DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

THE GEOMETRY OF VOTING

Norman Schofield

SOCIAL SCIENCE WORKING PAPER 485

July 1983

THE GEOMETRY OF VOTING*

Norman Schofield

ABSTRACT

For any non collegial voting game, σ, there exists a stability dimension $v^*(\sigma)$, which can be readily computed. If the policy space has dimension no greater than $v^*(\sigma)$ then no local σ-cycles may exist, and under reasonable conditions, a σ-core must exist. It is shown here, that there exists an <u>open</u> set of profiles, V, in the C^1 topology on smooth profiles on a manifold W of dimension at least $v^*(\sigma)+1$, such that for each profile in V, there exist local σ-cycles and no σ-core.

* Thanks are due to Jeff Strnad, at the University of Southern California Law Center, for making available some of his unpublished work. The result presented here as Theorem 1 is much influenced by Strnad's work.

THE GEOMETRY OF VOTING

Norman Schofield

Introduction

Consider a spatial voting game, σ, without vetoers on a topological space W of dimension w. Recent research has lead to results on the classification of any such game in terms of two integers: the stability and instability dimensions, $v^*(\sigma)$ and $w(\sigma)$ respectively, where $v^*(\sigma) < w(\sigma)$.

For a q-majority voting game, where q players out of n win, Greenberg (1979) showed that when W was convex compact of dimension $w < {}^{q}/_{n-q}$, and all individuals had convex preference then a core for the game was non empty. Strnad (1981,1982) essentially showed that for a general game σ without vetoers that this result held as long as $w \leq v^*(\sigma)$ (for the q majority game $v^*(\sigma)$ is the largest integer strictly less than ${}^{q}/_{n-q}$). Indeed for majority rule in general $v^*(\sigma) = 1$. Schofield (1983a,1983b) developed this further to show that with arbitrary smooth preferences and in dimension $w \leq v^*(\sigma)$, no local cycles for the game could occur.

Greenberg also showed that if the dimension of W exceeded ${}^{q}/_{n-q}$ then a smooth profile could be constructed such that no core existed. Strnad (1981,1982) showed that this was true for an arbitrary voting game σ as long as dimension $(W) \geq v^*(\sigma) + 1$. In this paper it is shown that in dimension $v^*(\sigma) + 1$ a smooth profile can always be constructed so that the σ-voting cycles fill the pareto set.

Moreover the features that this profile possesses are unchanged under small perturbations in the C^1-topology on all smooth profiles.

We also show that if $v^*(\sigma)+1 \leq w^*(\sigma)-1$, then there is an open set in the C^1 topology on profiles, such that every profile in this set admits a non empty "local" σ-core.

Statement of the Result

We consider a voting game σ, defined in terms of a set $\mathcal{D}$ of winning coalitions. A coalition M is a subset of the society $N = \{1,\ldots,n\}$. Suppose that $u = (u_1,\ldots,u_n): W \to \mathbb{R}^n$ is a continuous profile for the society, where W is a topological space of alternatives.

Given a profile u, the preference correspondence $P_i: W \to W$ of player i is defined by $y \in P_i(x)$ iff $u_i(y) > u_i(x)$. For a coalition $M \subset N$, let $P_M: W \to W$ be defined by $P_M(x) = \bigcap_{i \in M} P_i(x)$, where Φ stands for the empty set. Write $GO(M, \;) = \{x \in W: P_M(x) = \Phi\}$.
The preference correspondence of (σ,u) is then $P_\sigma: W \to W$ where $y \in P_\sigma(x)$ iff $y \in P_M(x)$ for some $M \in \mathcal{D}$. Then the core or global optimum set of (σ,u) is $GO(\sigma,u) = \bigcap_{M \in \mathcal{D}} GO(M,u)$.
Under certain conditions the core of (σ,u) can be shown to be non empty for a certain class of profiles. For example let $\sigma(u)$ be the preference relation $x\sigma(u)y$ iff $x \in P_\sigma(x)$.

Say there exists a $\sigma(u)$-cycle through a point $x \in W$ iff there is a finite subset $\{x_1,\ldots,x_r\}$ in W such that

$$x\sigma(u)x_1\sigma(u) \;\ldots\; \sigma(u)x_r\sigma(u)x.$$

In this case say x belongs to the global cycle set $GC(\sigma,u)$ of $\sigma(u)$. It is known that if u is continuous, and $\sigma(u)$ is acyclic (ie. $GC(\sigma,u) = \Phi$), then $GO(\sigma,u)$ must be non empty. In general however it is not known that the preference relation $\sigma(u)$ is acyclic.

