

# Elliptic double affine Hecke algebras

Eric M. Rains\*

Department of Mathematics, California Institute of Technology

September 7, 2017

## Abstract

We give a construction of an affine Hecke algebra associated to any Coxeter group acting on an abelian variety by reflections; in the case of an affine Weyl group, the result is an elliptic analogue of the usual double affine Hecke algebra. As an application, we use a variant of the  $\tilde{C}_n$  version of the construction to construct a flat noncommutative deformation of the  $n$ th symmetric power of any rational surface with a smooth anticanonical curve.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Line bundles on <math>E^n</math> and their sections</b>	<b>8</b>
<b>3</b>	<b>Coxeter group actions on abelian varieties</b>	<b>23</b>
<b>4</b>	<b>Elliptic analogues of affine Hecke algebras</b>	<b>30</b>
<b>5</b>	<b>Infinite groups</b>	<b>51</b>
<b>6</b>	<b>The (double) affine case</b>	<b>64</b>
<b>7</b>	<b>The <math>C^\vee C_n</math> case</b>	<b>67</b>
<b>8</b>	<b>The (spherical) <math>C^\vee C_n</math> Fourier transform</b>	<b>85</b>

## 1 Introduction

The origin of this paper was a question of P. Etingof which was conveyed to the author by A. Okounkov at a 2011 conference<sup>1</sup>, to wit whether the author knew of a way to construct noncommutative deformations of symmetric powers of the complement of a smooth cubic plane curve. Although the answer was “no” (at the time, see below!), it seemed likely that it should be possible to extend the approach of [17] (which the author and S. Ruijsenaars had developed earlier that month) to multivariate difference operators; although this would not answer the question as posed,

---

\*Supported by grants from the National Science Foundation, DMS-1001645 and DMS-1500806

<sup>1</sup>Affine Hecke Algebras, the Langlands Program, Conformal Field Theory and Super Yang-Mills Theory, Centre International de Rencontres Mathématiques (Luminy), June 2011

it *would* give analogous deformations associated to the complement of a smooth biquadratic curve in  $\mathbb{P}^1 \times \mathbb{P}^1$ , represented as algebras of elliptic difference operators in  $n$  variables. The author continued to develop this approach while on a sabbatical that fall at MIT, eventually coming up with a construction for such deformations for *any* rational surface equipped with a smooth anticanonical curve and a rational ruling.

There were, however, a couple of significant issues. One was that the spaces of operators were cut out by a number of conditions, including in particular certain residue conditions that only made sense for generic values of the parameters. This would have been merely an annoying technicality, except that the conditions specifically failed to make sense in the commutative case, making it rather difficult to consider the family as actually being a deformation. This could be worked around by considering the family as a whole, or in other words only considering those operators that could be extended to an open subset of parameter space. However, although this would indeed give a well-defined family of algebras, it would make the question of flatness even more difficult, and would in principle even allow the representation in difference operators to fail to be faithful!

Despite these difficulties, the definition was still well-behaved enough to allow a fair amount of experimentation. One thing that became clear was that the construction directly led to some spaces of operators that had been considered in the literature, in particular those associated to the difference equations for interpolation and biorthogonal functions [22, 21], of particular interest in the latter case since no satisfying construction was yet known for the full space of operators. In addition, since these functions degenerate to more familiar functions, to wit the Macdonald and Koornwinder polynomials, this suggested that the algebras of elliptic difference operators should degenerate to algebras related to those polynomials. In particular, the latter algebras can be constructed as spherical algebras of appropriate double affine Hecke algebras, and P. Etingof suggested to the author that the same might hold at the elliptic level, and give a possible approach to flatness.

Indeed, the same approach to constructing the algebra of operators (as operators preserving (locally) holomorphic functions and satisfying appropriate vanishing conditions on the coefficients) could be fairly easily extended to give a construction of elliptic double affine Hecke algebras. The resulting residue conditions turned out to be essentially those of [9], with again the caveat that they only make sense when the noncommutative parameter  $q$  is non-torsion. In fact, something slightly stronger is true: the residue conditions are well-behaved as long as one only considers a sufficiently small interval relative to an appropriate (Bruhat) filtration, with the constraint on the interval being simply that it act faithfully. This led the author to investigate that filtration more carefully, leading eventually to the realization that (a) the residue conditions always make sense on rank 1 subalgebras (which are very special cases of the construction of [9]), and (b) those rank 1 subalgebras always generate a flat algebra, even when  $q$  is torsion. As a result, one could avoid the residue conditions entirely and simply consider the algebra generated by the rank 1 algebras. It is then relatively straightforward to show that the resulting family of algebras is flat (and the representation as difference-reflection operators is faithful), and not too difficult to show that this flatness is inherited by the spherical algebra. Moreover, much of the theory can be developed for quite general actions of Coxeter groups on abelian varieties, so that the DAHAs are just the special cases in which the Coxeter group is affine.

In the above discussion, we have neglected a few technical issues. The first is that the deformations of symmetric powers are not, in general, *algebras* of operators. The difficulty is that, with the exception of complements of smooth anticanonical curves in del Pezzo surfaces, none of the surfaces we wish to deform are actually affine! Since they are only quasiprojective in general, it is easier to simply deform the symmetric power of the original projective surface (and then take the appropriate localization if desired). This still requires a choice of ample divisor, and one then encounters the difficulty that twisting a noncommutative variety by a line bundle tends to change

the noncommutative variety. As a result, the object actually being deformed is the category of line bundles on the symmetric power.

When it comes to the DAHA, however, the situation is even more complicated. The problem is that the elliptic DAHA essentially arises by replacing one of the two commutative subalgebras of the usual DAHA by the structure sheaf of an abelian variety of the form  $E^n$ . In the affine case considered in [9], the commutative subalgebra is finite over the center of the affine Hecke algebra, and thus one is naturally led to consider sheaves over the center, or in other words sheaves on the quotient  $E^n/W$  where  $W$  is the relevant finite Weyl group. Unfortunately, in the double affine setting,  $W$  is replaced by an affine Weyl group, and there *is* no such quotient scheme! As a result, the trickiest part of our construction turns out to be simply figuring out what kind of object we should be constructing. The key idea, coming from earlier work in noncommutative geometry [2, 31] is that since the elliptic DAHA should have  $\mathcal{O}_{E^n}$  as a subalgebra, it should have a natural *bimodule* structure over  $\mathcal{O}_{E^n}$ , and thus correspond to a quasicoherent sheaf on the product  $E^n \times E^n$ . Subject to some finiteness conditions (satisfied for any sheaf of meromorphic difference operators), such bimodules form a monoidal category, and thus we can construct the elliptic DAHA as a monoid object (“sheaf algebra”) in that category.

A final technical issue is that we wish to deform symmetric powers of arbitrary blowups of  $\mathbb{P}^1 \times \mathbb{P}^1$  or the Hirzebruch surface  $F_1$ . Each time we blow up a point, we acquire a new parameter, and thus our construction needs to admit an unbounded number of parameters. This is an issue from the standpoint of traditional double affine Hecke algebras, where one normally has precisely one parameter per root (which must be constant on orbits), plus an overall parameter  $q$ . One partial exception is the  $C^\vee C_n$  case, where one has a total of 5+1 parameters. This is normally explained by taking a nonreduced root system, so that one has 5 orbits of roots, but from the standpoint of the actual algebra, this is actually quite artificial: there is an action of  $S_2 \times S_2$  on the parameters that has no effect on the algebra, but moves degrees of freedom between corresponding short and long roots. It is thus much more natural to view those four parameters as assigning an unordered pair to each orbit of short roots of the *reduced* root system. This turns out to generalize easily to the elliptic setting: we obtain an elliptic DAHA for every assignment of an effective divisor on  $E$  to each orbit of roots. This causes some difficulties in constructing the spherical algebra, as the usual construction via idempotents fails even generically, but one can show that the spherical algebra still continues to inherit flatness in this general case.

As we mentioned above, the construction of deformations of  $\mathrm{Sym}^n(X)$  where  $X$  is a projective rational surface with a choice of smooth anticanonical curve depends on a choice of rational ruling on  $X$ . One consequence is that we cannot directly obtain the case  $X = \mathbb{P}^2$  from our construction. This can be worked around by blowing up a point but then only considering those line bundles coming from  $\mathbb{P}^2$ , but this approach leads to a nontrivial question of showing that the result is independent of the choice of point. Similarly, if  $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ , then there are two choices of ruling on  $X$ , but deformation theory suggests that the resulting deformations should be the same (both have the maximum number of parameters). Both questions turn out to reduce to the existence of a certain generalized Fourier transform in the  $\mathbb{P}^1 \times \mathbb{P}^1$  case, which is also key to proving the most general form of the flatness result. (The DAHA only tells us that certain sheaves are flat, so gives only an asymptotic flatness result for their global sections.) We find that not only is our deformation of the category of symmetric powers of line bundles on  $X$  flat (modulo some possibility of bad parameters in codimension  $\geq 2$  not including the original symmetric-power-of-commutative-surface case), but it is invariant under the action of a Coxeter group of type  $W(E_{m+1})$ . (In other words, to first approximation, the construction only depends on the underlying surface  $X$  and two points  $q$  and  $t$  of the Jacobian of the anticanonical curve.) Note that both facts are actually *not* true for the full original family of commutative categories, but hold for the subcategory in which

we only allow those morphisms that extend to a neighborhood in the family.

The plan of this paper is as follows. First, in section 2, we deal with a largely notational issue that arises due to the fact that we wish to deal with the construction from a purely algebraic standpoint. The issue is that we need in many cases to deal with *twisted* versions of our algebras, in which the coefficients of the operators lie in nontrivial line bundles. To make sense of this, one must not only describe those line bundles but various maps between tensor products of pullbacks of those bundles through elements of the Coxeter group, with associated concerns about compatibility. In the analytic setting, one can avoid most of those issues by constructing the line bundle via an appropriate automorphy factor on the universal cover, and our objective in section 2 is to do something similar in the algebraic setting. The key idea is to replace the individual curve  $E$  by the universal curve over the moduli stack of elliptic curves; it turns out that one can compute the Picard group of the corresponding stack  $\mathcal{E}^n$  in general, with only very mild rigidification required to make pullbacks and tensor products behave well. In addition, certain of the line bundles come with natural global sections, which one can use to construct sections of more general bundles; our most significant result along these lines gives conditions for a function on the analytic locus described via theta functions to extend to the full moduli stack. We also partially consider the case of varieties which are isogenous to powers of elliptic curves (i.e., over a moduli stack of elliptic curves with a cyclic subgroup), and as an application give some results on spaces of invariant sections of equivariant line bundles on  $\mathcal{E}^n$ . The main result along those lines states that the dimension of the space of invariants is independent of the curve  $E$  with only finitely many possible exceptions (supersingular curves of characteristic dividing the order of the group).

In section 3, we give some structural results on the main scenario we consider in the sequel, namely a Coxeter group acting on an abelian variety “by reflections”. In particular, we show that under reasonable conditions one can associate a “coroot” morphism to an elliptic curve to every root of the Coxeter group, compatibly with the linear relations between roots in the standard reflection representation (and satisfying suitable notions of positivity!). This is a key ingredient in our construction, as our parameters will correspond to effective divisors on those curves. We also show that in the case of a finite Coxeter group, the invariant theory is better behaved than suggested by the results of section 2: as long as a certain isogeny (which is an isomorphism in the most natural cases) has diagonalizable kernel, the invariant theory continues to behave well even for supersingular curves. This flatness of invariants is a crucial ingredient in proving flatness of the spherical algebra, and in particular means the  $C^\vee C_n$  case will be flat over any field.

Section 4 is largely a recapitulation of the construction of [9], in which we associate to any finite Coxeter group  $W$  acting on a family  $X$  of abelian varieties a family (the “elliptic affine Hecke algebra”) of sheaves of algebras on  $X/W$  parametrized by effective divisors on the “coroot” curves. The main difference, apart from allowing arbitrary numbers of parameters and slightly more general abelian varieties, is that we replace the residue conditions of [9] by the (equivalent) condition that the operators preserve the spaces local sections of the structure sheaf of  $X$  on  $W$ -invariant open subsets, as the latter is easier to generalize from a conceptual standpoint. Our main new tool for studying these algebras is a natural filtration by Bruhat order on  $W$ , which allows us to express various subsheaves as extensions of line bundles in natural ways. In particular, this makes it easy to show that the algebra is generated by the subalgebras corresponding to simple roots, which will be key to the extension to infinite Coxeter Groups, as well as giving a construction which is easily seen to respect base change. We also prove an important technical lemma on the space of “invariants” of a module over the elliptic AHA, giving fairly general conditions on a family of modules guaranteeing that the invariants are well-behaved; this will be the key lemma in proving flatness of spherical algebras. In addition, we give an analogue of Mackey’s theorem for the case of

parabolic subgroups in which the usual sum over double cosets is replaced by a filtration.

Section 5 begins with a discussion of sheaf algebras, which we use in place of a sheaf of algebras on the quotient. Since the corresponding tensor product of bimodules is somewhat tricky to deal with, we discuss approaches to dealing with this issue (and in particular ways to describe the maps  $A \otimes B \rightarrow C$  we need in order to discuss algebras or categories). This makes it relatively straightforward to construct analogues of the affine Hecke algebra in which  $W$  is replaced by an infinite Coxeter group: simply take the sheaf subalgebra of the sheaf algebra of meromorphic reflection operators generated by the rank 1 algebras. (The only tricky aspect comes when we consider twisted forms of the algebra, where we are rescued by the fact that although we do not understand the relations explicitly, we do know they live exclusively on finite rank 2 parabolic subgroups.) Most of the results extend immediately from the finite case (with the caveat that any parabolic subgroups considered should be finite); in particular, the infinite analogue of the Mackey result is precisely the remaining result we need to show that the spherical algebras are flat.

Section 6 has a (very brief) discussion of the special case in which  $W$  is an affine Hecke algebra, so that the sheaf algebras constructed in section 5 are analogues of double affine Hecke algebras. Apart from some mild issue about viewing  $q$  as a parameter (not the case for the standard construction), this mainly consists of some observations about the spherical algebra (relative to the associated finite Weyl group), namely the fact that the fibers are (sheaf) algebras of difference operators, and are thus in a natural sense domains. In addition, we briefly discuss the consequences of the fact that the action of  $\tilde{W}$  fails to be faithful when  $q$  is torsion: not only does the sheaf algebra come from a sheaf of algebras on the quotient by the image of  $\tilde{W}$ , but it has a  $2n$ -dimensional center (over which it is presumably finite).

Section 7 considers in detail the case that  $W$  is the affine Weyl group of type  $C_n$  and  $X$  is a particularly nice action of that group, with a view to constructing deformations of symmetric powers of rational surfaces. In addition to the spherical algebras themselves, one must also consider certain intertwining bimodules. We prove that these are always flat *as sheaf algebras*, and show that in the case  $t = 0$  the result is indeed a symmetric power of the univariate case (which was constructed in [20] without reference to DAHAs). This enables us to at least partially extend the flatness as sheaves to flatness of global sections, by giving a number of cases in which the sheaves are acyclic. In addition to these general results, our main result in this section is showing that if we blow up 8 points of  $\mathbb{P}^1 \times \mathbb{P}^1$ , then there is a hypersurface in parameter space on which the “anticanonical” algebra is an integrable system: it is generated by  $n + 1$  commuting (and self-adjoint) elliptic difference operators in  $n$  variables. (One can verify that this is precisely the integrable system of [6, 33, 12]. In addition, the geometry strongly suggests the existence of other integrable systems with the same number of parameters, but higher-order operators.)

Section 8 deals with the question of showing that the construction is mostly independent of the way in which we represented our rational surface as a blowup of a ruled surface. The key ingredient is a certain “Fourier transform”. Analytically, this should be represented by the integral operator with kernel constructed in [19], but there are difficulties in showing this is well-defined in general (as well as showing that it respects the additional conditions associated to any points we have blown up). As a result, we construct the transform in several steps. First, we give a construction that is manifestly well-defined (and a homomorphism) as long as  $q$  is not torsion, but lives on a certain completion of the algebra of meromorphic difference operators. Although the construction is in general quite complicated, it is sufficiently well-behaved to allow us to compute a few special cases. The result in those special cases turns out to be well-defined even when  $q$  is torsion, and this allows us to show that this formal transform extends in general. Moreover, the special cases we understand are sufficiently close to generating the full algebras that we can prove that the formal transform restricts to an actual transform. This also gives us some ability to explore

how the algebras behave when we degenerate the elliptic curve, as it is easy to take limits of the almost-generators. In addition to allowing us to prove a much stronger flatness result, the Fourier transform also allows us to construct a large collection of “quasi-integrable” systems, in particular including the aforementioned operators associated to interpolation and biorthogonal functions.

We close with a summary of some of the various open problems that arose in the course of this work. (Of course, this is only a small sampling of such problems, as nearly any existing result on double affine Hecke algebras suggests the existence of a generalization to the elliptic case! We are also omitting some questions discussed in sections 2 and 3, as they are peripheral to the main thrust of the work.) One big collection of questions has to do with the fact that, although we show that the spherical algebras of the elliptic DAHAs are flat in significant generality, we can prove almost nothing else about them in general. In particular, we cannot even show that they are Noetherian (even in the specific  $C^\vee C_n$ -type cases for which we have such strong flatness results). The approach to such questions in [1] suggests that one should be able to reduce this to the case in which everything is defined over a finite field, when one expects the spherical algebra to be finite over its center (which should itself be Noetherian); unfortunately, understanding the center (when  $q$  is torsion) is itself an open problem. In addition, the usual DAHA is at least generically Morita equivalent to its spherical algebra, and one expects the same to hold for the elliptic analogue, together with Morita equivalences involving shifting the  $t$  parameter by integer multiples of  $q$ .

Another natural question is whether the Fourier transform of the usual DAHA extends to the elliptic level. The construction of section 8 can be viewed as a partial affirmative answer to this question in the  $C^\vee C_n$  case, as it constructs a Fourier transform on the spherical algebra. For  $n = 1$ , this at least implicitly leads to a Fourier transform on the elliptic DAHA by using the appropriate Morita equivalence, but even there it is unclear how to make the transformation explicit. (There is also a philosophical question, as our work suggests that the transform should really be viewed as living on the spherical algebra, as the analogous transform on the DAHA is not as well behaved relative to the natural filtration.)

In fact, even in the  $C_n$  case, there are still open questions about the Fourier transform, as the construction of section 8 only applies to the case in which we have assigned precisely one parameter to the  $D_n$ -type roots. There is evidence suggesting the existence of some other Fourier transforms related to the versions of the DAHAs of types  $A_n$  and  $C_n$  that do not have this “ $t$ ” parameter. Indeed, the paper [29] discusses several integral transformations; one appears to relate two versions of the  $A_n$  spherical algebra, while the other two appear to relate the  $A_n$  spherical algebra to the  $C_n$  spherical algebra. The main obstruction to understanding these cases is that the lack of a  $t$  parameter makes it difficult to control the spaces of global sections, as we can no longer view the spherical algebras as deformations of a symmetric power. In any event, this suggests that the existence of isomorphisms between spherical algebras is a much more subtle question than one might have thought based on the classical theory of double affine Hecke algebras. Another source of questions about the Fourier transform is the analytic version constructed in [19]. In addition to the “interpolation kernel”, that paper constructed a few other functions with some similar properties (the “Littlewood”, “dual Littlewood” and “Kawanaka” kernels), and it is natural to ask whether those functions interact with the  $C^\vee C_n$  spherical algebra in any interesting way. That such an interaction should exist is strongly suggested by the fact that the quadratic transforms proved in that paper were generalizations of results first proved using the action of affine Hecke algebras on Laurent polynomials.

There are also a number of questions about our deformations of symmetric powers having to do with taking global sections. One fundamental question has to do with the fact that our construction, although (mostly) flat and highly symmetrical, does not quite correspond directly to

deformations of symmetric powers of surfaces: we must still make a choice of ample line bundle in order to obtain an actual projective deformation in the commutative case or a deformation of the category of sheaves in the noncommutative case. For  $n = 1$ , it was shown in [20] that all such choices give the same result, but it will be difficult to extend those techniques to  $n > 1$ . In addition, one would like to show that the corresponding family of commutative quasiprojective surfaces is at least generically smooth (which experiment suggests is the case). A somewhat related problem would be to promote the deformations of symmetric powers to corresponding deformations of the Hilbert scheme of points. (This may be related to the issue of Morita equivalences shifting  $t$ , as one would expect that twisting by a class in the Picard group of the Hilbert scheme that doesn't come from the symmetric power would produce a Morita equivalent algebra with a different value of  $t$ .) In particular, a different construction of a family of commutative deformations of the Hilbert scheme of points was given in [20] (as moduli spaces of rank 1 sheaves on noncommutative rational surfaces), and one would like to understand how the two constructions are related. In addition, one would like to have a proof that our family of algebras (or noncommutative varieties) actually depends on all of the parameters (and is not, say, simply a base change from a lower-dimensional family); this suggests trying to understand how the family relates to the infinitesimal deformation theory of the symmetric power. (Note, however, that the deformation associated to the  $t$  parameter is almost certainly not locally trivial. Also, the fact that replacing  $t$  by  $q - t$  gives an isomorphic algebra implies that the corresponding Kodaira-Spencer map will vanish unless we first descend the family to the quotient by this symmetry, and it is unclear how to do so without breaking the representation via difference operators.)

In any event, a choice of ample divisor allows one to translate each original line bundle into an actual sheaf on the (noncommutative) variety, and thus produces a saturated version of the original Hom space (by taking all morphisms between the sheaves). In the univariate case, the saturated morphisms were still difference operators, and there is a primarily combinatorial algorithm for computing the resulting dimensions. The question is more subtle in the multivariate setting, with two main issues arising. For some line bundles on  $F_1$  (including those coming from line bundles on  $\mathbb{P}^2$ ), we can only prove flatness away from a possible bad locus of codimension  $\geq 2$ , and some new approach to constructing global sections is likely needed to eliminate this possibility. The other tricky case arises from the elliptic pencil on a deformation of a non-Jacobian elliptic surface (i.e., in which the elliptic fibration does not have a section). There, not only do we want to know how many global sections there are (with the conjecture being that on the hypersurface where the algebra of global sections is nontrivial, it is flat), but also, by analogy with the Jacobian case, expect that the algebra of global sections will give a new integrable system, associated to the same parameters as the van Diejen/Komori-Hikami system, with the addition of a choice of nontrivial torsion point on the elliptic curve.

Finally, the condition that a projective rational surface have a *smooth* anticanonical curve is quite restrictive, and in particular excludes a number of cases in which deformations were already known. Although the strong version of flatness cannot be expected to extend in general, one can still expect to have flatness for *ample* bundles. There are already issues before blowing up any points, as it appears one must give an analogue of the elliptic DAHA in which the elliptic curve becomes singular, reducible, or even nonreduced, and this can cause issues with the Bruhat filtration as well as the generation in rank 1. Beyond that, although it should be fairly straightforward (especially if the base curve remains integral) to consider blowups in *smooth* points of the base curve, this is again a pretty restrictive condition, while blowing up singular points quickly leads to a combinatorial explosion without some more conceptual approach.

**Acknowledgements** The author would particularly like to thank P. Etingof both for asking the original seed question (with an important assist from A. Okounkov!) and hosting the author's

sabbatical at MIT (which the author would also like to thank, naturally) where much of the basic approach was worked out, with great assistance from conversations with not only Etingof but also (regarding various geometrical issues) B. Poonen. Thanks also go to T. Graber and E. Mantovan for helpful and encouraging conversations regarding the constructions of section 2 as well as various general algebraic geometric questions.

## 2 Line bundles on $E^n$ and their sections

In the sequel, we will quite frequently need to specify line bundles on a power  $E^n$  or (meromorphic) sections thereof. At first glance, the problem of specifying a line bundle appears nearly trivial. Indeed, given any principally polarized abelian variety  $A$ , we have a short exact sequence

$$0 \rightarrow A \rightarrow \text{Pic}(A) \rightarrow \text{NS}(A) \rightarrow 0, \quad (2.1)$$

where the Néron-Severi group  $\text{NS}(A)$  is naturally isomorphic to the group of endomorphisms of  $A$  which are symmetric under the Rosati involution. If  $\text{End}(E) = \mathbb{Z}$  (which holds generically), then this becomes

$$0 \rightarrow E^n \rightarrow \text{Pic}(E^n) \rightarrow \text{NS}(E^n) \rightarrow 0, \quad (2.2)$$

where  $\text{NS}(E^n)$  is the group of symmetric  $n \times n$  integer matrices. Moreover, it turns out (as we will discuss in more detail below) that this short exact sequence splits, giving us canonical labels for line bundles *up to isomorphism*.

This last caveat is quite significant, however; if  $\mathcal{L}_1$  and  $\mathcal{L}_2$  represent given classes in  $\text{Pic}(E^n)$  and  $\mathcal{L}_3$  represents their sum, then there exists an isomorphism  $\mathcal{L}_1 \otimes \mathcal{L}_2 \cong \mathcal{L}_3$ , but this is only determined up to an overall scalar multiple. Another consequence is that if we have (as we will below) a group  $G$  acting on  $E^n$ , a  $G$ -invariant class in  $\text{Pic}(E^n)$  need not specify an *equivariant* line bundle.

If  $E$  is an analytic curve  $\mathbb{C}/\langle 1, \tau \rangle$ , there is a standard way to avoid these difficulties, namely the theory of theta functions. Indeed, given a cocycle  $z \in Z^1(\pi_1(E^n); \mathbb{A}(\mathbb{C}^n)^*)$ , we can construct a corresponding line bundle  $\mathcal{L}_z$ , and these bundles satisfy  $\mathcal{L}_{z_1} \otimes \mathcal{L}_{z_2} \cong \mathcal{L}_{z_1 z_2}$  and  $g^* \mathcal{L}_z \cong \mathcal{L}_{g^* z}$ . Moreover, since  $H^1(\mathbb{Z}, \mathbb{A}(\mathbb{C}^n)^*) = 0$ , we can arrange for our cocycles to have trivial restriction along  $\mathbb{Z}^n \subset \langle 1, \tau \rangle^n \cong \pi_1(E^n)$ . We thus obtain the following description of line bundles on  $E^n$ : given a symmetric integer matrix  $Q$  and constants  $C_1, \dots, C_n \in \mathbb{C}^*$ , we could consider the line bundle on  $(\mathbb{C}/\langle 1, \tau \rangle)^n$  with local sections given by functions on  $\mathbb{C}$  satisfying

$$f(z_1, \dots, z_{i-1}, z_i + 1, z_{i+1}, \dots, z_n) = f(z_1, \dots, z_n) \quad (2.3)$$

$$f(z_1, \dots, z_{i-1}, z_i + \tau, z_{i+1}, \dots, z_n) = C_i e\left(-\sum_j Q_{ij} z_j\right) f(z_1, \dots, z_n) \quad (2.4)$$

where  $e(x) := \exp(2\pi\sqrt{-1}x)$ . These line bundles behave well under tensor product, but in slightly odd ways under pullback; it turns out that we should not quite trivialize the cocycle along  $\mathbb{Z}^n$  in general, but instead merely insist that it restrict to an appropriate morphism  $\mathbb{Z}^n \rightarrow \{\pm 1\}$ . This leads us to define the line bundle  $\mathcal{L}_{Q, \vec{C}}$  on  $(\mathbb{C}/\langle 1, \tau \rangle)^n$  as the sheaf with local sections consisting of holomorphic functions satisfying

$$f(\dots, z_i + x_i \tau + y_i, \dots) = \prod_{1 \leq i \leq n} (-1)^{Q_{ii}(x_i + y_i)} C_i^{x_i} \prod_{1 \leq i, j \leq n} e(-Q_{ij} x_i (z_j + x_j \tau / 2)) f(\dots, z_i, \dots) \quad (2.5)$$

for  $\vec{x}, \vec{y} \in \mathbb{Z}^n$ . Note that this does not quite solve the problem as stated, since it gives multiple representatives for each line bundle (multiplying  $f$  by the nowhere vanishing entire function  $e(z_i)$

multiplies  $C_i$  by  $e(\tau)$ ), but still makes it straightforward to control equivariant structures on line bundles, as well as some of the more gerbe-like structures we need to consider below.

