] \\
&= \left[\sum_{j=1}^{t} w_j \sum_{i=1}^{t} r_i t_i^U |\hat{q}_i(s^j) - p_i| \right] + \left[\sum_{i=1}^{t} r_i t_i^U |p_i - \hat{q}_i(s^*)| \right] \\
&\le \left[\sum_{j=1}^{t} w_j t^* \sum_{i=1}^{t} r_i |\hat{q}_i(s^j) - p_i| \right] + \left[t^* \sum_{i=1}^{t} r_i |p_i - \hat{q}_i(s^*)| \right] \\
&= \left[\sum_{j=1}^{t} w_j t^* ||\hat{q}(s^j) - p|| \right] + \left[t^* ||p - \hat{q}(s^*)|| \right] \\
&\le \left[t^* \sum_{j=1}^{t} w_j ||\hat{q}(s^*) - p|| \right] + \left[t^* ||p - \hat{q}(s^*)|| \right] \\
&= 2t^* ||p - \hat{q}(s^*)|| < ||p - \hat{q}(s^*)||
\end{aligned}
$$

Hence, any fixed point p to $T(p)$ must have $||p - \hat{q}(s^*)|| = 0$, or $p = \hat{q}(s^*)$. But, then p is consistent, so it follows from Lemma 4.3 that the voter equilibrium $(b,\tilde{s})$ extracts all information.

Q.E.D.

5. Examples

We consider now a general class of preferences to which the above development applies, namely the class of so called "intermediate preferences." See e.g., Grandmont [ ] and Kramer [ ]. Let K be a positive integer and let f_0: $X \to \mathbb{R}$ and $f = (f_1,\dots,f_L) : X \to \mathbb{R}^L$ be continuous functions on X. Then for any $\beta = (\beta_1,\dots,\beta_L) \in \mathbb{R}^L$, define v_β: $X \to \mathbb{R}$ by $v_\beta = f_0 + \beta' f$. I.e., for any $x \in X$,

$$v_\beta(x) = f_0(x) + \beta' f(x) = f_0(x) + \sum_{i=1}^{L} \beta_i f_i(x) \qquad (5.1)$$

Then the class of intermediate preferences based on f_0, f, written $U(f_0, f)$ is defined as

$$U(f_0, f) = \{v_\beta = f_0 + \beta' f \mid \beta \in \mathbb{R}^L\}. \qquad (5.2)$$

It is easily verified that many standard classes of preferences are representable as classes of intermediate preferences. For example, the space of Cobb Douglas utility functions is generated by setting $f_0(x) \equiv 0$ and $f_i(x) = \ln x_i$. The space of Euclidian, or type I, preferences is generated by setting $L = m$, $f_0(x) = -\frac{1}{2} x'x$, and $f_i(x) = x_i$. The space of quadratic utility functions where each voter has an idiosyncratic salience matrix and ideal point is generated by setting $L = (m + 1)m$, $f_0(x) \equiv 0$, $f_i(x) = x_i$ for $1 \le i \le m$, and $f_{m^j+i}(x) = x_i x_j$ for $1 \le i \le m$, $1 \le j \le m$.

For our purposes, we need to identify voters not only by their utility functions, but also by their information class and voter type. So we let $N = \{0,1\}^{t+1} \times \mathbb{R}^L$, where L is a positive integer. So each voter $\alpha \in N$ is described by a vector of attributes $\alpha = (\gamma,\beta)$, where $\gamma \in \{0,1\}^{t+1}$ and $\beta \in \mathbb{R}^K$. The vector $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_t)$ describes the voter type (i.e., his information class and the subgroups to which he belongs). We define $I = \{\alpha = (\gamma,\beta) \in N | \gamma_0 = 1\}$, $U = \{\alpha \in N | \gamma_0 = 0\}$ and $N_i = \{\alpha \in N | \gamma_i = 1\}$. The vector $\beta = (\beta_1,\dots,\beta_L)$ gives the parameters of the voter utility function. We let f_0: $X \to \mathbb{R}$ and $f : X \to \mathbb{R}^L$ be defined as above. Then for each $\alpha = (\gamma,\beta) \in N$, we assume $u_\alpha = v_\beta = f_0 + \beta' f$. I.e., for all $x \in X$,

$$u_\alpha(x) = f_0(x) + \beta' f(x) = f_0(x) + \sum_{i=1}^{L} \beta_i f_i(x) \tag{5.3}$$

Clearly, $u_\alpha \in U(f_0, f)$ for all $\alpha \in N$.

