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EXISTENCE OF EQUILIBRIUM ON A MANIFOLD

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Forthcoming in Mathematics of Operation Research

SOCIAL SCIENCE WORKING PAPER 482

July 1983

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ABSTRACT

Existence of equilibrium of a continuous preference relation p or correspondence P on a compact topological space W can be proved either by assuming acyclicity or convexity (no point belongs to the convex hull of its preferred set). Since both properties may well be violated in both political and economic situations, this paper considers instead a "local" convexity property appropriate to a "local" preference relation or preference field. The local convexity property is equivalent to the non existence of "local" cycles. When the state space W is a convex set, or is a smooth manifold of a certain topological type, then the "local" convexity property is sufficient to guarantee the existence of a set of critical optima.

#An earlier version of this paper was presented at the International Conference on the Economics of Information, Luminy, Marseille, September 1981, and the final version prepared while the author was Hallsworth Research Fellow in Political Economy at Manchester University. Support from the Nuffield Foundation is also gratefully acknowledged. Discussion with Mike Martin of Essex University was extremely helpful.

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1. Introduction

A fundamental question in both economic theory and social choice is whether a preference relation p on some set of feasible states or alternatives admits an equilibrium, choice or maximal element. The relation p might be defined in terms of the preferred choices of a number of winning coalitions, or result from the combination of preferred choices of individuals in the society.

In social choice theory if the set W of alternatives is finite and the preference relation p is acyclic (admitting no cycle $x\ p\ y_1\ py_2 \ldots px$) then there is a choice $\bar{x}$ such that $y\ p\ \bar{x}$ for no $y \in W$. When W is a compact topological space, and the preference correspondence is continuous, in the sense that for each x the set $\{y \in W: x\ p\ y\}$ is open in W, then again there is a choice.

The Ky Fan proof of existence of a choice assumes these two topological properties (compactness of W and continuity of p) but instead of acyclicity assumes that W is convex, and p is semi-convex (a point never belongs to the convex hull of its preferred set).

In social choice theory, and indeed in the theory of voting games, it is not generally the case that acyclicity can be assumed, whereas in economic theory convexity, particularly of feasible or production sets, cannot readily be assumed.

However, when the preference is not convex, if the state space W is a smooth manifold, then it is often possible to approximate the preference relation by a preference field H from W into the tangent space TW. At a point x the preference cone H(x) is the set of "small" preferred directions of change. It is shown that the appropriate convexity property is directly analogous to the property that at no point x in W does the origin, in the tangent space above x, belong to the convex hull of H(x). This property implies that, in some neighbourhood of the point x, there exists no "local" cycles of the underlying preference relation. When W is a convex set then this property implies the existence of a critical optimum for the preference field.

As an application consider the pareto preference field for a society, defined in terms of small changes that benefit all individuals. When the space is convex then there exists a set of critical optima, where the field is empty. The set corresponds, in the multi-objective problem, to the set of singular points of a single smooth function.

The result on existence of a critical optima set is also valid when W is a smooth compact manifold with non-zero Euler characteristic (as for example a topologically complete and contractible manifold).

Theoretically at least the results suggest a procedure for examining existence of equilibrium in a political economic environment. Suppose the economy has given resources and production functions for private goods, and a smooth production function able to transform private goods to public goods. It seems natural to suppose that this model defines a smooth manifold W of possible states.

The preferences of the society are then made up of individual preferences for private goods and political preferences for public goods. Local properties of these preferences then define a preference field, and the topological features of this field and of the state space W can then be used to determine whether local cycles exist and thus whether critical optima exist.

2. Existence of Equilibrium

Let W be a set of alternatives, and $p \subset W \times W$ a binary (preference) relation. We write xpy to mean x is preferred to y, where x and y are members of W.

An important problem in economic theory and social choice concerns those properties of p and W which are sufficient to guarantee the existence of a choice, or equilibrium, for p on W. Here an element $\bar{x} \in W$ is maximal or a choice for p on W iff $\bar{x}$ belongs to the choice set

$$Cp(W) = \{x \in W : ypx \text{ for no } y \in W\}.$$

In social choice theory the question is often posed in terms of existence of a family $\mathcal{J}$ of subsets of W, such that for each $V \in \mathcal{J}$ the choice set $Cp(V) = \{x \in V: ypx \text{ for no } y \in V\}$ is non empty. In this case $Cp: \mathcal{J} \to 2^W$ (where 2^W is the power set of W) is known as a choice function.

When W is a finite set of alternatives then acyclicity of p is sufficient to guarantee that Cp is a choice function for all subsets of W (Sen, 1970).

Here the relation p is acyclic on the set W iff it is impossible to find a subset $\{x_1,\ldots,x_r\}$ in W such that $x_1px_2px_3\ldots x_rpx_1$.

Notice that acyclicity implies both asymmetry (xpy implies not (ypx)) and irreflexivity (xpx for no $x \in W$).

When W is not finite, but has a topology, then a continuity property for p is required for existence of a choice. Given the relation define the upper and lower (point to set) preference correspondences by:

$$P : W \to W: x \mapsto \{y \in W: ypx\}$$
$$P^{-1}: W \to W: x \mapsto \{y \in W: xpy\}$$

Say that the preference relation p (or equivalently its preference correspondence P) is lower demi-continuous on W iff W is a topological space, and for every $x \in W$, the set $P^{-1}(x)$ is open in the topology on W.

Walker's Theorem (1977)

If W is a compact topological space, and p an acyclic and lower demi-continuous preference then Cp(W) is non empty.

Proof Suppose that $Cp(W) = \Phi$. Then for every $y \in W$ there exists some $x \in W$ such that xpy, or $y \in P^{-1}(x)$. Hence $K = \{P^{-1}(x): x \in W\}$ is an open cover of W. Since W is compact, there exists a finite subcover $\{P^{-1}(x): x \in A\}$ of K, where A is a finite subset of W. But then, for every $y \in A$ there exists some $x \in A$ such that $y \in P^{-1}(x)$, or xpy. Hence $Cp(A) = \Phi$. Since A is finite, this contradicts the result that an acyclic preference relation on a finite set has a choice. Thus Cp(W) is non empty. ∎

Moreover if we let $\mathcal{J}$ be the set of compact subsets of W, then $Cp: \mathcal{J} \to 2^W$ will be a choice function. To see this we endow each member

V of $\mathcal{J}$ with the relative topology, and define $Cp(V) = \{x \in V : P(x) \cap V = \Phi\}$. Clearly $P^{-1}(x) \cap V$ will be open in the relative topology on V, and so, by the previous result, $Cp(V) \neq \Phi$.

The same result can be obtained if acyclicity of p is weakened to the <u>finite maximality property</u>, that for each finite set $A \subset W$, there exists some point x_A in A such that $P(x_A) \cap A = \Phi$.

(See Bergstrom 1975a, and Birchenhall, 1977, for further discussion of the relationship between acyclicity and existence of a choice).

An alternative proof of existence of a choice assumes convexity of the preference correspondence P and the space W instead of acyclicity of the preference relation.