Although this suffices for many purposes, we would like to have an *algebraic* solution to this problem. Not only is our construction below essentially algebraic in nature, there are also some indications that it may prove useful in later work to be able to consider versions defined over finite fields. (In particular, see the use in [1] of finite field instances of the Sklyanin algebra in proving the latter is Noetherian, of particular interest given that we do not yet have a proof that our algebras are Noetherian. . . ; see also unpublished work of Bezrukavnikov and Okounkov.) A first step towards this is to observe that we are not *really* interested in constructing things over a particular curve; rather, we wish to have constructions that apply to *all* curves. In other words, what we truly want to understand are line bundles on the  $n$ th fiber power  $\mathcal{E}^n$  of the universal curve  $\mathcal{E}$  over the moduli stack  $\mathcal{M}_{1,1}$  of elliptic curves.

The theory of Jacobi forms [13, 10, 7] gives us an approach to this at the analytic level. Analytically,  $\mathcal{E}^n$  is the quotient of  $\mathbb{C}^n \times \mathbb{H}$  by the appropriate action of  $\mathbb{Z}^{2n} \rtimes \mathrm{SL}_2(\mathbb{Z})$ , and thus again we may specify line bundles via cocycles. This leads to the following definition: Given a symmetric integer matrix  $Q$  (the “level”) and an integer  $w$  (the “weight”), we define the line bundle  $\mathcal{L}_{Q,w}$  on the complex locus of  $\mathcal{E}^n$  as the sheaf with local sections consisting of functions  $f(z_1, \dots, z_n; \tau)$  such that

$$f(\vec{z} + \vec{x}\tau + \vec{y}; \tau) = (-1)^{\vec{x}^t Q \vec{x} + \vec{y}^t Q \vec{y}} e(-\vec{x}^t Q(\vec{z} + \vec{x}\tau/2)) f(\vec{z}; \tau) \quad (2.6)$$

$$f(\vec{z}/(c\tau + d); (a\tau + b)/(c\tau + d)) = (c\tau + d)^w e(c\vec{z}^t Q \vec{z}/2(c\tau + d)) f(\vec{z}; \tau), \quad (2.7)$$

with  $\vec{x}, \vec{y} \in \mathbb{Z}^n$  and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . (This differs from the usual notion of Jacobi form by virtue of our not imposing any condition at the cusp; we have also allowed  $Q$  to have odd diagonal.)

**Lemma 2.1.** *The line bundle  $\mathcal{L}_{1,-1}(-[0])$  on the complex locus of  $\mathcal{E}$  is trivial, where the divisor  $[0]$  is the image of the identity section  $0 : \mathcal{M}_{1,1} \rightarrow \mathcal{E}$ .*

*Proof.* We first observe that the function

$$\begin{aligned} \vartheta(z; \tau) &:= \frac{(e(z/2) - e(-z/2)) \prod_{1 \leq j} (1 - e(j\tau + z))(1 - e(j\tau - z))}{\prod_{1 \leq j} (1 - e(j\tau))^2} \\ &= \frac{\sum_{k \in 1/2 + \mathbb{Z}} (-1)^{k-1/2} e(kz + k^2\tau/2)}{\sum_{k \in 1/2 + \mathbb{Z}} (-1)^{k-1/2} k e(k^2\tau/2)} \end{aligned} \quad (2.8)$$

is a global section of  $\mathcal{L}_{1,-1}$ . This follows immediately from the standard transformation law for Jacobi theta functions together with the transformation law for the Dedekind eta function  $e(\tau/24) \prod_{1 \leq j} (1 - e(j\tau))$ . (Note that in each case, the overall transformation law involves complicated arithmetic characters, but these turn out to cancel). This is holomorphic on  $\mathbb{C} \times \mathbb{H}$ , and for each  $\tau \in \mathbb{H}$  is a nonzero function vanishing only on the lattice  $\langle 1, \tau \rangle$ ; thus the corresponding section of  $\mathcal{L}_{1,-1}$  has divisor  $[0]$ , establishing triviality as required.  $\square$

*Remark.* The function  $\vartheta(z; \tau)$  may be expressed in the standard multiplicative notation for theta functions in elliptic special function theory as  $\vartheta(z; \tau) = \frac{x^{-1/2} \theta_p(x;p)}{(p;p)_\infty}$  where  $x = e(z)$ ,  $p = e(\tau)$ .

It follows in particular that we can extend  $\mathcal{L}_{1,-1}$  to the entirety of  $\mathcal{E}$ : simply take the line bundle  $\mathcal{O}([0])$ . Pulling this back through a homomorphism  $g : \mathcal{E}^n \rightarrow \mathcal{E}$ ,  $(x_1, \dots, x_n) \mapsto \sum_i g_i x_i$ , gives an algebraic version of  $\mathcal{L}_{g^t g, -1}$ , and thus by taking tensor products an algebraic version of  $\mathcal{L}_{Q,w}$  in general. Of course, there is a potential issue here of uniqueness. Up to isomorphism, this is given by the following.

**Proposition 2.2.** *The Picard group of  $\mathcal{E}^n$  is a canonically split extension of  $\text{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12\mathbb{Z}$  by the Néron-Severi group of the generic fiber.*

*Proof.* That  $\text{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12\mathbb{Z}$  is standard (see [8] for an extension to fairly general base changes), as is the fact that we can represent the restrictions of such line bundles to the complex locus via sheaves of modular forms of given weight. (The reduction mod 12 then comes from the fact that the discriminant  $\Delta(z)$  is a holomorphic form of weight 12, nowhere vanishing on the smooth locus.) Moreover, the identity section  $\mathcal{M}_{1,1} \rightarrow \text{Pic}(\mathcal{E}^n)$  gives rise to a splitting of the natural pullback morphism  $\text{Pic}(\mathcal{M}_{1,1}) \rightarrow \text{Pic}(\mathcal{E}^n)$ . The above construction moreover shows that the natural map from  $\text{Pic}(\mathcal{E}^n)$  to the Néron-Severi group of the generic fiber is surjective. (Note that since  $\mathcal{M}_{1,1}$  is a stack, “the generic fiber” does not quite make sense, but it suffices to base change to some smooth curve covering  $\mathcal{M}_{1,1}$ , say by imposing a full level 3 or full level 4 structure.)

It thus remains only to show that if  $\mathcal{L}$  is a line bundle on  $\mathcal{E}^n$  which on the generic fiber is algebraically equivalent to the trivial divisor, then every fiber of  $\mathcal{L}$  is trivial, and thus  $\mathcal{L}$  is the pullback of a line bundle on  $\mathcal{M}_{1,1}$ . Since an algebraically trivial line bundle on an abelian variety gives rise to a point of the dual variety, and  $\mathcal{E}^n$  is principally polarized via the product polarization, we find that  $\mathcal{L}$  induces a section  $\mathcal{M}_{1,1} \rightarrow \mathcal{E}^n$ , and we need merely show that this is the identity section. The elliptic curve  $y^2 + txy = x^3 + t^5$  over  $\mathbb{C}(t)$  has trivial Mordell-Weil group, and thus the pullback of any section  $\mathcal{M}_{1,1} \rightarrow \mathcal{E}^n$  to this elliptic curve is trivial. Since the corresponding map  $\text{Spec}(\mathbb{C}(t)) \rightarrow \mathcal{M}_{1,1}$  is dominant, it follows that any section  $\mathcal{M}_{1,1} \rightarrow \mathcal{E}^n$  is generically trivial, and thus everywhere trivial since this is a closed condition.  $\square$

Let us now choose for each symmetric integer matrix  $Q$  and integer  $w \in \mathbb{Z}$  a line bundle  $\mathcal{L}_{Q,w}$  (unique up to isomorphism) which restricts to the bundle of weight  $w$  on the identity section and induces the symmetric endomorphism  $Q$  on the generic fiber of  $\mathcal{E}^n$ . This very nearly solves our problem of constructing line bundles with consistent isomorphisms under tensor product and pullbacks, by virtue of the fact that there are very few global units on  $\mathcal{E}^n$ . Indeed, since  $\mathcal{E}^n$  is proper over  $\mathcal{M}_{1,1}$ , the global units on  $\mathcal{E}^n$  are just the global units on  $\mathcal{M}_{1,1}$ , and these are easily seen to consist precisely of elements of the form  $\pm\Delta^l$  for  $l \in \mathbb{Z}$ . But in fact the same thing that gives us a splitting of the Picard group gives us a natural way to rigidify things completely.

**Definition 2.1.** For an integer  $w$ , a *weight  $w$  trivialization* of a line bundle  $\mathcal{L}$  on  $\mathcal{E}^n$  is an isomorphism

$$0^*\mathcal{L} \cong (0^*\omega_{\mathcal{E}/\mathcal{M}_{1,1}})^w. \quad (2.9)$$

We now enhance our data as follows:  $\mathcal{L}_{Q,w}$  is not just a line bundle in the appropriate isomorphism class; rather, it is such a line bundle together with a choice of weight  $w$  trivialization. We also insist that  $\mathcal{L}_{0,0} = \mathcal{O}_{\mathcal{E}^n}$  with the obvious trivialization. (Here we now take  $w$  an integer rather than a class mod 12, as trivializing  $0^*\omega_{\mathcal{E}/\mathcal{M}_{1,1}}^{12}$  requires choosing one of  $\Delta$  or  $-\Delta$ . This is not a significant issue, however.)

**Theorem 2.3.** *There is a natural isomorphism*

$$\mathcal{L}_{Q_1,w_1} \otimes \mathcal{L}_{Q_2,w_2} \cong \mathcal{L}_{Q_1+Q_2,w_1+w_2} \quad (2.10)$$

and, for any linear transformation  $g : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$  (with induced homomorphism  $g : \mathcal{E}^n \rightarrow \mathcal{E}^m$ ), a natural isomorphism

$$g^*\mathcal{L}_{Q,w} \cong \mathcal{L}_{g^t Q, w}, \quad (2.11)$$

satisfying all of the obvious compatibility conditions.

*Proof.* In either case, not only are the bundles isomorphic, but so are their pullbacks through 0 (e.g., since  $0^*g^* = 0^*$ ), and the respective trivializations actually induce a specific choice of isomorphism of the pullbacks. Since both isomorphisms are determined up to a global unit on  $\mathcal{M}_{1,1}$ , rigidifying the pullback suffices to rigidify the desired isomorphisms. Similarly, if we construct an automorphism from such maps which induces the identity on the fiber over 0, it will actually be the identity, giving the desired compatibility.  $\square$

To specify (meromorphic) sections of such line bundles, we would like to have an analogue of the Jacobi theta function. Here we have the following.

**Lemma 2.4.** *The line bundle  $\mathcal{O}([0])$  on  $\mathcal{E}$  has a natural weight  $-1$  trivialization.*

*Proof.* We need an isomorphism

$$0^*\mathcal{O}([0]) \otimes 0^*\omega_{\mathcal{E}/\mathcal{M}_{1,1}} \cong \mathcal{O}_{\mathcal{M}_{1,1}} \quad (2.12)$$

or equivalently

$$0^*(\omega_{\mathcal{E}/\mathcal{M}_{1,1}}([0])) \cong \mathcal{O}_{\mathcal{M}_{1,1}}, \quad (2.13)$$

but this is just adjunction.  $\square$

*Remark.* Note that the isomorphism coming from adjunction simply takes a differential with simple pole at 0 to its residue.

**Definition 2.2.** Let  $\mathcal{L}_{1,-1}$  be the chosen line bundle with trivialization on  $\mathcal{E}$ . Then the global section  $\vartheta \in \Gamma(\mathcal{L}_{1,-1})$  is the image of 1 under the isomorphism  $\mathcal{O}([0]) \cong \mathcal{L}_{1,-1}$  respecting the trivialization.

Note that the function  $\vartheta$  considered above was normalized so that

$$\text{Res}_{z=0} \frac{dz}{\vartheta(z; \tau)} = 1, \quad (2.14)$$

and thus the two definitions agree on the analytic locus. This gives us the following consequence, a very powerful way of constructing functions on powers  $\mathcal{E}^n$ .

**Theorem 2.5.** *Let  $c_{ij}$ ,  $1 \leq i \leq l$ ,  $1 \leq j \leq n$  and  $m_i$ ,  $1 \leq i \leq l$  be integers such that  $\sum_{1 \leq i \leq l} m_i = 0$  and  $\sum_{1 \leq i \leq l} m_i c_{ij} c_{ik} = 0$  for  $1 \leq j, k \leq n$ . Then there is a (unique) meromorphic function on  $\mathcal{E}^n$ , defined on every fiber, which on the complex locus restricts to  $\prod_{1 \leq i \leq l} \vartheta(\sum_j c_{ij} z_j; \tau)^{m_i}$ .*

*Proof.* We may view each  $c_i$  as a morphism  $\mathcal{E}^n \rightarrow \mathcal{E}$ , and find that  $\prod_{1 \leq i \leq l} (c_i^* \vartheta)^{m_i}$  is a meromorphic section of the line bundle

$$\bigotimes_{1 \leq i \leq l} (c_i^* \mathcal{L}_{1,-1})^{m_i} \cong \mathcal{L}_{\sum_i m_i c_i^t c_i, -\sum_i m_i} \cong \mathcal{O}_{\mathcal{E}^n}, \quad (2.15)$$

with both maps canonical. In other words, the given product of sections of line bundles determines a meromorphic function on  $\mathcal{E}^n$ , the divisor of which manifestly has no vertical components.  $\square$

*Remark 1.* By a very mild abuse of notation, we will write the functions constructed in this way as  $\prod_{1 \leq i \leq l} \vartheta(\sum_j c_{ij} z_j)^{m_i}$ , and similarly for the analogous meromorphic sections of line bundles  $\mathcal{L}_{Q,w}$ .

*Remark 2.* Of course, if we multiply by a suitable power of  $\Delta$ , we can construct similar functions under the weaker assumption that  $\sum_i m_i$  is a multiple of 12. On the other hand, with the constraints as given, there is a natural limit at the cusp of  $\mathcal{M}_{1,1}$ , namely the rational function

$$\prod_j x_j^{-\sum_i m_i c_{ij}/2} \prod_{1 \leq i \leq l} (1 - \prod_j x_j^{c_{ij}})^{m_i} \quad (2.16)$$

on  $(\mathbb{C}^*)^n = e(\mathbb{C})^n$ . Note that  $\sum_i m_i c_{ij}$  is even since  $\sum_i m_i c_{ij}^2 = 0$ , so this is indeed well-defined. This gives a useful method for sanity-checking calculations, by verifying that the result is correct in this limit.

It is worth noting that there is an alternate approach to the Theorem which, while it does not help us deal with line bundles, is more powerful in one important respect: it gives a reasonable algorithm for *evaluating* such functions at specific points (i.e., on  $n$ -tuples of points of specific elliptic curves, say over a finite field). We choose (to make the appropriate induction work) a differential  $\omega$  on  $E$ , allowing us to eliminate the constraint  $\sum_i m_i = 0$ . The corresponding trivialization of the line bundle  $\mathcal{O}(d[0])$  can then be computed as follows: choose any uniformizer  $u$  at 0 such that  $\omega/u$  has residue 1, and then for any function  $f$  with multiplicity  $d$  at 0, express  $f = c(f, \omega)u^d + O(u^{d+1})$  to obtain a “leading coefficient”  $c(f, \omega)$ . It is easy to see that this leading coefficient is independent of the choice of  $u$ , and that  $c(f, \omega)$  scales in the appropriate way with  $\omega$ .

Now, we can characterize the function  $\prod_{1 \leq i \leq l} \vartheta(\sum_j c_{ij} z_j)^{m_i}$  up to a scalar multiple by viewing it as a function of  $z_n$  with specified divisor. If there are no factors depending only on  $z_n$ , then the value of the function along  $z_n = 0$  is then a function of the same form, which can be computed by induction, giving us the requisite scale factor and letting us plug in the specific value of  $z_n$  desired. If the only factors depending only on  $z_n$  are powers of  $\vartheta(z_n)$ , then we use the leading coefficient instead and proceed as before. (The correct leading coefficient is computed by removing powers of  $\vartheta(z_n)$  and setting  $z_n = 0$ .) Note that in general we can compute the leading coefficient as long as every factor  $\vartheta(kz_n)^m$  that arises has  $k$  invertible (since  $\vartheta(kz)$  has leading term  $ku$  in characteristic 0). Thus if we can construct functions of the form  $\frac{\vartheta(kz)}{\vartheta(z)^{k^2}}$  directly, we could eliminate those factors before computing the leading coefficient. Since  $\vartheta(kz) = -\vartheta(-kz)$ , we reduce to the case  $k > 1$ . Divisor considerations then tell us that such a function must be proportional to the  $k$ -division polynomial (as defined in [28, Ex. 3.7]), and we find in fact that the  $k$ -division polynomial has the correct leading term in characteristic 0, so gives the desired function on general  $E$ .

In addition to line bundles and their sections, we will also need some gerbe-ish structures. The objects we want to consider are “equivariant gerbes”: a system of line bundles  $\mathcal{Z}_g$  associated to  $g \in G \subset \mathrm{GL}_n(\mathbb{Z})$  along with morphisms  $\zeta_{gh} : \mathcal{Z}_g \otimes (g^{-1})^* \mathcal{Z}_h \cong \mathcal{Z}_{gh}$  satisfying the obvious compatibility condition. This, of course, is easy to construct: given any cocycle valued in pairs  $(Q, w)$ , we may take  $\mathcal{Z}_g$  to be  $\mathcal{L}_{Q_g, w_g}$  and  $\zeta_{gh}$  to be the natural morphism.

The situation becomes more complicated if we want to also construct meromorphic sections of such gerbes: i.e., a system of meromorphic sections  $z_g \in \mathcal{Z}_g$  such that  $\zeta_{gh}(z_g \otimes (g^{-1})^* z_h) = z_{gh}$ . In this case, we do not have a completely general solution, but there is an important special case.

Consider first the case  $G = \mathbb{Z}$ , with generator acting on  $\mathcal{E}^2$  by  $(z, q) \mapsto (z + q, q)$ . There is a natural equivariant gerbe with meromorphic section such that  $\mathcal{Z}_1 = \mathcal{L}_{z^2/2, -1}$  and  $z_1 = \vartheta(z)$ . (Here we are representing the symmetric bilinear form  $Q$  by the corresponding quadratic form  $zQz^t/2$ .) Indeed, we then find more generally that  $\mathcal{Z}_k$  is the natural line bundle of weight  $-k$  with polarization

$$(z, q)Q(z, q)^t/2 = k \frac{z^2}{2} + \frac{k(k-1)}{2} qz + \frac{k(2k-1)(k-1)}{6} \frac{q^2/2}{}, \quad (2.17)$$

and

$$z_k = \begin{cases} \prod_{0 \leq i < k} \vartheta(iq + z) & k \geq 0 \\ \prod_{1 \leq i \leq -k} \vartheta(-iq + z)^{-1} & k \leq 0. \end{cases} \quad (2.18)$$

We will denote this meromorphic section of an equivariant gerbe by  $\Gamma_q(z)$ , which we refer to as an ‘‘elliptic Gamma function’’. Of course, this is even less a function than  $\vartheta$ , but in the analytic setting one can indeed replace  $\Gamma_q(z)$  by a suitable meromorphic solution of the functional equation  $\Gamma_q(q+z) = \vartheta(z)\Gamma_q(z)$ . Of course, this only determines  $\Gamma_q$  up to multiplication by invertible  $q$ -periodic functions, so the resulting meromorphic function is far from unique. One such solution (for  $q$  in the upper half-plane) is the doubly infinite product

$$\Gamma_q(z; \tau) := \left( - \prod_{1 \leq j} (1 - e(j\tau))^2 \right)^{-z/q} e(-z(z-q)/4q) \prod_{0 \leq j, k} \frac{1 - e((j+1)\tau + (k+1)q - z)}{1 - e(j\tau + kq + z)}; \quad (2.19)$$

this depends on a choice of  $\log(-\prod_{1 \leq j} (1 - e(j\tau))^2)$ , but this will not matter for our purposes. Here the double product is just the usual elliptic Gamma function [24].

More generally, for given morphisms  $q, z : \mathcal{E}^n \rightarrow \mathcal{E}$ , we may pull this formal symbol back to  $\mathcal{E}^n$ . This is tricky to deal with in complete generality, but if we fix  $q$  and vary  $z$ , we may consider general products

$$\prod_{1 \leq i \leq l} \Gamma_q(\vec{\alpha}_i \cdot \vec{z})^{m_i}. \quad (2.20)$$

If the corresponding element  $\sum_i m_i [\vec{\alpha}_i]$  of  $\mathbb{Z}[\mathbb{Z}^n/q]$  is trivial, then this formal product may be resolved into a product of  $\vartheta$  functions using the functional equation. Moreover, since for each congruence class the corresponding subproduct may be pulled back from  $\mathcal{E}^2$ , we find that the resulting product of  $\vartheta$  functions is independent of any choices that may have been made.

More generally, if  $G \subset \mathrm{GL}_n(\mathbb{Z})$  fixes  $q$  and the element  $\sum_i m_i [\vec{\alpha}_i] \in \mathbb{Z}[\mathbb{Z}^n/q]$ , then the ratio of the formal product and its pullback under  $g \in G$  will always resolve to a product of  $\vartheta$  functions, and thus gives an equivariant-gerbe-with-meromorphic-section which we refer to as the *coboundary* of the formal symbol. Note in particular that the hypothesis *always* holds for the (translation) subgroup of  $\mathrm{GL}_n(\mathbb{Z})$  that acts trivially on both  $q$  and the quotient  $\mathbb{Z}^n/q$ ; in the cases of interest (affine Weyl groups), the intersection of  $G$  with this subgroup will be cofinite in both groups, and it will be relatively straightforward to check that individual instances give rise to sections of gerbes.

We should caution the reader that there is a mild subtlety when it comes to the reflection principle. Although the product  $\Gamma_q(z)\Gamma_q(q-z)$  corresponds to the trivial equivariant gerbe (it has polarization 0 and weight  $-1$  (see below), both of which are invariant under  $z \mapsto z + q$ ), the corresponding meromorphic section is not quite trivial: one finds that  $\Gamma_q(z)\Gamma_q(q-z)$  is negated by the translation  $z \mapsto z + q$ .

On the other hand, the multiplication principle  $\Gamma_q(z) = \prod_{0 \leq j < k} \Gamma_{kq}(z + jq)$  for integer  $k > 0$  does work, as both sides truly do correspond to the same gerbe-with-section. This also works for negative  $k$ :  $\Gamma_q(z) = \prod_{1 \leq j \leq -k} \Gamma_{kq}(z - jq)^{-1}$ . Using this, one can make sense of products of elliptic Gamma functions with varying  $q$  as long as the different  $q$  that appear have a common multiple. Note in particular the special case  $\Gamma_q(z) = \Gamma_{-q}(z - q)^{-1}$ .

We will of course want to know the polarizations and weights of the line bundles associated to a given such gerbe section, which reduces to knowing the polarization and weight when such a product resolves to a product of  $\vartheta$  functions. There is, in fact, a very simple bookkeeping procedure for determining this. Define the polarization of  $\Gamma_q(z)$  to be  $\frac{z(z-q)(2z-q)}{12q}$ , and the weight of  $\Gamma_q(z)$  to be  $-\frac{z}{q}$ , extended to products of pullbacks in the obvious way. The polarization of the formal product  $\Gamma_q(qz)/\Gamma_q(z)$  is  $z^2/2$  and the weight is  $-1$ , agreeing with the polarization and weight of

$\vartheta(z)$ . It follows more generally that the polarization and weight of a product of powers of elliptic Gamma functions is consistent with the usual notion whenever the product resolves to a product of  $\vartheta$  functions. (In particular, if the product resolves and the polarization and weight are trivial, then it resolves to an honest function on  $\mathcal{E}^n$ .) Of course, when considering the associated gerbe-with-section, only the  $z$ -dependent portion of the polarization matters.

One should note here that not every cocycle valued in polarizations and weights of line bundles comes from a product of  $\Gamma_q$  symbols; for instance, any product of symbols  $\Gamma_q(az + bq)$  with trivial polarization has weight of the form  $12cz/q + d$ , so that every line bundle in the coboundary has weight a multiple of 12. (There are also some additional parity issues for  $n > 1$ .) Of course, it is conceivable that the cocycles violating these congruence conditions do not have *any* consistent family of meromorphic sections.

Note that if one wishes to convert a product of standard elliptic Gamma functions into a product of symbols  $\Gamma_q$ , one must first use the reflection principle to eliminate appearances of  $p$  from the arguments (which, if there is a balancing condition that involves  $p$  will require shifting some of the variables by  $p$  first). If the result has  $\sum_i m_i (\sum_i \vec{\alpha}_i \cdot z_i)^2 = \sum_i m_i (\sum_i \vec{\alpha}_i \cdot z_i) = 0$ , then replacing the standard Gamma functions by the explicit meromorphic solution given above for the functional equation of  $\Gamma_q$  will have no effect on the resulting meromorphic function.

Since we now have a method for constructing equivariant line bundles on  $\mathcal{E}^n$ , it is natural to ask about the corresponding representations of  $G$  on global sections; in particular, we will want to understand the space of  $G$ -invariant global sections. It turns out that if we want to extend the standard analytic approach to this question to algebraic curves, we will need to extend the above construction to cover certain abelian varieties *isogenous* to  $\mathcal{E}^n$ .

To be precise, for a positive integer  $N$  let  $\mathcal{X}_0(N)$  denote the moduli stack of elliptic curves  $E_1$  equipped with a cyclic  $N$ -isogeny  $\phi : E_1 \rightarrow E_N$ , with universal curves denoted by  $\mathcal{E} = \mathcal{E}_1, \mathcal{E}_N$ . (Here “cyclic” is in the sense of [11]; in particular, note that in characteristic  $p$ , *any* isogeny of degree  $p^k$  is cyclic.) For each divisor  $d|N$ , there is a corresponding factorization  $\phi = \phi_{N,d} \circ \phi_{d,1}$  where  $\phi_{d,1} : \mathcal{E}_1 \rightarrow \mathcal{E}_d$ ,  $\phi_{N,d} : \mathcal{E}_d \rightarrow \mathcal{E}_N$  are cyclic isogenies of degrees  $d$  and  $N/d$  respectively. For  $d = p$  prime, this is constructed as follows: on the locus where the  $p$ -part of  $\ker \phi$  is étale,  $\mathcal{E}_p$  is the quotient by the  $p$ -torsion of  $\ker \phi$ ; on the complementary locus, where the  $p$ -part of  $\ker \phi$  is nonreduced,  $\phi_{1,p}$  is the Frobenius isogeny. (Per [11], this rule gives the correct limit to make  $\mathcal{E}_p$  a flat family.) In either case,  $\ker \phi_{1,p} \subset \ker \phi$ , and thus  $\phi$  factors as required, and we find that  $\phi_{p,N}$  is again cyclic. We thus have induced factorizations for every  $d|N$ , and it is easy to see that they are all compatible. More precisely, for any pair  $d_1|d_2|N$ , we obtain an isogeny  $\phi_{d_2,d_1} : \mathcal{E}_{d_1} \rightarrow \mathcal{E}_{d_2}$ , and  $\phi_{d_3,d_2} \circ \phi_{d_2,d_1} \cong \phi_{d_3,d_1}$ . We also, of course have similarly compatible isogenies  $\phi_{d_1,d_2} : \mathcal{E}_{d_2} \rightarrow \mathcal{E}_{d_1}$  obtained by dualizing  $\phi_{d_2,d_1}$ .

**Lemma 2.6.** *For any  $d_1, d_2|N$ , we have  $\text{Hom}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) \cong \mathbb{Z}$ , generated by the composition*

$$\phi_{d_2,d_1} := \phi_{d_2, \text{gcd}(d_1, d_2)} \circ \phi_{\text{gcd}(d_1, d_2), d_1} = \phi_{d_2, \text{lcm}(d_1, d_2)} \circ \phi_{\text{lcm}(d_1, d_2), d_1}. \quad (2.21)$$

*Proof.* Let  $d_0 = \text{gcd}(d_1, d_2)$  and  $d_3 = \text{lcm}(d_1, d_2)$ , and let  $f : \mathcal{E}_{d_1} \rightarrow \mathcal{E}_{d_2}$  be any homomorphism. Then  $\phi_{d_0, d_2} \circ f \circ \phi_{d_1, d_0}$  is an endomorphism of  $\mathcal{E}_1$ , so must be multiplication by some integer. By degree considerations, that integer must be a multiple of  $d_2/d_0$  and  $d_1/d_0$ , and thus (since these are relatively prime) of  $d_1 d_2 / d_0^2 = d_3 / d_0$ . Since  $\phi_{d_0, d_2} \circ \phi_{d_2, d_1} \circ \phi_{d_0, d_1} = [d_1 d_2 / d_0^2]$ , the first claim follows. We moreover find that

$$\phi_{d_0, d_2} \circ \phi_{d_2, d_3} \circ \phi_{d_3, d_1} \circ \phi_{d_1, d_0} = \phi_{d_0, d_3} \circ \phi_{d_3, d_0} = [d_3 / d_0] \quad (2.22)$$

from which the other factorization follows.  $\square$

For any sequence of divisors  $d_i$  of  $N$ , we may consider the corresponding fiber product of curves  $\mathcal{E}_{d_i}$  over  $\mathcal{X}_0(N)$ ; we will generally omit  $\mathcal{X}_0(N)$  from the product notation. As in the  $N = 1$  case, morphisms between such products may be expressed as matrices, with the only difference being that the  $ij$  entry is now a multiple of the natural isogeny  $\phi_{d_i, d_j}$ .