Now, for any $s \in S$, we have that, for $k \in K$,

$$\begin{aligned}\hat{V}_k(s) &= \{\alpha \in N | u_\alpha(s_k) > u_\alpha(s_{\underline{k}})\} = \{\alpha \in N | f_0(s_k) + \beta' f(s_k) > f_0(s_{\underline{k}}) + \beta' f(s_{\underline{k}})\} \\ &= \{\alpha \in N | \beta'(f(s_k) - f(s_{\underline{k}})) > f_0(s_{\underline{k}}) - f_0(s_k)\} \\ &= \{\alpha \in N | \beta' h_k(s) > c_k(s)\}\end{aligned} \tag{5.4}$$

where

$$\begin{aligned} h_k(s) &= f(s_k) - f(s_{\underline{k}}) \in \mathbb{R}^L \\ c_k(s) &= f_0(s_{\underline{k}}) - f_0(s_k) \in \mathbb{R} \end{aligned} \tag{5.5}$$

So $\hat{V}_k(s)$ is simply the set of voters in N who's parameters β lie in a half space in $\mathbb{R}^L$ defined by the vector $h_k(s)$ and $c_k(s)$. A poll $p \in \Delta^t$ is consistent for $E \subseteq N$ iff $\exists$ $s \in \underline{S}$ such that for all $1 \leq i \leq t$ and $k \in K \cup \{0\}$,

$$p_{ik} = \hat{q}^E_{ik}(s) = \mu^E_i(\hat{V}_k(s)) \tag{5.6}$$

Now, for Assumption 4.1 to be met, for any $s, s' \in \underline{S}$ with $\hat{q}^E(s) = \hat{q}^E(s')$, we must have $s \sim s'$. Thus, we must have

$$\hat{q}^E(s) = \hat{q}^E(s') \Rightarrow \hat{V}_k(s) = V_k(s') \tag{5.7}$$

for all $k \in K \cup \{0\}$, $E \in \{I, U, N\}$. I.e., we must have, for all $s, s' \in S$, $k \in K \cup \{0\}$, $E \in \{I, U, N\}$,

$$\mu^E_i(\hat{V}_k(s)) = \mu^E_i(\hat{V}_k(s)) \text{ for } 1 \leq i \leq t \Rightarrow \hat{V}_k(s) = \hat{V}_k(s') \tag{5.8}$$

We conjecture that a sufficient condition for (5.8) to be generically satisfied (with respect to an appropriate topology on the set of possible measures μ on N) is that $t \geq L + 1$. I.e. there must be more sources of information than there are free parameters in the class of utility functions. The intuition behind this conjecture is illustrated in Figures 1 and 2 for the case when $L = 2$. Here the marginal density functions of the μ^E_i over the parameter space are assumed to be continuous density functions with support equal to $\mathbb{R}^2$. The figures give contours of the marginal density functions for the μ^E_i over the parameter space, $\mathbb{R}^2$. If $t = L$, then we note that any consistent poll can be generated by two different voting coalitions. However, if another group is added, as in figure 3, then as long as the density function for the third group is not "colinear" with that of the first two, then the additional information provided by the poll in the third group identifies the correct voting coalition, so that Assumption 1 is satisfied. A similar argument seems to hold for larger dimensions as well. With an appropriate topology on the set μ of allowable measures on N, any measure will be arbitrarily close to one for which μ^E_i is continuous with support equal to $\mathbb{R}^L$, and for which the μ^E_i are not "colinear." The above argument is obviously very heuristic, and we leave it for future research to study the validity