However Greenberg (1979) was able to show, with <u>semi-convex</u> preferences and certain constraints on the alternative space W, that $GO(\sigma,u)$ would be non empty. Say a preference correspondence $P:W \to W$ is semi-convex iff W is a convex subset of euclidean space, and for no x in W, does $x \in \mathrm{Con}P(x)$, where $\mathrm{Con}P(x)$ is the convex hull in W of the set $P(x)$. When W is a convex compact subset of $\mathbb{R}^w$ say W is an <u>admissible</u> space of dimension w.

Consider now a q-majority game σ_q, whose set of winning coalitions is given by $\mathcal{D}_q = \{M \subset N: |M| \geq q\}$. Greenberg showed that when W was admissible of dimension $< {}^{q}/_{n-q}$, and individual preference was semi convex as well as continuous, then the core $GO(\sigma_q,u)$, of the q-majority game was non empty.

Recently Strnad (1981,1982) and Schofield (1983a) have extended this result to the general class of non-collegial voting games.

Let $\mathcal{D}$ by any family of subsets of N. The <u>$\mathcal{D}$ -collegium</u> $C(\mathcal{D})$ of $\mathcal{D}$ is defined by $C(\mathcal{D}) = \bigcap_{M \in \mathcal{D}} M$.

If the family $\mathcal{D}$ of winning coalitions of the voting game, σ, is non empty then σ is called <u>collegial</u> and the Nakamura (1978) number, $v(\sigma)$, of σ is defined to be ∞. If σ is not collegial then define the Nakamura number to be

$$v(\sigma) = \min\{|\mathcal{D}'| : \mathcal{D}' \subset \mathcal{D} \text{ and } C(\mathcal{D}') = \Phi\}.$$

Thus any subfamily $\mathcal{D}'$ of $\mathcal{D}$ with at most $v(\sigma)-1$ members must have

a non empty intersection (collegium), whereas it is always possible to find a subfamily with $v(\sigma)$ members and empty collegium.

For the game σ, define the <u>stability dimension</u> $v^*(\sigma)$ of σ to be $v^*(\sigma)-2$. It is an easy matter to show that for a q-majority game σ_q $v^*(\sigma_q)$ is the largest integer which is strictly less than ${}^{q}/_{n-q}$.

Strnad (1981,1982) has shown essentially, for continuous and semi-convex individual preference, that if W is admissible of dimension $w \leq v^*(\sigma)$ then $GO(\sigma,u)$ must be non empty. Schofield (1983b) extended this result in order to show how to deal with non-convex but smooth preferences.

Suppose now that W is a smooth manifold of dimension w. At any point x, the tangent space T_xW is the space of tangent vectors at x. For an interior point x of W, T_xW may be identified with a copy of $\mathbb{R}^w$ with origin O at x. For a boundary point x, T_xW may be thought of as a half space

$$\{v \in \mathbb{R}^w: v.\, n(x) > 0\}$$

where $n(x)$ is the normal to the boundary pointing into W. When $u_i:W \to \mathbb{R}$ is the smooth utility function of player i, then $H_i(x)$ is the set of vectors $\{v \in T_xW: du_i(x)(v) > 0\}$ where $du_i(x): T_xW \to \mathbb{R}$ is the differential of u_i at x. We call $H_i:W \to TW$ the <u>preference field</u> of i, where $TW = \bigcup_{x \in W} T_xW$.

In the case W is a subset of $\mathbb{R}^w$, then H_i may be regarded as a correspondence $H_i:W \to \mathbb{R}^w$, and therefore as an approximation to the preference correspondence $P_i:W \to W$.

For coalition $M \subset N$, define $H_M:W \to TW$ by $H_M(x) = \bigcap_{i \in M} H_i(x)$, and for a voting game σ define $H_\sigma: W \to TW$ by $H_\sigma(x) = \bigcup_{M \in \mathcal{L}} H_M(x)$.

Write $IO(M,u) = \{x \in W: H_M(x) = \Phi\}$ and define the <u>infinitesimal optima set</u>, $IO(\sigma,u)$, of (σ,u) to be $IO(\sigma,u) = \bigcap_{M \in \mathcal{L}} IO(M,u)$.