Call the space W an <u>admissible space</u> if it is a convex compact (subset of a) topological linear space Y. Say the preference relation p (or alternatively the preference correspondence P) is i) <u>convex</u> iff for all $x \in W$, either P(x) is empty or is a convex set in W, ii) <u>semi-convex</u> iff for no x in W is it the case that $x \in ConP(x)$, where ConP(x) is the convex hull of the set P(x) in the convex set W.

Ky Fan (1961) has shown the existence of a choice for a lower demi-continuous, semi-convex preference correspondence on an admissible space.

The proof of the Ky Fan Theorem depends on the following theorem by Knaster-Kuratowski-Mazurkiewicz (1929)

First of all let $\{e_j : j = 1,\dots,n\}$ be the standard basis of $\mathbb{R}^n$, and for any subset M of $N = \{1,\dots,n\}$ let Δ_M be the convex hull of

$\{e_j : j \in M\}$. In particular

$$\Delta_N = \{(x_1,\dots,x_n) \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_j = 1\}.$$

<u>KKM Theorem</u>

Let $\{K_j : j \in N\}$ be a family of closed subsets of Δ_N. For each subset M of $\{1,\dots,n\}$ let

$$K_M = \bigcup_{j \in M} K_j .$$

Suppose for each $M \subset N$, $\Delta_M \subset K_M$, then there exists $v \in \Delta_N$ such that

$$v \in \bigcap_{j \in N} K_j$$ ■

Ky Fan has extended this theorem. Suppose first of all that X is a subset of a topological vector space Y. For any set of points $\{x_i \in X : i \in M\}$ indexed by a finite index set let X_M be the convex hull of $\{x_i \in X : i \in M\}$.

<u>Ky Fan Lemma</u>

Let $K: X \to Y$ be a (point set) correspondence, where for each $x \in X$, K(x) is a closed set in Y, and there exists $y \in X$ such that K(y) is compact. Suppose that for any finite set of points $\{x_j \in X : j \in M\}$ the convex hull of this set, X_M, belongs to $K_M = \bigcup_{j \in M} K(x_j)$.

Then $\bigcap_{x \in X} K(x)$ is non empty.

<u>Outline of Proof</u> Directly from the KKM theorem it is possible to show that for each finite set of points $\{x_j \in X : j \in M\}$ the intersection $\bigcap_{j \in M} K(x_j)$ is non empty. For each $x \in X$, let $P(x) = X \setminus K(x)$, noting that

P(x) is open.

Now $\bigcup_{x \in X} [K(y) \cap P(x)] = K(y) \setminus \bigcap_{x \in X} K(x)$,

so when $\bigcap_{x \in X} K(x)$ is empty, $\{K(y) \cap P(x) : x \in X\}$ is an open cover for the compact set K(y).

But then there exists a finite index set M such that $\bigcup_{j \in M} [K(y) \cap P(x_j)] = K(y)$, implying that $\bigcap_{j \in M} K(x_j) = \Phi$.

By contradiction, $\bigcap_{x \in X} K(x)$ is non empty. ∎

Ky Fan used the above lemma to prove the following theorem.

Ky Fan Theorem (1961)

Let W be an admissible space and $P:W \to W$ a preference correspondence such that i) P is irreflexive (ie. $x \in P(x)$ for no $x \in W$ ii) P is convex iii) the graph of P is open in W x W. Then there exists a choice $\bar{x} \in W$ such that $\bar{x} \in Cp(W) = \{x \in W : P(x) = \Phi\}$.

The proof procedure of Ky Fan was used by Bergstrom (1975b) to somewhat strengthen this theorem, by weakening the assumption of irreflexibility and convexity of P to semi-convexity, and weakening the requirement of an open graph to lower demi-continuity.

Bergstrom Theorem (1975b)

Let P be a lower demi-continuous and semi-convex correspondence on an admissible space W. Then there exists a choice $\bar{x} \in W$ such that

$\bar{x} \in Cp(W) = \{x \in W : P(x) = \Phi\}$.

Outline of Proof

Define the (weak) preference correspondence $R:W \to W$ by $R(x) = W \setminus P^{-1}(x)$. Since P is lower demi-continuous, R(x) is closed in W for each $x \in W$. Now $y \in R(x)$ iff not(xpy) where p is the preference relation defined by P (i.e. xpy iff $x \in P(y)$). Thus $Cp(W) = \bigcap_{x \in W} R(x)$.

Let A be any finite set $\{x_j : j \in M\}$ of points in W, and suppose that the convex hull of A is not contained in

$$R_M = \bigcup_{j \in M} R(x_j).$$

Then there exists some $y \in Con(A)$ such that $y \in R(x_j)$ for no $j \in M$. Thus $y \in P^{-1}(x_j)$ for all $j \in M$, and so $x_j \in P(y)$ for all $j \in M$. Thus the convex hull of $\{x_j : j \in M\}$ belongs to ConP(y) and so $y \in ConP(y)$. This violates semi-convexity of P. By contradiction

$$Con(A) \subset R_M.$$

By the Ky Fan Lemma, $\bigcap_{x \in W} R(x)$ is non empty. Thus there exists a choice $\bar{x} \in Cp(W)$ for P in W, such that $P(\bar{x}) = \Phi$. ∎

Closer examination of the proof shows that the result can be somewhat strengthened. If $A = \{x_j : j \in M\}$ is a finite set of points, then under the conditions of the theorem $\bigcap_{x \in A} R(x)$ is non empty and contained in the convex hull of A. Thus for any such finite set, A, in W, there exists a choice $y \in Con(A)$ such that

$$P(y) \cap Con(A) = \Phi.$$

Since the continuity and convexity properties of P are preserved on the family $\mathcal{J}$ of admissible subsets of W, Bergstrom's Theorem shows that Cp is a choice function on $\mathcal{J}$.

Sonnenschein (1971) independently obtained a result similar to the Ky Fan Theorem and used it to show existence of demand functions in economic environments in the absense of individual preference transitivity (viz. xPy and yPz implies xPz for any x,y,z).

As Sonnenschein noted, his result could be used to show existence of a choice in the absence of acyclicity. Sonnenschein's version of the theorem has also been used by Shafer and Sonnenschein (1975) to generalise the classical results of Debreu (1952) and Arrow and Debreu (1954) on existence of a competitive economic equilibrium, and by Borglin and Keiding (1976) to show existence of Nash Equilibrium. Arrow (1969) has also used the result to show existence of a voting equilibrium in a majority rule game with an infinite electorate.

Finally Aliprantis and Brown (1982) have exploited the fact that the Ky Fan theorem is true when W has arbitrary dimension to show the existence of a price equilibrium associated with a demand correspondence satisfying Walras' Law, when the price simplex is infinite dimensional.

In case W is finite dimensional then the continuity requirement can be further weakened. Say that the preference correspondence $P:W \to W$ is <u>lower hemi-continuous</u> (lhc) iff for every open set U in W the set

$\{x \in W: P(x) \cap U \neq \Phi\}$ is also open in W.

Clearly P is lhc. iff whenever $y \in W$ and U is an open neighbourhood of y in W, st. $P(x) \cap U \neq \Phi$ then there is a nbd. V of x, such that $P(x') \cap U \neq \Phi$ for all x' in V. A preference correspondence which is

lower demi-continuous is also lower hemi-continuous. To see this suppose $P(x) \cap U \neq \Phi$ for some $x \in W$, and some open set U in W. Then there exists $y \in U$ such that $x \in P^{-1}(y)$.