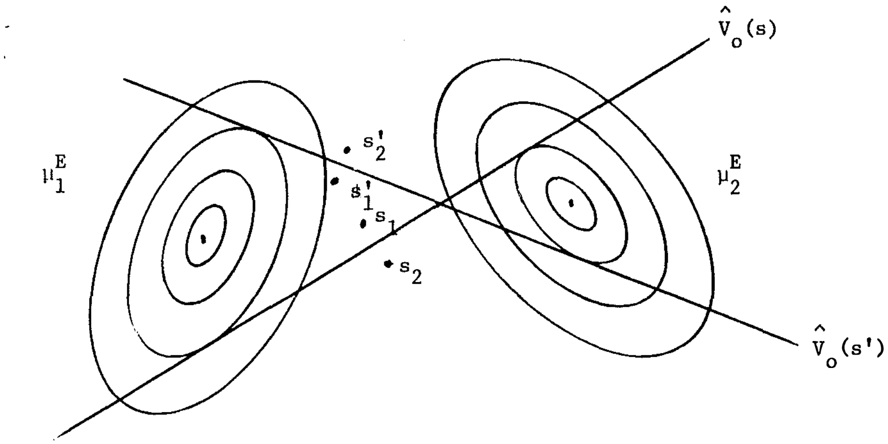

FIGURE 1

Two consistent candidate positions for the poll p=((0,.8,.2),(0,.3,.7))

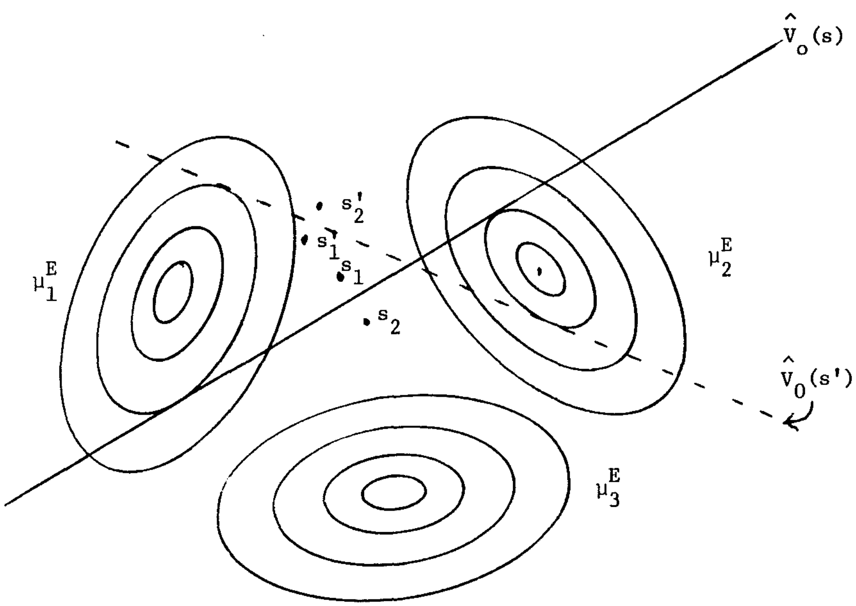

FIGURE 2

With three groups, candidate strategy pair s' is no longer consistent for the poll p=((0,.8,.2),(0,.3,.7),(0,.01,.99))

of this conjecture in more detail. In any case it should be clear that with t sufficiently large, and the μ_i^E continuous with support equal to $\mathbb{R}^L$ and sufficiently dispersed, Assumption 1 will be met.

To illustrate some of the ideas which have been developed above, we present an example. We let $X = \mathbb{R}^2$, and assume preferences are Euclidian.

So for all $\alpha = (\gamma,\beta) \in N$, $u_\alpha \in U(f_0,f)$, where $f_0(x) = -\frac{1}{2}x'x$ and $f(x) = x$. So $u_\alpha(x) = \beta'x - \frac{1}{2}x'x = (\beta - \frac{x}{2})'x$ for some $\beta \in \mathbb{R}^2$. (The parameter β represents α's ideal point). Using (5.4) and (5.5), we can write

$$\hat{V}_k(s) = \{\alpha \in N \mid \beta' h_k(s) > c_k(s)\} \tag{5.9}$$

where

$$\begin{aligned} h_k(s) &= f(s_k) - f(s_{-k}) = s_k - s_{-k} \\ c_k(s) &= f_0(s_{-k}) - f_0(s_k) = (s_k - s_{-k})'\frac{(s_k + s_{-k})}{2} \end{aligned} \tag{5.10}$$

So

$\alpha \in \hat{V}_k(s) \iff \beta \in G_k(s)$, where

$$G_k(s) = \{\beta \in \mathbb{R}^L \mid \beta'(s_k - s_{-k}) > \frac{(s_k + s_{-k})'(s_k - s_{-k})}{2}\} \tag{5.11}$$

Hence $\hat{V}_k(s)$ consists of exactly those voters whose parameter β (which corresponds to the voter's ideal point) is closer to s_k than to s_{-k}.