Under the assumption that u is smooth, the preference field $H: W \to TW$ can be integrated in the following way. Suppose that, for coalition M, at some point $x \in W$, $H_M(x) \neq \Phi$. Then it is possible to construct a smooth path from x to some nearby point $y \in W$ such that utility for each member of the coalition increases along the path. In this case we write $y \in \hat{P}_M(x)$. We also write $y\hat{\sigma}(u)x$ iff $y \in \hat{P}_M(x)$ for some $M \in \mathcal{L}$. Under the definitions it follows that $IO(M,u) = \{x \in W: y \in \hat{P}_M(x) \text{ for no } y \in W\}$

$$IO(\sigma,u) = \{x \in W: y\hat{\sigma}(u)x \text{ for no } y \in W\}.$$

Moreover it is the case that for any $x,y \in W$, $y \in \hat{P}_M(x)$ implies that $y \in P_M(x)$. Thus

$$GO(M,u) \subset IO(M,u)$$

and

$$GO(\sigma,u) \subset IO(\sigma,u).$$

(See Schofield, 1980, for further details).

Suppose now that W is convex subset of $\mathbb{R}^w$. Say that i has convex preference iff for each $x \in W$, it is the case that $P_i(x) \subset \{y \in W: du_i(x)(y-x) > 0\}$. With convex preference the global and infinitesimal optima sets will be identical.

In a number of papers Schofield (1978,1980) has examined a set of "local voting cycles".

More specifically for a voting game σ, and smooth profile u, say that a point x belongs to the local cycle set $LC(\sigma,u)$ iff for any neighbourhood U of x there is a finite subset $\{x_1,\ldots,x_r\}$ in W such that

$$x\hat{\sigma}(u)\ x_1\hat{\sigma}(u)\ \ldots\ \hat{\sigma}(u)\ x_r\hat{\sigma}(u)x,$$

and this voting trajectory belongs to U.

For the profile u, let $p_i(x)$ be the direction gradient of u_i at x (i.e. $p_i(x)$ is a vector at the point x, normal to the indifference surface $u_i^{-1}[u_i(x)]$ in the direction of increasing utility). For coalition M, let $p_M(x)$ be the convex hull of $p_i(x)$.

Given σ, let $\mathcal{D}(x) = \{M \in \mathcal{L}: H_M(x) \neq \Phi\}$.

Define the <u>directional core</u> at x to be

$$p_\sigma(x) = \bigcap_{M \in \mathcal{D}(x)} p_M(x),$$

and define the <u>infinitesimal cycle set</u> of (σ,u) to be

$$IC(\sigma,u) = \{x \in W: p_\sigma(x) = \Phi\}.$$

It was shown in Schofield (1978) that, for any point in the interior of W, if $p_\sigma(x) = \Phi$ then $x \in LC(\sigma,u)$. Moreover if, for every $x \in W, p_\sigma(x) \neq \Phi$ then $LC(\sigma,u) = \Phi$ (Schofield, 1983c).

Suppose now that W is a smooth manifold of dimension $w \leq v^*(\sigma)$, and choose any point x in W. Suppose $\mathcal{D}'$ is a subfamily of $\mathcal{D}(x)$ of cardinality at most $v^*(\sigma)+1$. Since the cardinality of $\mathcal{D}'$ is less than the Nakamura number $v^*(\sigma)+2$, this implies there exists some individual belonging to $C(\mathcal{D}')$.

Thus $p_i(x) \in \bigcap_{M \in \mathcal{D}'} p_M(x) \neq \Phi$.

Since the dimension of W is no greater than $v^*(\sigma)$, and each $p_M(x)$ is a convex compact set, Helly's Theorem implies that

$$p_\sigma(x) = \bigcap_{M \in \mathcal{D}(x)} p_M(x) \neq \Phi.$$

(See Schofield, 1983a, for further details).

When W is admissible and u is a smooth profile then $IC(\sigma,u) = \Phi$ implies $IO(\sigma,u)$ must be non empty (Schofield 1983b).

Thus when W is admissible of dimension $\leq v^*(\sigma)$ then $IC(\sigma,u)$ and thus $LC(\sigma,u)$, must be empty. As a consequence $IO(\sigma,u)$ must be non empty.

If we suppose further that preferences are convex, then $GO(\sigma,u)$ must be non empty.

The purpose of this paper is to show that the dimensionality constraint is necessary, as well as sufficient for the non existence of local cycles. We prove the following result in the next section of this paper.

Theorem 1

Suppose that σ is a voting game and W is a smooth manifold of dimension $w \geq v^*(\sigma)+1$. Then there exists a smooth profile $u = (u_1,\ldots,u_n):W \to \mathbb{R}^n$ such that $LC(\sigma,u)$ is non empty.