By assumption $P^{-1}(y)$ is open, amd so there is a neighbourhood V of x such that $V \subset P^{-1}(y)$. Then, for all $x' \in V$, $y \in P(x')$ and so $P(x') \cap U \neq \Phi$. Hence P is lhc.

In the case that W is an admissible space in the finite dimensional euclidean space $\mathbb{R}^W$ we may weaken the assumption of lower demi-continuity of P to lower hemi-continuity in the Ky Fan Theorem. First of all, if $\psi:W \to W$ is a lhc. correspondence on a subset W of $\mathbb{R}^n$ such that for each $x \in W$, $\psi(x)$ is a non empty and convex set, then by Michael's selection theorem (1956) there exists a continuous function $f:W \to W$ such that, for each $x \in W$, $f(x) \in \psi(x)$.

Suppose now that P is a semi-convex and lsc correspondence on an admissible space W in $\mathbb{R}^W$. Since P is lsc, the correspondence $\psi:W \to W$, where $\psi(x) = \text{Con}P(x)$, is also lhc. If $P(x) = \Phi$ for no x in W, then by Michael's selection theorem, there exists a continuous selection $f:W \to W$ of ψ. By Brouwer's fixed point theorem, since W is compact and convex, there exists a fixed point $\bar{x} = f(\bar{x})$. But then $\bar{x} \in \text{Con}P(\bar{x})$ contradicting semi-convexity of P. Thus there exists a choice $\bar{x}$ of P from W such that $P(\bar{x}) = \Phi$.

A related result by Gale and MasCollel (1975) shows the existence of a <u>Nash equilibrium</u> for a society $N = \{1,\ldots,n\}$. Suppose each player $i \in N$ has an admissible strategy space W_i in $\mathbb{R}^W$. Let $W = \prod_i W_i$ and let $P_i:W \to W_i$ be the i^{th} players convex valued lower demi-continuous preference correspondence (ie. for each $x \in W_i$, $P_i^{-1}(x)$ is open in W).

Let $U_i = \{x \in W : P_i(x) \neq \phi\}$. Then $P_i : U_i \to W_i$ is lower hemi-continuous on U_i and by Michael's selection theorem has a continuous selection $f_i : U_i \to W_i$. Let $\theta_i : W \to W_i$ be the correspondence taking the value $f_i(x)$ for $x \in U_i$, and W elsewhere.

Then $\theta = \Pi_i \theta_i : W \to W$ is an upper hemi continuous correspondence (ie if $\theta(x)$ belongs to an open set V in W, then there is an open set U of x in W such that $\theta(x')$ belongs to V for all $x' \in U$). By the Kakutani (1941) theorem, there exists a fixed point $\bar{y} \in \theta(\bar{y})$. If we assume that for no $x \in W$ is $x \in \Pi_i P_i(x)$, then the fixed point $\bar{y}$ must be a Nash equilibrium satisfying $P_i(\bar{y}) = \Phi$ for all i.

Clearly this result can be obtained directly from the Ky Fan theorem, illustrating the close relationship between the Ky Fan Theorem and fixed point theorems for compact convex sets.

Although the Ky Fan-Sonnenschein-Bergstrom theorem has wide application in economic theory and game theory, one obvious difficulty is the assumption that the preference correspondence be semi-convex. In the usual applications the preference correspondence P at x may be defined to be a subset of $\psi(x)$, where $\psi(x)$ is a feasible set dependent on the state of the world, x. In many instances $\psi(x)$ will be a production set, and it would be useful to be able to extend the analysis to non convex production sets.

This difficulty is even more pronounced in social choice applications. Consider for example a simple majority voting game with three individuals, on a compact policy space W of two dimensions. Suppose that the preferences

of the three individuals are convex and represented by smooth utility functions. In figure 1 are drawn indifference curves for the individuals through a point x. Since any two individuals form a winning coalition, the socially preferrred set P(x) at x is the union of the three shaded sets in the diagram. Clearly x belongs to the convex hull of P(x), violating the semi-convexity requirement of the Ky Fan Theorem. If we let p_i represent the preference relation of player i, then we see that ap_1bp_2c, bp_2cp_2a and cp_3ap_3b. Thus we obtain a social preference cycle apbpcpa (see Kramer, 1973, and Schofield, 1977 for further discussion of this example).

[Insert Figure 1 about here]

This indicates that there is a connection between acyclicity and semi-convexity of a social preference correspondence.

The purpose of the rest of this paper is to obtain an equilibrium theorem which does not implicitly make use of global convexity or acyclicity properties. Instead we consider a preference field, which at each point specifies the direction of preferred change. The continuity property which we assume for this field allows us to examine a "small" preference correspondence, or deformation of the space W. The construction permits the use of general topological arguments to show existence of a choice or equilibrium when the space W has a certain topological nature.

In particular it is shown that on an admissible space, the non existence of "local" cycles is equivalent to a "local" convexity property, and thus sufficient to guarantee the existence of a "local" equilibrium. The procedure is analogous to the proof of existence of a singular point for a smooth function.

3. Equilibrium of a Preference Field

Let W be a smooth manifold and TW the tangent bundle, with $\Pi = TW \to W$ the bundle map. At a point $x \in W$, $\Pi^{-1}(x)$ is the tangent space at x, isomorphic to $\mathbb{R}^w$, where w is the dimension of W. A point in $\Pi^{-1}(x)$ is called a tangent vector at x. (See for example Hirsch, 1976, for a full discussion). The tangent space is locally trivial in the sense that for a neighbourhood U of x, $\Pi^{-1}(U) \simeq U \times \mathbb{R}^w$. If $u:W \to \mathbb{R}$ is a smooth function, then the differential du(x) of u at x may be regarded as a linear function $du(x):\Pi^{-1}(x) \to \mathbb{R}$. In the case that W is a subset of euclidean space, then the differential du is a continuous function $du:W \to L(\mathbb{R}^w,\mathbb{R})$ where $L(\mathbb{R}^w,\mathbb{R})$ is the space of linear functions from $\mathbb{R}^w$ to $\mathbb{R}$ with the usual topology.

Definition 1

i) A vector field, X, is a smooth function $X:W \to TW$ such that for each $x \in W$ at which x is defined, $X(x) \in \Pi^{-1}(x)$.

ii) A preference field $H:W \to TW$ is a correspondence which satisfies, whenever H(x) is non empty, the following properties

a) $H(x) \in \Pi^{-1}(x)$

b) H(x) is a cone, i.e. if $v \in H(x)$ is a tangent vector, then $\lambda v \in H(x)$ for all real $\lambda > 0$.

c) $0 \notin H(x)$.

iii) A vector field X is integral for a preference field H on an open set $V \subset W$ iff $X(x) \in H(x)$ for all $x \in V$ (note that $X(x) \neq 0$ for all $x \in V$).

iv) A preference field H is continuous iff for each $x \in W$ such that $H(x) \neq \Phi$ it is the case that for each $v \in H(x)$ there is a neighbourhood V(x) of x in W and a vector field X, integral for H on V(x) with X(x) = v.

v) A preference field H is S-continuous iff H is continuous and, for each $x \in W$ it is the case that a) H(x) is open in $\Pi^{-1}(x)$ and b) if $v \in H(x)$ and X is a vector field such that X(x) = v, then X is integral for H on a neighbourhood V(x) of x.

vi) A preference field is half open iff, for each $x \in W$, either H(x) is empty, or there exists a linear function $f:\Pi^{-1}(x) \to \mathbb{R}$ such that f(v) > 0 for all $v \in H(x)$ ∎

The property called S-continuity here was first introduced by Smale (1973). Its significance is that it permits integration of the field to give a preference relation intrinsic to the field, and defined in the following way.

