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ABSTRACT

One proof of existence of general equilibrium assumes convexity and continuity of a preference correspondence on a compact convex feasible set W. Here we show the existence of a local equilibrium for a preference field which satisfies, not convexity, but the weaker local acyclicity. The theorem is then applied to a voting game, σ, without veto players. It is shown that if the dimension of the policy space is no greater than $v(\sigma)-2$, where $v(\sigma)$ is the Nakumura number of the game, then no local cycles may occur and a local equilibrium must exist. With convex preferences then there will exist a choice of the game from W.

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INTRODUCTION

Proof of the existence of a general economic equilibrium, without the assumption of transitivity of individual preference was obtained by Sonnenschein [21], essentially using a version of Ky Fan's Theorem [6]. The method of proof was to construct a preference correspondence satisfying certain convexity and continuity properties. In general social choice processes, involving coalitions of players, the assumption of individual convex preference is insufficient to guarantee convex social preference. However Greenberg [9] showed that, for a q-majority game (where any coalition of q or more players from the society of size n is a winning coalition) on a compact convex subset of euclidean space, of dimension w, if $w < q/_{(n-q)}$ then convex individual preference guarantees convex social preference. Consequently Sonnenschein's theorem can be used to prove existence of a voting equilibrium or choice.

In this paper we extend Greenberg's Theorem in two directions. First of all we demonstrate how to extend the theorem to deal with an arbitrary weighted voting game σ without vetoers. For any such game a "stability dimension" $v^*(\sigma)$ can be computed. We show that if individual preferences are smooth, though not necessarily convex, and the policy space W has dimension no greater than $v^*(\sigma)$, then there can exist no "local voting cycles". Secondly we use the Ky Fan Theorem to show that if the policy space is compact convex, and if the voting game has no local cycles, then there must exist a "local" voting equilibrium. When we reimpose the assumption of convexity of individual preference, then the "local" equilibrium must be a choice, or core, of the game in W.

Since the stability dimension $v^*(\sigma)$ for a q-majority game is the largest integer strictly less than $q/_{(n-q)}$, we obtain an alternative proof of Greenberg's theorem.

Moreover the result on non-existence of local cycles is significant because of the relationship that this property has to the impossibility of agenda manipulation [11] and to the existence of well behaved "gradient" planning procedures [12,25].

1. AN EQUILIBRIUM THEOREM

Let W be a set of alternatives, and $P:W \to W$ a preference correspondence, where P(x) is the set of points in W preferred to the point x.

The choice of P from the set W is the set

$$C(P,W) = \{x \in W : P(x) = \phi\}.$$

An important question concerns those properties of P and W which are sufficient to guarantee the nonemptiness of the choice set C(P,W).

Let $p \in W \times W$ be the preference relation induced on W by P, where xpy iff $x,y \in W$ and $x \in P(y)$. Then the preference correspondence P (or equivalently the preference relation p) is acyclic iff there exists no finite subset $\{x,y_1,\ldots,y_r\}$ of W such that $xpy_1py_2\ldots py_rpx$. When W is a finite set and P is acyclic then, by [18], there exists a nonempty choice set C(P,W).

More generally suppose that W is a topological space. The preference correspondence $P:W \to W$ is

a) lower demi-continuous (ldc) iff, for each $x \in W$ the lower preference set $P^{-1}(x) = \{y \in W : x \in P(y)\}$ is open in W

b) upper demi-continuous (udc) iff for each $x \in W$, the upper preference set $P(x) = \{y \in W : y \in P(x)\}$ is open in W

c) continuous iff P is both ldc. and udc.

Walker [24] has shown that if W is compact, and P is acyclic and ldc. then there exists a choice of P from W.

In many contexts in economic theory and social choice there is no reason for the preference correspondence under examination to be acyclic. However both Ky Fan [6] and Sonnenschein [21] have shown that an appropriate convexity property for the preference correspondence can be substituted for acyclicity. Say that the preference correspondence P is semi-convex iff W is a convex set in a topological vector space Y and for no $x \in W$ does x belong to ConP(x), the convex hull of P(x) in W. By Ky Fan [6], if W is admissible (both convex and compact) and P is lower demi-continuous and semi-convex then there exists a choice $\bar{x}$ of P from W such that $P(\bar{x}) = \phi$.