Existence of Cycles

Proof of Theorem 1

By the definition of the Nakamura number there exists a subfamily $\mathcal{D}'$ of $\mathcal{D}$ with cardinality $v^*(\sigma)+2$ such that the collegium of $\mathcal{D}'$ is empty.

Let Δ be the abstract simplex in $\mathbb{R}^w$ of dimension $v^*(\sigma)+1$. A face F_j of this simplex is a subsimplex of dimension $v^*(\sigma)$, and can be identified with the convex hull of $v^*(\sigma)+1$ vertices. All together there are $v^*(\sigma)+2$ faces, and we identify each face F_j with a coalition M_j in $\mathcal{D}'$.

For each $i \in N$, let $\mathcal{D}_i = \{M_j \in \mathcal{D}' : i \in M_j\}$ be the subfamily of $\mathcal{D}'$ of coalitions including i. Let $T = \{i \in N : \mathcal{D}_i \neq \Phi\}$.

For each $i \in T$, i cannot belong to every member of $\mathcal{D}'$ since $C(\mathcal{D}') = \Phi$.

Thus $1 \leq |\mathcal{D}_i| \leq v^*(\sigma)+1$.

Since Δ is a simplex, any family of at most $v^*(\sigma)+1$ faces has a non empty intersection.

Let $S_i = \bigcap_{j \in \mathcal{D}_i} F_j$.

Now S_i is a subsimplex of Δ of dimension at most $v^*(\sigma)$. If S_i is a vertex (an 0-dimensional subsimplex) define $v_i = S_i$.

If S_i has dimension $(S_i) \geq 1$ then define the interior of S_i to be the set of points in S_i but in no subsimplex (of dimension strictly less than $\dim(S_i)$). In this case define v_i to be some point in the interior of S_i.

By this definition, if $i \in T$, then $v_i \in F_j$ for any F_j such that $i \in M_j$.

Let Δ' be the simplex generated by the vertices $\{v_i : i \in T\}$.

For each coalition $M_j \in \mathcal{D}'$, let p_{M_j} be the convex hull of $\{v_i : i \in M_j\}$.

By definition $p_{M_j} \subset F_j$.

Now let θ be the barycentre of the simplex Δ', and for each $i \in T$, let p_i be the vector from θ to the vertex v_i. Note that for each $M_j \in \mathcal{D}'$ the vectors $\{p_i : i \in M_j\}$ are not semipositively dependent (i.e. it is impossible to find non negative numbers $\{\lambda_i \in \mathbb{R} : i \in M_j\}$ such that $\sum \lambda_i p_i = 0$). Thus for no $M_j \in \mathcal{D}'$ does θ belong to p_{M_j}.

By definition, the family $\{F_j : j=1,\ldots,v^*(\sigma)+2\}$ of faces has empty intersection, and thus

$$p_\sigma = \bigcap_{\mathcal{D}'} p_{M_j}$$

has empty intersection.

Now let x be a point in the interior of the manifold W. For each $i \in T$, define the vector $p_i(x) \in \mathbb{R}^w$ to be the vector p_i as defined above. It is then possible to define $u_i : W \to \mathbb{R}$ such that with respect to some coordinate chart at x, the direction gradient of u_i at x can be identified with $p_i(x)$. For each $i \in N \setminus T$, define u_i to be the constant function on W.

It is well known (Smale 1974) that at a point x in the interior of W, then $H_M(x) = \Phi$ for coalition M iff $\{p_i(x) : i \in M\}$ are semipositively dependent.

By the construction above, for each $M_j \in \mathcal{L}'$, $H_{M_j}(x) \neq \Phi$.

Thus $p_\sigma(x) = \bigcap_{\mathcal{L}'} p_{M_j} = \bigcap_{\mathcal{L}(x)} p_M = \Phi$.

Hence $x \in IC(\sigma,u)$. But by the construction x is an interior point of W and u is a smooth profile.

Thus $x \in LC(\sigma,u)$ and so $LC(\sigma.u)$ is non empty. ∎

The proof procedure presented here is adapted from Strnad's construction of a smooth profile with empty global optima set.

More specifically Strnad assumes that W is admissible of dimension $v^*(\sigma)+1$. Call Δ' the σ-simplex: since this simplex is of dimension at most $v^*(\sigma)+1$, it can be embedded in W, with the barycentre θ corresponding to point $x \in W$.