**Proposition 2.7.** *For any  $d_1, d_2 | N$ , there is an isomorphism*

$$\mathcal{E}_{d_1} \times \mathcal{E}_{d_2} \cong \mathcal{E}_{\gcd(d_1, d_2)} \times \mathcal{E}_{\text{lcm}(d_1, d_2)}. \quad (2.23)$$

*Proof.* Let  $d_0 = \gcd(d_1, d_2)$ ,  $d_3 = \text{lcm}(d_1, d_2)$ , and choose  $a, b$  so that  $ad_1 - bd_2 = d_0$ . We then easily verify that the morphisms

$$\begin{pmatrix} a\phi_{d_0, d_1} & b\phi_{d_0, d_2} \\ \phi_{d_3, d_1} & \phi_{d_3, d_2} \end{pmatrix} : \mathcal{E}_{d_1} \times \mathcal{E}_{d_2} \rightarrow \mathcal{E}_{d_0} \times \mathcal{E}_{d_3} \quad (2.24)$$

and

$$\begin{pmatrix} \phi_{d_1, d_0} & -b(d_2/d_0)\phi_{d_1, d_3} \\ -\phi_{d_2, d_0} & a(d_1/d_0)\phi_{d_2, d_3} \end{pmatrix} : \mathcal{E}_{d_0} \times \mathcal{E}_{d_3} \rightarrow \mathcal{E}_{d_1} \times \mathcal{E}_{d_2} \quad (2.25)$$

are inverses, giving the desired isomorphism.  $\square$

It follows immediately that any product of curves  $\mathcal{E}_d$  is isomorphic to one of the form  $\prod_i \mathcal{E}_{d_i}$  where  $1|d_1| \cdots |d_n|N$ .

If we attempt to repeat our  $N = 1$  construction for general  $N$ , we encounter two difficulties. The first is that for each  $d|N$ , we may obtain a line bundle on  $\mathcal{X}_0(N)$  by taking the fiber at 1 of the sheaf of differentials on  $\mathcal{E}_d$ , but these line bundles are not quite the same. If  $E_1 \rightarrow E_d$  is an *étale* isogeny, there is no problem:  $\phi_{1,d}^*$  induces an isomorphism of  $\omega_{E_d}$  and  $\omega_{E_1}$ , so in particular of their fibers over 1. However, if  $E_1 \rightarrow E_d$  is inseparable, then  $\phi_{1,d}^*$  actually annihilates  $\omega_{E_d}$ . As a result,  $\omega_{E_d}|_0$  and  $\omega_{E_1}|_0$  differ by a linear combination of components of the fibers of  $\mathcal{X}_0(N)$  over primes dividing  $d$ . Of course, if all we want to do is construct *line bundles*, this is not an issue, but this does mean that the construction of *functions* in this way will be nontrivial.

The more serious difficulty is that it is no longer the case that  $\prod_i \mathcal{E}_{d_i} \rightarrow \mathcal{X}_0(N)$  has only the trivial section. Indeed, we have the following.

**Lemma 2.8.** *Let  $N$  be a positive integer, and  $d|N$ . Then the group of sections of  $\mathcal{E}_d$  over  $\mathcal{X}_0(N)$  consists entirely of 2-torsion. If both  $d$  and  $N/d$  are odd, the group is trivial; if precisely one is even, it has rank 1, and if both are even, it has rank 2.*

*Proof.* Since the stack  $\mathcal{X}_0(N)$  has nontrivial stabilizers, any section of  $\mathcal{E}_d$  must be invariant under the action of the stabilizer, and is thus preserved by  $[-1]$ , so is 2-torsion. The structure of the 2-torsion of  $\mathcal{E}_d$  then follows by considering the image of the corresponding congruence subgroup in  $\text{SL}_2(\mathbb{Z}/2\mathbb{Z})$ .  $\square$

*Remark.* For more general level structures, it follows from [27, Thm. 5.5] that for any subgroup  $\Gamma \subset \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ , the group of sections over the corresponding quotient stack  $\mathcal{X}(N)/\Gamma$  is  $N$ -torsion, and isomorphic to the subgroup  $((\mathbb{Z}/N\mathbb{Z})^2)^\Gamma$ .

If  $N$  is odd, we may thus conclude as before that the pullback of  $\mathcal{O}([0])$  through  $\phi_{d_1, d_2}$  is indeed fiberwise isomorphic to  $\mathcal{O}([0])^{\deg(\phi_{d_1, d_2})}$ . However, if  $N$  is even, this is no longer the case; indeed, the pullback of  $\mathcal{O}([0])$  through a 2-isogeny is the tensor product of  $\mathcal{O}(2[0])$  by the corresponding 2-torsion line bundle.

It turns out that we can fix this, at the cost of imposing some additional level structure. Let  $\mathcal{X}_0(2N, 2)$  be the slight reinterpretation of the stack  $\mathcal{X}_0(4N)$  obtained by dividing all of the subscripts of the isogenous curves by 2; that is  $\mathcal{X}_0(2N, 2)$  classifies cyclic  $4N$ -isogenies  $\mathcal{E}_{1/2} \rightarrow \mathcal{E}_{2N}$  in terms of the curve  $\mathcal{E}_1$ . Now, it follows from the Lemma that for each  $d|N$ , the curve  $\mathcal{E}_d$  has full 2-torsion over  $\mathcal{X}_0(2N, 2)$ ; in addition to the generators of the kernels  $\mathcal{E}_d \rightarrow \mathcal{E}_{d/2}$  and  $\mathcal{E}_d \rightarrow \mathcal{E}_{2d}$ , it also has their sum, which we denote by  $\omega$ . We thus obtain a line bundle  $\hat{\mathcal{L}}_{1, \mathcal{E}_d} := \mathcal{O}([\omega]) \otimes 0^* \mathcal{O}([\omega])^{-1}$  on  $\mathcal{E}_d$  with trivial fiber over 0. Note that again  $0^* \mathcal{O}([\omega])$  is a nontrivial line bundle, though it is actually far better-behaved than  $0^* \mathcal{O}([0])$ . Indeed,  $0^* \mathcal{O}([\omega])$  is trivial away from the locus where  $\omega = 0$ . This can only happen in characteristic 2, and only when the 4-isogeny  $\mathcal{E}_{d/2} \rightarrow \mathcal{E}_{2d}$  is multiplication by 2 (corresponding to one of the three components of  $\mathcal{X}_0(2, 2)$  in characteristic 2).

The merit of using this 2-torsion point is that it makes everything compatible.

**Lemma 2.9.** *If  $d_1|d_2|N$  then there are natural isomorphisms*

$$\begin{aligned} \phi_{d_2, d_1}^* \hat{\mathcal{L}}_{1, \mathcal{E}_{d_2}} &\cong \hat{\mathcal{L}}_{1, \mathcal{E}_{d_1}}^{d_2/d_1} \\ \phi_{d_1, d_2}^* \hat{\mathcal{L}}_{1, \mathcal{E}_{d_1}} &\cong \hat{\mathcal{L}}_{1, \mathcal{E}_{d_2}}^{d_2/d_1} \end{aligned}$$

*Proof.* The two claims are equivalent via the (Atkin-Lehner) involution  $\mathcal{X}_0(2N, 2) \cong \mathcal{X}_0(2N, 2)$  that replaces  $\mathcal{E}_{1/2} \rightarrow \mathcal{E}_{2N}$  by its dual, so it suffices to prove the first. Since the fibers at the origin of the two bundles are trivial, it suffices to show that the line bundles are isomorphic on each fiber over  $\mathcal{X}_0(2N, 2)$ , at which point we may take the isomorphism respecting the fibers at the origin. We may also, for convenience, reduce to the case  $d_1 = 1, d_2 = N$ .

Being isomorphic is a closed condition, so we may exclude primes dividing  $4N$ , and in particular assume that the isogenies are all separable. On a given such curve, the line bundle  $\phi_{N,1}^* \hat{\mathcal{L}}_{1, \mathcal{E}_N}$  is represented by the divisor

$$\sum_{x \in \phi_{N,1}^{-1}(\omega_N)} [x], \tag{2.26}$$

and thus we need to show

$$\sum_{x \in \phi_{N,1}^{-1}(\omega_N)} x = N\omega_1, \tag{2.27}$$

or equivalently

$$\sum_{x \in \phi_{N,1}^{-1}(\omega_N - \phi_{N,1}(\omega_1))} x = 0. \tag{2.28}$$

If  $N$  is odd, then  $\phi(\omega_N) = \omega_1$ , and this sum becomes

$$\sum_{x \in \ker \phi_{N,1}} x = 0 \tag{2.29}$$

as required. Otherwise, we may factor through  $\mathcal{E}_2$  and thus reduce to the case  $N = 2$ . Then we find that  $\phi_{2,1}\omega_1$  is the generator of the kernel of  $\phi_{1,2}$ , and thus  $\omega_2 - \phi_{2,1}\omega_1$  is the generator of the kernel of  $\phi_{2,4}$ . It follows that the preimage of  $\omega_2 - \phi_{2,1}\omega_1$  consists of the two generators of the kernel of  $\phi_{1,4}$ , and these add to 0 as required.  $\square$

*Remark.* It is worth noting here that there are additional curves in the isogeny class of  $\mathcal{E}_1$ , since after all each of the 2-torsion points  $\omega_d$  itself determines a 2-isogeny. It is unclear whether we can extend the above family of line bundles to the other curves arising in this way.

**Corollary 2.10.** *For any integer  $a$ ,  $[a]^* \mathcal{L}_{1, \mathcal{E}_d} \cong \mathcal{L}_{1, \mathcal{E}_d}^{a^2}$ .*

*Proof.* This is clearly true for  $a = 0$  and  $a = -1$ , so we may assume  $a > 0$ . Over  $\mathcal{X}_0(2aN, 2)$ , we have  $[a] = \phi_{d, ad} \circ \phi_{ad, d}$ , so that the claim follows immediately from the Lemma. Since the isomorphism is natural, it descends to  $\mathcal{X}_0(2N, 2)$  as required.  $\square$

Since  $\mathcal{L}_{1, \mathcal{E}_d}$  represents the standard principal polarization on  $\mathcal{E}_d$ , we also have the following.

**Lemma 2.11.** *For any  $d|N$ , the bundle*

$$[x_1 + x_2]^* \hat{\mathcal{L}}_{1, \mathcal{E}_d} \otimes [x_1]^* \hat{\mathcal{L}}_{1, \mathcal{E}_d}^{-1} \otimes [x_2]^* \hat{\mathcal{L}}_{1, \mathcal{E}_d}^{-1} \quad (2.30)$$

*is naturally isomorphic to the Poincaré bundle  $\mathcal{P}_{\mathcal{E}_d}$  on  $\mathcal{E}_d \times \mathcal{E}_d$ .*

**Theorem 2.12.** *For any sequence  $1|d_1|\cdots|d_n|N$ , suppose  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are two line bundles on  $\prod_i \mathcal{E}_{d_i}$  obtained as tensor products of pullbacks of bundles  $\hat{\mathcal{L}}_{1, \mathcal{E}_d}$  through morphisms  $\prod_i \mathcal{E}_{d_i} \rightarrow \mathcal{E}_d$ . If  $\mathcal{L}_1$  and  $\mathcal{L}_2$  represent the same polarization, then they are naturally isomorphic.*

*Proof.* By the theorem of the cube, it suffices to prove this for  $n = 2$ . In this case, it is clear that the images of the bundles  $(1 \times 0)^* \hat{\mathcal{L}}_{1, \mathcal{E}_{d_1}}$ ,  $(0 \times 1)^* \hat{\mathcal{L}}_{1, \mathcal{E}_{d_2}}$  and  $(1 \times \phi_{d_1, d_2})^* \mathcal{P}_{\mathcal{E}_{d_1}}$  span the group of symmetric endomorphisms of  $\mathcal{E}_{d_1} \times \mathcal{E}_{d_2}$ , and thus it will suffice to show that any pullback of  $\hat{\mathcal{L}}_{1, \mathcal{E}_d}$  is isomorphic to the appropriate product of these bundles (again using the triviality at 0 to fix the isomorphism).

Thus consider a morphism  $\psi := a\phi_{d, d_1} + b\phi_{d, d_2} : \mathcal{E}_{d_1} \times \mathcal{E}_{d_2} \rightarrow \mathcal{E}_d$ . We then have

$$\psi^* \hat{\mathcal{L}}_{1, \mathcal{E}_d} \cong (a\phi_{d, d_1})^* \hat{\mathcal{L}}_{1, \mathcal{E}_d} \otimes (b\phi_{d, d_2})^* \hat{\mathcal{L}}_{1, \mathcal{E}_d} \otimes (a\phi_{d, d_1} \times b\phi_{d, d_2})^* \mathcal{P}_{\mathcal{E}_d}. \quad (2.31)$$

Now,

$$(a\phi_{d, d_1})^* \hat{\mathcal{L}}_{1, \mathcal{E}_d} = [a]^* \phi_{\gcd(d, d_1), d_1}^* \phi_{d, \gcd(d, d_1)}^* \hat{\mathcal{L}}_{1, \mathcal{E}_d} = \hat{\mathcal{L}}_{1, \mathcal{E}_{d_1}}^{a^2 \gcd(d, d_1)^2 / d_1 d} \quad (2.32)$$

and similarly for the second term. We also have

$$(a\phi_{d, d_1} \times b\phi_{d, d_2})^* \mathcal{P}_{\mathcal{E}_d} \cong (1 \times a\phi_{d_1, d} b\phi_{d, d_2})^* \mathcal{P}_{\mathcal{E}_d} = (1 \times abc\phi_{d_1, d_2})^* \mathcal{P}_{\mathcal{E}_d} \cong ((1 \times \phi_{d_1, d_2})^* \mathcal{P}_{\mathcal{E}_d})^{abc} \quad (2.33)$$

for a suitable integer  $c$ , so that the claim follows. (The first step here is essentially the definition of the dual isogeny.)  $\square$

Thus for each symmetric  $Q \in \text{End}(\prod_i \mathcal{E}_{d_i})$ , we obtain a line bundle  $\hat{\mathcal{L}}_{Q; d_1, \dots, d_n}$  on  $\prod_i \mathcal{E}_{d_i}$ , and these line bundles satisfy the same compatibility relations as for our earlier construction on  $\mathcal{E}^n$ . In general, our two constructions do not agree, but there is one important special case.

**Proposition 2.13.** *Suppose  $Q \in \text{Mat}_n(\mathbb{Z})$  is a symmetric matrix with even diagonal entries. Then  $\hat{\mathcal{L}}_{Q; 1, \dots, 1}$  descends to  $\mathcal{M}_{1, 1}$ , where it is canonically isomorphic to  $\mathcal{L}_{Q, 0}$ .*

*Proof.* Any symmetric integer matrix with even diagonal is not only in the span of pullbacks of 1, but in fact is in the span of pullbacks of  $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , the symmetric endomorphism associated to the Poincaré bundle  $\mathcal{P} \cong \hat{\mathcal{L}}_{H; 1} \cong \mathcal{L}_{H, 0}$ .  $\square$

*Remark.* More generally, if  $Q_{ii}$  is even whenever  $d_i$  is odd, then  $\hat{\mathcal{L}}_{Q; \vec{d}}$  descends to  $\mathcal{X}_0(2N)$ ; similarly, if  $Q_{ii}$  is even whenever  $N/d_i$  is odd, then  $\hat{\mathcal{L}}_{Q; \vec{d}}$  descends to  $\mathcal{X}_0(N, 2)$ .

It will be convenient to have a somewhat more functorial version of the constructions of  $\mathcal{E}^n$  or the products  $\prod_i \mathcal{E}_{d_i}$  above. For the first, if  $B$  is a finitely generated free abelian group, then we may consider the family of group schemes  $\mathcal{E} \otimes B$ . For a specific curve  $E$ , we may also construct  $E \otimes B$ , which is simply the corresponding fiber of  $\mathcal{E} \otimes B$ .

This clearly extends to a functor, which is exact on short exact sequences of free abelian groups. Note, however, that if we extend it in the obvious way to a functor on the category of *all* finitely generated abelian groups, then it is no longer exact. We readily compute the special cases

$$\begin{aligned} E \otimes \mathbb{Z} &\cong E \\ \mathrm{Tor}_p(E, \mathbb{Z}) &= 0, \quad p > 0 \\ E \otimes \mathbb{Z}/N\mathbb{Z} &\cong 0 \\ \mathrm{Tor}_1(E, \mathbb{Z}/N\mathbb{Z}) &= E[N] \\ \mathrm{Tor}_p(E, \mathbb{Z}/N\mathbb{Z}) &= 0, \quad p > 0 \end{aligned}$$

using the obvious projective resolution of  $\mathbb{Z}/N\mathbb{Z}$ . (This, of course, is the expected behavior for tensoring with a divisible group.)

**Proposition 2.14.** *Let  $\phi : B \rightarrow C$  be a morphism of finitely generated free abelian groups. Then  $E \otimes \phi$  is surjective iff  $\mathrm{coker}(\phi)$  is finite, and injective iff  $\ker(\phi) = 0$  and  $\mathrm{coker}(\phi)$  is free.*

*Proof.* By choosing isomorphisms  $B \cong \mathbb{Z}^n$ ,  $C \cong \mathbb{Z}^m$  such that  $\phi$  takes basis elements to basis elements, we may reduce to the cases that  $B = C = \mathbb{Z}$ ,  $B = 0$ , or  $C = 0$ , each of which are straightforward.  $\square$

*Remark.* Equivalently, since  $E \otimes^L$  has homological dimension 1, we find that the cokernel is  $E \otimes \mathrm{coker} \phi$ , while the kernel is an extension of  $E \otimes \ker \phi$  by  $\mathrm{Tor}_1(\mathcal{E}, \mathrm{coker} \phi)$ .

**Proposition 2.15.** *If  $E$  does not have complex (or quaternionic) multiplication (in particular if  $E = \mathcal{E}$ ), then  $\mathrm{Hom}(B, C) \rightarrow \mathrm{Hom}(E \otimes B, E \otimes C)$  is an isomorphism.*

*Proof.* Again, this reduces to the case  $B = \mathbb{Z}^n$ ,  $C = \mathbb{Z}^m$ , and thus to the case  $B = C = \mathbb{Z}$ , where it is essentially by definition.  $\square$

**Corollary 2.16.** *If  $E$  does not have complex multiplication, then the natural map  $B \rightarrow \mathrm{Hom}(E, E \otimes B)$  is an isomorphism, as is the natural map  $E \otimes \mathrm{Hom}(E, E \otimes B) \rightarrow E \otimes B$ .*

The dual variety is then easy to compute.

**Proposition 2.17.** *For  $B$  free, there is a natural isomorphism  $(E \otimes B)^\vee \cong E \otimes B^*$ .*

*Proof.* Indeed, we have  $E \otimes B \cong E^n$ , so dually  $(E \otimes B)^\vee \cong E^n$ . Thus for  $E$  without complex multiplication, the natural map  $E \otimes \mathrm{Hom}(E, (E \otimes B)^\vee) \rightarrow (E \otimes B)^\vee$  is an isomorphism. Duality gives  $\mathrm{Hom}(E, (E \otimes B)^\vee) \cong \mathrm{Hom}(E \otimes B, E) \cong \mathrm{Hom}(B, \mathbb{Z})$ , and thus  $E \otimes \mathrm{Hom}(B, \mathbb{Z}) \cong (E \otimes B)^\vee$  as desired. The isomorphism holds for  $E = \mathcal{E}$ , and thus for all fibers  $E$ .  $\square$

This gives the following description of the Néron-Severi group of  $\mathcal{E} \otimes B$ :  $NS(\mathcal{E} \otimes B)$  consists of symmetric morphisms  $\mathcal{E} \otimes B \rightarrow (\mathcal{E} \otimes B)^\vee \cong \mathcal{E} \otimes B^*$ , and thus of symmetric pairings  $Q : B \otimes B \rightarrow \mathbb{Z}$ . We then find as above that any such symmetric pairing induces a line bundle  $\mathcal{L}_{Q,w}$  on  $\mathcal{E} \otimes B$ .

Now, suppose  $B \rightarrow C$  is an injective morphism with finite kernel. Then we have a short exact sequence

$$0 \rightarrow \mathrm{Tor}_1(\mathcal{E}, C/B) \rightarrow \mathcal{E} \otimes B \rightarrow \mathcal{E} \otimes C \rightarrow 0. \quad (2.34)$$

where the kernel is a product of groups of the form  $E[d_i]$ . If  $C/B$  has exponent  $N$ , then we have  $\mathrm{Tor}_1(\mathcal{E}, C/B) \cong \mathcal{E}[N] \otimes_{\mathbb{Z}/N\mathbb{Z}} C/B$ , which in turn suggests that we consider the subgroup  $\kappa_N \otimes_{\mathbb{Z}/N\mathbb{Z}} C/B$  where  $\kappa_N$  is the kernel of a the cyclic  $N$ -isogeny corresponding to a point of  $\mathcal{X}_0(N)$ . This, it turns out, does not quite behave correctly in characteristic dividing  $N$ , but we do have the following.

**Proposition 2.18.** *Let  $N$  be a positive integer, and let  $B$  be a finitely generated abelian group of exponent  $N$ . Then the group scheme  $\kappa_N \otimes_{\mathbb{Z}/N\mathbb{Z}} B$  on  $\mathcal{X}_0(N) \times \mathrm{Spec}(\mathbb{Z}[1/N])$  extends in a natural way to a flat group scheme on  $\mathcal{X}_0(N)$ .*

*Proof.* If we choose an isomorphism  $B \cong \bigoplus \mathbb{Z}/d_i\mathbb{Z}$ , then we certainly have such an extension: away from  $N$ , the group is just  $\prod_i \kappa_{d_i}$  where  $\kappa_{d_i}$  is the kernel of the isogeny  $E_1 \rightarrow E_{d_i}$ , and this product makes sense in all characteristics. If  $B \rightarrow B'$  is a morphism of  $N$ -torsion groups, then the morphism  $\kappa_N \otimes_{\mathbb{Z}/N\mathbb{Z}} B \rightarrow \kappa_N \otimes_{\mathbb{Z}/N\mathbb{Z}} B'$  is simply the restriction of the morphism  $E[N] \otimes_{\mathbb{Z}/N\mathbb{Z}} B \rightarrow E[N] \otimes_{\mathbb{Z}/N\mathbb{Z}} B'$ . The latter morphism is defined in all characteristics, and the requirement that it restrict to a specific morphism is a closed condition, so is inherited from the generic case.  $\square$

This allows us to define families of abelian varieties over  $\mathcal{X}_0(N)$  as follows: given abelian groups  $B, C$  such that  $NC \subset B \subset C$ , we define  $\mathcal{E}_{B,C}$  to be the quotient of  $\mathcal{E} \otimes B$  by the subgroup scheme extending  $\kappa_N \otimes_{\mathbb{Z}/N\mathbb{Z}} C/B$ . (We will denote this extension by the same tensor product notation, but caution the reader that it is *not* the actual tensor product in general.)

**Proposition 2.19.** *If  $NC_1 \subset B_1 \subset C_1$ ,  $NC_2 \subset B_2 \subset C_2$  are pairs of f.g. free abelian groups and  $\phi : C_1 \rightarrow C_2$  is a morphism such that  $\phi(B_1) \subset B_2$ , then there is an induced morphism  $\mathcal{E}_{B_1, C_1} \rightarrow \mathcal{E}_{B_2, C_2}$ , making the construction functorial.*

*Proof.* The condition on  $\phi$  implies that it induces a morphism  $C_1/B_1 \rightarrow C_2/B_2$ , and thus we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \kappa_N \otimes_{\mathbb{Z}/N\mathbb{Z}} C_1/B_1 & \longrightarrow & \mathcal{E} \otimes B_1 & \longrightarrow & \mathcal{E}_{B_1, C_1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \kappa_N \otimes_{\mathbb{Z}/N\mathbb{Z}} C_2/B_2 & \longrightarrow & \mathcal{E} \otimes B_2 & \longrightarrow & \mathcal{E}_{B_2, C_2} \longrightarrow 0 \end{array} \quad (2.35)$$

Since the rows are exact and the given vertical arrows are functorial, the claim follows.  $\square$

Of course, given an isomorphism  $C/B \cong \prod_i \mathbb{Z}/d_i\mathbb{Z}$ , there is a corresponding isomorphism  $\mathcal{E}_{B,C} \cong \prod_i \mathcal{E}_{d_i}$ . This makes the following straightforward to verify.

**Proposition 2.20.** *If  $E \rightarrow E'$  is a cyclic  $N$ -isogeny between curves with no complex multiplication, then the morphisms between corresponding fibers of  $\mathcal{E}_{B_1, C_1}$  and  $\mathcal{E}_{B_2, C_2}$  are precisely those coming from the previous Proposition.*

**Corollary 2.21.** *For  $NC \subset B \subset C$ , we have*

$$\begin{aligned} \mathrm{Hom}(\mathcal{E}, \mathcal{E}_{B,C}) &\cong B \\ \mathrm{Hom}(\mathcal{E}_N, \mathcal{E}_{B,C}) &\cong C \end{aligned}$$

*in such a way that composition with the dual isogeny  $\mathcal{E}_N \rightarrow \mathcal{E}$  induces the inclusion  $B \subset C$ .*

**Proposition 2.22.** *There is a natural isomorphism  $\mathcal{E}_{B,C}^\vee \cong \mathcal{E}_{C^*, B^*}$ .*

*Proof.* The previous corollary allows us to canonically identify anything isomorphic to a product of curves  $\mathcal{E}_{d_i}$  with a variety of the form  $\mathcal{E}_{B,C}$ . Since  $\mathcal{E}_{B,C}$  is isomorphic to such a product, its dual is also of that form, and thus it remains only to compute  $\text{Hom}(\mathcal{E}, \mathcal{E}_{B,C}^\vee)$  and  $\text{Hom}(\mathcal{E}_N, \mathcal{E}_{B,C}^\vee)$ .  $\square$

It follows that the Néron-Severi group of  $\mathcal{E}_{B,C}$  (or of any fiber without complex multiplication) consists of pairings  $B \otimes C \rightarrow \mathbb{Z}$  which become symmetric when restricted to  $B \otimes B$ ; equivalently, it consists of symmetric pairings  $Q : B \otimes B \rightarrow \mathbb{Z}$  such that  $Q(B, NC) \subset N\mathbb{Z}$ . From our construction above, we find that if we base change to  $\mathcal{X}_0(2N, 2)$ , then any such symmetric pairing induces a natural line bundle  $\hat{\mathcal{L}}_{Q;B,C}$  on  $\mathcal{E}_{B,C}$ , satisfying the appropriate compatibility relations.

Suppose now that  $Q$  is a positive definite pairing of “level” dividing  $N$  (i.e., such that  $NB^* \subset QB$ ). Then we have a chain of free abelian groups  $Q^{-1}NB^* \subset B \subset C \subset Q^{-1}B^*$ , giving rise to an isogeny  $\pi : \mathcal{E}_{B,C} \rightarrow \mathcal{E}_{B,Q^{-1}B^*}$ . The pairing  $Q$  still induces an element of the Néron-Severi group of the quotient, which by degree considerations is now a principal polarization, represented by the line bundle  $\hat{\Theta}_B := \hat{\mathcal{L}}_{Q;B,Q^{-1}B^*}$ . We moreover find that  $\pi^*\hat{\Theta}_B \cong \hat{\mathcal{L}}_{Q;B,C}$ .