Ky Fan's Theorem is valid for W a subset of a topological linear space Y of arbitrary dimension. However if Y is euclidean space $\mathbb{R}^w$ of finite dimension w, then the continuity requirement can be weakened. Say a correspondence $P:W \to W$ is lower hemi-continuous (lhc) iff for all $x \in W$, and any open set $U \subset W$ such that $P(x) \cap U \neq \phi$, then there is an open neighbourhood V of x in W such that $P(x') \cap U \neq \phi$ for all $x' \in V$. Clearly if a correspondence is ldc. then it is lhc. A fixed point argument due to Bergstrom,[2], shows that for the finite dimensional case, lower demi-continuity can be weakened to lower hemi-continuity in the Ky Fan Theorem (see [17] for further discussion).

Greenberg, [9], has used the Ky Fan-Sonnenschein theorem to show existence of a choice in a particular voting game. More specifically let $N = \{1,\ldots,n\}$ be a society with n members, and suppose the i^{th} individual has a preference correspondence $P_i:W \to W$. For coalition M in N, define $P_M:W \to W$ by $P_M(x) = \bigcap_{i \in M} P_i(x)$.

Note that if each P_i, for $i \in M$, is ldc. then so is P_M. However if each P_i is lhc then P_M need not be lhc. For this reason we assume lower demi-continuity of preference. A simple voting game σ is defined in terms of a class $\mathcal{D}$ of winning coalitions. The social preference correspondence $P_\sigma:W \to W$ of the voting game is then given by

$$P_\sigma(x) = \bigcup_{M \in \mathcal{D}} P_M(x).$$

Thus $y \in P_\sigma(x)$ iff there exists some $M \in \mathcal{D}$ such that $y \in P_M(x)$.

Greenberg examined a q-majority rule, σ_q, where the integer $q \leq n$ "specifies the smallest number of individuals that can enforce a change". In other words the class of winning coalitions $\mathcal{D}_q$ for σ_q is given by $\mathcal{D}_q = \{M \subset N: |M| \geq q\}$.

Suppose now that W is a compact, convex subset of $\mathbb{R}^w$, and that each individual has a continuous and semi-convex preference correspondence. Greenberg [9, Theorem 2] showed that when $w < \frac{q}{n-q}$ then the q-majority preference correspondence P_{σ_q} is also continuous and semi-convex. By the Ky Fan Theorem there exists a choice $\bar{x}$ in W such that $P_{\sigma_q}(\bar{x}) = \phi$. In other words there exists no alternative $y \in W$ such that y is preferred to $\bar{x}$ by at least q individuals.

Note that Greenberg's Theorem is only valid for an anonymous simple voting game, in which it is precisely the number of players in a coalition that determines whether the coalition is winning or not.

We can generalize Greenberg's Theorem to cover an arbitrary (weighted) voting game σ by introducing the Nakamura [13] number of σ. Let $\mathcal{D}$ be a family of subsets of N. Define the $\mathcal{D}$-collegium $C(\mathcal{D})$ of $\mathcal{D}$ by

$C(\mathcal{D}) = \bigcap_{M \in \mathcal{D}} M$.

If the class $\mathcal{D}$ of winning coalitions of a voting game, σ, has nonempty collegium then σ is called collegial, and the Nakamura number is defined to be ∞. If the game is collegial, then the members of the collegium are also called veto players. If σ is not collegial, then define $v(\sigma)$ by

$$v(\sigma) = \min\{|\mathcal{D}'| : \mathcal{D}' \subset \mathcal{D} \text{ and } C(\mathcal{D}') = \phi\}.$$

In other words $v(\sigma)$ is the cardinality of the smallest subfamily $\mathcal{D}'$ of the family $\mathcal{D}$ of winning coalitions of σ such that $\mathcal{D}'$ has an empty collegium.

When σ is a simple voting game with Nakamura number $v(\sigma)$, define the stability dimension of σ to be $v^*(\sigma) = v(\sigma) - 2$.

Call a nonempty compact and convex subset of W of $\mathbb{R}^m$ an admissible space. The dimension, w, of W is the dimension of the affine manifold spanned by W.

Theorem 1 Let σ be a voting game for a society $N = \{1,\ldots,n\}$ on an admissible space W of dimension w.