To each $i \in T$, assign the smooth "euclidean" utility function $u_i : W \to \mathbb{R}$ to i by $u_i(x) = -\|x - v_i\|$. Thus u_i has a maximum point at the vertex v_i, and at any other point x, $p_i(x) = \lambda(v_i - x)$ where $\lambda > 0$.

For each $i \notin T$ assign a constant utility function. Let $\mathcal{D}'$ be as in the proof of the theorem. Then for each $M_j \in \mathcal{D}'$ it is the case that $IO(M_j,u)$ is the convex hull of $\{v_i : i \in M_j\}$, and can be identified with the convex set p_{M_j}.

Moreover

$$IO(\sigma,u) = \bigcap_{M \in \mathcal{L}} IO(M,u) \subset \bigcap_{\mathcal{L}'} p_{M_j} = \Phi.$$

Thus $IO(\sigma,u)$ is empty.

Furthermore each utility function defines a convex preference, and so for each $M \in \mathcal{L}$, $IO(M,u) = GO(M,u)$. Thus $GO(\sigma,u)$ is empty.

Note also that at the point x located at the barycentre of the σ-simplex the analysis in the proof of Theorem 1 is valid. That is to say that $p_\sigma(x) = \Phi$ and so $x \in IC(\sigma,u)$.

Indeed it should be clear that with this "euclidean" profile, there exists an open dense set X in the interior of the σ-simplex such that any point in X also belongs to $IC(\sigma,u)$.

Note also that the σ-simplex corresponds to the optima set $IO(N,u) = GO(N,u)$ for the whole society N.

Thus Strnad's method shows that $IC(\sigma,u)$ is open dense in $IO(N,u)$, so that the closure of $IC(\sigma,u)$ includes $IO(N,u)$.

Suppose now that dimension $(W) = v^*(\sigma)+1$. In this case Schofield (1983a, Theorem 1) has shown that $IC(\sigma,u) \subset IO(N,u)$.

In dimension $v^*(\sigma)+1$ we therefore see that $\text{closure}(IC(\sigma,u)) = IO(N,u)$.

The proof of Theorem 1 relies heavily on the fact that the $v^*(\sigma)+2$ distinct faces, of a simplex of dimension $v^*(\sigma)+1$, do not intersect. Notice however that any $v^*(\sigma)+1$ distinct faces do intersect. Suppose now that we project the simplex onto the $v^*(\sigma)$-dimensional plane. Then by Helly's Theorem the projections of the $v^*(\sigma)+2$ faces will intersect. This is the essence of the proof that there can be no local cycles in dimension at most $v^*(\sigma)$.

To give an example of the construction in Theorem 1, consider a weighted voting game σ with four players, $N = \{a,b,c,d\}$, where the winning coalitions are $\{a,b\}$ $\{a,c\}$, $\{a,d\}$, $\{b,c,d\}$.
Clearly $v(\sigma) = 3$, or $v^*(\sigma) = 1$, since $\mathcal{D}' = \{\{a,b\},\{a,c\},\{b,c,d\}\}$ has empty collegium. Let Δ be the two dimension simplex, whose three faces correspond to these three coalitions.

[Insert Figure 1 about here].

The three vertices are identified with the players $\{a,b,c\}$ and the fourth player with a point in the bottom face. The four vectors $\{p_a,p_b,p_c,p_d\}$ are thus semipositively dependent. Consequently if utility functions are assigned to these players on a two dimensional space W such that at a point x, the direction gradients of the players are given by these vectors, then the point itself belongs to the pareto set IO(N,u) as well as the cycle set $IC(\sigma,u)$.

Theorem 1 can be strengthened somewhat.
Note that the vectors $\{p_i(x): i \in N\}$ may be perturbed slightly from the assigned values and yet still give an empty directional core at x, and thus a non empty cycle set.

Let $U(W)^N$ be the set of smooth utility profiles. We endow

$U(W)^N$ with the Whitney C^1-topology. In this topology a neighbourhood $N(u,\delta)$ of the profile u includes any profile $v \in U(W)^N$ such that for all $x \in W$ and all $i \in N$,

$$|u_i(x) - v_i(x)| < \delta \quad \text{and} \quad |du_i(x)-dv_i(x)| < \delta.$$

Clearly the profile u constructed in the proof of theorem 1 has a neighbourhood $N(u,\delta)$ such that $IC(\sigma,u') = \Phi$ for all $u' \in N(u,\delta)$. Thus we obtain:

<u>Theorem 2</u>

Let σ be a voting game and W a smooth compact manifold of dimension $w \geq v^*(\sigma)+1$.