We also note that the action of  $\mathcal{E}[N] \otimes (Q^{-1}B^*/B) \cong \text{Tor}_1(\mathcal{E}, Q^{-1}B^*/B)$  on  $\mathcal{E} \otimes B$  descends to an action of the quotient  $(\mathcal{E}[N] \otimes (Q^{-1}B^*/B)) / (\kappa_N \otimes (Q^{-1}B^*/B))$  on  $\mathcal{E}_{B,Q^{-1}B^*}$ , which by the Weil pairing may be identified with an action of the Pontryagin dual  $\text{Hom}(\kappa_N \otimes (Q^{-1}B^*/B), \mu_N)$ . If  $\kappa_N$  is diagonalizable, then this dual is discrete, and may be identified with  $\text{Hom}(Q^{-1}B^*/B, \text{Hom}(\kappa_N, \mu_N))$ .

**Lemma 2.23.** *On the locus of  $\mathcal{X}_0(2N, 2)$  where the  $N$ -isogeny  $E_1 \rightarrow E_N$  has diagonalizable kernel  $\kappa_N$ , we have natural identifications*

$$\Gamma(E_{B,C}; \hat{\mathcal{L}}_{Q;B,C}) \cong \bigoplus_{g \in \text{Hom}(Q^{-1}B^*/C, \text{Hom}(\kappa_N, \mu_N))} g^* \Gamma(E_{B,Q^{-1}B^*}; \hat{\Theta}_B) \quad (2.36)$$

in which each (1-dimensional!) summand on the right is an eigenspace for the induced action of  $\ker(E_{B,C} \rightarrow E_{B,Q^{-1}B^*})$ .

*Proof.* We have

$$\Gamma(E_{B,C}; \hat{\mathcal{L}}_{Q;B,C}) \cong \Gamma(E_{B,C}; \pi^*\hat{\Theta}_B) \cong \Gamma(E_{B,Q^{-1}B^*}; \pi_*\pi^*\hat{\Theta}_B) \quad (2.37)$$

The natural map  $\hat{\Theta}_B \rightarrow \pi_*\pi^*\hat{\Theta}_B$  selects a particular eigenspace of the kernel of the isogeny, and the decomposition as claimed then follows by the structure theory of representations of Heisenberg groups, see [26] as well as the exposition in [16]. Here we use the fact that  $\Gamma(E_{B,C}; \hat{\mathcal{L}}_{Q;B,C})$  is the unique irreducible representation of the Heisenberg group  $\mathcal{G}(\hat{\mathcal{L}}_{Q;B,C})$  on which the central  $\mathbb{G}_m$  acts with weight 1, together with the fact that the commutator pairing on the Heisenberg group is precisely the Weil pairing, so the different isotypic components for the diagonalizable kernel are related via the complementary translation subgroup.  $\square$

**Corollary 2.24.** *Suppose that the finite group  $G$  acts on  $C$ , preserving the subgroup  $B$  and the polarization  $Q : B \rightarrow B^*$  so that  $G$  acts on  $\mathcal{E}_{B,C}$  as automorphisms fixing the identity, with an induced equivariant structure on  $\hat{\mathcal{L}}_{Q;B,C}$ . On the locus of  $\mathcal{X}_0(2N, 2)$  where  $E_1 \rightarrow E_N$  has diagonalizable kernel, the  $G$ -module  $\Gamma(E_{B,C}; \hat{\mathcal{L}}_{Q;B,C})$  is isomorphic to the permutation module arising from the action of  $G$  on  $\text{Hom}(Q^{-1}B^*/C, \mathbb{Q}/\mathbb{Z})$ .*

*Remark.* Here we note that for general free abelian groups  $B \subset C$  with finite quotient,

$$\text{Hom}(C/B, \mathbb{Q}/\mathbb{Z}) \cong \text{Tor}_1(B^*/C^*, \mathbb{Q}/\mathbb{Z}) \cong B^*/C^*, \quad (2.38)$$

and thus

$$\text{Hom}(Q^{-1}B^*/C, \mathbb{Q}/\mathbb{Z}) \cong C^*/QB \cong Q^{-1}C^*/B. \quad (2.39)$$

**Corollary 2.25.** *Let  $B$  be a f.g. free abelian group and  $Q : B \otimes B \rightarrow \mathbb{Z}$  an even symmetric pairing of level dividing  $N$ , and let  $E$  be an elliptic curve which, if supersingular, has characteristic prime to  $N$ . Then  $\Gamma(E \otimes B; \mathcal{L}_{Q;B})$  is isomorphic as a  $G$ -module to the permutation module coming from the action of  $G$  on  $Q^{-1}B^*/B$ .*

*Proof.* The condition on  $E$  ensures that we may choose a point of  $\mathcal{X}_0(2N, 2)$  lying over it such that the cyclic  $N$ -isogeny  $E \cong E_1 \rightarrow E_N$  has diagonalizable kernel. We may then identify the  $G$ -module  $\Gamma(E \otimes B; \mathcal{L}_{Q,0})$  with  $\Gamma(E_{B,B}; \hat{\mathcal{L}}_{Q;B,B})$  and thus apply the previous Corollary.  $\square$

**Corollary 2.26.** *With the same hypotheses, the dimension  $\dim(\Gamma(E \otimes B; \mathcal{L}_{Q,0})^G)$  is equal to the number of orbits of  $G$  in  $Q^{-1}B^*/B$ .*

*Remark.* Both conclusions remain valid for supersingular curves of characteristic prime to  $|G|$ ; indeed, any 1-parameter family of  $G$ -modules over an algebraically closed field containing  $1/|G|$  is trivial, so the claims follow from the case of ordinary curves.

It turns out that the exclusion of certain supersingular curves above is indeed necessary. For instance, suppose that  $B = \mathbb{Z}^8$  and  $Q$  is the Gram matrix of the lattice  $Q_8(1)$  of [5]. This is a symmetric matrix with even diagonal and elementary divisors 1, 1, 1, 1, 5, 5, 5, 5, and the corresponding automorphism group  $\mathrm{GO}_4^+(5)$  acts in the natural way on  $\mathrm{coker}(Q)$ . If  $E$  is the (geometrically unique) supersingular curve of characteristic 5, then one finds that the induced  $\mathrm{GO}_4^+(5)$ -module structure on  $\Gamma(E^8; \mathcal{L}_{Q,0})$  is *not* isomorphic to the given permutation representation. In this case, the actual invariant subspace does not jump, but we find that

$$\dim(\Gamma(E^{16}; \mathcal{L}_{Q \oplus Q,0})^{\mathrm{GO}_4^+(5)}) \cong \dim((\Gamma(E^8; \mathcal{L}_{Q,0})^{\otimes 2})^{\mathrm{GO}_4^+(5)}) = 160, \quad (2.40)$$

while for all other curves, the invariant space has dimension 156. To compute the action of  $G$  in such supersingular cases, we may use the fact that  $\mathcal{L}_{Q,0}$  always descends to the appropriate quotient and gives an equivariant isomorphism  $\Gamma(E^n; \mathcal{L}_{Q,0}) \cong \Gamma(\ker \psi_Q^\vee; \mathcal{L}')$ ; what fails is that we no longer have a natural basis (which in the case of the Theorem are an eigenbasis for the action of the diagonalizable group  $\ker \psi_Q$ ) of the latter. There is, however, an isomorphism

$$\Gamma(\ker \psi_Q^\vee; \mathcal{L}') \cong \Gamma(\ker \psi_Q^\vee; \mathcal{O}_{\ker \psi_Q^\vee}), \quad (2.41)$$

since  $\ker \psi_Q^\vee$  is 0-dimensional, but this is not in general equivariant; in general, the action of  $G$  is twisted by some cocycle with values in the unit group of the coordinate ring. When  $\det(Q)$  is odd, however, we can use the description of  $\mathcal{L}'$  in terms of pullbacks of  $\mathcal{O}([\omega])$  to compute this cocycle. In the  $Q_8(1)$  case, this further simplifies, since we only need to know what happens on  $\alpha_5^4$ , allowing us to reduce to an evaluation of functions on  $E_5^4$  in an appropriate formal neighborhood of the identity.

A similar calculation applies to the Gram matrix of  $\sqrt{3}\Lambda_{E_6}^\perp$ , with its automorphism group  $O_5(3)$ ; in this case, there are also subgroups of  $O_5(3)$  for which the corollary fails on the supersingular curve of characteristic 3. We can also obtain a characteristic 2 counterexample from  $\sqrt{2}\Lambda_{E_7}^\perp$ ; in this case, it is unclear how to compute the cocycle, but we can simply check that none of the 16 elements of  $H^1(\mathrm{Sp}_6(2); \mu_2(\alpha_2^6))$  give rise to the permutation module. (There is also a subgroup with too many invariants, namely the preimage in  $W(D_6)$  of the transitive  $\mathrm{Alt}_5 \subset S_6$ , which has too many invariants in each of the 16 possible cases.)

We should further note that the requirement that  $Q$  have even diagonal is also necessary; indeed, otherwise the claim already fails for the case  $Q = 1$ ,  $G = \mathrm{GL}_1(\mathbb{Z})$  for *any* curve of characteristic not 2.

Even when  $E$  is supersingular of characteristic dividing  $N$ , there may still be isogenies of the form

$$E_{B,C} \rightarrow E_{B,C'} \quad (2.42)$$

with diagonalizable kernel, which as an abstract group scheme can be (geometrically) identified with  $\mu_N \otimes_{\mathbb{Z}/N\mathbb{Z}} C'/C$ . Indeed, the only requirement is that  $|C'/C|$  be prime to the characteristic of  $E$ . Making this a canonical identification is somewhat trickier, as the kernel is only naturally described as the quotient

$$(\kappa_N \otimes_{\mathbb{Z}/N\mathbb{Z}} C'/B)/(\kappa_N \otimes_{\mathbb{Z}/N\mathbb{Z}} C/B). \quad (2.43)$$

Moreover, the translations moving between the different eigenspaces are only defined up to the kernel of the descended polarization. We find in general that

$$\Gamma(E_{B,C}; \mathcal{L}_{Q;B,C}) \cong \bigoplus_g g^* \Gamma(E_{B,C'}; \mathcal{L}_{Q;B,C'}) \quad (2.44)$$

where  $g$  ranges over the *quotient* group

$$\mathrm{Hom}(Q^{-1}B^*/C, \mathrm{Hom}(\kappa_N, \mu_N))/\mathrm{Hom}(Q^{-1}B^*/C', \mathrm{Hom}(\kappa_N, \mu_N)). \quad (2.45)$$

We can thus only use this decomposition in understanding group actions when this quotient group has an equivariant splitting. Luckily, there is an important case when this happens: if  $C'/C$  is the  $l$ -part of  $Q^{-1}B^*/C$  for some prime  $l$  which is invertible on  $E$ , then the quotient may be identified with the  $l$ -part of  $\mathrm{Hom}(Q^{-1}B^*/C, \mathrm{Hom}(\kappa_N, \mu_N))$ .

**Lemma 2.27.** *With  $B, C, Q$  as above and  $l$  a prime invertible on  $E$ , there is a  $G$ -equivariant isomorphism*

$$\Gamma(E_{B,C}; \mathcal{L}_{Q;B,C}) \cong \bigoplus_H \mathrm{Ind}_H^G \mathrm{Res}_H^G \Gamma(E_{B,C'}; \mathcal{L}_{Q;B,C'}) \quad (2.46)$$

where  $H$  ranges over the point stabilizers in the different orbits of the action of  $G$  on the  $l$ -part of  $Q^{-1}C^*/B$  and

$$C' = \bigcup_k Q^{-1}B^* \cap l^{-k}C \quad (2.47)$$

Here we should note that we have such a reduction for every prime dividing  $N$ ; in particular, if the polarization does not have prime power degree, then we can always choose a prime dividing the degree of the polarization which is invertible on  $E$ , and use the corresponding reduction.

Although we have seen that there can indeed be (finitely many) bad curves for such invariant theory questions, it turns out that our hypotheses are in fact slightly more restrictive than they need to be. Suppose we have a finite group  $G$  acting on an abelian variety  $A$  (fixing the identity). There are two natural induced abelian subvarieties. The subgroup scheme  $A^G$  is still projective, and thus (up to a possible inseparable base change) we may take its reduced identity component  $A^{G^0}$ . Equivalently (and without need for base change), we could instead define  $A^{G^0}$  to be the image of the endomorphism  $\sum_{g \in G} g \in \mathrm{End}(A)$ . There is also an almost complementary subvariety  $A_G$  giving by the image of the endomorphism  $|G| - \sum_{g \in G} g$ . Both subvarieties are clearly preserved by  $G$ , and since the sum of the endomorphisms is an isogeny, it follows that we have a natural  $G$ -equivariant isogeny  $A_G \times A^{G^0} \rightarrow A$ . Any  $G$ -equivariant line bundle on  $A$  (with trivial action on the fiber over the identity) pulls back to a  $G$ -equivariant line bundle on  $A_G \times A^{G^0}$ , namely  $\mathcal{L}|_{A_G} \boxtimes \mathcal{L}|_{A^{G^0}}$ . Moreover, the action of  $G$  on the second factor is trivial, since it is trivial at the identity.

It turns out that even though this isogeny can fail to have diagonalizable kernel, we can still use it to understand the  $G$ -module structure of  $\Gamma(A; \mathcal{L})$ .

**Lemma 2.28.** *With  $A, G, \mathcal{L}$  as above, assume that  $\mathcal{L}$  is ample. Then there is a  $G$ -module isomorphism*

$$\Gamma(A; \mathcal{L})^d \cong \Gamma(A_G; \mathcal{L}|_{A_G})^e \quad (2.48)$$

where  $d = |A_G \cap A^{G^0}|$  and  $e = \dim \Gamma(A^{G^0}; \mathcal{L}|_{A^{G^0}})$ .

*Proof.* Let  $K = A_G \cap A^{G^0}$  be the kernel of the isogeny  $A_G \times A^{G^0} \rightarrow A$ . Then we have a natural isomorphism

$$\Gamma(A; \mathcal{L}) \cong \Gamma(A_G \times A^{G^0}; \mathcal{L}|_{A_G} \boxtimes \mathcal{L}|_{A^{G^0}})^K \cong (\Gamma(A_G; \mathcal{L}|_{A_G}) \otimes \Gamma(A^{G^0}; \mathcal{L}|_{A^{G^0}}))^K. \quad (2.49)$$

Let  $H$  be the preimage of  $H$  in the Heisenberg group  $\mathcal{G}(\mathcal{L}_{A^{G^0}}^{-1})$ . This certainly acts naturally on  $\Gamma(A^{G^0}; \mathcal{L}|_{A^{G^0}})^*$ , but the fact that the line bundle descends to  $A$  implies that it also acts on  $\Gamma(A_G; \mathcal{L}|_{A_G})$ . We thus have a natural isomorphism

$$(\Gamma(A_G; \mathcal{L}|_{A_G}) \otimes \Gamma(A^{G^0}; \mathcal{L}|_{A^{G^0}}))^K \cong \mathrm{Hom}_H(\Gamma(A^{G^0}; \mathcal{L}|_{A^{G^0}})^*, \Gamma(A_G; \mathcal{L}|_{A_G})), \quad (2.50)$$

which by Frobenius reciprocity further becomes

$$\mathrm{Hom}_H(\Gamma(A^{G^0}; \mathcal{L}|_{A^{G^0}})^*, \Gamma(A_G; \mathcal{L}|_{A_G})) \cong \mathrm{Hom}_{\mathcal{G}(\mathcal{L}_{A^{G^0}}^{-1})}(\Gamma(A^{G^0}; \mathcal{L}|_{A^{G^0}})^*, \mathrm{Ind}_H^{\mathcal{G}(\mathcal{L}_{A^{G^0}}^{-1})} \Gamma(A_G; \mathcal{L}|_{A_G})). \quad (2.51)$$

By the structure of Heisenberg representations, we may then conclude that there is a functorial isomorphism

$$\Gamma(A; \mathcal{L}) \otimes \Gamma(A^{G^0}; \mathcal{L}|_{A^{G^0}}) \cong \mathrm{Ind}_H^{\mathcal{G}(\mathcal{L}_{A^{G^0}}^{-1})} \Gamma(A_G; \mathcal{L}|_{A_G}) \quad (2.52)$$

of  $\mathcal{G}(\mathcal{L}_{A^{G^0}}^{-1})$ -modules. Moreover, the splitting  $K \rightarrow H$  is  $G$ -invariant, since it could be computed inside  $A^{G^0}$ , and thus we may rewrite this as a  $G \times \mathcal{G}(\mathcal{L}_{A^{G^0}}^{-1})$ -module isomorphism:

$$\Gamma(A; \mathcal{L}) \otimes \Gamma(A^{G^0}; \mathcal{L}|_{A^{G^0}}) \cong \mathrm{Ind}_{G \times H}^{G \times \mathcal{G}(\mathcal{L}_{A^{G^0}}^{-1})} \Gamma(A_G; \mathcal{L}|_{A_G}) \quad (2.53)$$

Moreover, the induction functor is exact (since the homogeneous space is affine), as is restriction to  $G$ , and thus if we forget the action of the Heisenberg group, we obtain a  $G$ -module isomorphism

$$\Gamma(A; \mathcal{L})^e \cong \Gamma(A_G; \mathcal{L}|_{A_G})^{e^2/d}, \quad (2.54)$$

from which the claim follows.  $\square$

*Remark.* Note that although  $d$  always divides  $e^2$ , it need not divide  $e$ , and vice versa  $e$  can fail to divide  $d$ .

### 3 Coxeter group actions on abelian varieties

One of the major ingredients in the construction of elliptic analogues of double affine Hecke algebras is a suitable action of an affine Weyl group on a power of an elliptic curve (or more generally on a variety isogenous to such a power). It will be convenient to work somewhat more abstractly, and begin with the finite case.

With this in mind, let  $A$  be an abelian variety, and suppose the finite Weyl group  $W$  acts faithfully on  $A$  (fixing the identity) in such a way that for any reflection  $r \in W$ , the corresponding fixed subgroup scheme has codimension 1. We will naturally refer to such an action as an action “by reflections”.

For each reflection  $r \in W$ , the subgroup  $\langle r \rangle$  splits  $A$  (up to isogeny) as discussed above; in this case, we have natural subvarieties  $A^{r^0} := \text{im}(1 + r)$  and  $A_r := \text{im}(1 - r)$ , and an induced isogeny  $A^{r^0} \times A_r \rightarrow A$  (with kernel contained in  $A_r[2]$ ). Since  $A^{r^0}$  by assumption has codimension 1, we see that  $A_r$  is a 1-dimensional abelian variety. In other words, each reflection in  $W$  induces a corresponding elliptic curve  $E_r = A_r$  contained in  $A$ , as the image of the endomorphism  $1 - r$ . We call such a curve the “root curve” associated to  $r$ . Applying the same construction to the dual variety  $A^\vee$  gives root curves  $E'_r \subset A^\vee$ , the duals of which we refer to as “coroot curves”. Note that the coroot curve associated to  $r$  can be described directly as the cokernel of the endomorphism  $1 + r$ . In particular, the endomorphism  $1 - r$  factors through  $E'_r$ , giving rise to a natural map  $E'_r \rightarrow E_r$  such that the composition  $A \rightarrow E'_r \rightarrow E_r \rightarrow A$  is  $1 - r$  and the composition  $E_r \rightarrow A \rightarrow E'_r \rightarrow E_r$  is multiplication by 2.

Fix a system of simple roots  $S = \{\alpha_1, \dots, \alpha_n\}$  in  $W$ , and let  $E_1, \dots, E_n; E'_1, \dots, E'_n$  be the corresponding root and coroot curves, with induced maps  $\iota_i : E_i \rightarrow A$ ,  $\iota'_i : A \rightarrow E'_i$ ,  $\nu_i : E'_i \rightarrow E_i$ . The action of  $s_i$  on  $E_j$  can be described quite simply:

$$s_i \circ \iota_j = \iota_j + (s_i - 1) \circ \iota_j = \iota_j - \iota_i \circ \nu_i \circ \iota'_i \circ \iota_j \quad (3.1)$$

This suggests that we should define a morphism  $\mu_{ij} : E_j \rightarrow E_i$  as the composition  $-\nu_i \circ \iota'_i \circ \iota_j$ ; that is, it is the morphism  $E_j \rightarrow E_i$  induced by  $s_i - 1$ .

**Lemma 3.1.** *The curves  $E_1, \dots, E_n$  are distinct.*

*Proof.* Suppose otherwise, and reorder the simple roots so that  $E_1 = E_2$ . Then  $s_1 s_2 \neq 1$ , but

$$(s_1 s_2 - 1) = (s_1 - 1)(s_2 - 1) + (s_1 - 1) + (s_2 - 1) \quad (3.2)$$

so that  $s_1 s_2 - 1$  has image  $E_1 = E_2$ . Since  $s_1 s_2$  fixes  $E_1 = E_2$ , we find that  $(s_1 s_2)^k - 1 = k(s_1 s_2 - 1)$  for all  $k \geq 1$ , and thus  $s_1 s_2$  has infinite order, contradicting finiteness of  $W$ .  $\square$

**Lemma 3.2.** *For  $i \neq j$ , the composition  $\mu_{ji} \circ \mu_{ij}$  is multiplication by  $k \in \{0, 1, 2, 3\}$ , and if the composition is 0, then  $\mu_{ij} = \mu_{ji} = 0$ .*

*Proof.* Since  $E_i \neq E_j$ , the product  $E_i \times E_j$  is isogenous with its image in  $A$ . Define an action of  $s_i, s_j$  on  $E_i \times E_j$  by

$$\begin{aligned} s_i(x_i, x_j) &= (-x_i + \mu_{ji}(x_j), x_j), \\ s_j(x_i, x_j) &= (x_i, -x_j + \mu_{ij}(x_i)). \end{aligned}$$

The elements  $s_i, s_j$  clearly act as involutions on the product, and the actions are compatible with the actions on  $A$ , so that the action of  $(s_i s_j)^{m_{ij}} - 1$  induces a homomorphism from  $E_i \times E_j$  to the kernel of the map  $E_i \times E_j \rightarrow A$ . Since  $E_i \times E_j$  is proper, reduced, and connected, and said kernel is finite, we see that this description actually gives an action of the rank 2 Weyl group  $\langle s_i, s_j \rangle$ . Moreover, since this group is finite, there is a  $W$ -invariant polarization on  $E_i \times E_j$ , of the form

$$\begin{pmatrix} 2r_i & \psi_{ji} \\ \psi_{ji}^\vee & 2r_j \end{pmatrix}, \quad (3.3)$$

with  $4r_i r_j - \deg(\psi_{ji}) > 0$ . We then find that  $\psi_{ji} = r_j \mu_{ji} = r_i \mu_{ij}^\vee$ , so that

$$\deg(\psi_{ji}) = \psi_{ji} \psi_{ji}^\vee = r_j \mu_{ji} r_i \mu_{ij}, \quad (3.4)$$

and thus  $\mu_{ji} \mu_{ij}$  is multiplication by a nonnegative integer less than 4. Moreover, since  $r_j \mu_{ji} = r_i \mu_{ij}^\vee$ , we see that if one of  $\mu_{ij}, \mu_{ji}$  vanishes, then so does the other.  $\square$

*Remark.* We then readily see that the order of  $s_i s_j$  is equal to 2, 3, 4, 6 when  $\mu_{ij} \mu_{ji} = \mu_{ji} \mu_{ij}$  is equal to 0, 1, 2, 3 respectively.

**Proposition 3.3.** *Let  $(W, S)$  be a finite Weyl group, and suppose that  $E_1, \dots, E_n$  is a system of elliptic curves and  $\mu_{ij} : E_i \rightarrow E_j$ ,  $i \neq j$ , a system of morphisms such that  $\mu_{ij} \mu_{ji} = 4 \cos(\pi/m_{ij})^2$ , with  $\mu_{ij} = \mu_{ji} = 0$  whenever  $m_{ij} = 2$ . Then there is a faithful action of  $W$  on  $\prod_i E_i$  such that*

$$s_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, -x_i + \sum_{j \neq i} \mu_{ji}(x_j), x_{i+1}, \dots, x_n). \quad (3.5)$$

*Proof.* The action of  $s_i$  is clearly an involution, and the braid relations are straightforward to verify. (This is easy when  $m_{ij} = 2$ , and for  $m_{ij} > 2$  it suffices to show that  $((s_i s_j)^2 + (2 - \mu_{ij} \mu_{ji})(s_i s_j) + 1)(s_i s_j - 1)$  vanishes, which reduces to a computation in  $E_i \times E_j$ .) So this certainly gives an action of  $W$ , and it remains only to show that it is faithful.

Since the construction clearly respects products, we may as well assume that  $W$  is irreducible. For any path in the Coxeter diagram of  $W$ , we may take the corresponding composition of morphisms  $\mu_{ij}$ ; since  $\mu_{ij} = 0$  iff  $m_{ij} = 2$ , any such composition will be an isogeny. Moreover, since the Coxeter diagram is a tree by finiteness of  $W$ , we see that any two isogenies  $E_i \rightarrow E_j$  arising in this way will differ by a *positive* factor (any time the path backtracks introduces a factor  $\mu_{ij} \mu_{ji} > 0$ ). In particular, for any element  $w \in W$ , the induced map  $E_i \rightarrow E_j$  (apply  $w$  then project onto the  $j$ th factor) is an integer linear combination of such isogenies, and in particular has a corresponding notion of positivity. This allows us to turn any element of  $w$  into a *real* matrix by taking each such morphism to the appropriately signed square root of its degree. The consistency of sign ensures that this will give rise to an actual representation of  $W$ , and we can then verify that up to a diagonal change of basis, this is precisely the standard reflection representation of  $W$ .  $\square$

**Corollary 3.4.** *Let the finite Weyl group  $W$  act faithfully on the abelian variety  $A$  by reflections, with simple root curves  $E_1, \dots, E_n$ . Then the induced morphism  $\prod_i E_i \rightarrow A$  is made  $W$ -equivariant by the above action, and its kernel is finite and fixed by  $W$ .*

*Proof.* The equivariance is obvious by construction, so it remains only to show that the kernel  $K$  is fixed by  $W$ . Otherwise, some simple reflection  $s_i$  will act nontrivially on  $K$ , and thus  $(s_i - 1)K \subset K$  is nonzero. But  $(s_i - 1)K \subset E_i$ , contradicting the assumption that  $E_i$  injects in  $A$ .  $\square$

The proof of the Proposition suggests an extension of this construction to more general crystallographic Coxeter groups (in particular to affine Weyl groups). Certainly, one could consider an action of the above form associated to any system of morphisms  $\mu_{ij}$ , but to relate it to the standard reflection representation of a Coxeter group, we must make the following assumptions:

- The composition  $\mu_{ij} \mu_{ji}$  is multiplication by  $k_{ij} \in \{0, 1, 2, 3, 4\}$ .
- There is a system of positive integers  $r_i$  such that  $r_j \mu_{ji} = r_i \mu_{ij}^\vee$  for each  $i \neq j$ .
- Any composition  $\mu_{i_1 i_2} \mu_{i_2 i_3} \cdots \mu_{i_m i_1}$  is multiplication by a nonnegative integer.

We call such a system of curves and morphisms an “elliptic root datum”.