If each individual has semi-convex and continuous preferences on W, and if $w \leq v^*(\sigma)$ then there exists a choice $\bar{x} \in W$ for P_σ from W. ∎

This theorem, proved below, has also been obtained by Strnad [22,23].

To show that Theorem 1 is an extension of Greenberg's result we need to compute the Nakamura number of a q-majority rule σ_q with $q < n$. As Greenberg himself showed, a necessary and sufficient condition for any subfamily $\mathcal{D}'$ of $\mathcal{D}_q$, of cardinality $|\mathcal{D}'| = r$, to have nonempty intersection is $q > \left(\frac{r-1}{r}\right) n$.

This inequality can be written $r < \left[\frac{q}{n-q}\right] + 1$.

Let $v(n,q)$ be the largest integer which is strictly less than $\left[\frac{q}{n-q}\right]$.

Then $|\mathcal{D}'| \leq v(n,q) + 1$ implies that $C(\mathcal{D}') \neq \phi$.

Consequently the Nakamura number of σ_q is $v(n,q)+2$, and so the stability dimension for the q-majority rule is $v(n,q)$. By definition $w \leq v(n,q)$ iff $w < q/_{(n-q)}$.

Greenberg also showed that if $A = \{v_1,\ldots,v_{w+1}\}$ was a finite set of points, and $p = \{p_1,\dots,p_n\}$ a profile of transitive preferences on A, then the profile p could be extended over the abstract w-dimensional simplex Δ generated by A. As long as $|A| = w+1 \leq v(n,q)+1$, then the preference correspondence for the q-majority rule would have a choice from Δ. Indeed Greenberg's method gave a q-majority choice from A.

More generally Nakamura [13] has shown that if A is a finite set of cardinality $|A| \leq v^*(\sigma)+1$ then the preference correspondence P_σ, for the arbitrary voting game σ, will be acyclic. Moreover if $|A| > v^*(\sigma)+1$ then an acyclic profile on A can always be constructed is such a way that P_σ itself is cyclic (See also [4],[5],[7]).

These results for the finite alternative case suggest that Theorem 1 can be extended to a more general result involving the non existence of voting cycles in dimension below $v^*(\sigma)$. In the next section of the paper we obtain Theorem 1 essentially by showing that a voting game σ exhibits no "local" cycles on a topological space W of dimension $w \leq v^*(\sigma)$ as long as individual preferences are continuous. Moreover the extension of Theorem 1 that we obtain provides a method for dealing with non-convex individual preference.

Consider again the case of q-majority rule, σ_q, for a society of size n. If q is strictly greater than $\frac{n}{2}$, then clearly any two coalitions in $\mathcal{D}_q$ must intersect. Thus $v(\sigma_q) \geq 3$ and so $v^*(\sigma_q) \geq 1$. Thus a direct corollary of Theorem 1 is that if W is a convex compact subset of the real line and individual preference is semi convex and continuous then there is a choice for such a q-majority rule. Thus Theorem 1 may be

thought of as a generalization of proofs on the existence of a majority equilibrium when preferences are single peaked [3].

Consider the situation however when individual preferences on a one dimensional admissible space W are not semi-convex. As Kramer and Klevorick [10] have shown, it is possible to construct a profile of such preferences on W such that the majority rule preference relation is cyclic. Consequently neither the theorems of Walker nor Greenberg can be used to show existence of an equilibrium. Kramer and Klevorick however showed, under the assumption of smooth preferences, that a local choice set was nonempty.

Here a point $\bar{x}$ is a local choice of P_σ on W iff there is some neighbourhood U of $\bar{x}$ such that $P_\sigma(\bar{x}) \cap U = \phi$.

We shall use an adaptation of the Ky Fan Theorem to show that non existence of "local" cycles essentially implies the existence of a "local" choice set. This result holds in the absence of any convexity assumptions. If one imposes convexity then the "local" choice set and "global" choice set coincide.

Greenberg also showed that if W was admissible but of dimension greater than $q/(n-q)$ then it was always possible to construct a profile of continuous semi-convex individual preferences such that the preference correspondence of the q-majority game had no choice.