If $c:(-1,1) \to W$ is a smooth curve in W, then the derivative [c](x) at a point $x = c(t), t \in (-1,1)$, is a tangent vector in $\Pi^{-1}(x)$. If H is a preference field on W and there is a smooth curve $c:(-1,1) \to W$ such that $[c](x) \in H(x)$ whenever $x = c(t)$, $t \in (-1,1)$, then c is called an <u>integral curve</u> of H. An integral curve, c, of the field H from $x = c(0)$ to $y = \lim_{t \to 1} c(t)$ we shall call a <u>preference curve</u> of H from x to y.

A <u>local preference</u> relation p on W is a binary relation on W such that ypx iff there is a smooth curve $c:(-1,1) \to W$ such that $x = c(0)$, $y = \lim_{t \to 1} c(t)$ and $c(t)pc(t')$ whenever $-1 < t' < t < 1$.

Suppose now that H is a continuous preference field, and that $H(x) \neq \Phi$. Then there exists a vector field, X, integral for H on a neighbourhood V of x, and therefore an integral curve $c:(-1,1) \to W$ of H such that $c(0) = x$. We shall write $y\rho_H x$ whenever there is a preference curve for H from x to a point y in the neighbourhood of x. In this way we obtain a local preference relation ρ_H which is <u>intrinsic</u> to the preference field H.

<u>Definition 2</u>

Let H be a continuous preference field.

i) A point $x \in W$ belongs to the <u>critical</u> (or infinitesimal) <u>cycle set</u> IC(H), of H iff there exist a finite set of tangent vectors $\{v_j \in H(x): j=1,\dots,r\}$ and a set of non negative real numbers $\{\lambda_j \in \mathbb{R}: j=1,\dots,r\}$ such that

$$0 = \sum_{j=1}^{r} \lambda_j v_j$$

(The vectors $v_1,\dots v_r$ are called semipositively dependent).

ii) A point $x \in W$ belongs to the <u>local cycle set</u>, LC(H), of H iff for <u>any</u> neighbourhood V(x) of x there exists a finite set of alternatives $\{y_1,\dots,y_r\}$ and a ρ_H cycle

$$x\rho_H y_1 \,_H y_2 \dots \rho_H y_r x,$$

where the preference paths making up this cycle belong to the neighbourhood V(x).

iii) A point $x \in W$ belongs to the <u>critical</u> (or <u>infinitesimal</u>) <u>optima</u> set, IO(H), of H iff $H(x) = \Phi$.

■

A previous paper (Schofield, 1978) showed, for the S-continuous preference field H_σ defined by a voting game, when individuals have smooth preferences, that the critical cycle set belonged to the local cycle set.

For example suppose that the individual preferences of a society $N = \{1,\dots,n\}$ are represented by a smooth utility profile $u = (u_1,\dots,u_n): W \to \mathbb{R}^n$. Define the <u>preference cone</u> of i at x to be

$$H_i(x) = \{v \in \Pi^{-1}(x) : du_i(x)(v) > 0\}.$$

For a coalition $M \subset N$, let $H_M(x) = \bigcap_{i \in M} H_i(x)$ be the <u>pareto</u> preference field. Clearly each preference field H_i will be S-continuous, and so therefore will be $H_M(x)$. A voting game σ for N is characterised by a collection $\mathcal{D}$ of winning coalitions. The preference field H_σ of the game is then given by $H_\sigma(x) = \bigcup_{M \in \mathcal{D}} H_M(x)$, and so $H_\sigma: W \to TW$ is S-continuous. (See Schofield, 1980, for further details).

Consider again the three person voting game on $\mathbb{R}^2$ considered in the previous section, and represented by Figure 1.

Figure 2 shows the direction gradients $\{du_i(x): i=1,2,3\}$ for the three players. For any two player coalition $\{i,j\}$, let $H_{ij}(x)$ represent the preference cone at x. It is clearly possible to find three vectors $v_a \in H_{13}(x)$, $v_b \in H_{12}(x)$ and $v_c \in H_{23}(x)$ such that $\{v_a, v_b, v_c\}$ are semipositively dependent. Moreover it is possible to find a preference curve for coalition $\{1,3\}$ from x to a nearly point a such that both 1 and 3 prefer a to x. In the same way coalition $\{1,2\}$ prefers b to x, and coalition $\{2,3\}$ prefers c to x. By this method a social preference cycle is constructed.

[Insert Figure 2 about here]

Note that the point x belongs to the critical cycle set $IC(H_\sigma)$ of H_σ where σ is the game with winning coalitions $\{1,2\},\{1,3\},\{2,3\}$. Schofield (1978) has shown that wherever a point x belongs to $IC(H_\sigma)$ then in any neighbourhood of x there exists a preference cycle of the underlying local preference relation.

Here we show that IC(H) and LC(H) are essentially identical, for any arbitrary S-continuous field.

<u>Lemma 1</u>

Let $H: W \to TW$ be a continuous preference field on a smooth manifold W. Then

$$\text{Int}\, IC(H) \subset LC(H).$$

If H is S-continuous, then IC(H) is open and

$$IC(H) \subset LC(H) \subset \text{Clos}\, IC(H).$$

(Here Int and Clos stand for interior and closure).

<u>Outline of proof</u> i) If $x \in IC(H)$ then clearly H(x) is not half open, since there can exist no linear function $f: \Pi^{-1}(x) \to \mathbb{R}$ such that $f(v) > 0$ for all $v \in H(x)$. When H is S-continuous then by the results of Schofield (1978) there exists an open neighbourhood V(x) of x in IC(H). A ρ_H cycle can then be constructed within V(x).

If H is continuous and $x \in \text{Int}IC(H)$, then there exists a neighbourhood V(x) of x in IC(H), and again a ρ_H-cycle can be constructed within V(x).

ii) Suppose now that there is a neighbourhood V(x) of x such that $V(x) \cap IC(H) = \Phi$. We seek to show that $x \notin LC(H)$.

If $x \in LC(H)$ then there exists a piecewise differentiable ρ_H - cycle $\sum = [x, y_1) \cup [y_1, y_2) \cup \ldots \cup [y_r, x)$.

By careful choice of coordinate charts we may choose the tangent vector $v_j(z)$ at any point z on the j^{th} component of this path to be fixed in $\mathbb{R}^w$ and equal to v_j, for $j = 1,\ldots,r+1$. This gives a sequence of vectors $v_1,\ldots,v_{r+1}$ in $\mathbb{R}^w$, forming a piecewise differentiable cycle in $\mathbb{R}^w$. Consequently $\{v_1,\ldots,v_{r+1}\}$ must be semipositively dependent. Moreover by S-continuity, each v_j belongs to H(x). Thus x belongs to IC(H). By contradiction $x \notin LC(H)$.