**Theorem 3.5.** *Any elliptic root datum gives rise to a faithful action on  $\prod_i E_i$  of the Coxeter group with multiplicities  $m_{ij}$  given by  $k_{ij} = 4 \cos(\pi/m_{ij})^2$ , such that  $s_i$  acts as above.*

*Proof.* The conditions on the morphisms ensure that we can faithfully translate the action into one on a real vector space, taking each morphism to the appropriately signed square root of its degree. Conjugating by the diagonal matrix with entries  $\sqrt{r_i}$  turns this into the standard reflection representation of the given Coxeter group.  $\square$

**Corollary 3.6.** *For each conjugacy class  $C$  of reflections, there is a corresponding elliptic curve  $E_C$  equipped with isomorphisms  $E_C \cong E_r$ ,  $r \in C$  such that the action of  $W$  on the set of compositions  $E_C \cong E_r \rightarrow A$  and their negatives can be identified with the action of  $W$  on the corresponding set of roots, with the compositions  $E_C \cong E_r \rightarrow A$  corresponding to the positive roots.*

*Proof.* Indeed, for each simple root  $\alpha_i$ , we may consider the set of compositions  $w \circ \iota_i$  for  $w \in W$ , and find that each such composition has the form  $(\beta_{i1}, \beta_{i2}, \dots, \beta_{in})$  in which either every entry is a nonnegative rational multiple of the appropriate composition of  $\mu_{jk}$  or every entry is a nonpositive such multiple. If  $r = ws_iw^{-1}$ , then the image of  $1 - r$  is  $w$  times the image of  $1 - s_i$ , and thus  $w \circ \iota_i$  identifies  $E_i$  with  $E_r$ . Each  $E_r$  is afforded with precisely two such identifications, of which we naturally choose the one corresponding to a positive root. We furthermore see that conjugate simple reflections give rise to equivalent systems of identifications of root curves.  $\square$

**Corollary 3.7.** *Relative to the action of  $W$  on  $\prod_i E_i$  arising in this way, any two reflections have distinct root curves.*

*Proof.* Assuming WLOG that  $W$  is irreducible, we find that each  $E_i$  is isogenous to  $E_1$  in an essentially canonical way (choose the isogeny of smallest degree among the “positive” isogenies), and then see that two root curves agree iff the corresponding maps  $E_1 \rightarrow \prod_i E_i$  correspond to proportional real vectors, making the two reflections agree.  $\square$

More generally, if  $B$  is an abelian variety with trivial action of  $W$ , we could consider the image  $B \times E_1 \times \dots \times E_n$  under a  $W$ -equivariant isogeny. We will say that the abelian variety  $A$  arising in this way has an action of  $W$  of “root type”. Note that as in the finite case, we may always arrange for the kernel of the isogeny to be not just preserved by  $W$  but fixed elementwise by  $W$ , as otherwise there will be kernel elements contained in root curves. We will also need the dual notion: an abelian variety with an action of  $W$  is of “coroot type” if its dual is of root type. These are equivalent for finite groups, or more generally for Coxeter groups with nondegenerate Cartan matrices, but in the affine case the two notions do not agree. Note that in the coroot type case, rather than having well-behaved positivity for roots, we have well-behaved positivity for coroots: for each conjugacy class of reflections, we can choose isomorphisms between the corresponding coroot curves and a fixed curve  $E$  in such a way that the resulting set of maps to  $E$ , together with their negatives, are in equivariant, sign-preserving bijection with the corresponding set of root vectors.

Consider the case of the affine Weyl group of type  $\tilde{A}_2$ . Since  $m_{ij} = 3$ ,  $k_{ij} = 1$ , we see that each  $\mu_{ij}$  is an isomorphism, and the positivity assumption forces the isomorphisms to be consistent. We thus obtain the following faithful action on  $E^3$ :

$$s_0(x_0, x_1, x_2) = (x_1 + x_2 - x_0, x_1, x_2) \tag{3.6}$$

$$s_1(x_0, x_1, x_2) = (x_0, x_0 + x_2 - x_1, x_2) \tag{3.7}$$

$$s_2(x_0, x_1, x_2) = (x_0, x_1, x_0 + x_1 - x_2). \tag{3.8}$$

This action fixes the diagonal copy of  $E$ , but does not fix any morphism to  $E$ . It follows that the corresponding action on the dual variety fixes a morphism to  $E$ , but does not fix any curve. In fact, we see that the dual action takes the form

$$s_0(x_0, x_1, x_2) = (-x_0, x_0 + x_1, x_0 + x_2) \tag{3.9}$$

$$s_1(x_0, x_1, x_2) = (x_0 + x_1, -x_1, x_1 + x_2) \tag{3.10}$$

$$s_2(x_0, x_1, x_2) = (x_0 + x_2, x_1 + x_2, -x_2) \tag{3.11}$$

from which we may see that the corresponding root curves do not even generate  $E^3$ .

For our purposes, we will in fact prefer actions of coroot type. The main issue with actions of root type in the affine case is that since there are only finitely many distinct coroots, the kernel of any given coroot map is fixed by infinitely many reflections. For instance, in the above  $\tilde{A}_2$  example, both  $s_0$  and  $s_1s_2s_1$  fix the hypersurface  $x_1 + x_2 = 2x_0$  pointwise. However, the dual of the standard model, though of coroot type, is badly behaved for other reasons: the product of root curves corresponding to the finite Weyl group does not inject.

Suppose  $\tilde{W} = \langle s_0, \dots, s_n \rangle$  is an affine Weyl group (with associated finite Weyl group  $W = \langle s_1, \dots, s_n \rangle$ ), and that the abelian variety  $A$  is equipped with an action of  $\tilde{W}$  of coroot type. The  $\tilde{W}$ -invariant subvariety of  $A^\vee$  has codimension  $n$ , and induces by duality a universal equivariant morphism  $A \rightarrow B$  such that  $\tilde{W}$  acts trivially on  $B$  and the fibers have dimension  $n$ . In contrast, the invariant subvariety of  $A$  has codimension  $n + 1$ , and thus its image in  $B$  has codimension 1. Thus if we base change by a suitable isogeny, we may arrange for  $B$  to be the product of  $A^{\tilde{W}^0}$  by an elliptic curve, allowing us to split off that factor and reduce to the case that  $B$  is an elliptic curve  $E$ . Now, since  $W$  is finite,  $A^{\tilde{W}^0}$  has codimension  $n$ , and is thus itself an elliptic curve, which necessarily surjects onto  $E$ . Although this curve  $A^{\tilde{W}^0}$  is not preserved by  $\tilde{W}$ , we may still base change by it, and thus find that the natural action of  $W$  on  $A^{\tilde{W}^0} \times A_W$  extends to an action of  $\tilde{W}$  in such a way that the isogeny  $A^{\tilde{W}^0} \times A_W \rightarrow A$  is equivariant. (Note, however, that the factorization itself is *not* equivariant; the projection to  $A_W$  is not an equivariant map.)

We can describe this action explicitly on generators. Of course, for  $1 \leq i \leq n$ ,  $s_i(z, x) = (z, s_i(x))$ , so only  $s_0$  is nontrivial. The root curve associated to  $s_0$  is the same as the root curve associated to the reflection  $r$  in the highest root of  $W$ , and the action on  $0 \times A_W$  is the same as that of  $r$ . We thus see that  $s_0(z, x) = (z, r(x) + \zeta(z))$  for some (nonzero) morphism  $\zeta : A^{\tilde{W}^0} \rightarrow E_r$ . Conversely, it is easy to see that any action of this form has coroot type.

We may view this action as a family of actions of  $\tilde{W}$  on  $A_W$  parametrized by  $z$ , with the one caveat being that the action no longer preserves the identity; indeed, the translation subgroup of  $\tilde{W}$  acts (unsurprisingly) as translations of  $A_W$ . Of course, the action on a given fiber depends only on the point  $q := \zeta(z)$ , and we easily see that it is faithful precisely when  $q$  is non-torsion. (It follows from the above considerations that this is the typical form of an action of coroot type, up to base change and twisting by a  $(A_W)^W$ -torsor.)

Returning to the finite case, suppose that  $A/S$  is a family of abelian varieties (over an integral base  $S$ ) equipped with a faithful action of the finite Weyl group  $W$  by reflections, and suppose moreover that we are given a  $W$ -invariant ample line bundle  $\mathcal{L}$  on  $A$ . This can be made equivariant by taking the action on the fiber at 1 to be trivial, and we may then ask when the map  $s \mapsto \dim \Gamma(A_s; \mathcal{L})^W$  is constant on  $S$ . By the reductions of the previous section, this reduces to considering the corresponding question for  $A_W$ , which is very nearly a variety of the form we considered above. To be precise, the root curves in each irreducible component of  $W$  are isogenous, and since indecomposable finite Weyl groups have at most two conjugacy classes of reflections, we see that each component is associated to a point of  $\mathcal{X}_0$ ,  $\mathcal{X}_0(2)$ , or  $\mathcal{X}_0(3)$ . To ensure that we can apply our previous results, we must insist that the induced line bundles on the root curves be suitable; to wit, we insist that for each reflection,  $\mathcal{L}|_{E_r} \cong \mathcal{L}_{2d_r, 0; E_r}$  for some positive integer  $d_r$ , clearly constant on conjugacy classes of involutions. We thus see that the only possible issues arise when (a) one of the curves  $E_r$  is supersingular of characteristic dividing  $W$ , or (b) the ‘‘root kernel’’, i.e., the kernel of  $\prod_i E_i \rightarrow A$ , fails to be diagonalizable. In fact, the first condition turns out not to be necessary.

**Lemma 3.8.** *Let  $\mathcal{L}$  be a  $W$ -invariant ample line bundle on an abelian variety  $A$  such that there are positive integers  $d_i$  such that  $\mathcal{L}|_{E_i} \cong \mathcal{L}_{2d_i, 0; E_i}$  for each  $i$ . Then for any section  $f \in \Gamma(A; \mathcal{L})$ , the antisymmetrization  $\sum_{w \in W} \sigma(w)wf$  vanishes along the divisor  $\sum_{r \in R(W)} [\ker(r - 1)]$ .*

*Proof.* We may write  $\sum_{w \in W} \sigma(w)w = (1 - r) \sum_{w \in W_0} w$ , where  $W_0$  is the even subgroup of  $W$ . Since the divisors  $\ker(r - 1)$  are transverse for distinct  $r$ , it thus suffices to show that  $(1 - r)f$  vanishes along the divisor  $[\ker(r - 1)]$ . We thus reduce to the case that  $W$  has rank 1. In other words,  $A$  is a quotient of a variety  $B \times E$  (with  $r$  acting trivially on  $B$ ) obtained by identifying some subgroup  $K \subset E[2]$  with a subgroup of  $B$ . The action of  $r$  lifts to  $B \times E$ , and we conclude (by considering how  $[-1]$  acts on sections of  $\mathcal{L}_{2d}$ ) that the antisymmetrization of any section of the pulled back line bundle must vanish on the divisor  $B \times E[2]$ . This is the preimage of the divisor  $(B \times E[2])/K$ , which in turn is precisely the kernel of  $r - 1$  as required.  $\square$

This gives us the following possible approach to controlling invariants in such bundles. Let  $\mathcal{L}_\Delta$  be the line bundle  $\mathcal{O}(\sum_{r \in R(W)} [\ker(r - 1)])$ , but equipped with the equivariant structure which is trivial at the identity. If there is a section  $g \in \Gamma(A_W; \mathcal{L}_\Delta)$  with nontrivial antisymmetrization, then the operation  $f \mapsto \frac{\sum_{w \in W} \sigma(w)w(gf)}{\sum_{w \in W} \sigma(w)w(g)}$  induces an idempotent on any  $\Gamma(A; \mathcal{L})$  which projects onto the symmetric subspace. More generally, if we have a family of such varieties such that such a section  $g$  exists locally (or, equivalently, on every fiber), then we obtain such idempotents locally, and thus the spaces  $\Gamma(A_s; \mathcal{L})^W$  are fibers of a vector bundle, implying that their dimensions are constant.

**Theorem 3.9.** *Suppose  $A/S$  is a family of abelian varieties equipped with a faithful action by reflections of the finite Weyl group  $W$ , and let  $\mathcal{L}$  be a  $W$ -invariant ample line bundle on  $A$  such that the restriction to every root curve of every fiber is isomorphic to an even power of  $\mathcal{L}_1$ . If the root kernel of  $A$  is diagonalizable, then the functions  $s \mapsto \dim \Gamma(A_s; \mathcal{L})^W$  and  $s \mapsto \dim((\sum_{w \in W} \sigma(w)w)\Gamma(A_s; \mathcal{L}))$  are constant on  $S$ .*

*Proof.* We first observe that by Lemma 2.28, the claims hold for  $A$  iff they hold for  $A_W$ , a.k.a. the image of  $\prod_i E_i \rightarrow A$ . We may thus WLOG assume that the morphism  $\prod_i E_i \rightarrow A$  is an isogeny, such that  $W$  acts trivially on the root kernel  $K$ . By assumption,  $K$  is diagonalizable, so that  $\Gamma(\prod_i E_i; \mathcal{L})$  decomposes into  $K$ -eigenspaces, and this decomposition is compatible with the action of  $W$ . It then follows by semicontinuity that the claims hold for  $A$  if they hold for  $\prod_i E_i$ . Since this is a product over the components of  $W$ , we may assume WLOG that  $W$  is indecomposable.

We may then reduce as discussed to showing that for any elliptic root datum corresponding to an indecomposable finite Weyl group, the corresponding line bundle  $\mathcal{L}_\Delta$  contains a section with nontrivial antisymmetrization. (There is also the technical, but easy to verify, condition that  $\mathcal{L}_\Delta|_{E_r} \cong \mathcal{L}_{2d_r, 0; E_r}$  for suitable positive integers.)

Here we may use the classification. The simplest case is  $W = A_n$ , in which case we may identify  $\prod_i E_i$  with the subvariety of  $E^{n+1}$  on which  $\sum_i z_i = 0$ . By induction in  $n$  (with trivial base case  $n = 0$ ), the result holds for  $n - 1$ , and thus any  $S_n$ -antiinvariant section can be obtained by antisymmetrization over  $S_n$ . It thus suffices to show that there is an  $S_n$ -antiinvariant section that when summed over coset representatives of  $S_{n+1}/S_n$  with appropriate sign gives a nonzero result. Equivalently, by dividing by the appropriate product of  $\vartheta$  functions, we need to find an  $S_n$ -invariant function with suitable poles that symmetrizes to a nonzero constant. For auxiliary parameters  $y_1, \dots, y_{n+2}$ , we may consider the function

$$\frac{\prod_{1 \leq i \leq n+2} \vartheta(z_{n+1} - y_i) \prod_{1 \leq i \leq n} \vartheta(Y - z_i)}{\prod_{1 \leq i \leq n} \vartheta(z_{n+1} - z_i)}, \quad (3.12)$$

where  $Y = \sum_{1 \leq i \leq n+2} y_i$ . Summing this over  $S_{n+1}/S_n$  gives a function with no poles, which must therefore be constant; on the other hand, of the  $n + 1$  terms that result, all but one vanishes when

$z_{n+1} = Y$ . We thus find that

$$\sum_{w \in S_{n+1}/S_n} w \cdot \frac{\prod_{1 \leq i \leq n+2} \vartheta(z_{n+1} - y_i) \prod_{1 \leq i \leq n} \vartheta(Y - z_i)}{\prod_{1 \leq i \leq n} \vartheta(z_{n+1} - z_i)} = \prod_{1 \leq i \leq n+2} \vartheta(Y - y_i), \quad (3.13)$$

which is generically nonzero. (Note that the case  $n = 1$  is a version of the standard addition law for theta functions.) Note that this identity is a disguised form of a classical theta function identity; see the discussion around [23, (1.22)].

For types  $B/C/D$ , we may similarly reduce to lower rank cases, noting that  $D_2$  and  $B_1$  both follow from the result for  $A_1$ . There are four cases to consider: the action of  $D_n$  on  $E \otimes \Lambda_{D_n}$ , the action of  $B_n$  on the same variety, the action of  $C_n$  on  $E \otimes \mathbb{Z}^n$  (following [14], we label the cases by the dual root system), and the action of  $BC_n$  on the variety  $E_{\Lambda_{D_n}, \mathbb{Z}^n}$  associated to a point of  $\mathcal{X}_0(2)$  lying over  $E$ . In each case, there is a natural isogeny to  $E \otimes \mathbb{Z}^n$ , and it turns out we can choose the function being symmetrized to be the pullback of a function on  $E \otimes \mathbb{Z}^n$ . The simplest identity corresponds to the  $C_n$  case, valid for all  $n \geq 1$ :

$$\sum_{w \in C_n/C_{n-1}} w \cdot \frac{\prod_{1 \leq i \leq 2n+1} \vartheta(z_n - y_i) \prod_{1 \leq i \leq n} \vartheta(Y + z_i) \prod_{1 \leq i \leq n} \vartheta(Y - z_i)}{\vartheta(2z_n) \prod_{1 \leq i \leq n} \vartheta(z_n + z_i) \vartheta(z_n - z_i)} = \prod_{1 \leq i \leq 2n+1} \vartheta(Y - y_i), \quad (3.14)$$

with  $Y = \sum_{1 \leq i \leq 2n+1} y_i$ ; if we expand this out as a sum of  $2n$  terms, we find that it is simply the special case  $(z_1, \dots, z_{2n}) \mapsto (-z_1, \dots, -z_n, z_1, \dots, z_n)$  of the  $S_{2n}/S_{2n-1}$  identity. In characteristic not 2, we may set  $y_{2n-2}, \dots, y_{2n+1}$  to be the four points of  $E[2]$  to obtain an identity for  $D_n$ ,  $n > 2$ :

$$\sum_{w \in D_n/D_{n-1}} w \cdot \frac{\prod_{1 \leq i \leq 2n-3} \vartheta(z_n - y_i) \prod_{1 \leq i \leq n} \vartheta(Y + z_i) \prod_{1 \leq i \leq n} \vartheta(Y - z_i)}{\prod_{1 \leq i \leq n} \vartheta(z_n + z_i) \vartheta(z_n - z_i)} = \vartheta(2Y) \prod_{1 \leq i \leq 2n-3} \vartheta(Y - y_i), \quad (3.15)$$

where  $Y = \sum_{1 \leq i \leq 2n-3} y_i$ . Since this identity is expressed entirely in terms of  $\vartheta$ , it continues to hold in characteristic 2. If we only specialize three parameters to the nonzero 2-torsion points, we instead obtain (for  $n \geq 2$ ):

$$\sum_{w \in B_n/B_{n-1}} w \cdot \frac{\prod_{1 \leq i \leq 2n-2} \vartheta(z_n - y_i) \prod_{1 \leq i \leq n} \vartheta(Y + z_i) \prod_{1 \leq i \leq n} \vartheta(Y - z_i)}{\vartheta(z_n) \prod_{1 \leq i \leq n} \vartheta(z_n + z_i) \vartheta(z_n - z_i)} = \frac{\vartheta(2Y)}{\vartheta(Y)} \prod_{1 \leq i \leq 2n-2} \vartheta(Y - y_i). \quad (3.16)$$

We omit the analogous identity for  $BC_n$  (obtained from the  $C_n$  identity by setting two of the  $y_i$  to be the 2-torsion points not in the kernel of  $\phi$ ) due to notational difficulties with using  $\vartheta$  in the presence of isogenies, but note that for purposes of the reduction there is no reason we cannot simply use the  $B_n$  identity.

For the seven exceptional cases (two each of  $G_2$  and  $F_4$  along with the simply laced cases  $E_6$ ,  $E_7$ ,  $E_8$ ), we observe that the relevant line bundle comes from a polarization of degree a multiple of 6, and we may thus use Lemma 2.27 to reduce to a smaller group. That is, we obtain an equivariant isomorphism

$$\Gamma(A; \mathcal{L}_\Delta) \cong \bigoplus_H \text{Ind}_H^W \text{Res}_H^W \Gamma(A'; \mathcal{L}_\Delta) \quad (3.17)$$

where  $H$  ranges over the point stabilizers in the different orbits of the action of  $G$  on an appropriate diagonalizable 2- or 3-group. The image of antisymmetrization on the left is thus the direct sum of terms

$$\left( \sum_{h \in H} \sigma(h) h \right) \text{Res}_H^W \Gamma(A'; \mathcal{L}_\Delta), \quad (3.18)$$

and since  $H$  is a reflection group in each case, we may apply induction. (In fact, only that term which is nonzero in characteristic 0 has any hope of contributing).

For instance, for  $E_8$ , the variety is  $E^8$  with polarization given by 30 times the Cartan matrix of  $E_8$ . In characteristic not 2, we may use the 2-part of  $\Lambda_{E_8}/30\Lambda_{E_8}$  to split into eigenspaces, of which only one term survives. We thus reduce to showing that the descended line bundle on  $E_{\Lambda_{E_8}, 2\Lambda_{E_8}} \cong E_2^8$  (with nonproduct polarization) has a nontrivial antisymmetrization under the stabilizer  $W(D_8)$ . This is 2-isogenous to the standard  $D_8$  model, and thus (since the characteristic is not 2) the image of antisymmetrization has the same dimension as in characteristic 0. A similar reduction using the 3-part reduces to antisymmetrization for  $W(A_8)$  and proves the result for any characteristic other than 3.  $\square$

*Remark.* Results of [14, 25] in characteristic 0 actually compute the structure of the invariant ring (i.e.,  $\bigoplus_d \Gamma(E_{B,C}; \mathcal{L}_Q^d)$  where  $Q$  is the minimal invariant polarization satisfying the evenness requirement) and find that in each case the result is a free polynomial ring in generators of degrees that can be read off of the coefficients of the highest short (co)root. This suggests that something similar should hold in arbitrary characteristic. It would be natural in this context to also consider actions of complex reflection groups on varieties isogenous to  $E^n$  where  $j(E) \in \{0, 1728\}$ , or even quaternionic reflection groups in the case of supersingular curves of characteristic 2 or 3.

*Remark.* The diagonalizability hypothesis is necessary, at least as far as the antisymmetrization claim is concerned. For example, suppose  $x \in E[3]$  is a nontrivial 3-torsion point, and consider the quotient  $A$  of the sum 0 subvariety of  $E^3$  by the subgroup generated by  $(x, x, x)$ . The image in  $A$  of the point  $(0, x, -x)$  is negated by every reflection, and is thus not contained in any reflection hypersurface, but still has nontrivial stabilizer  $C_3 \subset S_3$ . It follows that in characteristic 3, the antisymmetrization of any section of an ample line bundle will vanish at this point, and thus for *no* ample line bundle is the dimension of the image of antisymmetrization the same as in characteristic 0 (except, of course, when there are no antisymmetric elements in characteristic 0). The invariants remain well-behaved, however, as the invariant ring is the same as that of  $\langle x \rangle$  on  $\mathbb{P}^2$ ; this is a permutation representation, so has Hilbert series independent of the characteristic.

## 4 Elliptic analogues of affine Hecke algebras

Before proceeding to the construction of Hecke algebras associated to general elliptic root data, it will be helpful to consider the finite case, a generalization of the construction of [9]. Note that although we work with a finite group, the resulting Hecke algebras are most naturally thought of as elliptic analogues of *affine* Hecke algebras, as they include multiplication operators in addition to reflection operators. (In particular, [9] constructs affine Hecke algebras as degenerations of a special case of the construction given below.)

Although the approach in [9] via residue conditions can (mostly) be extended to the infinite case, there are two alternate approaches for which the generalization is more straightforward: one as a space of operators preserving appropriate holomorphy conditions, the other as the subalgebra of the algebra of operators generated by the rank 1 subalgebras. Since we will need to understand the rank 1 case to give the second construction, we begin with the first.

In our application to noncommutative rational varieties below, we will need to be able to attach an arbitrary finite set of parameters to the endpoint roots of the affine  $C_n$  diagram; we will thus give a version of the general construction in which each conjugacy class of reflections can be given arbitrarily many parameters. This, of course, includes a case without any parameters at all, which we consider first.

This “master” Hecke algebra has a third alternate description which does not generalize well to the infinite case, but is simplest of all to give (and extend to actions of arbitrary finite groups). Let  $X$  be a regular integral scheme, and let  $G$  be a finite group acting faithfully on  $X$ . Then we define the master Hecke algebra  $\mathcal{H}_G(X)$  to be the sheaf of algebras on  $X/G$  given by  $\mathcal{H}_G(X) := \mathcal{E}nd(\pi_*\mathcal{O}_X)$ , where  $\pi : X \rightarrow X/G$  is the quotient map.

If  $\pi$  is flat, then  $\mathcal{H}_G(X)$  is the endomorphism ring of a vector bundle, so is in particular an Azumaya algebra on  $X/G$ , and the category of quasicoherent  $\mathcal{H}_G(X)$ -modules is equivalent to the category of quasicoherent sheaves on  $X/G$ . However, this condition holds only rarely; even in the case when  $G$  is a finite Weyl group acting on an abelian variety, this morphism can easily fail to be flat. For instance, consider the case  $G = G_2$  acting on the sum zero subvariety  $X$  of  $E^3$  (as permutations and global negation). In characteristic not 2, consider the point  $(\tau_1, \tau_2, \tau_1 + \tau_2) \in E^3$ , where  $\tau_1, \tau_2$  generate  $E[2]$ . This point has stabilizer  $Z(G)$ , and is isolated in the subvariety fixed by the central element of  $G$ , and thus we see that its image in  $X/G$  is a singular point (of type  $A_1$ ), and that the quotient morphism fails to be flat in a neighborhood of that orbit.

There are two prominent cases in which we do have flatness, namely the action of  $A_n$  on the sum 0 subvariety of  $E^{n+1}$  and the action of  $C_n$  on  $E^n$ ; in each case, the quotient morphism is flat because it is a finite morphism between regular schemes.

In general, although  $\mathcal{H}_G(X)$  may not be an Azumaya algebra, we at least know that it is torsion-free, and thus may be viewed as contained in its generic fiber  $\text{End}_{k(X/G)}(k(X))$ . Since  $k(X)$  is Galois over  $k(X/G)$ , the generic fiber has an alternate description as a twisted group algebra  $k(X)[G]$ , giving rise to the following description of  $\mathcal{H}_G(X)$ . Denote the natural action of  $\text{Aut}(X)$  on  $k(X)$  by  ${}^g f := (g^{-1})^* f$ ; we will also use a similar notation for the actions on line bundles and divisors.

**Proposition 4.1.** *The master Hecke algebra  $\mathcal{H}_G(X)$  is the subsheaf of the twisted group algebra  $k(X)[G]$  such that for any  $G$ -invariant open subset  $U$ ,  $\Gamma(U/G; \mathcal{H}_G(X))$  consists of the operators  $\sum_i c_i g$  such that for any  $G$ -invariant open  $V$  and  $f \in \Gamma(V; \mathcal{O}_X)$ ,  $\sum_i c_i {}^g f \in \Gamma(U \cap V; \mathcal{O}_X)$ .*

*Proof.* Indeed,  $\mathcal{H}_G(X)$  is the subsheaf of  $\text{End}_{k(X/G)}(k(X))$  which on  $U$  consists of endomorphisms preserving  $\mathcal{O}_X|_U$ , or equivalently preserving global sections of  $\mathcal{O}_X|_V$  for all invariant  $V \subset U$ . The claim then follows by using the twisted group algebra description of the endomorphism ring and observing that  $\Gamma(V; \mathcal{O}_X) \subset \Gamma(U \cap V; \mathcal{O}_X)$ .  $\square$

One consequence is that if  $H \subset G$ , then there is a natural inclusion  $\mathcal{H}_H(X) \subset \mathcal{H}_G(X)$  (where we conflate  $\mathcal{H}_H(X)$  with its direct image under  $X/H \rightarrow X/G$ ). In addition, if  $\alpha$  is an automorphism of  $X$  that normalizes  $G$ , then there is a corresponding automorphism of  $\mathcal{H}_G(X)$ , either by pulling back through the induced automorphism of  $X/G$ , or on operators as  $\sum_g c_g g \mapsto \sum_g {}^\alpha c_{\alpha^{-1}g} g$ .

Note that  $\mathcal{H}_G(X)$  clearly contains a copy of the structure sheaf  $\mathcal{O}_X$  as well as the operators  $g$  for each  $g \in G$ , and thus contains a copy of the twisted group algebra  $\mathcal{O}_X[G]$ . It can, however, be bigger than the twisted group algebra. Consider the case of  $G = \mu_n = \langle s \rangle$  acting on  $X = \mathbb{A}^1$  in characteristic prime to  $n$ . Applying the operator  $1 + \zeta_n s + \zeta_n^2 s^2 + \dots + \zeta_n^{n-1} s^{n-1}$  to any function which is holomorphic at the origin gives a function which vanishes to order  $n-1$  at the origin, and thus  $\mathcal{H}_{\mu_n}(\mathbb{A}^1)$  contains the operator  $x^{1-n}(1 + \zeta_n s + \zeta_n^2 s^2 + \dots + \zeta_n^{n-1} s^{n-1})$  not contained in  $\mathcal{O}_X[G]$ .

It turns out that this is the typical case in which the coefficients may have poles; more precisely, the only poles are associated to “complex reflections” in  $G$  (relative to the action on  $X$ ). It will be useful to consider a more general setting.