Strnad ([22],[23]) extended Greenberg's method of proof to show that for an arbitrary non-collegial voting game σ, if dimension(W) $\geq v^*(\sigma)+1$ then a smooth profile u can be constructed on the admissible space W such that there exists no choice of P_σ from W. Indeed Strnad's procedure can be adapted to show that the cycle set $IC(\sigma,u)$ will also be non empty, (see also [16]). Thus the constraint that dimension(W) $\leq v^*(\sigma)$ is both necessary and sufficient for the non existence of local cycles and the existence of a choice, when individual preferences are both semi-convex and continuous.

2. EXISTENCE OF AN OPTIMUM

We restrict attention first of all to the case where W is an open set in $\mathbb{R}^W$ and preferences of the members of the society N are given by a profile of smooth utilities

$$u = (u_1,\ldots,u_n):W \to \mathbb{R}^n$$

(The following analysis except where indicated is also valid when W is a smooth manifold; see [15] whose notation we follow here). At each point x, the tangent space T_xW can be identified with $\mathbb{R}^W$, and for each $i \in N$, the derivative $du_i(x)$ of u_i at x belongs to the topological space $L(\mathbb{R}^W,\mathbb{R})$ of linear maps from $\mathbb{R}^W$ to $\mathbb{R}$. Moreover $du_i:W \to L(\mathbb{R}^W,\mathbb{R})$ is continuous. If $du_i(x)$ is non-zero it may be represented by a vector $p_i(x)$ of unit length. Let

$$IO(i) = \{x \in W: du_i(x) = 0\}$$

be the optima set of player i. Clearly IO(i) is closed in W, and $p_i:W \setminus IO(i) \to \mathbb{R}^W$ is continuous.

For each i, let $H_i(x) = \{v \in \mathbb{R}^W: du_i(x)(v) > 0\}$ be the preference cone of i at x and $H_i:W \to \mathbb{R}^W$ the preference field. For coalition M, let

$$H_M:W \to \mathbb{R}^W$$

be the preference field defined by $H_M(x) = \bigcap_{i \in M} H_i(x)$.

The critical pareto set for M is $IO(M,u) = \{x \in W: H_M(x) = \phi\}$.

For a voting game σ, with $\mathcal{D}$ its family of winning coalitions, the preference field $H_\sigma:W \to \mathbb{R}^W$ is defined by $H_\sigma(x) = \bigcup_{M \in \mathcal{D}} H_M(x)$.

The preference correspondence $P_i:W \to W$ for individual i is given by $P_i(x) = \{y \in W: u_i(y) > u_i(x)\}$ for each $x \in W$. This in turn gives the preference correspondences P_M, for $M \in \mathcal{D}$, and P_σ as in the previous section. Because of the smoothness assumption on preference, the preference fields so constructed satisfy a continuity property which we call S-continuity. A preference field $H:W \to \mathbb{R}^W$ is called S-continuous iff for each $x \in W$ such that $H(x) \neq \phi$ the following properties are satisfied: (a) H(x) is open in $\mathbb{R}^W$ and does not contain 0, (b) if $v \in H(x)$ then there is a neighbourhood U of x and a continuous function (also called a vector field) $X:U \to \mathbb{R}^W$ such that $X(x') = v$ for all $x' \in U$, (c) if $X:U \to \mathbb{R}^W$ is a vector field, where U is a neighbourhood of x, and $X(x) \in H(x)$, then there is a neighbourhood U' of x in U such that $X(x') \in H(x')$ for all $x' \in U'$.

A second property that concerns us is whether the field is half open. A set θ in $\mathbb{R}^W$ is half open iff there exists a linear function $f:\mathbb{R}^W \to \mathbb{R}$ such that $f(v) > 0$ for all $v \in \theta$. A set $\{v_1,\ldots,v_r\}$ of vectors in $\mathbb{R}^W$ is called semipositively dependent iff there exists a semipositive vector $\lambda = (\lambda_1\ldots,\lambda_r)$ such that $\sum_{j=1}^{r} \lambda_j v_j = 0$. If

θ contains a semipositively dependent set of vectors then it cannot be half open. A preference field is called half open iff for all x such that $H(x) \neq \phi$, $H(x)$ is half open.

Clearly the preference field $H_i : W \rightarrow \mathbb{R}^W$ for each $i \in N$ is S-continuous and half open, and therefore for each coalition $M \subset N$, H_M is also S-continuous and half open. For any voting game H_σ will also be S-continuous but need not be half open.