We assume there are 3 subpopulations, N_1, N_2, and N_3, which partition N. We assume that $\mu(N_i) = \mu(N)/3$ for all i, and for E = I or E = U, we assume that, for any borel set $C \subseteq X$,

$$\mu_i^E(\{\alpha = (\gamma,\beta) \in N \mid \beta \in C\}) = \int_C \frac{1}{\sqrt{2\pi}} e^{-1/2(x - x_i^E)'(x - x_i^E)} dx \tag{5.12}$$

where

$$\begin{array}{ll} x_1^I = (-2.0, -1.0) & x_1^U = (-2.0, 0.0) \\ x_2^I = (2.0, 1.0) & x_2^U = (2.0, 2.0) \\ x_3^I = (1.0, -2.0) & x_3^U = (3.0, 2.0) \end{array} \tag{5.13}$$

We assume that for each i, $\mu_i(U) = \mu_i(I) = 1/2$, so that for each i,

$$\mu_i(C) = 1/2\ \mu_i^I(C) + 1/2\mu_i^U(C). \tag{5.14}$$

Now, we assume that the candidates adopt the positions $s^* = (\ , \ )$, and we consider the initial voter strategy $(b,\tilde{s})$ which is defined as follows: For $\alpha \in I$, voters vote correctly. I.e.,

$$\alpha \in \hat{V}_k(s^*) \Rightarrow b_\alpha = k \tag{5.15}$$

or equivalently, $\beta \in G_k(s^*) \Rightarrow b_\alpha = k$, whereas for $\alpha \in U$,

$$\begin{array}{l} \alpha \in N_1 \Rightarrow b_\alpha = 2 \\ \alpha \in N_2 \Rightarrow b_\alpha = 1 \\ \alpha \in N_3 \Rightarrow b_\alpha = 1 \end{array} \tag{5.16}$$

(Thus, the uninformed voters vote in such a way as to create a worst case—i.e. the uninformed voters vote contrary to how they would tend to vote under full information). The resulting poll, p', is given in Table 1.

We now consider a sequence of polls $\{p^t\}_{t=1}^{\infty}$ generated by choosing $p^{t+1} \in T(p^t)$ for each $t \geq 1$. This is the sequence of polls which would result if voters, in period t, adjusted their beliefs to be consistent with p^t (say to $s^t \in \underline{S}^0(p^t)$), and then vote optimally according to this belief. I.e., for $\alpha \in I$, in period t,

$$\alpha \in \hat{V}_k(s^*) \Rightarrow b_\alpha^t = k \tag{5.17}$$

and for $\alpha \in U$, in period t,

$$\alpha \in \hat{V}_k(s^{t-1}) \Rightarrow b_\alpha^t = k \tag{5.18}$$

Equivalently, in light of (5.9)-(5.11), we can write, for $\alpha \in I$, $\beta \in G_k(s^*) \Rightarrow b_\alpha = k$, whereas for $\alpha \in U$, $\beta \in G_k(s^{t-1}) \Rightarrow b_\alpha = k$. So for all i,

$$p_i^t = 1/2\ \hat{q}_i^U(s^{t-1}) + 1/2\hat{q}_i^I(s^*). \tag{5.19}$$

(Compare to equation (4.10)).

Table 1 gives, for each period, $\hat{q}^U(s^{t-1})$, $\hat{q}^I(s^*)$, p^t, and the best fitting poll to p^t, namely $\hat{q}(s^t)$. Figure 1 graphs the best fitting hyperplanes $G(s^t)$ for $1 \leq t \leq 8$. We see that for this example, that the p^t converge to $\hat{q}(s^*)$ and that $s(p^t)$ converges to s^*.