**Lemma 4.2.** *Let  $X$  be a regular integral scheme, let  $g_1, \dots, g_n$  be a finite sequence of distinct automorphisms of  $X$ , and consider the subsheaf  $\mathcal{M}_{\vec{g}}$  of  $k(X)^n$  which on an open subset  $U \subset X$*

consists of those  $n$ -tuples  $(c_1, \dots, c_n) \in k(X)^n$  such that for any open subset  $V \subset X$  and any  $f \in k(X)$  which is holomorphic in  $V$ , the function  $\sum_i c_i g_i f$  is holomorphic in  $U \cap \bigcap_i g_i V$ . Then there is an  $n$ -tuple of divisors  $\Delta_i \in \text{Div}(X)$  such that  $\mathcal{M}_{\vec{g}} \subset \bigoplus_i \mathcal{O}(\Delta_i)$ , where each  $\Delta_i$  is supported on those hypersurfaces which are fixed pointwise by automorphisms of the form  $g_j g_i^{-1}$  for some  $j \neq i$ .

*Proof.* Let  $Y$  be a reduced irreducible hypersurface in  $X$ ; we need to understand the possible singularities of the coefficients along  $Y$ . Note that if  $U_1, U_2$  are two open subsets meeting  $Y$ , then  $U_1 \cap U_2$  also meets  $Y$ , and any bound on singularities holding on  $U_1 \cap U_2$  also holds for global sections along  $U_1, U_2$ . We may thus take a limit along those open subsets meeting  $Y$ . Similarly, since we are only considering holomorphy along  $Y$ , we may take a limit over  $V$  such that each  $g_i^* V$  contains the generic point of  $Y$ . In other words, the condition for  $(c_1, \dots, c_n)$  to be a section of the base change of  $\mathcal{M}_{\vec{g}}$  to the local ring at  $k(Y)$  is that for any function  $f$  which is holomorphic along  $g_1^{-1}(Y), \dots, g_n^{-1}(Y)$ , the image  $\sum_i c_i g_i f$  is holomorphic along  $Y$ .

Now, given such an  $n$ -tuple, let  $d$  be the maximum order of pole of a section  $c_i$  along  $Y$ . The condition that  $\sum_i c_i g_i f$  be holomorphic only depends on the value of  $f$  modulo the intersections of the  $d$ -th powers of the maximal ideals at the divisors  $g_i^{-1}(Y)$ , and strong approximation means that the reductions corresponding to distinct divisors may be chosen independently. We thus see that if  $\sum_i c_i g_i$  preserves holomorphy, then so does the operator  $\sum_{i: g_i^{-1}(Y)=g_j^{-1}(Y)} c_i g_i$  for any  $j$ .

We may as well assume, therefore, that the divisors  $g_i^{-1}(Y)$  are all equal to the same divisor  $Y'$ . We then find that each  $g_i$  induces an isomorphism  $\sigma_i : k(Y') \cong k(Y)$ . Suppose that  $\sigma_1 \neq \sigma_n$ . Then we may choose an element  $\bar{h} \in k(Y')$  such that  $\sigma_1(\bar{h}) - \sigma_n(\bar{h})$  is a unit, and then choose a lift of  $\bar{h}$  to the local ring. Then for any holomorphic  $f$ , both

$$\sum_i c_i g_i f \quad \text{and} \quad \sum_i c_i g_i (fh) \quad (4.1)$$

are holomorphic, and thus

$$\sum_i (g^n h - g^i h) c_i g_i f \quad \text{and} \quad \sum_i (g^1 h - g^i h) c_i g_i f \quad (4.2)$$

are holomorphic. Moreover, we can recover  $(c_1, \dots, c_n)$  from  $(\dots, (g^n h - g^i h) c_i, \dots)$  and  $(\dots, (g^1 h - g^i h) c_i, \dots)$ , since  $g^n h - g^1 h$  is a unit. In this way, we may reduce to the case that every  $g_i$  with nonzero coefficient acts in the same way on the residue fields. In fact, we could apply a similar reduction even in this case, with the only change being that dividing by  $g^n h - g^1 h$  may introduce poles. Since such a step can only add a finite amount to the order of the pole, however, the end result has bounded order and the claim follows.  $\square$

**Corollary 4.3.** *The sheaf  $\mathcal{M}_{\vec{g}}$  as above is coherent.*

Note that we can also right-multiply operators by local sections of  $\mathcal{O}_X$ , and thus obtain an  $(\mathcal{O}_X, \mathcal{O}_X)$ -bimodule structure on  $\mathcal{M}_{\vec{g}}$ . Such a structure is equivalent to an  $\mathcal{O}_X \otimes \mathcal{O}_X$ -module structure, or equivalently an  $\mathcal{O}_{X \times X}$ -module structure. We will consider this structure in more detail when discussing the infinite case, but for the moment we observe the following.

**Corollary 4.4.** *The induced sheaf  $\mathcal{M}_{\vec{g}}$  on  $X \times X$  is a coherent subsheaf of a direct sum*

$$\bigoplus_i (1, g_i^{-1})_* \mathcal{O}(D_i), \quad (4.3)$$

where each  $D_i$  is an effective Cartier divisor supported on the irreducible hypersurfaces fixed pointwise by automorphisms  $g_j g_i^{-1}$ .

**Corollary 4.5.** *The algebra  $\mathcal{H}_G(X)$ , viewed as an  $\mathcal{O}_{X \times X}$ -module, is coherent, and contained in the sum  $\sum_{g \in G} (1, g)_* \mathcal{O}(\Delta)$ , where  $\Delta$  is an effective divisor supported on the “reflection hypersurfaces” of  $G$  on  $X$ : the irreducible hypersurfaces which are fixed pointwise by some  $g \in G$ .*

Note that “bounded” is the most we can hope for in general. Indeed, even in the simplest elliptic case  $A_1$  acting on  $E$ , the divisor  $D$  is precisely the divisor corresponding to the subscheme  $E[2]$ . In characteristic not 2, this subscheme is reduced, and thus the coefficients have at most simple poles at the 2-torsion points, but in characteristic 2, the coefficients can have double poles at the 2-torsion points of an ordinary curve, and a quadruple pole at the origin of a supersingular curve. This, of course, is an artifact of wild ramification; without that, a “reflection” of order  $n$  will admit poles of order at most  $n - 1$  along the corresponding reflection hypersurfaces.

There is an important variation arising from the interpretation as an  $\mathcal{O}_{X \times X}$ -module: simply consider the sheaf  $\mathcal{H}_{G; \mathcal{L}_1, \mathcal{L}_2}(X) := \mathcal{H}_G(X) \otimes_{X \times X} \mathcal{L}_2 \boxtimes \mathcal{L}_1^{-1}$  for invertible sheaves  $\mathcal{L}_2, \mathcal{L}_1$  on  $X$ . Since both left- and right-multiplication by sections of  $\mathcal{O}_{X/G}$  agree in  $\mathcal{H}_G(X)$ , this twisted version still descends to a sheaf on  $\mathcal{O}_{X/G}$ , and the result moreover has induced compositions

$$\mathcal{H}_{G; \mathcal{L}_1, \mathcal{L}_2}(X) \otimes_{X/G} \mathcal{H}_{G; \mathcal{L}_2, \mathcal{L}_3}(X) \rightarrow \mathcal{H}_{G; \mathcal{L}_1, \mathcal{L}_3}(X). \quad (4.4)$$

In fact, as a sheaf on  $X/G$ , we have  $\mathcal{H}_{G; \mathcal{L}_1, \mathcal{L}_2}(X) \cong \mathcal{H}om_{G/X}(\pi_* \mathcal{L}_1, \pi_* \mathcal{L}_2)$ , with the obvious induced composition; this is most easily seen by representing  $\mathcal{L}_1, \mathcal{L}_2$  by Cartier divisors, and observing that this turns  $\mathcal{H}_{G; \mathcal{L}_1, \mathcal{L}_2}(X)$  into the subsheaf of the meromorphic twisted group algebra taking the subsheaf of  $k(X)$  corresponding to  $\mathcal{L}_1$  into the subsheaf corresponding to  $\mathcal{L}_2$ .

In addition to the isomorphisms of such sheaves arising from isomorphisms of  $\mathcal{L}_1, \mathcal{L}_2$ , there are also isomorphisms coming from twisting both sheaves by a suitable invertible sheaf. Let  $\mathcal{L}$  be a  $G$ -equivariant invertible sheaf on  $X$ , and for each reflection hypersurface  $H$  with inertia group (i.e., pointwise stabilizer)  $I_H$ , suppose that there is a  $I_H$ -invariant neighborhood  $U_H$  of the generic point of  $H$  such that the  $H$ -equivariant sheaf  $\mathcal{L}_{I_H}$  is equivariantly isomorphic to  $\mathcal{O}_{U_H}$ . Then there is a natural isomorphism

$$\mathcal{H}_{G; \mathcal{L}, \mathcal{L}}(X) \cong \mathcal{H}_G(X). \quad (4.5)$$

Indeed, since  $\mathcal{L}$  is  $G$ -equivariant, both algebras are naturally contained in  $k(X)[G]$ , and the conditions for any given reflection hypersurface are the same on both sides. This condition is automatically satisfied by the pullback of an invertible sheaf on  $X/G$ , but this is not necessary. For instance, in the case of  $W(G_2)$  acting on the sum 0 subvariety of  $E^3$ , the quotient is a weighted projective space with generators of degree 1, 1, 2. The pullback of  $\mathcal{O}_{\mathbb{P}^2}(1)$  does not descend to a line bundle on  $X/W(G_2)$  (since  $\mathcal{O}_{X/W(G_2)}(1)$  is not invertible on such a weighted projective space), but satisfies the requisite triviality condition along the reflection hypersurfaces.

In the case  $\mathcal{L}_1 = \mathcal{L}_2$ , the result is of course a sheaf of algebras on  $X/G$ , a twisted version of the master Hecke algebra. Since twisting by the pullback of an invertible sheaf on  $X/G$  has no effect, this suggests that the twist should not be specified by a line bundle but rather some more subtle datum. For the (meromorphic or naïvely holomorphic) twisted group algebra, the appropriate datum is easy to see: it is a  $G$ -equivariant structure on a trivial gerbe, i.e., a family of invertible sheaves  $\mathcal{Z}_g, g \in G$  along with explicit isomorphisms

$$\mathcal{Z}_g \otimes^g \mathcal{Z}_h \cong \mathcal{Z}_{gh} \quad (4.6)$$

satisfying the obvious compatibility conditions. Such a datum (which we will refer to mildly abusively as an “equivariant gerbe”, leaving the triviality of the underlying gerbe implicit) induces a natural “crossed product” algebra structure on the sheaf  $\bigoplus \mathcal{Z}_{g}$ . Note that the isomorphisms are an

important part of the data: one can modify the isomorphisms by any 2-cocycle in  $Z^2(G, \Gamma(X; \mathcal{O}_X^*))$ , which must be a coboundary for the resulting twisted algebra to be isomorphic to the original. There is also a notion of coboundary for equivariant gerbes as a whole: given a line bundle  $\mathcal{L}$ , there is a corresponding equivariant gerbe with  $\mathcal{Z}_g = \mathcal{L}^g \mathcal{L}^{-1}$ . Tensoring an equivariant gerbe by such a coboundary simply twists the corresponding sheaf of algebras by  $\mathcal{L}$ .

Unfortunately, there is no way to associate a twisted master Hecke algebra to an equivariant gerbe alone; any equivariant line bundle induces the trivial equivariant gerbe, but twisting by an equivariant line bundle changes the master Hecke algebra if the reflections do not act trivially on the restrictions to the corresponding reflection hypersurfaces. We will return to this when considering twists in the infinite case below.

A particularly important instance of twisting arises when we consider the natural involution on operators:  $\sum_g c_g g \mapsto \sum_g g^{-1} c_g$ .

**Proposition 4.6.** *There is a contravariant isomorphism  $\mathcal{H}_G(X)^{op} \cong \mathcal{H}_{G; \omega_X}(X)$  given by  $\sum_g c_g g \mapsto \sum_g g^{-1} c_g$ .*

*Proof.* We need to show that  $\sum_g c_g g$  preserves holomorphic functions iff  $\sum_g g^{-1} c_g$  preserves holomorphic  $n$ -forms. By Lemma 4.2, it suffices to prove that the conditions on individual hypersurfaces are the same, and thus we may fix a hypersurface  $D$  and restrict our attention to the case that every term in the sum gives the same divisor  $g^{-1}D = D'$ . By duality, a function  $f$  is holomorphic along  $D$  iff  $\text{Res}_D f \omega = 0$  for all  $n$ -forms  $\omega$  which are holomorphic along  $D$ , and we may similarly detect holomorphy of  $n$ -forms by taking residues against test functions. In particular, for any function  $f$  holomorphic along  $D$ , and any  $n$ -form  $\omega$  holomorphic along  $D'$ , we have

$$\text{Res}_{D'} \left( \sum_g c_g^g f \right) \omega = \sum_g \text{Res}_{D'} c_g^g f \omega = \sum_g \text{Res}_D g^{-1} c_g f g^{-1} \omega = \text{Res}_D f \cdot \left( \sum_g g^{-1} (c_g \omega) \right). \quad (4.7)$$

It follows that  $\sum_g c_g^g f$  is holomorphic along  $D'$  for all  $f$  iff  $\sum_g g^{-1} (c_g \omega)$  is holomorphic along  $D$  for all  $\omega$ .  $\square$

Let us now turn to the elliptic case, in which  $G$  is a finite Weyl group  $W$  acting by reflections on an abelian torsor  $X/S$  over an integral base  $S$ . That is, the flat family  $X/S$  is a torsor over an abelian variety  $A/S$ , and  $W$  acts on  $X$  in such a way that the induced action on  $A$  is an action by reflections. Note that since the subvariety  $X^W$  is the intersection of the simple reflection hypersurfaces, it has codimension at most  $n$ , so is nonempty and thus a torsor over  $A^W$ ; conversely, any  $A^W$ -torsor induces a corresponding family  $X/S$  by twisting. The action of  $W$  is faithful on every fiber, and this remains true if we view  $X$  as a family over the quotient  $S' := X/A_W$ . We will see that the fibers of  $\mathcal{H}_W(X)$  over  $S$  are identified with the master Hecke algebras of the fibers, in a fairly strong way.

Given a reflection  $r$ , let  $[X^r]$  denote the effective Cartier divisor cut out by the equation  $rx = x$ . Also, let  $C_r$  denote the quotient of  $X$  by the abelian subvariety  $(r+1)A$ ; this, of course, is a torsor over the corresponding coroot curve  $E'_r$ . The morphism  $(r-1) : X \rightarrow A$  factors through  $C_r$  and has image  $E_r$ , so that there is a morphism  $C_r \rightarrow E_r$  compatible with the isogeny  $E'_r \rightarrow E_r$ , and thus  $C_r$  corresponds to a class in  $H_{\text{fppf}}^1(S; \ker(E'_r \rightarrow E_r))$ . Note that in contrast to the coroot curve, we cannot expect to have a natural torsor over the root curve inside  $X$ .

For the rank 1 case, we have the following immediate consequence of Lemma 4.2. Here a ‘‘hyperelliptic curve of genus 1’’ is a smooth genus 1 curve  $C$  with a marked involution such that the quotient is (geometrically) rational; note that the torsor arising in the rank 1 case is always a family of such curves-with-involutions.

**Lemma 4.7.** *Let  $C/S$  be a flat family of hyperelliptic curves of genus 1 with  $G = A_1 = \langle s \rangle$  acting by the marked involution. Then for any  $G$ -invariant open set  $U$ ,  $\Gamma(U; \mathcal{H}_{A_1}(C))$  consists of operators  $f_0 + f_1(s - 1)$  such that  $f_0 \in \Gamma(U; \mathcal{O}_C)$  and  $f_1 \in \Gamma(U; \mathcal{O}([C^s]))$ .*

*Remark 1.* Note that

$$f_0 + f_1(s - 1) = (f_0 - f_1 - {}^s f_1) + (1 + s)^s f_1, \quad (4.8)$$

and thus (since  $f_1 + {}^s f_1 \in \Gamma(U; \mathcal{O}_C)$ ) we may also describe  $\Gamma(U; \mathcal{H}_{A_1}(C))$  as the space of operators  $f'_0 + (1 + s)f'_1$  such that  $f'_0 \in \Gamma(U; \mathcal{O}_C)$  and  $f'_1 \in \Gamma(U; \mathcal{O}([C^s]))$ . This also follows from the above description of the adjoint once we realize that  $\omega_C$  is the trivial line bundle with equivariant structure such that  $s$  acts as  $-1$ .

*Remark 2.* It will be useful in the sequel to know when twisting by an  $A_1$ -equivariant line bundle on  $C$  has no effect on  $\mathcal{H}_{A_1}(C)$ . If  $\mathcal{L}$  is equivariantly isomorphic to  $\mathcal{O}(D)$  for some symmetric Cartier divisor, then we may write  $D$  as a linear combination of divisors  $D'$  and  $D'' + {}^s D''$  with  $D'$ ,  $D''$  irreducible and  $D' = {}^s D'$ . The latter case is the pullback of the image of  $D''$  in  $C/A_1$ , and thus certainly has no effect on twisting, while if  $D'$  is not a component of  $[C^s]$ , then its image in  $C/A_1$  is twice a divisor, so that again  $D'$  is a pullback. We are thus left to consider the linear combinations of reflection hypersurfaces, and thus determine that the condition on  $D$  is precisely that the valuations along reflection hypersurfaces must be even (or no condition at all in characteristic 2 if the reflection hypersurface is inseparable over  $S$ ). This, of course, is for the standard equivariant structure on  $\mathcal{O}(D)$ ; in odd characteristic, we may twist by the sign character of  $A_1$ , in which case the condition becomes that the valuations along reflection hypersurfaces are odd. Note that in any event, a line bundle satisfying the descent conditions will restrict on the generic fiber to a power of the hyperelliptic bundle.

For rank  $n$ , we have the following.

**Lemma 4.8.** *The  $\mathcal{O}_{X/W}$ -algebra  $\mathcal{H}_W(X)$  is generated by the  $\mathcal{O}_{X/W}$ -subalgebras  $\mathcal{H}_{\langle s_i \rangle}(X)$  for  $1 \leq i \leq n$ .*

*Proof.* It follows from Lemma 4.2 that  $\mathcal{H}_W(X)$  is generated as a left  $\mathcal{O}_X$ -module by the twisted group algebra  $\mathcal{O}_X[W]$  along with the subsheaves arising from operators of the form  $c_w w + c_{rw} r w$  for some reflection  $r \in R(W)$ . Now, by the above explicit description, each subalgebra  $\mathcal{H}_{\langle s_i \rangle}(X)$  contains  $\mathcal{O}_X[s_i]$ , and thus the algebra they generate contains  $\mathcal{O}_X[W]$ . But then if we express  $r$  above as  $w_1^{-1} s_i w_1$  for some simple reflection  $s_i$ , we have

$$c_w w + c_{rw} r w = c_w w + c_{w_1^{-1} s_i w_1} w_1^{-1} s_i w_1 w = w_1^{-1} (w_1 c_w + w_1 c_{w_1^{-1} s_i w_1} s_i) w_1 w \quad (4.9)$$

and find that  $c_w w + c_{rw} r w$  is a local section of  $\mathcal{H}_W(X)$  iff

$$w_1 c_w + w_1 c_{w w_1 s_i w_1^{-1} s_i} \quad (4.10)$$

is a local section of  $\mathcal{H}_W(X)$ , iff it is a local section of  $\mathcal{H}_{\langle s_i \rangle}(X)$ . The claim follows.  $\square$

*Remark.* Of course, the same argument shows that if  $G$  is generated by a collection of cyclic groups meeting every conjugacy class of (generalized) reflections, then  $\mathcal{H}_G(X)$  is generated by the corresponding subalgebras.

We can actually say a great deal more in the case of interest; not only is  $\mathcal{H}_W(X)$  a flat sheaf in general, but we can in fact express it as an extension of invertible sheaves on  $X$ . The key ingredient is the fact that there is a natural partial order on  $W$  (the Bruhat order), the weakest partial order such that if  $w'w^{-1}$  is a reflection, then  $w$  and  $w'$  are comparable and ordered according to their

length. Note that omitting any set of reflections from a reduced word for  $w$  gives an element  $w' \leq w$ , and classical results on Coxeter groups give the converse: for any reduced word for  $w$ ,  $w' \leq w$  iff some word for  $w'$  (iff some *reduced* word for  $w'$ ) can be obtained by omitting reflections from the chosen reduced word.

Given an “order ideal”  $I$  with respect to Bruhat order (i.e., a subset  $I \subset W$  such that if  $w \in I$  and  $w' \leq w$ , then  $w' \in I$ ), we may consider the subsheaf  $\mathcal{H}_W(X)[I]$  of  $\mathcal{H}_W(X)$  consisting of those operators in which the coefficient of  $w$  is 0 for  $w \notin I$ . Any chain of order ideals induces in this way a filtration, and we will show that in the case of a maximal chain, the subquotients of the filtration are invertible sheaves on  $X$ . Let  $[\leq w]$  denote the order ideal consisting of elements  $\leq w$ .

For any element  $w \in W$ , define a divisor  $D_w := \sum_{r \in R(W), rw < w} [X^r]$ .

**Lemma 4.9.** *Let  $I$  be a Bruhat order ideal, and suppose that  $w$  is a maximal element of  $I$ . Then there is a short exact sequence*

$$0 \rightarrow \mathcal{H}_W(X)[I \setminus \{w\}] \subset \mathcal{H}_W(X)[I] \rightarrow \mathcal{O}(D_w) \rightarrow 0 \quad (4.11)$$

*Proof.* By definition,  $\mathcal{H}_W(X)[I \setminus \{w\}]$  is the kernel of the “coefficient of  $w$ ” map on  $\mathcal{H}_W(X)[I]$ , so we first need to show that the coefficient of  $w$  is contained in  $\mathcal{O}(D_w)$ . It follows from Lemma 4.2 that the coefficient has polar divisor bounded by  $\sum_{w' \in (I \setminus \{w\}) \cap R(W)_w} [X^{w'w^{-1}}]$ . If  $w'w^{-1}$  is a reflection, then  $w'$  is comparable to  $w$ , which since  $w$  is maximal in  $I$  implies that  $w' < w$ , and thus the bound on the divisor is  $D_w$  as required.

It remains only to show that the map is surjective. Choose a reduced word  $w = s_1 \cdots s_n$ , and consider the multiplication map

$$\mathcal{H}_{\langle s_1 \rangle}(X) \otimes \cdots \otimes \mathcal{H}_{\langle s_n \rangle}(X) \rightarrow \mathcal{H}_W(X). \quad (4.12)$$

Every term in the resulting expansion corresponds to an element in which some (possibly empty) subset of the simple reflections have been omitted, and thus the image of this multiplication map is contained in  $\mathcal{H}_W(X)[\leq w] \subset \mathcal{H}_W(X)[I]$ . The image under the leading coefficient map can then be determined by replacing each factor by its corresponding leading coefficient line bundle. It thus remains only to verify that

$$D_w = \sum_{1 \leq i \leq n} s_1 \cdots s_{i-1} [X^{s_i}] = \sum_{1 \leq i \leq n} [X^{s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1}]. \quad (4.13)$$

But this follows from the strong exchange property: the reflections  $r$  such that  $rw < w$  are precisely those of the form  $s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1$ .  $\square$

**Corollary 4.10.** *For any reduced word  $w = s_1 \cdots s_n$ , the multiplication map*

$$\mathcal{H}_{\langle s_1 \rangle}(X) \otimes \cdots \otimes \mathcal{H}_{\langle s_n \rangle}(X) \rightarrow \mathcal{H}_W(X)[\leq w] \quad (4.14)$$

*is surjective.*

*Proof.* It suffices to show that if  $sw < w$ , then

$$\mathcal{H}_{\langle s \rangle} \otimes \mathcal{H}_W(X)[\leq sw] \rightarrow \mathcal{H}_W(X)[\leq w] \quad (4.15)$$

is surjective. The image clearly contains the subsheaf  $\mathcal{H}_W(X)[\leq sw]$ , so it suffices to show surjectivity to the quotient  $\mathcal{H}_W(X)[\leq w]/\mathcal{H}_W(X)[\leq sw]$ . This, in turn, is an iterated extension of invertible sheaves  $\mathcal{O}(D_{w'})$  on  $X$ , and thus it suffices to show surjectivity for each subquotient. That

is, if  $[\leq sw] \subset I \subset [\leq w]$  is an order ideal and  $w'$  is a maximal element of  $I$  not contained in  $[\leq sw]$ , then we need to show that the intersection of the image with  $\mathcal{H}_W(X)[I]$  surjects onto  $\mathcal{L}_{D_{w'}}$ .

Since  $sw < w$ , we may choose a reduced word for  $w$  beginning with  $s$ , and the subword description of the Bruhat order then tells us that  $[\leq w] = [\leq sw] \cup s[\leq sw]$ . Since  $w' \not\leq sw$  and  $w' \neq w$ , it follows that  $sw' < sw$ . We may thus consider the composition

$$\mathcal{H}_{\langle s \rangle}(X) \otimes \mathcal{H}_W(X)[\leq sw'] \rightarrow \mathcal{H}_W(X)[\leq w'] \rightarrow \mathcal{H}_W(X)[I]. \quad (4.16)$$

The proof of the Lemma shows that the composition with the ‘‘coefficient of  $w'$ ’’ map is surjective as required.  $\square$

*Remark.* In particular, the closest thing to an analogue of the braid relations in this setting is the fact that if  $(s_i s_j)^{m_{ij}} = 1$ , then the products

$$\mathcal{H}_{\langle s_i \rangle}(X) \mathcal{H}_{\langle s_j \rangle}(X) \cdots = \mathcal{H}_{\langle s_j \rangle}(X) \mathcal{H}_{\langle s_i \rangle}(X) \cdots \quad (4.17)$$

(with  $m_{ij}$  terms on each side) agree as subsheaves of  $\mathcal{H}_W(X)$ . Indeed, both sides are equal to the order ideal generated by the longest element of  $\langle s_i, s_j \rangle$ , and thus equal  $\mathcal{H}_{\langle s_i, s_j \rangle}(X)$ .

**Corollary 4.11.** *The construction  $\mathcal{H}_W(X)$  respects base change  $T \rightarrow S$ .*

*Proof.* Let  $\pi_1 : X \times_S T \rightarrow X$  be the natural projection; we need to show that  $\pi_1^* \mathcal{H}_W(X) \cong \mathcal{H}_W(X \times_S T)$ . In the rank 1 case, this is immediate from the explicit description and the fact that  $\pi_1^*(\mathcal{O}([X^s])) \cong \mathcal{O}([(X \times_S T)^s])$ . Since the rank 1 subalgebras generate, this induces a morphism  $\pi_1^* \mathcal{H}_W(X) \rightarrow \mathcal{H}_W(X \times_S T)$ . (Normally one would need to check relations, but this is simply the restriction of the corresponding isomorphism for the algebra of meromorphic operators such that the common polar divisor does not contain any fiber; thus generators suffice.)

It remains only to show that this morphism is an isomorphism, but this follows from the existence of compatible filtrations (i.e., coming from a chain of Bruhat order ideals) such that the induced maps on subquotients are isomorphisms.  $\square$

We can also give an alternate description of the adjoint involution. Let  $w_0$  be the longest element of  $W$ .

**Proposition 4.12.** *There is a contravariant isomorphism  $\mathcal{H}_W(X)^{op} \cong \mathcal{O}(-D_{w_0}) \otimes \mathcal{H}_W(X) \otimes \mathcal{O}(D_{w_0})$ .*

*Proof.* It suffices to show that the naïve adjoint  $\sum_w c_w w \mapsto \sum_w w^{-1} c_w$  on the meromorphic twisted group algebra restricts to an isomorphism as claimed. Since this is an involution, it reduces to showing the analogous claim for each rank 1 subalgebra. Let  $U$  be an open subset on which the effective Cartier divisor  $D_{w_0}$  is cut out by an equation  $h = 0$ . We thus need to show (using the two descriptions of the rank 1 Hecke algebra and taking the adjoint on the left)

$$\Gamma(U; \mathcal{O}_X) + (s_i - 1)\Gamma(U; \mathcal{O}([C^s])) \subset h(\Gamma(U; \mathcal{O}_X) + (s_i + 1)\Gamma(U; \mathcal{L}[C^s]))h^{-1}. \quad (4.18)$$

Given an instance  $f_0 + (s_i - 1)f_1$  on the left, conjugating by  $h$  gives

$$f_0 - (1 + (h/s^i))f_1 + (s_i + 1)(h/s^i h)f_1. \quad (4.19)$$

Since  ${}^{s_i}D_{w_0} = D_{w_0}$ , we find that  $h/s^i h$  is a unit, and local considerations near  $[X^{s_i}]$  tell us that  $1 + (h/s^i h)$  vanishes on  $[X^{s_i}]$ .  $\square$

*Remark 1.* We could also show that  $\omega_X \otimes \mathcal{O}(D_{w_0})$  satisfies the conditions for twisting to not change  $\mathcal{H}_W(X)$ . The divisor  $D_{w_0}$  is certainly invariant under every reflection, and thus it remains only to verify the conditions along reflection hypersurfaces. In characteristic not 2,  $\omega_X$  is equivariantly isomorphic to the twist of  $\mathcal{O}_X$  by the sign character, and thus the condition is that the divisor must have odd valuation along the reflection hypersurfaces, while in characteristic 2, there is no need to twist, and the valuations of separable reflection hypersurfaces must be even. In either case, the condition is automatically satisfied.