S-continuity of a preference field permits integration of the field in the following way. A smooth curve $c:[0,1) \rightarrow W$ from $c(0) = x$ to $y = \lim_{t \to 1} c(t)$ is called an integral curve of the field H from x to y iff for each $t \in [0,1)$ the derivative $[c](z)$ at $z = c(t) \in W$ belongs to $H(z)$. By S-continuity, if $H(x) \neq \phi$ then there exists some integral curve from x to a point y in a neighbourhood of x.

For a voting game σ and smooth profile u, we write $y\rho_M x$ whenever there is an integral curve from x to y for the preference field H_M, and write $y\rho x$ whenever $y\rho_M x$ for some $M \in \mathcal{D}$.
Note in particular that if $H_M(x) = \phi$ then $y\rho_M x$ for no $y \in W$.
On the other hand if $H_M(x) \neq \phi$ then there is some $y \in W$ such that $y\rho_M x$, and thus $y\rho x$.

Our analysis is concerned with the following three sets.

Definition

Let σ be a voting game on W, and let u be a smooth profile. A point x belongs to

i) the infinitesimal optima set, $IO(\sigma,u)$, for (σ,u) iff $H_\sigma(x) = \phi$ (i.e. $x \in IO(M,u)$ for all $M \in \mathcal{D}$).

ii) the infinitesimal cycle set, $IC(\sigma,u)$, for (σ,u) iff $H_\sigma(x)$ is not half open.

iii) the local cycle set, $LC(\sigma,u)$, iff for any neighbourhood U of x there exists a local ρ-cycle $x\rho y_1 \rho y_2 \ldots y_r \rho x$ where the preference curve associated with each component of this cycle belongs to U.

In [14] it was shown that $IC(\sigma,u)$ is open and if $x \in IC(\sigma,u)$ then there exists a neighbourhood U of x and a voting trajectory

$$z\rho y_r \rho y_{r-1} \ldots . \rho x$$

from x to any point z in U. Consequently $IC(\sigma,u)$ belongs to $LC(\sigma,u)$. On the other hand with weaker continuity assumptions it can be shown [17] that $LC(\sigma,u)$ is contained in the closure of $IC(\sigma,u)$.
Note also that if a point x does not belong to $LC(\sigma,u)$ then there is a neighbourhood U of x such that the preference correspondence $P_\sigma : U \rightarrow U$, restricted to U, contains no cycles.

We shall now show with the appropriate dimension restriction on the space W, that the set $IC(\sigma,u)$, for any smooth profile u, must be empty. This in turn implies that there can exist no local cycles for the relation ρ.

We require an alternative characterization of a point in $IC(\sigma,u)$. For a voting game σ and smooth profile u, and point $x \in W$, let

$$\mathcal{D}(x) = \{M \in \mathcal{D} : H_M(x) \neq \phi\}.$$

For each $M \in \mathcal{D}(x)$ let $p_M(x)$ be the convex hull in $\mathbb{R}^w$ of the vectors $\{p_i(x): i \in M\}$. Since it is known that $H_M(x) = \phi$ iff $\{p_i(x): i \in M\}$ are semipositively dependent [19,20], the set $p_M(x)$ is compact, convex and does not contain the origin. Define $p_\sigma(x) = \bigcap_{M \in \mathcal{D}(x)} p_M(x)$.
By [14, lemma 2.1], $x \in IC(\sigma,u)$ iff $p_\sigma(x) = \phi$. The set $p_\sigma(x)$ has also been called the directional core or the set of undominated directions [12] of the game (σ,u) at x.

Theorem 2 Let σ be a voting game on an open set W in $\mathbb{R}^w$, where $w \leq v^*(\sigma)$. Then $IC(\sigma,u) = \phi$ for any smooth profile u.