Period t	Group i	$\hat{q}^U_{ik}(s(p^{t-1}))$ candidate k 1	2	$\hat{q}^I_{ik}(s^*)$ candidate k 1	2	p^t_{ik} candidate k 1	2	$\hat{q}_{ik}(s(p^t))$ candidate k 1	2	$\|\|p^t - \hat{q}(s(p^t))\|\|$
1	1	.000	1.000	.944	.056	.472	.528	.468	.532	
	2	1.000	.000	.196	.804	.598	.402	.601	.399	
	3	1.000	.000	.500	.500	.750	.250	.961	.039	.4352
2	1	.366	.634	.944	.056	.655	.345	.657	.343	
	2	.503	.497	.196	.804	.349	.651	.346	.654	
	3	.980	.020	.500	.500	.740	.260	.941	.059	.4128
3	1	.549	.451	.944	.056	.746	.254	.750	.250	
	2	.237	.763	.196	.804	.216	.784	.103	.897	
	3	.950	.050	.500	.500	.725	.275	.720	.280	.2414
4	1	.666	.334	.944	.056	.805	.195	.913	.087	
	2	.063	.937	.196	.804	.129	.871	.130	.870	
	3	.644	.356	.500	.500	.572	.428	.578	.422	.2284
5	1	.887	.113	.944	.056	.916	.084	.935	.065	
	2	.094	.906	.196	.804	.145	.855	.145	.855	
	3	.403	.597	.500	.500	.451	.549	.453	.547	.0430
6	1	.921	.079	.944	.056	.933	.067	.938	.062	
	2	.118	.882	.196	.804	.157	.843	.156	.844	
	3	.245	.755	.500	.500	.373	.627	.376	.624	.0180
7	1	.929	.071	.944	.056	.937	.063	.938	.062	
	2	.136	.864	.196	.804	.166	.834	.166	.834	
	3	.171	.829	.500	.500	.336	.664	.332	.668	.0088
8	1	.931	.069	.944	.056	.938	.062	.938	.062	
	2	.151	.849	.196	.804	.173	.827	.172	.828	
	3	.137	.863	.500	.500	.318	.682	.321	.679	.0096
∞	1	.936	.064	.944	.056	.940	.060	.940	.060	
	2	.174	.826	.196	.804	.185	.815	.185	.815	
	3	.121	.879	.500	.500	.311	.689	.311	.689	.0000

Table 1

Convergence of p^t to $\hat{q}(s^*)$.