*Remark 2.* It is worth noting that this operation is triangular, in the sense that the image of the subsheaf corresponding to an order ideal is always the subsheaf corresponding to an order ideal. This follows immediately from the fact that  $w \mapsto w^{-1}$  is an order-preserving automorphism of the Bruhat poset.

The proof of Theorem 3.9 has the following consequence for our algebras. Here and below, by the root kernel of  $X$ , we mean the root kernel of the corresponding abelian scheme  $A$ .

**Proposition 4.13.** *Let  $X/S$  be a flat family of abelian torsors equipped with a faithful action by reflections of the finite Weyl group  $W$ . If the root kernel of  $X$  is diagonalizable on  $S$ , then  $S$  may be covered by open subsets on which  $\mathcal{H}_W(X)$  has an idempotent global section which on each fiber has image  $\Gamma(X_x; \mathcal{O}_X)^W$ .*

*Proof.* For any fiber  $x$ , choose a global section  $h$  of  $\mathcal{O}(\Delta)$ ,  $\Delta = D_\omega = \sum_{r \in R(W)} [X^r]$  with nonzero antisymmetrization, extend it to a neighborhood of  $x \in S$ , and observe that the antisymmetrization will remain nonzero in a possibly smaller, but nonempty, neighborhood. (By the proof of Theorem 3.9, this is guaranteed to exist for any geometric fiber (over which we can equivariantly trivialize the torsor), but the existence of an element with nontrivial antisymmetrization in an extension field implies the existence of such an element over the ground field.) Dividing by the antisymmetrization gives a function  $f$  with poles at most  $\Delta$  such that  $\sum_{w \in W} w f = 1$ . The proof of Theorem 3.9 shows that the idempotent operator  $\sum_{w \in W} w f$  preserves the space of functions holomorphic on any given invariant open subset of  $X$ , and thus is a section of  $\mathcal{H}_W(X)$  over the given open subset of  $S$ .  $\square$

Since  $\mathcal{H}_W(X)$  contains the twisted group algebra of naïvely holomorphic operators, there is in particular an action of the group on any  $\mathcal{H}_W(X)$ -module, and thus for any such module  $M$  which is (quasi)coherent as an  $\mathcal{O}_{X/W}$ -module, we could consider the  $W$ -invariant subsheaf of  $M$ . This as it stands may not be well-behaved, say for torsion sheaves supported on the reflection hypersurfaces. To obtain a better notion, we note that if we view  $\mathcal{O}_X$  as a module over  $\mathcal{H}_W(X)$ , then there is a surjective  $\mathcal{H}_W(X)$ -module morphism  $\sum_w c_w w \mapsto \sum_w c_w$  from  $\mathcal{H}_W(X)$  to  $\mathcal{O}_X$ , with kernel containing the kernel of the natural morphism  $\mathcal{O}_X[W] \rightarrow \mathcal{O}_X$ . Thus for modules which are torsion-free as  $\mathcal{O}_X$ -modules, the sheaf  $\mathcal{H}om_{\mathcal{H}_W(X)}(\mathcal{O}_X, M)$  will agree with the sheaf of  $W$ -invariant sections of  $M$ . With this in mind, we define  $M^W$  as the image of the composition

$$\mathcal{H}om_{\mathcal{H}_W(X)}(\mathcal{O}_X, M) \rightarrow \mathcal{H}om_{\mathcal{H}_W(X)}(\mathcal{H}_W(X), M) \cong M. \quad (4.20)$$

**Corollary 4.14.** *If the root kernel of  $X$  is diagonalizable, then the functor  $-^W$  on  $\mathcal{H}_W(X)$ -modules is exact and commutes with base change.*

*Proof.* If  $\mathcal{H}_W(X)$  has an idempotent of the form  $(\sum_w w)c$ , then the map  $\mathcal{H}_W(X) \rightarrow \mathcal{O}_X$  splits as  $f \mapsto f(\sum_w w)c$ . Since such idempotents exist locally on  $S$ , it follows that  $\mathcal{O}_X$  is locally projective, and thus the corresponding sheaf Hom functor is exact. Moreover, it follows that  $M^W$  is precisely the image of the idempotent  $(\sum_w w)c$ , and this operation clearly commutes with base change.  $\square$

Of course, as it stands, the algebra  $\mathcal{H}_W(X)$  does not bear a terribly strong resemblance to the usual Hecke algebras, due to the lack of any parameters associated to the roots. Classically, one generally has one parameter for each orbit of roots, but in the classical  $C_n$  case (viewing the affine Hecke algebra as being specified by an action of the finite Hecke algebra on the space of Laurent polynomials), one effectively has two parameters associated to the endpoint of the Dynkin diagram. This is traditionally interpreted as arising from the nonreduced root system  $BC_n$ , in which the endpoint is associated to two orbits of roots (differing by a factor of 2). If one looks at the actual action on Laurent polynomials, however, one finds that there is more symmetry in the parameters than is suggested by this interpretation, making it far more natural to associate an unordered pair of parameters to the given simple reflection. In fact, as we mentioned above, for our application, we will need a place to put an unbounded number of parameters; since there is already an example in which one can assign two parameters to a root without breaking things, this suggests that we should be able to assign arbitrarily many parameters to each orbit of roots.

Let us first consider the case of rank 1, so that  $X$  is a flat family  $C/S$  of hyperelliptic curves of genus 1. By consideration of the classical  $A_1$  and  $C_1$  cases, we are led to consider the following algebra.

**Definition 4.1.** Let  $C/S$  be a flat family of hyperelliptic curves of genus 1 on which  $A_1 = \langle s \rangle$  acts as the marked involution, and let  $T$  be an effective Cartier divisor on  $C$  not containing any fiber of  $C$  over  $S$ . The *rank 1 Hecke algebra*  $\mathcal{H}_{A_1, T}(C)$  is the subsheaf of  $\mathcal{H}_{A_1}(C)$  such that the coefficient of  $s$  in a local section of  $\mathcal{H}_{A_1, T}(C)$  is a local section of  $\mathcal{O}([C^s] - T)$ .

To see that this is an algebra, we note that the local sections of  $\mathcal{H}_{A_1, T}(C)$  are precisely the operators of the form  $f_0 + f_1(s - 1)$  with  $f_0 \in \Gamma(U; \mathcal{O}_C)$ ,  $f_1 \in \Gamma(U; \mathcal{O}([C^s] - T))$ , or equivalently the operators of the form  $g_0 + (s + 1)^s g_1$  with  $g_0 \in \Gamma(U; \mathcal{O}_C)$ ,  $g_1 \in \Gamma(U; \mathcal{O}([C^s] - T))$ . Thus the general product of two local sections can be expressed as

$$(f_0 + f_1(s - 1))(g_0 + (s + 1)^s g_1) = f_0 g_0 + f_1 ({}^s g_0 - g_0) + f_0 (g_1 + {}^s g_1) + (f_1 {}^s g_0 + f_0 g_1)(s - 1), \quad (4.21)$$

so that the coefficient of  $s$  in the product is again a section of  $\mathcal{O}([C^s] - T)$ .

There is an alternate description which makes the algebra property clearer, at the cost of a mild loss of generality.

**Proposition 4.15.** *The algebra  $\mathcal{H}_{A_1, T}(C)$  is contained in the subalgebra of  $\mathcal{H}_{A_1}(C)$  which preserves the subsheaf  $\mathcal{O}(-T) \subset \mathcal{O}_C$ . If the divisors  $T$  and  ${}^s T$  have no component in common, then this subalgebra is equal to  $\mathcal{H}_{A_1, T}(C)$ .*

*Proof.* Let  $U$  be an invariant open subset on which  $T$  is cut out by a single equation  $h = 0$ . Then the space  $\Gamma(U; \mathcal{H}_{A_1, T}(C))$  can be described as the space of operators  $f_0 + (s + 1)f_1 {}^s h$  such that  $f_0 \in \Gamma(U; \mathcal{O}_C)$ ,  $f_1 \in \Gamma(U; \mathcal{O}([C^s]))$ . Similarly,  $\Gamma(U; \mathcal{O}(-T)) = h\Gamma(U; \mathcal{O}_C)$ , so we need to show that  $f_0 + (s + 1)f_1 {}^s h$  preserves  $h\Gamma(U; \mathcal{O}_C)$ , or equivalently that  $h^{-1}(f_0 + (s + 1)f_1 {}^s h)h \in \Gamma(U; \mathcal{H}_{A_1}(C))$ . Since

$$h^{-1}(f_0 + (s + 1)f_1 {}^s h)h = f_0 + h^{-1}(s + 1)f_1 ({}^s h h) = f_0 + {}^s h (s + 1)f_1, \quad (4.22)$$

and  $f_0, {}^s h, (s + 1)f_1 \in \Gamma(U; \mathcal{H}_{A_1}(C))$ , the first claim follows.

Conversely, if  $f_0 + s f_1 \in \Gamma(U; \mathcal{H}_{A_1}(C))$  also preserves  $\mathcal{O}(-T)|_U$ , then both  $f_0 + s f_1$  and  $h^{-1}(f_0 + s f_1)h$  are in  $\Gamma(U; \mathcal{H}_{A_1}(C))$ . The first condition implies  $f_1 \in \Gamma(U; \mathcal{O}([C^s]))$ , while the second implies  $f_1 \in \Gamma(U; \mathcal{O}([C^s] - T + {}^s T))$ . If  ${}^s T$  has no component in common with  $T$ , so that  $\mathcal{O}(T) \cap \mathcal{O}({}^s T) = \mathcal{O}_C$ , then  $\mathcal{O}([C^s]) \cap \mathcal{O}([C^s] - T + {}^s T) = \mathcal{O}([C^s] - T)$ . In other words,  $f_1 \in \Gamma(U; \mathcal{O}([C^s] - T))$ , so that  $f_0 + s f_1 \in \Gamma(U; \mathcal{H}_{A_1, T}(C))$  as required.  $\square$

*Remark.* It is likely that the condition on  $T$  here is slightly stronger than strictly necessary: the claim most likely continues to hold even when  $T$  has some components fixed by  $s$ , as long as those components have multiplicity 1.

Note that we could have used this to prove the algebra property, in the following way. For each nonnegative integer  $m$ , let  $S'$  be the relative symmetric  $m$ -th power of  $C$  over  $S$ , and let  $C'$  be the base change of  $C$  to  $S'$ . There is a corresponding tautological divisor  $T'$ , and our original data  $(C/S, T)$  (assuming  $T$  has degree  $m$  over  $S$ ) is the base change of  $(C'/S', T')$  by the section  $S \rightarrow S'$  corresponding to  $T$ . The space of operators as described respects base change, and thus it suffices to prove the algebra property in this larger family. Since  $T'$  and  ${}^sT'$  have no component in common, this follows from the above result. There is one caveat here, though: although our original description respects base change, the description from the Lemma does not. Indeed, if there is an effective divisor  $T_0$  such that  $T - T_0 - {}^sT_0$  is effective, then the subalgebra preserving  $\mathcal{O}(-T)$  is the same as that preserving  $\mathcal{O}(-T + T_0 + {}^sT_0)$ , but the corresponding rank 1 Hecke algebras are not the same.

In the above argument, we used the fact that  ${}^s hh$  is central. This means we could also have described  $\mathcal{H}_{A_1, T}(C)$  (subject to the given condition on  $T$ ) as the subalgebra of  $\mathcal{H}_{A_1}(C)$  preserving the *supersheaf*  $\mathcal{O}({}^sT)$ . This symmetry leads to the following.

**Proposition 4.16.** *There is a natural isomorphism*

$$\mathcal{O}(T) \otimes \mathcal{H}_{A_1, T}(C) \otimes \mathcal{O}(-T) \cong \mathcal{H}_{A_1, {}^sT}(C) \quad (4.23)$$

*Proof.* Replacing  $(C/S, T)$  by a larger family as necessary, we may assume that  $T$  and  ${}^sT$  have no component in common. We then have

$$\mathcal{H}_{A_1, T}(C) = \mathcal{H}_{A_1}(C) \cap \mathcal{O}(-T) \otimes \mathcal{H}_{A_1}(C) \otimes \mathcal{O}(T) \quad (4.24)$$

and thus, conjugating by  $\mathcal{O}(T)$ ,

$$\mathcal{O}(T) \otimes \mathcal{H}_{A_1, T}(C) \otimes \mathcal{O}(-T) = \mathcal{H}_{A_1}(C) \cap \mathcal{O}(T) \otimes \mathcal{H}_{A_1}(C) \otimes \mathcal{O}(-T) \quad (4.25)$$

Replacing  $T$  by  ${}^sT$  in the alternate description

$$\mathcal{H}_{A_1, T}(C) = \mathcal{H}_{A_1}(C) \cap \mathcal{O}({}^sT) \otimes \mathcal{H}_{A_1}(C) \otimes \mathcal{O}(-{}^sT) \quad (4.26)$$

tells us that

$$\mathcal{H}_{A_1}(C) \cap \mathcal{O}(T) \otimes \mathcal{H}_{A_1}(C) \otimes \mathcal{O}(-T) = \mathcal{H}_{A_1, {}^sT}(C) \quad (4.27)$$

as required.  $\square$

**Proposition 4.17.** *The adjoint isomorphism  $\mathcal{H}_{A_1}(C)^{op} \cong \mathcal{O}(-[C^s]) \otimes \mathcal{H}_{A_1}(C) \otimes \mathcal{O}([C^s])$  restricts to a contravariant isomorphism*

$$\mathcal{H}_{A_1, T}(C)^{op} \cong \mathcal{O}(-[C^s]) \otimes \mathcal{H}_{A_1, {}^sT}(C) \otimes \mathcal{O}([C^s]) \quad (4.28)$$

*inducing a contravariant isomorphism*

$$\mathcal{H}_{A_1, T}(C)^{op} \cong \mathcal{O}(T - [C^s]) \otimes \mathcal{H}_{A_1, T}(C) \otimes \mathcal{O}([C^s] - T), \quad (4.29)$$

With this construction in mind, let  $\vec{T}$  be a system of effective Cartier divisors  $T_\alpha$  on  $X$  associated to the roots  $\alpha \in \Phi(W)$ , such that  $T_\alpha$  never contains a fiber of  $X$  and  $w(T_\alpha) = T_{w\alpha}$  for all  $\alpha \in \Phi(W)$ ,  $w \in W$ . Clearly, to specify such a system, it suffices to specify  $T_\alpha$  for one representative of each orbit of roots, subject to the condition that  $w(T_\alpha) = T_\alpha$  whenever  $w\alpha = \alpha$ . Although the construction would work in this generality, we will also impose the further condition that  $T_\alpha$  descends to a divisor on the corresponding coroot curve (or, equivalently, is invariant under translation by any point in  $(1 + r_\alpha)A$ ). This makes the stabilizer condition automatic, and thus we may specify  $\vec{T}$  by specifying effective divisors on the coroot curves associated to a set of inequivalent simple roots.

We will call such a system  $\vec{T}$  of divisors a ‘‘system of parameters for  $W$  on  $X$ ’’.

**Definition 4.2.** Let  $W$  be a finite Weyl group acting on an abelian torsor  $X/S$  by reflections, and let  $\vec{T}$  be a system of parameters for  $W$  on  $X$ . Then the *Hecke algebra*  $\mathcal{H}_{W;\vec{T}}(X)$  is the subalgebra of  $\mathcal{H}_W(X)$  generated by the rank 1 algebras  $\mathcal{H}_{\langle s_i \rangle, T_i}(X)$ .

We again have a filtration by Bruhat order, inherited from  $\mathcal{H}_W(X)$ , and the subquotients are again explicit line bundles.

**Lemma 4.18.** *Let  $I$  be a Bruhat order ideal, and suppose that  $w$  is a maximal element of  $I$ . Then there is a short exact sequence*

$$0 \rightarrow \mathcal{H}_{W;\vec{T}}(X)[I \setminus \{w\}] \subset \mathcal{H}_{W;\vec{T}}(X)[I] \rightarrow \mathcal{O}(D_w(\vec{T})) \rightarrow 0, \quad (4.30)$$

where  $D_w(\vec{T}) := \sum_{r \in R(W), rw < w} ([X^r] - T_{\alpha_r})$ , with  $\alpha_r$  the positive root corresponding to  $r$ .

*Proof.* Suppose first that  $T_\alpha$  never has a component in common with the discriminant divisor  $D_{w_0}$ . Then an easy induction tells us that the left coefficient of  $w$  in any local section of  $\mathcal{H}_{W;\vec{T}}(X)$  vanishes on  $T_{\alpha_r}$  for every reflection  $r$  such that  $rw < w$ ; this is by a calculation as in Lemma 4.9 above, except that we must also argue that  $s_1 \cdots s_{i-1} \alpha_i$  is positive. But this is again standard Coxeter theory; if it were not positive, then  $s_1 \cdots s_i$  could not be a reduced word.

The claim then follows as in the no parameter case. To extend this to bad parameters, we observe (as in the rank 1 case, as we will discuss more precisely below) that we can always embed our family in a larger family which generically satisfies the condition on  $\vec{T}$ . On the one hand, since  $\mathcal{H}_{W;\vec{T}}(X)$  is generated by a flat family of submodules, its Hilbert polynomial is lower semicontinuous and is thus bounded above by the sum of the Hilbert polynomials of the line bundles of the subquotients of the *generic* Bruhat filtration. Since we can construct elements of Bruhat intervals with the desired leading coefficients, it follows that this bound must be tight, and the claim follows in general.  $\square$

*Remark.* We may also write the divisor as  $D_w(\vec{T}) = \sum_{\alpha \in \Phi^+(W) \cap w\Phi^-(W)} ([X^{r_\alpha}] - T_\alpha)$ .

Similarly, the corollaries carry over immediately.

**Corollary 4.19.** *For any reduced word  $w = s_1 \cdots s_n$ , the multiplication map*

$$\mathcal{H}_{\langle s_1 \rangle, \vec{T}}(X) \otimes \cdots \otimes \mathcal{H}_{\langle s_n \rangle, \vec{T}}(X) \rightarrow \mathcal{H}_{W;\vec{T}}(X)[\leq w] \quad (4.31)$$

*is surjective.*

**Corollary 4.20.** *The construction  $\mathcal{H}_{W;\vec{T}}(X)$  respects base change.*

We also have an immediate extension of the adjoint isomorphism.

**Proposition 4.21.** *The adjoint isomorphism  $\mathcal{H}_W(X)^{op} \cong \mathcal{O}(-D_{w_0}) \otimes \mathcal{H}_W(X) \otimes \mathcal{O}(D_{w_0})$  restricts to a contravariant isomorphism*

$$\mathcal{H}_{W;\vec{T}}(X)^{op} \cong \mathcal{O}(-D_{w_0}(\vec{T})) \otimes \mathcal{H}_{W;\vec{T}}(X) \otimes \mathcal{O}(D_{w_0}(\vec{T})) \quad (4.32)$$

*Proof.* Again, it suffices to prove that the adjoint identifies the corresponding rank 1 subalgebras, and one finds that twisting by  $\mathcal{O}(D_{w_0}(\vec{T}) - D_{s_i}(\vec{T}))$  has no effect, so the claim follows from the rank 1 case.  $\square$

One important special case is when  $T_\alpha = [X^{r_\alpha}]$  (which descends to the coroot curve since it is the preimage of the identity under the composition  $X \rightarrow E'_r \rightarrow E_r$ ). In that case, we find that the rank 1 subalgebras are just the twisted group algebras  $\mathcal{O}_X[\langle s \rangle]$ , and thus that the full algebra is itself simply equal to  $\mathcal{O}_X[W]$ .

One disadvantage of the approach via rank 1 subalgebras is that it is not particularly convenient when trying to determine whether a given operator is a (local) section of the Hecke algebra. For this, it will be helpful to have a generalization of Proposition 4.15.

**Proposition 4.22.** *The algebra  $\mathcal{H}_{W;\vec{T}}(X)$  is contained in the subalgebra of  $\mathcal{H}_W(X)$  preserving the subsheaf  $\mathcal{O}(-\sum_{\alpha \in \Phi^+(W)} T_\alpha) \subset \mathcal{O}_X$ , with equality holding unless there is a root  $\alpha$  such that  $T_\alpha$  and  $T_{-\alpha}$  have a common component.*

*Proof.* Containment reduces to showing that the rank 1 subalgebras preserve the given subsheaf. Since the simple reflection  $s_i$  permutes the positive roots other than  $\alpha_i$ , the divisor  $T_i - \sum_{\alpha \in \Phi^+(W)} T_\alpha$  is  $s_i$ -invariant, and has trivial valuation along the components of  $[X^{s_i}]$ . It follows that on the corresponding rank 1 subalgebra, preserving  $\mathcal{O}(-\sum_{\alpha \in \Phi^+(W)} T_\alpha)$  is equivalent to preserving  $\mathcal{O}(-T_i)$ , at which point the claim is just Proposition 4.15.

Using the Bruhat filtration, we see that equality holds whenever

$$\mathcal{O}(D_w(\vec{T})) = \mathcal{O}(D_w) \cap \mathcal{O}(D_w - \sum_{\alpha \in \Phi^+(W)} T_\alpha + w(\sum_{\alpha \in \Phi^+(W)} T_\alpha)). \quad (4.33)$$

Since

$$\sum_{\alpha \in \Phi^+(W)} T_\alpha - w(\sum_{\alpha \in \Phi^+(W)} T_\alpha) = \sum_{\alpha \in \Phi^+(W)} T_\alpha - \sum_{\alpha \in \Phi^+(W)} T_{w\alpha} = \sum_{\alpha \in \Phi^+(W) \cap w\Phi^-(W)} (T_\alpha - T_{-\alpha}), \quad (4.34)$$

we have equality as long as there is no cancellation, i.e., unless there is a positive root  $\alpha$  and a negative root  $\beta$  such that  $T_\alpha$  and  $T_\beta$  have a common component. If  $\beta \neq -\alpha$ , then the two divisors are pulled back through different coroot maps, and thus cannot have a common component, so only the case  $T_\alpha, T_{-\alpha}$  is relevant, and the claim follows.  $\square$

As in the rank 1 case, the restriction on the divisors is not particularly serious, as we can always obtain the algebra we want as the base change of a more general family. In particular, if  $S'$  is an appropriate product of relative symmetric powers of coroot curves, then there is a corresponding tautological system of parameters  $\vec{T}'$  on the base change to  $S'$ , and the original system  $\vec{T}$  is the pullback along a suitable section  $S \rightarrow S'$ .

**Corollary 4.23.** *There is a natural isomorphism*

$$\mathcal{O}(\sum_{\alpha \in \Phi^+(W)} T_\alpha) \otimes \mathcal{H}_{W;\vec{T}}(X) \otimes \mathcal{O}(-\sum_{\alpha \in \Phi^+(W)} T_\alpha) \cong \mathcal{H}_{W;-\vec{T}}(X) \quad (4.35)$$

where  $^{-}T_\alpha := T_{-\alpha}$ .

Another source of isomorphisms is diagram automorphisms.

**Corollary 4.24.** *Let  $\delta$  be an automorphism of  $X$  over  $S$  such that composition with  $\delta$  permutes the set of positive coroot maps. Then  $\delta$  normalizes  $W$ , and the induced action on  $\mathcal{H}_W(X)$  preserves  $\mathcal{H}_{W;\vec{T}}(X)$  for all  $\vec{T}$ .*

*Proof.* The assumption on  $\delta$  implies that  $\delta$  preserves the divisor  $\sum_{\alpha \in \Phi^+(W)} T_\alpha$ . □

**Corollary 4.25.** *Let  $w_0$  be the longest element of  $W$ . Then the action of  $w_0$  on  $\mathcal{H}_W(X)$  takes  $\mathcal{H}_{W;\vec{T}}$  to  $\mathcal{H}_{W;-\vec{T}}$ .*

*Proof.* Indeed,  $w_0$  takes the set of positive roots to the set of negative roots, and thus

$$w_0\left(\sum_{\alpha \in \Phi^+(W)} T_\alpha\right) = \sum_{\alpha \in \Phi^+(W)} T_{-\alpha}. \quad (4.36)$$

□

One important construction of modules comes from the fact that our algebras are generated by the rank 1 subalgebras, and thus any parabolic subgroup  $W_I$  induces a corresponding parabolic subalgebra  $\mathcal{H}_{W_I;\vec{T}|_{\Phi(W_I)}}(X) \subset \mathcal{H}_{W;\vec{T}}(X)$ , which by mild abuse of notation we denote by  $\mathcal{H}_{W_I;\vec{T}}(X)$ . As a result, given a (left)  $\mathcal{H}_{W_I;\vec{T}}(X)$ -module  $M$ , we may tensor with  $\mathcal{H}_{W;\vec{T}}(X)$  to obtain an induced  $\mathcal{H}_{W;\vec{T}}(X)$ -module which we denote by  $\text{Ind}_{W_I}^{W;\vec{T}} M$ , or by  $\text{Ind}_{W_I}^{W;0} M$  when considering the analogous construction for the master Hecke algebra.

Another construction arises from line bundles  $\mathcal{L}$  satisfying the descent conditions along reflection hypersurfaces. Since  $\mathcal{L} \otimes \mathcal{H}_{W;\vec{T}}(X) \otimes \mathcal{L}^{-1} \cong \mathcal{H}_{W;\vec{T}}(X)$ , we obtain an induced functor on left  $\mathcal{H}_{W;\vec{T}}(X)$ -modules which we denote by  $M \mapsto \mathcal{L} \otimes M$ . Note that on the underlying left  $\mathcal{O}_X$ -module, this is just tensoring with  $\mathcal{L}$ , and the left- and right-module constructions are compatible:  $M_1 \otimes (\mathcal{L} \otimes M_2) \cong (M_1 \otimes \mathcal{L}) \otimes M_2$ .

If we take the restriction of an induced module, we would ordinarily expect the result to split as a sum over double cosets. This fails even in the case of the regular representation, as  $\mathcal{H}_{W;\vec{T}}(X)$  does not naturally split as a direct sum of  $(\mathcal{H}_{W_I;\vec{T}}(X), \mathcal{H}_{W_J;\vec{T}}(X))$ -bimodules corresponding to double cosets. The case  $I = J = \emptyset$  is suggestive however: although the Hecke algebra does not split as a *sum* of line bundles indexed by  $W$ , our results on the Bruhat filtration come fairly close. It turns out that there is a natural Bruhat order on (parabolic) double cosets. Indeed, every double coset  $W_I w W_J$  has a unique minimal representative, and the restriction of Bruhat order to the set of such representatives is well-behaved. (See, e.g., [30] and references therein.) In particular, for any order ideal in the set  ${}^I W^J$  of minimal representatives, the corresponding union of double cosets is an order ideal in  $W$ . In particular, any order ideal in  ${}^I W^J$  induces a corresponding sub-bimodule of  $\mathcal{H}_{W;\vec{T}}(X)$ , and thus a subfunctor of  $\text{Res}_{W_I}^{W;\vec{T}} \text{Ind}_{W_J}^{W;\vec{T}}$ .

Given any element  $w \in {}^I W^J$ , the intersections  $W_I \cap w W_J w^{-1}$  and  $w^{-1} W_I w \cap W_J$  are both parabolic, giving subsets  $I(w) \subset I$ ,  $J(w) \subset J$  such that  $W_{I(w)} \cong W_{J(w)}$ , extending in an obvious way to an isomorphism of the corresponding Hecke algebras.