■

Thus when H is an S-continuous preference field, local cycles of the underlying preference relation do not exist iff the critical cycle set is empty. Note however that even when IC(H) is empty there may well be a "global" cycle of the preference relation ρ_H.

We now seek to show that when there exists no local cycles (ie when IC(H) = Φ) then there exists a critical optimum.

Note first of all that $x \in IO(H)$ iff $y\rho_H x$ for no $y \in W$. To see this, if $H(x) = \Phi$ there can be no integral curve through x. On the other hand if $H(x) \neq \Phi$ then there must exist an integral curve at x, and so $y\rho_H x$ for some $y \in W$.

When $H:W \to TW$ is a preference field on a smooth manifold W, let $H:W \to W$ be the preference correspondence induced from H by

$\hat{H}:W \to W:x \to \{y \in W: y\rho_H x\}$.

As before $\hat{H}^{-1}:W \to W$ is the inverse preference correspondence.

Lemma 2

If $H:W \to TW$ is an S-continuous preference field on a smooth manifold, then $\hat{H}:W \to W$ is lower demi-continuous.

Proof Consider $y \in \hat{H}^{-1}(x)$. By definition $x\rho_H y$, and so there is an integral curve of H from y to x. By S-continuity there is a neighbourhood V(y) of y such that for any $z \in V(y)$ there is an integral curve of H from z to x. Thus $x\rho_H z$ for all $z \in V(y)$. Thus $\hat{H}^{-1}(x)$ is open. ∎

In general when H is an S-continuous preference field on a manifold with IC(H) = Φ we cannot show that IO(H) = Φ. However by lemma 1,

the set LC(H) is empty. For any point x W, there will exist a neighbourhood U of x containing no ρ_H-cycles. For any compact set V within U, Walker's theorem together with the lower demi-continuity of $\hat{H}$ implies that there exists some $\bar{x}$ within V such that $\hat{H}(\bar{x}) \cap V = \Phi$. In other words at $\bar{x}$, the preferred set $\hat{H}(\bar{x})$ is either empty or the preferred directions point "out of V".

Since there may exist "global" ρ_H cycles, we cannot show $\hat{H}(\bar{x}) = \Phi$ for some $\bar{x} \in W$. In the case that W is an admissible set in euclidean space $\mathbb{R}^w$, however, we can show that the critical optima is non empty whenever the critical cycle set is empty.

To do this we now assume W is a smooth admissible set of dimension w in finite dimensional euclidean space. In particular this implies that for each point x in the interior of W, $\Pi^{-1}(x)$ can be identified with $\mathbb{R}^w$. Since W is assumed to be compact, convex there exists a smooth boundary ∂W. At any point x in ∂W there is a normal, n(x), pointing into W, and the tangent space at x is then the closed half space $\{v \in \mathbb{R}^w: v.n(x) \geq 0\}$ defined by this normal, where . is the usual scalar product. Given a preference field $H:W \to TW$ we define a preference correspondence $\tilde{H}:W \to W$ by

$\tilde{H}(x) = \Phi$ iff $H(x) = \Phi$

$\tilde{H}(x) = \{z \in W: z = x + v \text{ for } v \in H(x)\}$.

Note that in a sense, $\tilde{H}$ is an approximation to the preference correspondence $\hat{H}$ associated with ρ_H. By the properties of a preference field, H(x) is a cone in $\Pi^{-1}(x)$ and $0 \notin H(x)$. Thus for each $v \in H(x)$ there exists a ray $x + \lambda v$ in $\tilde{H}(x)$, where $\lambda > 0$. However $x \notin \tilde{H}(x)$. It is precisely when $x \in IC(H)$ that $x \in \text{Con}\tilde{H}(x)$.

Thus the property $IC(H) = \Phi$ for a preference field is essentially the analogue of the convexity property for a preference correspondence.

Theorem 1

If $H:W \to TW$ is a S-continuous preference field on a smooth admissible space W in $\mathbb{R}^w$ then $IC(H) = \Phi$ implies that $IO(H) \neq \Phi$.

Proof Suppose that $x \in \text{Con}\tilde{H}(x)$. Then there exists a solution to the equation

$$x = \sum_{j=1}^{r} \lambda_j z_j \quad \text{where} \quad \sum_{j=1}^{r} \lambda_j = 1,$$

each $\lambda_j \geq 0$, and $z_j \in \tilde{H}(x)$.

By definition $z_j = x + v_j$ where $v_j \in H(x)$.

Hence $\sum_{j=1}^{r} \lambda_j v_j = 0$ for semi-positive $\lambda = (\lambda_1, \ldots, \lambda_r)$.

Thus $x \in IC(H)$.

If $IC(H) = \Phi$ then for no $x \in W$ does $x \in \text{Con}\tilde{H}(x)$.

Just as in lemma 2, $\tilde{H}^{-1}(x) = \{y \in W: x \in \tilde{H}(y)\}$ is open in W for each $x \in W$.

By the Ky Fan Theorem, there exists some choice $\bar{x}$ such that $\tilde{H}(\bar{x}) = \Phi$.

If $H(\bar{x}) \neq \Phi$ then clearly there exists $v \in H(\bar{x})$ yet $z = \bar{x} + \lambda v \notin W$ for any $\lambda > 0$.

This implies that $\bar{x}$ belongs to the boundary of W, and v does not belong to $\Pi^{-1}(x)$.

Thus $H(\bar{x}) = \Phi$ and so $IO(H) \neq \Phi$. ∎

Consider again the case of a society $N = \{1, \ldots, n\}$ where each individual has smooth preference on W. The pareto preference field

$H_N:W \to TW$ is S-continuous and $IC(H_N) = \Phi$.

To see this note that at any point x in W, $du_i(x)(v) > 0$ for any $i \in N$ and any $v \in H_N(x)$. Thus $H_N(x)$ is half open. If W is admissible then by Theorem 1, $IO(H_N) \neq \Phi$.

Suppose that $x \notin IO(H_N)$. Then there exists a preference curve for H_N from x to a nearby point y. This implies that $u_i(y) > u_i(x)$ for all $i \in N$. If we write $y\, p_N x$, in this case, where p_N is the pareto (or unanimity) relation, and let the pareto set be the choice set under this relation, then we see that any point x outside $IO(H_N)$ cannot belong to the pareto set. In other words the pareto set must belong to the critical optima set $IO(H_N)$. Thus $IO(H_N)$ is the analogue, in this multi-objective optimization problem, of the set of critical points in the single objective optimization problem. When individual preferences are convex then $IO(H_N)$ will be identical to the pareto set. In the case of non-convex preferences, $IO(H_N)$ will contain the equivalent of societal minima and saddle points.

Smale (1974) has shown that an interior point, x, of W belongs to $IO(H_N)$ iff $\{du_i(x): i \in N\}$ are semi-positively dependent viz. $\sum \lambda_i \, du_i(x) = 0$ for semi-positive $\lambda = (\lambda_1, \ldots, \lambda_u)$. Moreover Smale (1982) and Porteous (1970) have shown that a Hessian can then be defined at x to determine the stability properties of the optima points.