a) Then there exists an open neighbourhood V in the Whitney C^1-topology for $U(W)^N$ such that $IC(\sigma,u) \neq \Phi$ for all $u \in V$.

b) Moreover if W is a smooth compact convex manifold in $\mathbb{R}^w$ of dimension $w \geq v^*(\sigma)+1$ then (a) can be strengthened to read

$$IO(\sigma,u) = \Phi \text{ for all } u \in V.$$

Let us now define

$$\mathcal{L}(\sigma) = \{u \in U(W)^N : IO(\sigma,u) = \Phi \text{ and } IC(\sigma,u) \neq \Phi\}.$$

Clearly theorem 2 implies that, for a smooth admissible manifold, $\mathcal{L}(\sigma)$ has a non empty interior in $U(W)^N$, as long as dimension $(W) \geq v^*(\sigma)+1$.

In an earlier paper (Schofield 1980) it was shown that for each non-collegial voting game there is an integer $w^*(\sigma)$ such that when dimension $(W) > w^*(\sigma)$ then $\mathcal{L}(\sigma)$ is open dense in $U(W)^N$. Indeed for each $u \in \mathcal{L}(\sigma)$ in this dimension range $IC(\sigma,u)$ was shown to be open dense in W.

Now define $K(\sigma) = \{u \in U(W)^N : IO(\sigma,u) \neq \Phi \text{ and } IC(\sigma,u) = \Phi\}$.

The earlier result implied that $K(\sigma)$ had empty interior if dimension $(W) > w^*(\sigma)$.

We may now show, in dimension $v^*(\sigma)+1$, that $K(\sigma)$ has a non empty interior.

To see this consider again the four person voting game in two dimensions considered above. From previous analysis it is easy to show that $w^*(\sigma) = 3$. Suppose now that the profile u is such that player a has a bliss point at x, and the direction gradients of the other players are semipositively dependent. See Figure 2.

[Insert Figure 2 about here].

Consequently $x \in IO(\sigma,u)$. Moreover it is easy to construct a profile u such that $IC(\sigma,u) = \Phi$ (for example take linear utilities for the players $\{b,c,d\}$).

Moreover the profile may be perturbed slightly in the C^1-topology, yet still retain these properties. It should be clear that we may construct a profile u in general in this way in dimension $v^*(\sigma)+1$ such that $IO(\sigma,u) \neq \Phi$. (See also Cox, 1983). Thus we obtain the following.

Theorem 3

Let σ be a voting game and W a smooth manifold of dimension $v^*(\sigma)+1$. If $v^*(\sigma)+1 \leq w^*(\sigma)-1$, then there exists an open neighbourhood U in the C^1-topology for $U(W)^N$ such that $IO(\sigma,u) \neq \Phi$ and $IC(\sigma,u) = \Phi$ for all $u \in U$.

Conclusion

The results presented here imply that a voting game σ can be classified in the following way.

The stability dimension $v^*(\sigma)$ is the smallest integer such that dimension $(W) \geq v^*(\sigma)+1$ implies that the set $\mathcal{L}(\sigma)$, of smooth profiles sustaining a cycle set, has non empty interior in the space $U(W)^N$ with the Whitney C^1-topology. In dimension $v^*(\sigma)+1$, the cycle set associated with any profile must be paretian.

Consider now the instability dimension $w(\sigma)$. It has been shown that in dimension at least $w(\sigma)+1$, there is a dense set V' in $U(W)^N$ such that $IO(\sigma,u) = \Phi$ and $IC(\sigma,u) \neq \Phi$ for all $u \in V'$. Indeed it was shown that $IC(\sigma,u)$ would be dense in W, for all $u \in V'$.

Consequently in this dimension range the set $K(\sigma)$ of profiles, sustaining a σ-core, must have empty interior. Moreover if W has no boundary, but is of dimension $w(\sigma)$, then $K(\sigma)$ must have empty interior.

Thus $w(\sigma)$ may be characterised to be the greatest integer such that if W is a smooth manifold of dimension w, then $w \leq w(\sigma)-1$ implies that $K(\sigma)$ has a non empty interior. At present upper bounds on $w(\sigma)$ are known to be (n-1) for a general non collegial game, q for a q-majority game, and 3 or 4 for strict majority rule. As Theorems 2 and 3 show, when the dimension range $[v^*(\sigma)+1, w(\sigma)-1]$ is non empty, then both $\mathcal{L}(\sigma)$ and $K(\sigma)$ have non empty interior.