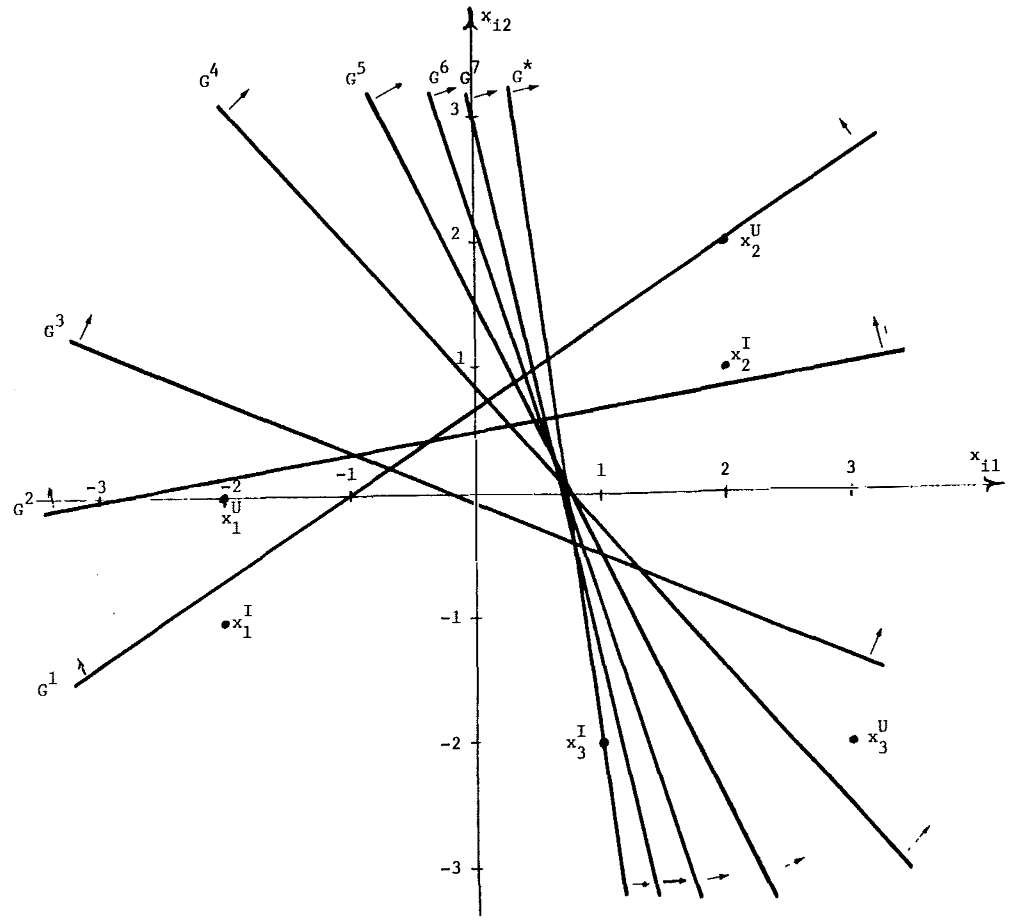

Figure 1

Convergence of $G(s(p^t))$ to $G(s^*)$.

*Here we use the notation $G^t = G(s(p^t))$. The half space $G_2(s(p^t))$ is on the side of G^t indicated by the arrows. Also $G^* = G(s^*)$.

This particular example suggests that the theorems of this paper can probably be strengthened. Here we note that the dynamic process described by $p^{t+1} \varepsilon T(p^t)$ converges to a voter equilibrium which extracts all available information even though the distributions of the uninformed and informed voters within each subgroup are different.

6. Results: Full Equilibrium

This section investigates the properties of full equilibria to the model of section 2, i.e., of equilibria $((s,b),(C,\tilde{s}))$ satisfying Definition 2.1. For a full equilibrium, then, both candidates and voters need to be in equilibrium. Again, we are concerned with conditions under which such equilibria correspond to what would happen under the case of full information.

Definition 6.1 The strategy pair $s \varepsilon \underline{S}$ is a *full information candidate equilibrium* if, whenever $k \varepsilon K$ and $s' \varepsilon \underline{S}$ satisfy $s_{\bar{k}} = s'_{\bar{k}}$,

$$M_k(s',\hat{b}(s'|N)) \leq M_k(s,\hat{b}(s|N)) \tag{6.1}$$

Note that equivalently, because of the symmetry of the game, we can write the equation of Definition 6.1 as

$$M_k(s',\hat{b}(s'|N)) \leq 0 \tag{6.2}$$

or

$$\mu(\hat{V}_k(s')) \leq \mu(\hat{V}_{\bar{k}}(s')) \tag{6.3}$$

for all $k \varepsilon K$, $s' \varepsilon S$ with $s_{\bar{k}} = s'_{\bar{k}}$.

Throughout this section, we also make another aasumption.

Assumption 6.1 For all $s \varepsilon S$ with $s_1 \neq s_2$, $\mu(\hat{V}_0(s)) = 0$.

Lemma 6.1 We assume $\bigcup_{i=1}^{t} N_i = N$ and that Assumption 6.1 holds. If, for all $s^* \varepsilon \underline{S}$ with $s^*_k \neq s^*_{\bar{k}}$, any voter equilibrium based on s^* extracts all information, then if $((s,b),(C,\tilde{s}))$ is an equilibrium, then either s is a full information candidate equilibrium or $s_k = s_{\bar{k}}$.

Proof Suppose $((s,b),(C,\tilde{s}))$ is an equilibrium with $s_k \neq s_{\bar{k}}$. Then, for the voters to satisfy (V1) and (V2) of the equilbrium definition, $(b,\tilde{s})$ must be a voter equilibrium with respect to s. But then, by assumption, $(b,\tilde{s})$ extracts all information, so $p(s,b) = \hat{q}(s)$. But now, since candidate beliefs satisfy C2,

$$\begin{aligned} C^k &\varepsilon \arg \min_{C \subseteq N} [||p(s,b) - \hat{p}(s|C)||_r] \\ &= \arg \min_{C \subseteq N} [||\hat{q}(s) - \hat{p}(s|C)||_r] \end{aligned} \tag{6.4}$$

Now for all $1 \leq i \leq t$, $k \varepsilon K \cup \{0\}$,

$$\hat{q}_{ik}(s) = \mu_i(\hat{V}_k(s)) \tag{6.5}$$

and

$$\hat{p}_{ik}(s) = \begin{cases} \mu_i(\hat{V}_k(s) \cap C) & \text{for } k \in K \\ \mu_i(N - C) & \text{for } k = 0 \end{cases} \tag{6.6}$$