In the more general case of a voting game, σ, Figures 1 and 2 show that even with convex individual preferences on an admissible space, there is no reason to expect $IO(H_\sigma) = \Phi$. As Figure 2 makes evident the point x belongs to the convex hull of $\tilde{H}_\sigma(x)$.

However it is possible to show that for a voting game σ, there is an integer $v(\sigma)$ such that if the policy space W is of dimension less than $v(\sigma)$ then $IC(H_\sigma)$ must be empty (Schofield 1983a,1983b).

The motivation of the final section of this paper is to determine what global topological features of a smooth manifold W are sufficient to ensure that local acyclicity implies the existence of a choice of the preference field.

4. Equilibrium on a Manifold

Section 2 of this paper showed the close relationship between the Ky Fan Theorem for a preference correspondence on a convex compact set, and the Kakutani fixed point theorem. In Section 3 it was shown that if a preference field H satisfied what was essentially a local convexity property on an admissible set then a local equilibrium IO(H) exists. The motivation for this section of the paper is to parallel the extension of the Kakutani fixed point theorem to a wider class of topological spaces. See Eilenberg-Montgomery (1946), Begle (1950), Gorniewicz and Granas (1970) and Powers (1970) for extensions of the Kakutani fixed point theorem.

We shall call a topological space W trivial if the homology groups $H_q(W)$ are zero, for all $q > 0$. Two continuous maps $f,g:X \to Y$ are homotopic iff there is a continuous map $F:X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$, $\forall x \in X$.

Homotopic maps induce identical maps at the homology level $f_*, g_*: H_*(X) \to H_*(Y)$.

If the identity map on W is homotopic to the constant map $W \to x_o, x_o \in W$, then W is contractible and therefore trivial.

If $f:X \to Y$ is a continuous map then define the Lefshetz number of f to be

$$\lambda(f) = \sum_q (-1)^q \text{ trace } (f_q)$$

where $f_q: H_q(X) \to H_q(Y)$.

When W is a compact manifold and $f: W \to W$ then $\lambda(f) \neq 0$ implies that f has a fixed point $\bar{x}$ s.t. $\bar{x} \in f(\bar{x})$. See Brown (1970). For example if W is trivial then the only non zero homology group is $H_o(W) = Q$, where Q is the coefficient field. In this case $\lambda(f) \neq 0$ and so f must have a fixed point.

This result can be extended to cover a correspondence $P:W \to W$ when P(x) is trivial for all x (see Eilenberg-Montogomery).

Of particular interest is the identity map $Id: W \to W$ where $\lambda(Id) = \sum_q (-1)^q \dim (H_q(W))$ is also called the Euler characteristic $\chi(W)$ of W.

For example $\chi(S^w) = 1 + (-1)^w = 0$ if w is odd
2 if w is even.

A function $f:W \to W$, homotopic to the identity, is called a deformation of W, and has Lefshetz number $\chi(W)$. If $\chi(W) = \lambda(Id) = \lambda(f) \neq 0$ then the deformation f must have a fixed point.

Thus any deformation f of the even dimensional sphere S^{2k} has $\lambda(f) = 2$ and thus must have a fixed point. On the other hand S^{2k+1} admits a fixed point free deformation. In the same way the torus $T^2 = S^1 \times S^1$ has $\chi(T^2) = 0$ and admits fixed point free deformations.

Theorem 2

If $H: W \to TW$ is an S-continuous preference field on a smooth w-dimensional compact manifold W with non zero Euler characteristic, then $IC(H) \cup IO(H)$ is non empty.

Proof

Let $\theta: U \subset W \to V \subset \mathbb{R}^w$ be a chart on W and let $H_\theta : V \to TV$ be the projection under this chart of the preference field H. If $IO(H) = \Phi$ then $H(x) \neq \Phi$, all $x \in W$. As in the proof of Theorem 1, let $\tilde{H}_\theta : V \to V$ be the associated lower demi-continuous preference correspondence. By Michael's selection theorem there exists a selection $f: V \to V$ for $Con\tilde{H}_\theta$. Moreover since $IC(H) = \Phi$, at no point $x \in U$ is it the case that $\theta(x) \in Con\tilde{H}_\theta(\theta(x))$. Thus $f(\theta(x)) \neq \theta(x)$.
From S-continuity, $f: V \to V$ can be chosen homotopic to the identity. Thus f can be pulled back to U to give a deformation $f_U: U \to U$. Finally the deformations for a family of charts can be pieced together to give a deformation $f: W \to W$, such that for no $x \in W$ does $x = f(x)$.

However if $\chi(W) \neq 0$, then W admits no fixed point free deformation. Consequently it cannot be the case that both IC(H) and IO(H) are empty.

As an illustration of the requirement that $\chi(W) \neq 0$, consider the preference field H on the one-dimensional sphere S^1 which assigns the tangent vector +1 at each point. Clearly $IC(H) = \Phi$ yet there is a global ρ_H-cycle obtained by integrating the flow round the circle. This is associated with a deformation $f: \alpha \to \alpha + \delta$ (in radial coordinates). Thus the preference field has no equilibrium.

[Insert figure 3 about here]

Consider now a preference field H on the two dimensional sphere, where the field at a point x is the half open cone pointing in a northerly direction but twisted slightly a few degrees of longitude east (as in figure 4).

[Insert figure 4 about here]

Clearly there is a ρ_H-cycle round each latitude of the sphere. However since IC(H) is empty, and $\chi(S^2) = 0$ there exists an optima set IO(H). In fact this set will consist of the two poles: the south pole is an unstable optimum (the ρ_H flow leaves this point) and there is a stable optimum at the north pole.

A field of this kind could characterise a pattern of winds which tended to blow in a general north easterly direction. Note that neither Walker's acyclicity theorem nor Ky Fan's convexity theorem can be used to show existence of optima or singularity points of the field in this example.