References

[1] Cox, G.W. (1983), Non Collegial Simple Games and the Nowhere Denseness of the Set of Preference Profiles having a Core. Mimeographed, University of Texas, at Austin.

[2] Greenberg, J. (1979), Consistent Majority Rule over Compact Sets of Alternatives, Econometrica, 47, 627-636.

[3] Nakamura, K. (1978), The Vectors in a Simple Game with Ordinal Preference, Int. J. Game Th., 8, 55-61.

[4] Schofield, N. (1978), Instability of Simple Dynamic Games, Rev. Econ. Stud., 45, 575-594.

[5] Schofield, N. (1980), Generic Profiles of Simple Bergson Samuelson Welfare Functions, J. Math. Econ., 7, 175-192.

[6] Schofield, N. (1983a) Equilibrium in Simple Dynamic Games, in Social Choice and Welfare (P. Pattanaik and M. Salles, Eds.) North Holland, New York.

[7] Schofield, N. (1983b), Social Equilibrium and Cycles on Compact Sets. Forthcoming, J.Econ.Theory.

[8] Schofield, N. (1983c), Existence of Equilibrium on a Manifold, Forthcoming, Math.Op.Res.

[9] Smale, S. (1974) Sufficient Conditions for an Optimum, in Dynamical Systems: Warwick (A. Manning, Ed.) Springer Verlag, New York.

[10] Strnad, J. (1981). The Structure of Continuous Neutral Monotonic Social Functions, Mimeographed, University of Southern California.

[11] Strnad, J. (1982), The Structure and Nash Implementation of Neutral Monotonic Social Functions, Ph.D. Dissertation, Yale University, December 1982.