Proof We show that, under the dimension restriction, at every point $x \in W$ it is the case that $p_\sigma(x) \neq \phi$.
By Helly's Theorem [1], if $\{A_1,\ldots,A_{w+k}\}$, with $k > 0$, is a family of convex compact sets in $\mathbb{R}^w$ such that the intersection of every subfamily of cardinality (w+1) is nonempty, then the intersection of the entire family is nonempty.
Consider a subfamily $\mathcal{D}'$ of $\mathcal{D}(x)$ with cardinality $|\mathcal{D}'| \leq v^*(\sigma)+1$.
By the definition of the Nakamura number, there exists a collegium $C(\mathcal{D}')$.
Pick $i \in C(\mathcal{D}')$. Since $i \in \mathcal{D}' \subset \mathcal{D}(x)$, $p_i(x) \neq 0$. Moreover $p_i(x) \in p_M(x)$ for each $M \in \mathcal{D}'$. Thus

$$\phi \neq \bigcap_{\mathcal{D}'} p_M(x).$$

Furthermore each $p_M(x)$, for $M \in \mathcal{D}(x)$, is a compact convex set. Since any subfamily of $\{p_M(x): M \in \mathcal{D}(x)\}$, of cardinality at most $v^*(\sigma)+1$, has non empty intersection, and $w \leq v^*(\sigma)$ by assumption, the entire family has nonempty intersection, and so $p_\sigma(x) \neq \phi$. Since this argument is valid at every point in W, $IC(\sigma,u) = \phi$.

In the proof of this theorem no convexity assumptions on preference are required. Indeed the theorem is valid when W is a smooth manifold; the modification to the proof being that at any point on the boundary the tangent space T_xW is defined to be isomorphic to a closed half space of $\mathbb{R}^w$. Although the theorem shows non existence of local cycles, it might well be the case that the preference correspondence $P_\sigma: W \rightarrow W$ is cyclic. Consequently Walker's [24] theorem on the existence of a choice for an acyclic preference correspondence on a compact space cannot be used to show existence of an equilibrium. However if W is admissible then we can adapt the Ky Fan Theorem to show existence of optima.

When W is an admissible space of dimension w, define the tangent space T_xW at x to be the subset of $\mathbb{R}^w$ s.t. $v \in T_xW$ iff there exists $y \in W$ and $\lambda \in \mathbb{R}, \lambda \geq 0$, such that $v = \lambda(y-x)$. A preference field $H: W \rightarrow \mathbb{R}^w$ is now required to satisfy $H(x) \subset T_xW$ for all $x \in W$. The previous definitions of S-continuity and of the sets $IO(\sigma,u)$ etc. then go through with this modification. Since $p_\sigma(x) \neq \phi$ implies that $x \notin IC(\sigma,u)$, Theorem 2 is valid when W is an admissible space.
We now show that when W is admissible and $IC(\sigma,u)$ is empty then $IO(\sigma,u)$ must be nonempty.

Given the voting game σ, and a smooth profile u, let $H_\sigma: W \rightarrow \mathbb{R}^w$ be the preference field and define the preference correspondence $\bar{H}_\sigma: W \rightarrow W$ by

i) if $H_\sigma(x) = \phi$ then $\bar{H}_\sigma(x) = \phi$

ii) if $H_\sigma(x) \neq \phi$ then $\bar{H}_\sigma(x) = \{z \in W: z = x+v, \text{any } v \in H_\sigma(x)\}$.

Note first of all that at any point $x \in W$, T_xW is a cone in $\mathbb{R}^W$ (i.e. if $v \in T_xW$ then $\lambda v \in T_xW$ for any strictly positive $\lambda \in \mathbb{R}$). Thus $H_M(x)$, and hence $H_\sigma(x)$, are cones in T_xW. Moreover if $H_\sigma(x) \neq \Phi$ then there is some $y \in W$ such that $\lambda(y-x) \in H_\sigma(x)$ and so $y \in \bar{H}_\sigma(x)$.

Theorem 3 If σ is a voting game on an admissible set W and u is a smooth profile then $IC(\sigma,u) = \phi$ implies that $IO(\sigma,u) \neq \phi$.

Proof i) We show first of all that the preference correspondence $\bar{H}_\sigma$ is lower demi-continuous. Suppose that $y \in \bar{H}_\sigma^{-1}(x)$ for some $x \in W$. Thus $x \in \bar{H}_\sigma(y)$ and $x = y+v$ for some $v \in H_\sigma(y)$.

Let $c:[0,1) \to W$ be defined by

$$c(t) = y + tv .$$

Clearly $[c](y) = v$,

Consider the vector field $X(z) = x-z$. Then

$$X(y) = (x-y) = v = [c](y).$$

Since $v \in H_\sigma(y)$, by S-continuity of H_σ there is a neighbourhood V of y such that $X(z) \in H_\sigma(z)$ for all $z \in V$. Thus for all $z \in V$, $x-z \in H_\sigma(z)$ and so $x = z+(x-z) \in \bar{H}_\sigma(z)$. Hence $V \subset \bar{H}_\sigma^{-1}(x)$.