So clearly, if $C = N$, then $\hat{p}_{ik}(s|C) = \hat{q}_{ik}(s)$, or $\hat{q}(s) = \hat{p}(s|C)$. Hence, any C^k solving (6.4) must satisfy $\hat{q}(s) = \hat{p}(s|C^k)$. This means, in particular, that $\hat{q}_{i0}(s) = \hat{p}_{i0}(s|C^k)$, or

$$|\mu_i(\hat{V}_0(s)) - \mu_i(N - C^k)| = 0 \tag{6.7}$$

But, by Assumption 6.1, $\mu(\hat{V}_0(s)) = 0$ for all s, so $\mu_i(\hat{V}_0(s)) = 0$. Also $\mu_i(N - C^k) = \mu(N_i - C^k)$. So the above implies $\mu(N_i - C^k) = 0$ for all i. It follows that

$$\begin{aligned} 0 &= \sum_{i=1}^{t} \mu(N_i - C^k) \geq \mu(\bigcup_{i=1}^{t} (N_i - C^k)) \\ &= \mu((\bigcup_{i=1}^{t} N_i) - C^k) = \mu(N - C^k) \end{aligned} \tag{6.8}$$

or,

$$\mu(C^k) = \mu(N) \tag{6.9}$$

Now assume that s is not a full information candidate equilibrium. Then there is a $k \in K$ and an $s' \in S$ with $s'_{\bar{k}} = s_{\bar{k}}$ such that

$$M_k(s', \hat{b}(s'|N)) > M_k(s, \hat{b}(s|N)) \tag{6.10}$$

But now, using the fact that $\mu(C^k) = \mu(N)$, it is easy to show that $M_k(s', \hat{b}(s'|N)) = M_k(s', \hat{b}(s'|C^k))$ for all s'. Hence, we have

$$M_k(s', \hat{b}/(s'|C^k)) > M_k(s, \hat{b}(s|C^k)). \tag{6.11}$$

It follows that s does not satisfy (C1) of the equilibrium definition, hence is not an equilibrium.

Q.E.D.

Unfortunately, the above Lemma leaves open the possibility that we could have equilibria where both candidates adopt the same policy positions, but where these policy positions are not at a full information candidate equilibrium. In this case, of course, the informed voters would abstain, but the uninformed voters might still believe there is information in the polls, and vote in a way such that they cue off of the information provided to the pollster by other uninformed voters.

Any belief by the candidates about who the concerned voters are would be consistent with observed data. So if the candidates were both positioned at a full information equilibrium of the voters who they believed to be concerned voters, then this could be an equilibrium as defined in Definition 2.1. However, equilibria of this sort are quite unstable, because if either candidate makes a slight miscalculation or an error in his choice of strategy, then the beliefs of both candidates will be inconsistent with the poll data which results from all voters adopting equilibrium strategies. To banish the above type of equilibria, we introduce a somewhat stronger notion of equilibrium. This stronger version requires that candidate beliefs must be consistent not only with the information that is generated when the candidates adopt their equilibrium strategies, but also with the information that is generated when they make small errors.

<u>Definition</u> An equilibrium $((s,b),(C,\tilde{s}))$ is said to be <u>informationally stable</u> if there is a neighborhood $N(s)$ of s such that whenever $s^* \varepsilon N(s)$, and $(b^*,\tilde{s}^*)$ is a voter equilibrium based on s^*, then the candidate beliefs C are consistent with the data $p(s^*,b^*)$.

<u>Theorem 6.1</u> Under the conditions of Lemma 6.1, if there is a full information candidate equilibrium, there exists an informationally stable equilibrium. Further if $((s,b),(C,\tilde{s}))$ is an informationally stable equilibrium, s is a full information candidate equilibrium.

<u>Proof</u> To prove existence, we assume s^* is a full information candidate equilibrium with $s_1^* = s_2^*$. By symmetry it follows such an s^* exists. Then we define $((s,b),(C,\tilde{s}))$ by setting $s = s^*, C = (N,N)$, snd for all $\alpha \varepsilon N$, set $b^\alpha = 0$, and $\text{supp}(\tilde{s}^\alpha) = \{s\}$.

To show this is an equilibrium, we verify each condition in turn.