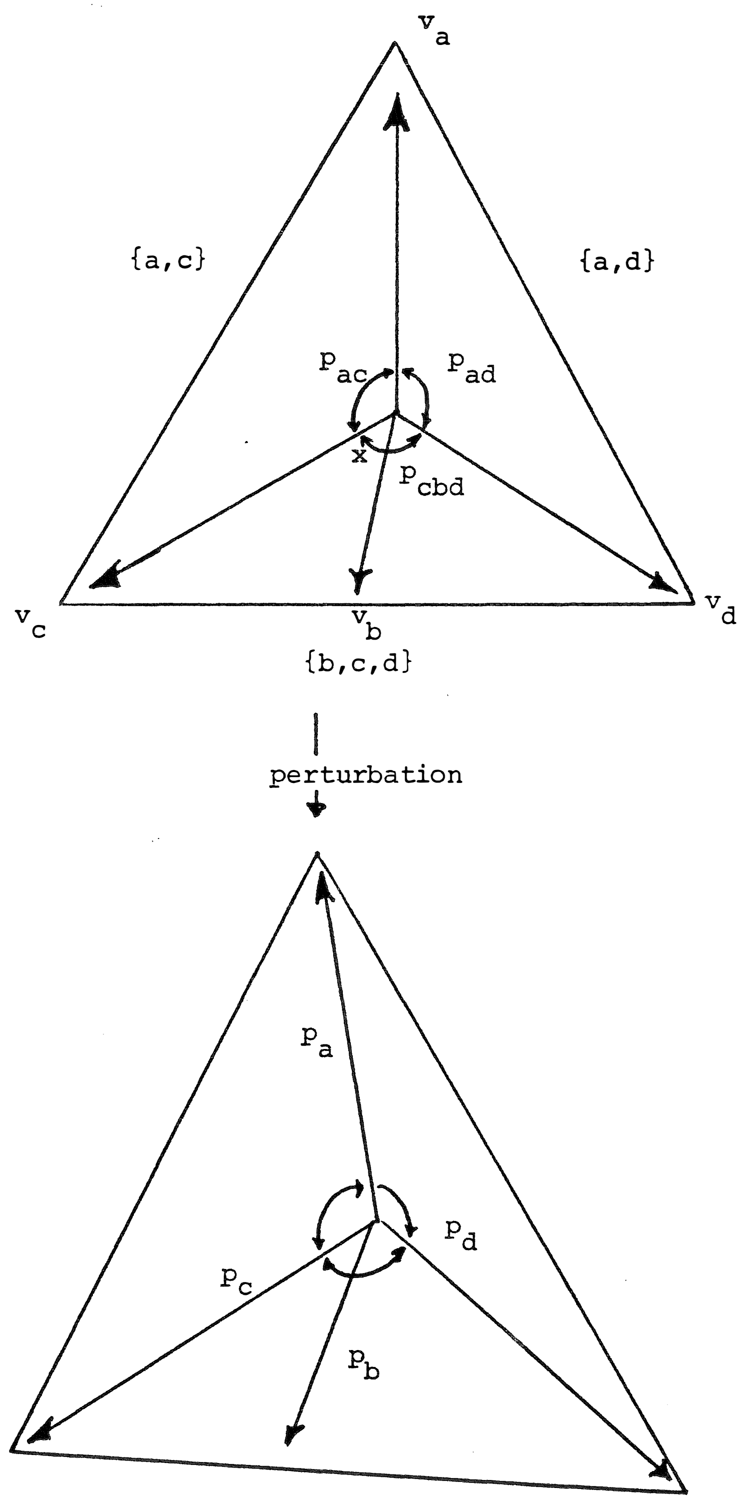

Figure 1: A point x in the cycle set IC(σ,u) in a two dimension space for voting game σ with players {a,b,c,d}, before and after perturbation.

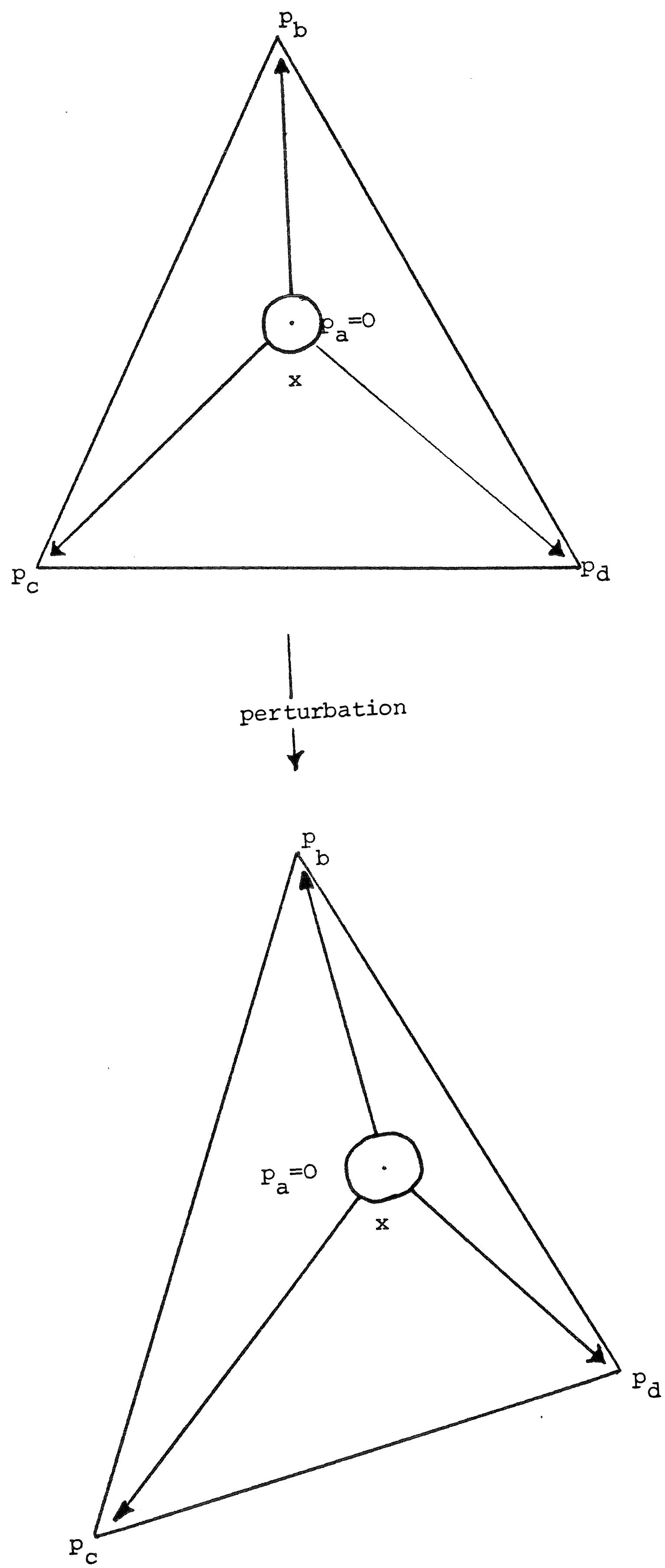

Figure 2: A point x in the optima set IO(σ,u) in a two dimension space for the voting game σ with players {a,b,c,d} before and after perturbation.