Hence $\bar{H}_\sigma^{-1}(x)$ is open and $\bar{H}_\sigma$ is ldc.

ii) By the definition of H_σ, at no point x is $0 \in H_\sigma(x)$. Thus $x \notin \bar{H}_\sigma(x)$.

If $x \in \text{Con}\bar{H}(x)$ then there exists a solution to $x = \sum_{j=1}^{r} \lambda_j z_j$ where $\sum_{j=1}^{r} \lambda_j = 1$ and $\lambda_j \geq 0$, $z_j \in \bar{H}_\sigma(x)$ for $j=1,\ldots,r$.

By definition $z_j = x + v_j$ where each $v_j \in H_\sigma(x)$.

Hence $\sum_{j=1}^{r} \lambda_j v_j = 0$ for semipositive $\lambda = (\lambda_1,\ldots,\lambda_r)$.

By [14, lemma 3.5] this can be the case iff $x \in IC(\sigma,u)$.

Consequently if $IC(\sigma,u) = \phi$ then $\bar{H}_\sigma$ is semi-convex.

By the Ky Fan Theorem there exists a choice $\bar{x} \in W$ such that $\bar{H}_\sigma(\bar{x}) = \phi$.

But $\bar{H}_\sigma(\bar{x}) = \phi$ iff $H_\sigma(\bar{x}) = \phi$. Hence $IO(\sigma,u) \neq \phi$. ∎

Corollary 1

Let σ be a voting game on an admissible set W in $\mathbb{R}^W$, where $w \leq v^*(\sigma)$. Then $IO(\sigma,u) \neq \phi$ for any smooth profile u.

Proof By Theorem 2, $IC(\sigma,u) = \phi$, and thus by Theorem 3, $IO(\sigma,u) \neq \phi$. ∎

Note that if $x \notin IO(\sigma,u)$, then from the definitions, there exists a point $y \in W$ such that $y \rho_M x$ for some $M \in \mathcal{D}$, and so $y \in P_\sigma(x)$. Consequently x cannot belong to the choice set $C(P_\sigma,W)$. Hence $C(P_\sigma,W)$ belongs to $IO(\sigma,u)$. Corollary 1 does not show existence of the choice set $C(P_\sigma,W)$ unless we assume additional convexity properties on preference.

Say that a profile $P = (P_1,\ldots,P_n)$ of preference correspondences for society N satisfies the convexity property iff for each $i \in N$, then for $x \in W$ either $P_i(x) = \Phi$ or there exists a vector $p_i(x)$ of x, of unit length, such that $P_i(x)$ belongs to the half space

$$\bar{H}_i(x) = \{y \in W: p_i(x).(y-x) > 0\},$$

where $p_i(x).(y-x)$ is the scalar product of $p_i(x)$ and $(y-x)$.

As above, when $P = (P_1,\ldots,P_n)$ satisfies the convexity property, define $\bar{H}_M(x)$, $p_M(x)$, for each $M \subset N$, and $\bar{H}_\sigma(x)$, $p_\sigma(x)$ in the obvious way. With the convexity property it is clear that $P_\sigma(x) \subset \bar{H}_\sigma(x)$, for each $x \in W$.

In the case of the smooth profile, $u = (u_1,\ldots,u_n)$ satisfying the convexity property for each $i \in N$, the cone $\bar{H}_i(x)$ may be identified with $\{y \in W: du_i(x)(y-x) > 0\}$.

For such a profile, if $x \in IO(\sigma,u)$ then $P_\sigma(x) = \phi$.

Thus we obtain the following corollary.

<u>Corollary 2</u>

Let σ be a voting game on an admissible set W in $\mathbb{R}^W$, where $w \le v^*(\sigma)$. Then, if u is a smooth profile satisfying the convexity property, the preference correspondence P_σ given by this profile has a choice $\bar{x}$ in W. ∎

We are now in a position to present a proof of Theorem 1 using the method of proof of Theorem 2.