V1: Since $\text{supp}(\tilde{s}^\alpha) = \{s^\alpha\}$ for all $\alpha \varepsilon N$, it follows that we need only find $b_\alpha \varepsilon B_\alpha$ to $\max[u_\alpha(s^*_{b_\alpha})]$. But $u_\alpha(s_1^*) = u_\alpha(s_2^*) = u_\alpha(s_0^*)$. Hence $0 \varepsilon \arg\max_{b_\alpha \varepsilon B_\alpha} [u_\alpha(s^*_{b_\alpha})]$.

C1: Since s^* is a full information candidate equilibrium with $s_k^* = s_{\bar{k}}^*$, it follows that for all $s \varepsilon S$ with $s_{\bar{k}} = s_{\bar{k}}^*$, that

$$M_k(s,\hat{b}(s|N)) \leq M_k(s^*,\hat{b}(s^*|N))$$

Since $C^k = N$, it follows that

$$s_k^* \varepsilon \arg\max_{s_k^* \varepsilon S_k} M_k(s^*,\hat{b}(s^*|N)).$$

V2: For all $1 \leq i \leq t$,

$$p_i(s,b) = (1,0,0) = (\mu(\hat{V}_0(s)),\mu(\hat{V}_1(s)),\mu(\hat{V}_2(s))) = \hat{p}(s|N)$$

hence $||p(s,b) - \hat{p}(s|N)||_r = 0$. So,

$\text{supp}(\tilde{s}^\alpha) = \{s\}$ solves V2

C2: As above, for $C^k = N$,

$$||p(s,b) - \hat{p}(s|C^k)||_r = 0$$

so $C^k = N$ solves C2.

Now to show that the equilibrium is informationally stable, we let $N(s)$ be a neighborhood of $s = s^*$, and pick s' s $N(s)$ with $s_k' \neq s_{\bar{k}}'$.

By assumption, we know that the resulting voter equilibrium $(b', \tilde{s})$ extracts all information. Hence, for all $\alpha \varepsilon N$, $k \varepsilon K \cup \{0\}$, $\alpha \varepsilon \hat{V}_k(s') \Rightarrow b_\alpha' = k$. I.e., $V_k(s', b') = \hat{V}_k(s')$. But then

$$\begin{aligned} p_{ik}(s',b') &= \mu_i(V_k(s',b')) = \mu_i(\hat{V}_k(s')) \\ &= \hat{p}_{ik}(s'|N) \end{aligned}$$

Hence $||p(s',b') - \hat{p}(s'|N)||_r = 0$, so for $C^k = N$, conditions C2 is met. So $C^k = N$ is consistent with the data $p(s'b')$, and it follows that the equilibrium is informationlly stable. This proves the first assertion in the theorem.

Now assume that $((s, b), (C, \tilde{s}))$ is an informationally stable equilibrium. From the previous lemma, we know that either the conclusion is true, or $s_1 = s_2$. So assume $s_1 = s_2$. Let $N(s)$ be a neighborhood of s, and pick $s' \varepsilon N(s)$ with $s_1' \neq s_2'$. Then by assumption, if $(b', \tilde{s}')$ is a voter equilibrium based on s', it extracts all information. So $\alpha \varepsilon \hat{V}_k(s') \Rightarrow b_\alpha' = k$ or $V_k(s', b') = \hat{V}_k(s')$. Further, by assumption 6.1, $\mu(\hat{V}_0(s')) = 0$. Then, using an argument similar to that in Lemma 6.1, we have $\mu(N - C^k) = 0$, or $\mu(C^k) = \mu(N)$. But now, if $((s,b),(C,\tilde{s}))$ is an informationally stable equilibrium, it must be an equilibrium. Thus, by C1,

$$M_k(s, \hat{b}(s|C^k)) \geq M_k(s', \hat{b}(s'|C^k))$$

for all s' with $s_{\bar{k}}' = s_{\bar{k}}$. But since $\mu(C^k) = \mu(N)$, it follows that for all s, $M_k(s, \hat{b}(s|C^k)) = M_k(s, \hat{b}(s|N))$. Hence

$$M_k(s, \hat{b}(s|N)) \geq M_k(s', \hat{b}(s'|N))$$

whenever $k \varepsilon K$ and $s' \varepsilon \underline{S}$ satisfies $s_{\bar{k}}' = s_{\bar{k}}$. But this implies that s is a full information candidate equilibrium.

Q.E.D.

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