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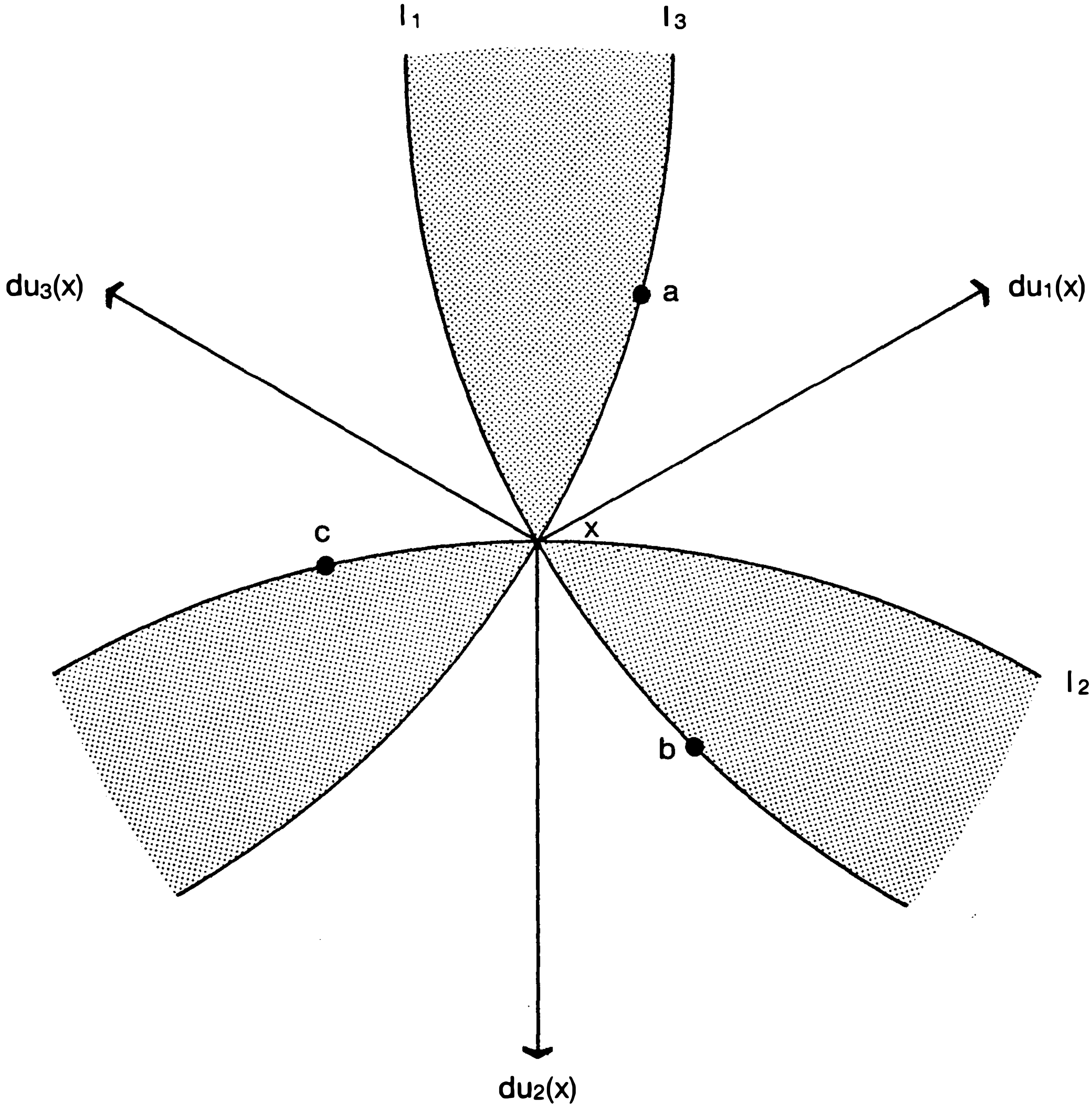

Figure 1

Caption: Acyclic and convex individual preferences, resulting in a social preference which violates acyclicity and semi-convexity.

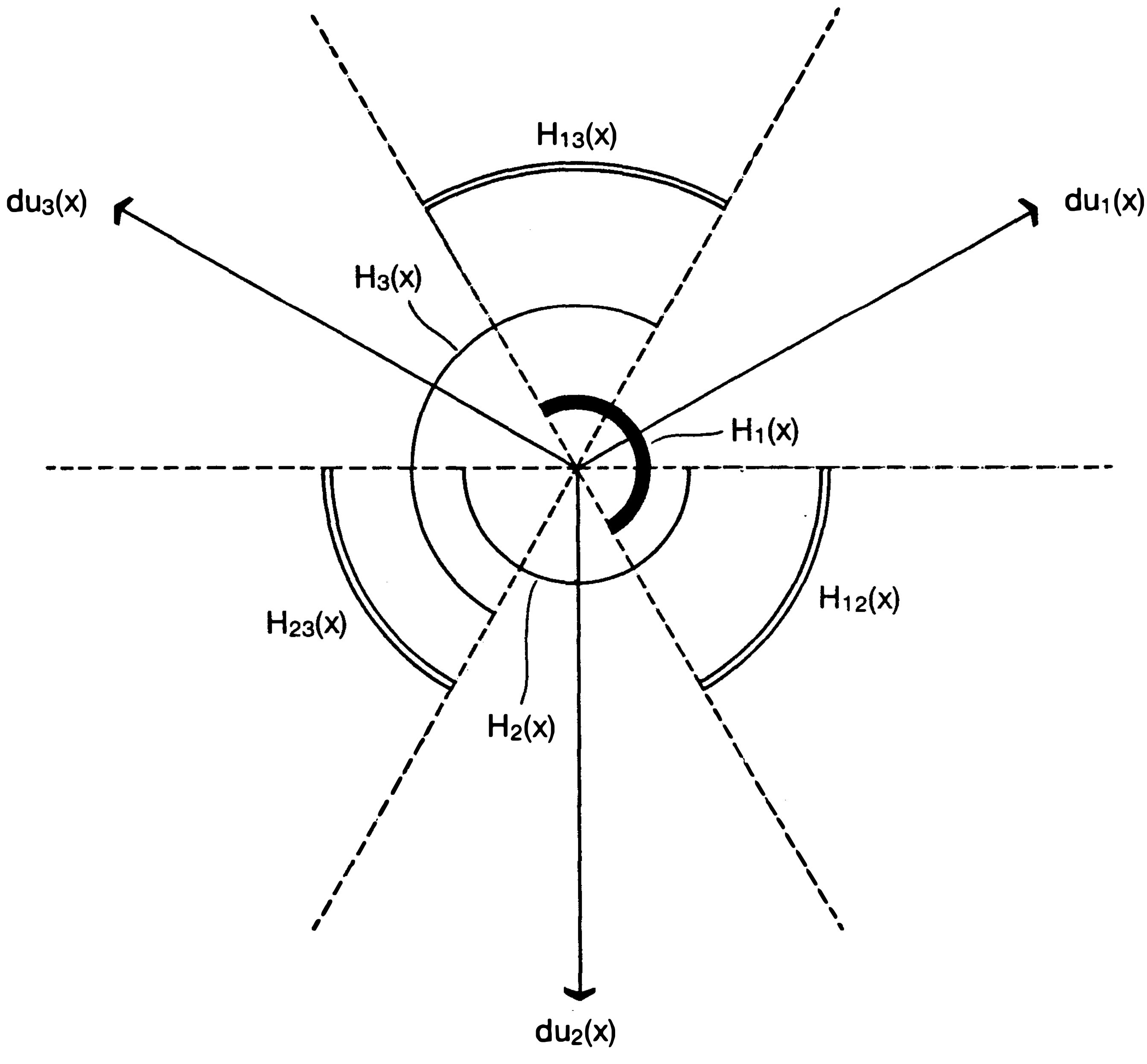

Figure 2

Caption: Three coalition preference cones at point x in the critical cycle set.