<u>Theorem 4</u> Let σ be a voting game on an admissible space W of dimension w, where $w \le v^*(\sigma)$. If $P = (P_1,\ldots,P_n)$ is a preference profile which satisfies the convexity property, and if each $P_i, i=1,\ldots,n$, is lower demi-continuous, then there exists a choice of P_σ from W. Moreover for any $x \in W$, there exists a neighbourhood U of x such that the preference correspondence $P_\sigma: U \to U$ has no cycles.

<u>Proof</u> We seek first of all to show that P_σ is lower demi-continuous.

Suppose that for some $x,y \in W$, it is the case that $y \in P_\sigma^{-1}(x)$. By definition this implies that $x \in P_\sigma(y)$ and therefore, for some $M \in \mathcal{D}$, that $x \in P_M(y)$, and so $x \in P_i(y)$ for all $i \in M$. For each $i \in M, P_i^{-1}(x)$ is open, and so there exists an open neighbourhood U_i of y such that $U_i \subset P_i^{-1}(x)$. Thus for all $z \in U = \bigcap_{i \in M} U_i$, it is the case that $z \in P_M^{-1}(x)$ for some $M \in \mathcal{D}$. Hence $U \subset P_\sigma^{-1}(x)$ and so P_σ is lower demi-continuous. We now show P_σ must be semi-convex.

By assumption, for each $i \in N$, and each $x \in W$, either $P_i(x) = \phi$ or there exists $p_i(x) \in \mathbb{R}^W$ such that

$$P_i(x) \subset \bar{H}_i(x) = \{y \in W: p_i(x).(y-x) > 0\}.$$

When $P_i(x) = \phi$ define $\bar{H}_i(x) = \phi$.

Define $\bar{H}_M(x)$, $\bar{H}_\sigma(x)$, $p_M(x)$ and $p_\sigma(x)$ as before and let

$$IC(\sigma,P) = \{x \in W: x \in \mathrm{Con}\bar{H}_\sigma(x)\}.$$

Proceed exactly as in the proof of Theorem 2 to show that at every point $x \in W$, $p_\sigma(x) \neq \phi$.

As in the proof of Theorem 3 this implies that $x \in \mathrm{Con}\bar{H}_\sigma(x)$ for no $x \in W$. Consequently $IC(\sigma,P)$ is empty and so both $\bar{H}_\sigma: W \to W$ and $P_\sigma: W \to W$ are semi convex.

By the Ky Fan Theorem, there exists a choice of $\bar{H}_\sigma$, and thus of P_σ, from W. Moreover since $IC(\sigma,P) = \phi$, at every point $x \in W$ it is the case that there exists a neighbourhood U of x such that the correspondence $P_\sigma: U \to U: x \mapsto P_\sigma(x) \cap U$ is acyclic. ∎

Proof of Theorem 1

By assumption each P_i is continuous and therefore lower demi-continuous.

If $P_i(x) \neq \phi$, then it is both open and semi convex. Thus there exists an open half space $\bar{H}_i(x)$, in $\mathbb{R}^W$, which contains $P_i(x)$. Thus $P = (P_1,\ldots,P_n)$ satisfies the conditions on preference of Theorem 4. The result follows .

CONCLUSION

The principal result of this paper is that, given a voting game, if the policy space is of dimension no greater than the stability dimension $v^*(\sigma)$, then at every point x there exists a nonempty directional core $p_\sigma(x)$.

The directional core $p_\sigma(x)$ at point x may also be thought of as a generalized gradient of the social welfare function associated with the voting procedure. The existence of this generalized gradient at every point is effectively a necessary and sufficient condition that the voting procedure is well behaved. It also turns out that various planning procedures that have been suggested to allocate public and private goods are associated with integral curves of this generalized gradient. See [12],[25] for an extensive discussion.

Although the analysis has been conducted in terms of a voting game, the procedures can be used for more general coalition decision making mechanisms. For example suppose we require a social preference correspondence $P:W \to W$ to be continuous. Then by [8],[23] the mechanism

must essentially be a local voting game, with a Nakamura number that may vary at different points in the policy space. Nonetheless the procedure used here can be used to determine whether a generalised social preference correspondence of this form exhibits an equilibrium.

Finally,when $p_\sigma(x) = \phi$ at some point x for such a procedure, manipulation of the agenda can lead to almost all points near enough to x [11]. Thus the dimensionality restriction presented here is one which is sufficient to prevent manipulation of this kind.

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