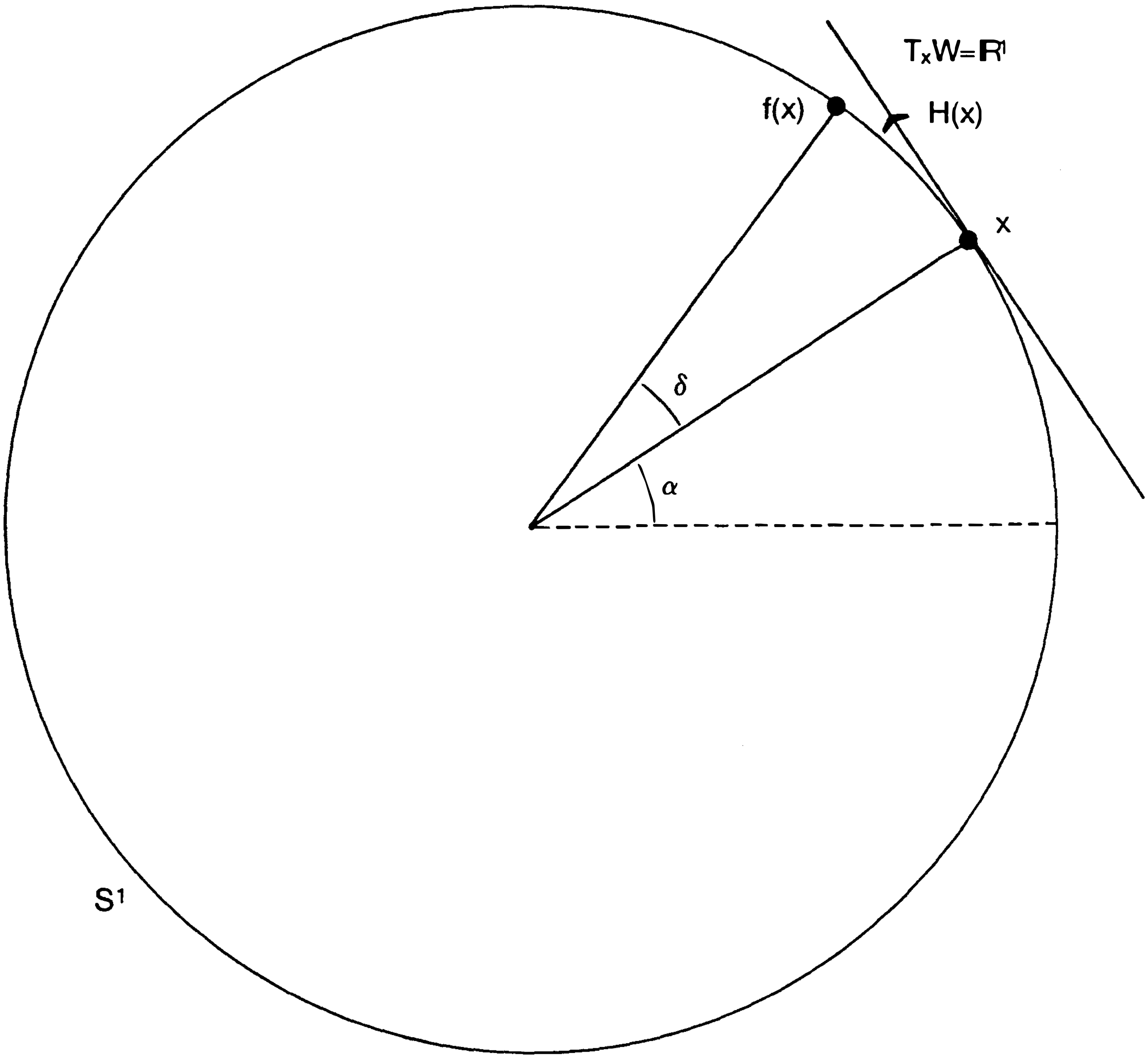

Figure 3

Caption: A preference field H and an associated deformation on the circle, S^1, showing no local cycles, no equilibrium, yet a global cycle.

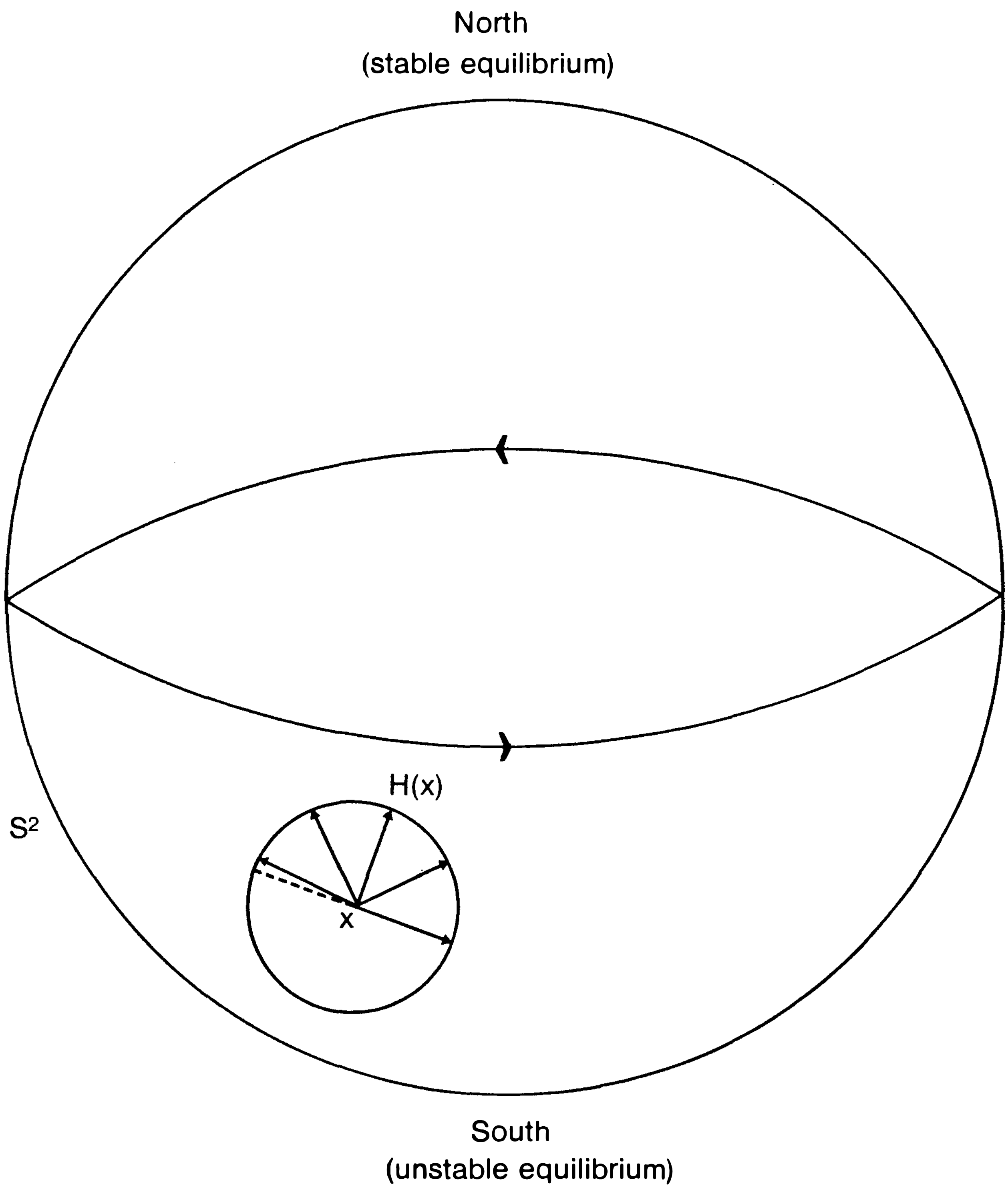

Figure 4

Caption: A preference field H on the sphere, S^2, showing no local cycles, a global cycle, and both stable and unstable equilibria.