**Lemma 4.26.** *For any  $w \in {}^I W^J$ ,  $D_w(\vec{T})$  is  $W_{I(w)}$ -invariant, and has trivial valuation along the corresponding reflection hyperplanes.*

*Proof.* We recall the expression

$$D_w(\vec{T}) = \sum_{\alpha \in \Phi^+(W) \cap w\Phi^-(W)} ([X^{r\alpha}] - T_\alpha). \quad (4.37)$$

The fact that  $w$  is  $W_I$ -minimal implies that no root of  $W_I$  appears in this sum, and thus in particular that no root of  $W_{I(w)}$  appears. It thus remains only to show  $W_{I(w)}$ -invariance, but this follows by comparing  $D_{s_i w}(\vec{T})$  and  $D_{ws_j}(\vec{T})$  for reflections  $s_i \in W_I$ ,  $s_j \in W_J$  such that  $s_i w = ws_j$ .  $\square$

This ensures that the twisting functor in the following Mackey-type result is well-defined.

**Proposition 4.27.** *Let  $I, J \subset S$ . Then for any  $\mathcal{H}_{W_J; \vec{T}}(X)$ -module  $M$  and any maximal chain in the Bruhat order on  ${}^I W^J$ , the subquotient corresponding to  $w \in {}^I W^J$  in the resulting filtration of  $\text{Res}_{W_I}^{W; \vec{T}} \text{Ind}_{W_J}^{W; \vec{T}} M$  is the  $\mathcal{H}_{W_I; \vec{T}}(X)$ -module*

$$\text{Ind}_{W_{I(w)}}^{W_I; \vec{T}} \left( \mathcal{O}(D_w(\vec{T})) \otimes w \text{Res}_{W_{J(w)}}^{W_J; \vec{T}} M \right), \quad (4.38)$$

where here  $w$  represents the induced isomorphism from the category of  $\mathcal{H}_{W_{J(w); \vec{T}}}(X)$ -modules to the category of  $\mathcal{H}_{W_{I(w); \vec{T}}}(X)$ -modules.

*Proof.* Since the description of the subquotient is functorial, it suffices to consider the case that  $M = \mathcal{H}_{W_J; \vec{T}}(X)$ , or in other words to consider the Bruhat filtration on  $\mathcal{H}_{W; \vec{T}}(X)$  viewed as a bimodule. Let  $O, O \cap \{w\}$  be the elements of the chosen maximal chain that differ by  $w$ , so that we need to understand the quotient of the subsheaf corresponding to  $W_I(O \cap \{w\})W_J$  by the subsheaf corresponding to  $W_I O W_J$ . Both of these are bimodules over the respective Hecke algebras, and the actions commute with projecting onto the vector space of meromorphic operators supported on  $W_I w W_J$ . We thus immediately see from Corollary 4.19 that the quotient is generated by the subsheaf supported on  $W_{I(w)} w W_{J(w)} = W_{I(w)} w$ , and is in fact induced from the corresponding  $(\mathcal{H}_{W_{I(w); \vec{T}}}(X), \mathcal{H}_{W_{J(w); \vec{T}}}(X))$ -bimodule structure. Moreover, one easily verifies that this bimodule induces the Morita equivalence  $M \mapsto \mathcal{O}(D_w(\vec{T})) \otimes w M$ , from which the result follows. Note that the fact that  $w \in {}^I W^J$  ensures that  $D_w(\vec{T})$  is  $W_{I(w)}$ -invariant and has trivial valuation along the reflection hypersurfaces corresponding to  $R(W_{I(w)})$ , so this twisting is indeed well-defined.  $\square$

Taking  $I = \emptyset$  gives the following, where we omit  $\emptyset$  from the notation in  ${}^\emptyset W^J$ .

**Corollary 4.28.** *Let  $I \subset S$ . Then for any  $\mathcal{H}_{W_I; \vec{T}}(X)$ -module  $M$  and any maximal chain in the Bruhat order on  $W^I$ , the subquotient corresponding to  $w \in W^I$  in the resulting filtration of  $\text{Ind}_{W_I}^{W; \vec{T}} M$  is the  $\mathcal{O}_X$ -module  $\mathcal{O}(D_w(\vec{T})) \otimes w M$ .*

As in the master Hecke algebra case, we again have a module  $\mathcal{O}_X$  coming from the action on operators, and the restriction to  $\mathcal{H}_{W; \vec{T}}(X)$  of the natural map  $\mathcal{H}_W(X) \rightarrow \mathcal{O}_X$  is still surjective. In particular, we may again define  $M^W$  to be the image of the natural injective morphism  $\text{Hom}_{\mathcal{H}_{W; \vec{T}}(X)}(\mathcal{O}_X, M) \rightarrow M$ .

**Proposition 4.29.** *The kernel of the natural morphism  $\mathcal{H}_{W; \vec{T}}(X) \rightarrow \mathcal{O}_X$  is generated as a left ideal sheaf by the subsheaves of the form  $\mathcal{O}([X^{s_i}] - T_i)(s_i - 1)$ .*

*Proof.* Let  $\mathcal{I}$  be the left ideal sheaf so generated. This is clearly contained in the kernel, so it remains to show that it contains the kernel. Let  $\sum_w c_w w$  be a local section of the kernel, and suppose  $w_1$  is Bruhat-maximal among the elements of  $W$  for which  $c_w \neq 0$ . Since by definition  $\sum_w c_w = 0$ ,  $w_1$  cannot be the identity, and thus has a reduced expression of the form  $w_1 = s_1 \cdots s_m$  with  $m > 0$ . We thus have a (surjective) multiplication map

$$\mathcal{H}_{\langle s_1 \rangle, \bar{T}}(X) \cdots \mathcal{H}_{\langle s_m \rangle, \bar{T}}(X) \rightarrow \mathcal{H}_{W; \bar{T}}(X)[\leq w_1]. \quad (4.39)$$

Restricting the last tensor factor to  $\mathcal{O}([X^{s_m}] - T_m)(s_m - 1)$  gives an image in  $\mathcal{I}$  without changing the leading coefficient line bundle, and thus there is an element  $\sum_{w \leq w_1} c'_w w$  of  $\mathcal{I}$  with  $c'_w = c_w$ . Subtracting this element makes the order ideal generated by the support of the operator smaller, and thus the result follows by induction.  $\square$

**Corollary 4.30.** *There is an exact sequence of  $\mathcal{O}_{X/W}$ -modules*

$$0 \rightarrow M^W \rightarrow M \rightarrow \bigoplus_{1 \leq i \leq n} \mathcal{O}(T_i - [X^{s_i}]) \otimes M \quad (4.40)$$

*Proof.* The Proposition gives a resolution of  $\mathcal{O}_X$ , and this is just the sheaf Hom from that resolution to  $M$ .  $\square$

If  $M$  is  $S$ -flat, then this tells us that  $M^W$  is the kernel of a morphism of  $S$ -flat sheaves.

**Lemma 4.31.** *Let  $Y/S$  be a projective scheme with relatively ample line bundle  $\mathcal{O}_Y(1)$ , and suppose  $\phi : M \rightarrow N$  is a morphism of  $S$ -flat coherent sheaves on  $Y$ . If the Hilbert polynomial of  $\ker(\phi_s)$  is independent of the point  $s \in S$ , then the kernel, image, and cokernel of  $\phi$  are all flat, and the natural map  $\ker(\phi)_s \rightarrow \ker(\phi_s)$  is an isomorphism for all  $s$ .*

*Proof.* If the Hilbert polynomial of  $\ker(\phi_s)$  is independent of  $s$ , then so is the Hilbert polynomial of  $\text{coker}(\phi_s) \cong \text{coker}(\phi)_s$ . It follows that  $\text{coker}(\phi)$  is  $S$ -flat, implying immediately that the image and kernel are also  $S$ -flat (as kernels of surjective morphisms of  $S$ -flat sheaves). The final claim follows using the four-term sequence

$$0 \rightarrow \text{Tor}_2(\text{coker}(\phi), \mathcal{O}_s) \rightarrow \ker(\phi)_s \rightarrow \ker(\phi_s) \rightarrow \text{Tor}_1(\text{coker}(\phi), \mathcal{O}_s) \rightarrow 0 \quad (4.41)$$

arising by comparing the two spectral sequences for tensoring the complex  $M \rightarrow N$  with  $\mathcal{O}_s$ .  $\square$

**Corollary 4.32.** *If  $M$  is a coherent  $S$ -flat  $\mathcal{H}_{W; \bar{T}}(X)$ -module such that the Hilbert polynomial of  $(M_s)^W$  is independent of  $s$ , then  $M^W$ ,  $M/M^W$  are flat and the natural map  $(M^W)_s \rightarrow (M_s)^W$  is an isomorphism for all  $s$ .*

If  $M$  satisfies the hypothesis, we say that  $M$  has “strongly flat invariants”.

Before introducing parameters, we could show that this functor respected base change and flatness by observing that (subject to diagonalizability of the root kernel) the Hecke algebra had (locally on  $S$ ) idempotents projecting onto  $M^W$ . Unfortunately, this fails, and quite badly, in cases with parameters. Indeed, if the divisors  $T_\alpha$  are of sufficiently large degree, then the subquotient corresponding to  $w$  in the Bruhat filtration will have negative degree unless  $w$  is the identity, and thus in such a case the fibers of the Hecke algebra cannot have *any* nonscalar global sections, let alone symmetric idempotents.

Luckily,  $S$ -flatness is local on the source, not the base, and thus the correct condition is not that there be *global* symmetric idempotents, but merely that there be *local* symmetric idempotents.

**Lemma 4.33.** *Let  $U$  be a  $W$ -invariant open subset. If  $h \in \Gamma(U; \mathcal{O}(D_{w_0}(\vec{T})))$ , then  $(\sum_w w)^{w_0} h \in \Gamma(U; \mathcal{H}_{W; \vec{T}}(X))$ .*

*Proof.* Suppose first that  $T_\alpha$  and  $T_{-\alpha}$  have no common component for any root  $\alpha$ , so that we may use Proposition 4.22 to characterize  $\mathcal{H}_{W; \vec{T}}(X)$ . The given operator clearly maps  $\Gamma(V; \mathcal{O}_X)$  to  $\Gamma(U \cap V; \mathcal{O}_X)^W$  for any invariant open  $V$ , so that  $(\sum_w w)^{w_0} h \in \mathcal{H}_W(X)$ . Similarly, if  $f \in \Gamma(V; -\sum_{\alpha \in \Phi^+(W)} T_\alpha)$ , then  ${}^{w_0} h f \in \Gamma(V; D_{w_0} - \sum_{\alpha \in \Phi^+(W)} T_\alpha)$ , so that the symmetrization vanishes along  $\sum_{\alpha \in \Phi^+(W)} T_\alpha$ . The claim follows in this case.

For the general case, we base change to the family with universal  $\vec{T}$ , and observe that since  $D_{w_0}(\vec{T})$  is a flat family of divisors,  $h$  extends to a local section of  $\mathcal{O}(D_{w_0}(\vec{T}))$  on a neighborhood of the original base. We thus find that there is a local section of the larger Hecke algebra that restricts to the desired local section, from which the result follows.  $\square$

We say that  $\mathcal{H}_{W; \vec{T}}(X)$  has a local symmetric idempotent at a point  $x \in X/W$  if the restriction of  $\mathcal{H}_{W; \vec{T}}(X)$  to the local ring at  $x$  contains an idempotent of the form  $(\sum_w w)h$ . By the lemma, this is equivalent to asking for the restriction of  $\mathcal{O}(D_{w_0}(\vec{T}))$  to the local ring at the orbit corresponding to  $x$  to contain an element  $h$  with  $\sum_{w \in W} {}^w h = 1$ . Moreover, if there is an element for which this sum is a unit, then we can divide by the sum to obtain an element symmetrizing to 1. It follows that if the condition holds on the fiber containing  $x$ , then we still have a local symmetric idempotent at  $x$ ; that is, the condition of having a local symmetric idempotent respects base change.

Similarly, we say that  $\mathcal{H}_{W; \vec{T}}(X)$  is covered by symmetric idempotents if it has a local symmetric idempotent at every point  $x \in X/W$ . This is too much to hope for even without imposing parameters, as the  $A_2$  example we considered at the end of Section 3 gives an explicit point where the master Hecke algebra fails to have a local symmetric idempotent. In general, the most we can say is that there is a (possibly empty) open subset of  $S$  such that the base change is covered by symmetric idempotents. Indeed, the condition to have a symmetric idempotent at  $x$  is open, and  $X/S$  is proper, so we can simply take the complement of the image of the complement of the locus with local symmetric idempotents.

**Lemma 4.34.** *Suppose that the root kernel of  $X$  is diagonalizable, and that for any nonnegative linear dependence  $\sum_i k_i \alpha_i = 0$  of roots, the intersection  $\cap_i T_{\alpha_i}$  is empty. Then  $\mathcal{H}_{W; \vec{T}}(X)$  is covered by idempotents.*

*Proof.* This is local in  $S$ , so we may restrict to an open subset over which  $\mathcal{H}_{W,0}(X)$  has a global symmetric idempotent  $\sum_w wh$ ; diagonalizability of the root kernel ensures that these open subsets cover  $S$ . For any point  $x \in X$ , let  $D_x$  be the corresponding decomposition group, and observe that

$$1 = \sum_{w \in W} {}^w h = \sum_{g \in D_x} g \left( \sum_{w \in D_x \setminus W} {}^w h \right), \quad (4.42)$$

and thus there is a section of  $\mathcal{O}(D_{w_0})$  in the local ring at  $x$  for which the sum over the decomposition group is 1.

Now, suppose that  $x$  is not contained in  $T_\alpha$  for any positive  $\alpha$ . Then this local section of  $\mathcal{O}(D_{w_0})$  at  $x$  is in fact a section of  $\mathcal{O}(D_{w_0} - \sum_\alpha T_\alpha)$  near  $x$ , and we can add a section that vanishes at  $x$  in such a way that the resulting section is holomorphic on the orbit  $Wx$  and vanishes at the points of the orbit other than  $x$ . It follows that the resulting function symmetrizes to a unit in the relevant local ring, and thus gives rise to a local symmetric idempotent at  $Wx$ .

In general, let  $\Phi_x$  be the set of roots such that  $x \in T_\alpha$ . Since  $x$  is contained in the corresponding intersection of divisors  $T_\alpha$ , we conclude that the elements of  $\Phi_x$ , viewed as real vectors, cannot

satisfy any nonnegative linear dependence. This implies that there is a real linear functional which is negative on  $\Phi_x$ , and thus (since all systems of positive roots in a finite Weyl group are equivalent) that  $w\Phi_x \subset \Phi^-(X)$  for some  $w \in W$ . This implies that  $wx$  satisfies the conditions for the construction of the previous paragraph to apply, and thus that there is a local symmetric idempotent in a neighborhood of the orbit  $Wx$ .  $\square$

It is not too hard to see that the empty intersection condition is satisfied on the generic fiber of the family with universal  $\bar{T}$ ; indeed, if we further base change to express each  $T_i$  as a sum of points, then the values of those parameters at a point of such an intersection must themselves satisfy a nonnegative linear dependence, and there are only finitely many minimal such dependences to consider. It follows that any family of Hecke algebras is the base change of a family which is generically covered by symmetric idempotents.

**Lemma 4.35.** *If  $\mathcal{H}_{W;\bar{T}}(X)$  is covered by symmetric idempotents, then  $-^W$  is exact and any coherent  $\mathcal{H}_{W;\bar{T}}(X)$ -module  $M$  has strongly flat invariants.*

*Proof.* Indeed,  $\mathcal{O}_X$  is locally a direct summand of  $\mathcal{H}_{W;\bar{T}}(X)$ , and is therefore locally projective as before. Exactness follows immediately. Any local section of  $(M_s)^W$  on an open subset supporting a symmetric idempotent extends to a local section of  $M$  which can then be projected to a section of  $M^W$  restricting to the given section of  $(M_s)^W$ . It follows that the natural map  $(M^W)_s \rightarrow (M_s)^W$  is an isomorphism. Since  $M^W$  is locally a direct summand of  $M$ , it is flat, and thus its fibers have constant Hilbert polynomial as required.  $\square$

It turns out that if the generic fiber is covered by symmetric idempotents, that this has consequences even on those fibers without such a covering.

**Lemma 4.36.** *Suppose that there is a nonempty open subset of  $S$  over which  $\mathcal{H}_{W;\bar{T}}(X)$  is covered by symmetric idempotents, and suppose that the module  $M$  admits a filtration such that each subquotient is  $S$ -flat with strongly flat invariants. Then  $M$  has strongly flat invariants.*

*Proof.* Fix a relatively ample bundle  $\mathcal{O}(1)$  on  $X/W$ . By semicontinuity, for any point  $s \in S$  and  $d \gg 0$ , we have

$$\begin{aligned} \dim(\Gamma(X/W, (M_{k(S)})^W(d))) &\leq \dim(\Gamma(X/W, (M_s)^W(d))) \\ &\leq \sum_i \dim(\Gamma(X/W, (M_s^i)^W(d))) \\ &= \sum_i \dim(\Gamma(X/W, (M_{k(S)}^i)^W(d))), \end{aligned}$$

where the  $M^i$  are the subquotients of the given filtration on  $M$ . Since the generic fiber of  $\mathcal{H}_{W;\bar{T}}(X)$  is covered by symmetric idempotents,  $-^W$  is exact and thus

$$\dim(\Gamma(X/W, (M_{k(S)})^W(d))) = \sum_i \dim(\Gamma(X/W, (M_{k(S)}^i)^W(d))) \quad (4.43)$$

for  $d \gg 0$ , implying

$$\dim(\Gamma(X/W, (M_{k(S)})^W(d))) = \dim(\Gamma(X/W, (M_s)^W(d))) \quad (4.44)$$

as required.  $\square$

To apply this, we will need a family of modules for which we can prove strongly flat invariants without resorting to idempotents. For  $I \subset S$ , let  $w_I$  denote the maximal element of  $W_I$ .

**Proposition 4.37.** *Let  $I \subset S$  and let  $\mathcal{L}$  be a  $W_I$ -equivariant line bundle satisfying the descent conditions along reflection hyperplanes. Then we have a natural isomorphism  $(\text{Ind}_{W_I}^W \mathcal{L})^W \cong (\mathcal{L} \otimes \mathcal{O}(D_{w_0}(-\vec{T}) - D_{w_I}(-\vec{T})))^{W_I}$  of  $\mathcal{O}_{X/W}$ -modules, where the right-hand side denotes the sheaf of  $W_I$ -invariant sections of the given line bundle.*

*Proof.* Any local section of  $\text{Ind}_{W_I}^W \mathcal{L}$  can be expressed in the form

$$\sum_{w \in W^I} f_w w W_I \quad (4.45)$$

where  $f_w$  is a meromorphic section of  ${}^w \mathcal{L}$ , and the condition to be in  $(\text{Ind}_{W_I}^W \mathcal{L})^W$  is that  $f_{w'w} W_I = {}^{w'} f_w W_I$ . In particular, the operator is uniquely determined by the function  $f_{w_0 W_I}$ , which must be a  $W_I$ -invariant section of  ${}^{w_0} \mathcal{L} \otimes \mathcal{O}(D_{w_0 w_I}(\vec{T}))$ , where  $w_I$  is the maximal element of  $W_I$ , so that  $w_0 w_I$  is the minimal representative of the coset  $w_0 W_I$ . Applying  $w_0$  gives an injective morphism

$$(\text{Ind}_{W_I}^W \mathcal{L})^W \rightarrow (\mathcal{L} \otimes \mathcal{O}(D_{w_0}(-\vec{T}) - D_{w_I}(-\vec{T})))^{W_I}, \quad (4.46)$$

since

$$\begin{aligned} w_0(D_{w_0 w_I}(\vec{T})) &= \sum_{\alpha \in \Phi^+(W) \cap w_0 \Phi^-(W)} w_0([X^{r\alpha}] - T_\alpha) \\ &= \sum_{w_0 \alpha \in \Phi^+(W) \cap w_I \Phi^-(W)} ([X^{r\alpha}] - T_\alpha) \\ &= \sum_{\alpha \in \Phi^-(W) \cap w_I \Phi^-(W)} ([X^{r\alpha}] - T_\alpha) \\ &= \sum_{\alpha \in \Phi^+(W) \cap w_I \Phi^+(W)} ([X^{r\alpha}] - T_{-\alpha}) \\ &= D_{w_0}(-\vec{T}) - D_{w_I}(-\vec{T}). \end{aligned}$$

It remains only to show that this map is an isomorphism, or in other words that for any invariant local section of the line bundle,

$$\sum_{w \in W^I} {}^w f w W_I \quad (4.47)$$

is a local section of  $\text{Ind}_{W_I}^W \mathcal{L}$ . If  $W_I$  has a local symmetric idempotent of the form  $\sum_{w \in W_I} w c$ , then we find that

$$\sum_{w \in W} w f c \quad (4.48)$$

is a local section of  $\mathcal{H}_{W, \vec{T}}(X)$  projecting to  $\sum_{w \in W^I} {}^w f w W_I$ . Thus the claim follows whenever  $\mathcal{H}_{W, \vec{T}}(X)$  is covered by local idempotents. More generally, if  $X'/S'$  is the base change to the family with universal parameters, then  $\mathcal{L}$  is the pullback of a line bundle on  $X'$ , namely the pullback to  $X'$  of  $\mathcal{L}$ . We can thus extend  $f$  to a local section in an appropriate neighborhood in  $X'$ , and conclude that  $\sum_{w \in W^I} {}^w f w W_I$  is a local section of  $\text{Ind}_{W_I}^W \mathcal{L}$  by virtue of this being a closed condition.  $\square$

**Corollary 4.38.** *Suppose that the root kernel of  $X$  is diagonalizable. Then any module of the form  $M = \text{Ind}_{W_I}^W \mathcal{L}$  has strongly flat invariants.*

**Corollary 4.39.** *Let  $I, J \subset S$  and let  $\mathcal{L}_I, \mathcal{L}_J$  be  $W_I, W_J$ -equivariant line bundles satisfying the descent conditions along reflection hypersurfaces. If the root kernel of  $X$  is diagonalizable, then*

$$\mathcal{H}om_{\mathcal{H}_{W;\bar{T}}(X)}(\text{Ind}_{W_I}^{W;\bar{T}} \mathcal{L}_I, \text{Ind}_{W_J}^{W;\bar{T}} \mathcal{L}_J) \quad (4.49)$$

is  $S$ -flat and respects base change.

*Proof.* By extending the family as appropriate (noting that  $\mathcal{L}_I$  and  $\mathcal{L}_J$  themselves extend), we may assume that there is an open subset  $U \subset S$  which is covered by symmetric idempotents. We observe that

$$\begin{aligned} \mathcal{H}om_{\mathcal{H}_{W;\bar{T}}(X)}(\text{Ind}_{W_I}^{W;\bar{T}} \mathcal{L}_I, \text{Ind}_{W_J}^{W;\bar{T}} \mathcal{L}_J) &\cong \mathcal{H}om_{\mathcal{H}_{W_I;\bar{T}}(X)}(\mathcal{L}_I, \text{Res}_{W_I}^{W;\bar{T}} \text{Ind}_{W_J}^{W;\bar{T}} \mathcal{L}_J) \\ &\cong (\mathcal{L}_I^{-1} \otimes \text{Res}_{W_I}^{W;\bar{T}} \text{Ind}_{W_J}^{W;\bar{T}} \mathcal{L}_J)^{W_I}. \end{aligned}$$

Each subquotient of the Bruhat filtration for

$$\mathcal{L}_I^{-1} \otimes \text{Res}_{W_I}^{W;\bar{T}} \text{Ind}_{W_J}^{W;\bar{T}} \mathcal{L}_J \quad (4.50)$$

has strongly flat invariants, and thus the same holds for the  $\mathcal{H}_{W_I;\bar{T}}(X)$ -module itself. It follows in particular that the module is  $S$ -flat and that the construction commutes with base change.  $\square$

**Proposition 4.40.** *Let  $\mathcal{L}_I, \mathcal{L}_J$  be  $W_I, W_J$ -equivariant line bundles satisfying the descent conditions along reflection hyperplanes. If the root kernel of  $X$  is diagonalizable, then there is a natural isomorphism*

$$\mathcal{H}om_{\mathcal{H}_W(X)}(\text{Ind}_{W_J}^{W;0} \mathcal{L}_J, \text{Ind}_{W_I}^{W;0} \mathcal{L}_I) \cong \mathcal{H}om((\pi_* \mathcal{L}_I)^{W_I}, (\pi_* \mathcal{L}_J)^{W_J}) \quad (4.51)$$

which is (contravariantly) compatible with composition.

*Proof.* Since the root kernel is diagonalizable, both  $\mathcal{H}_{W_I}(X)$  and  $\mathcal{H}_{W_J}(X)$  have symmetric idempotents  $e_I, e_J$  locally on  $S$ , and these embed as elements of  $\mathcal{E}nd(\pi_* \mathcal{L}_I)$  and  $\mathcal{E}nd(\pi_* \mathcal{L}_J)$  respectively. We may thus identify the left-hand side as the subspace  $e_J \mathcal{H}om(\pi_* \mathcal{L}_I, \pi_* \mathcal{L}_J) e_I$ , and this is contravariant with respect to composition. As an element of  $\mathcal{E}nd(\pi_* \mathcal{L}_I)$ ,  $e_I$  is a projection onto  $(\pi_* \mathcal{L}_I)^{W_I}$ , and similarly for  $e_J$ ; the claim follows immediately.  $\square$

*Remark.* Note that if  $\mathcal{L}_I$  descends to a line bundle on  $X/W_I$ , then  $(\pi_* \mathcal{L}_I)^{W_I}$  may be identified with the direct image of that line bundle.

There is an analogue of the adjoint in this setting.

**Corollary 4.41.** *If the root kernel of  $X$  is diagonalizable, then there is an isomorphism*

$$\mathcal{H}om_{\mathcal{H}_W(X)}(\text{Ind}_{W_J}^{W;0} \mathcal{O}_X, \text{Ind}_{W_I}^{W;0} \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{H}_W(X)}(\text{Ind}_{W_I}^{W;0} \mathcal{O}(D_{w_0} - D_{w_I}), \text{Ind}_{W_J}^{W;0} \mathcal{O}(D_{w_0} - D_{w_J})), \quad (4.52)$$

contravariant with respect to composition.

*Proof.* Embedding the left-hand side in  $\mathcal{E}nd(\pi_* \mathcal{O}_X)$  and taking the adjoint there gives an isomorphism to  $\mathcal{H}om(e_J^* \pi_* \mathcal{O}(D_{w_0}), e_I^* \pi_* \mathcal{O}(D_{w_0}))$ , where  $e_I^*, e_J^*$  are the adjoints of the corresponding idempotents. If  $e_I = (\sum_{w \in W_I} w) h_I$ , then  $e_I^* = h_I (\sum_{w \in W_I} w)$ , and we then find that

$$e_I^* \pi_* \mathcal{O}(D_{w_0}) = h_I \left( \sum_{w \in W_I} w \right) \pi_* \mathcal{O}(D_{w_0}) = h_I (\pi_* \mathcal{O}(D_{w_0} - D_{w_I}))^{W_I}, \quad (4.53)$$

where the second equality follows from Theorem 3.9. We thus have

$$\begin{aligned} \mathcal{H}om(e_J^* \pi_* \mathcal{O}(D_{w_0}), e_I^* \pi_* \mathcal{O}(D_{w_0})) &\cong \mathcal{H}om(h_J(\pi_* \mathcal{O}(D_{w_0} - D_{w_J}))^{W_J}, h_I(\pi_* \mathcal{O}(D_{w_0} - D_{w_I}))^{W_I}) \\ &\cong \mathcal{H}om((\pi_* \mathcal{O}(D_{w_0} - D_{w_J}))^{W_J}, (\pi_* \mathcal{O}(D_{w_0} - D_{w_I}))^{W_I}), \end{aligned}$$

from which the claim follows.  $\square$

*Remark.* There is, of course, a version with a pair of line bundles; we omit the details.

Define

$$\mathcal{H}_{W, W_I, W_J; \vec{T}}(X) := \mathcal{H}om_{\mathcal{H}_{W; \vec{T}}(X)}(\text{Ind}_{W_J}^{W; \vec{T}} \mathcal{L}_J, \text{Ind}_{W_I}^{W; \vec{T}} \mathcal{L}_I), \quad (4.54)$$

with composition law given by

$$\mathcal{H}_{W, W_J, W_K; \vec{T}}(X) \otimes \mathcal{H}_{W, W_I, W_J; \vec{T}}(X) \rightarrow \mathcal{H}_{W, W_I, W_K; \vec{T}}(X), \quad (4.55)$$

contravariant to the standard composition on Hom sheaves.

**Proposition 4.42.** *If the root kernel of  $X$  is diagonalizable, then  $\mathcal{H}_{W, W_I, W_J; \vec{T}}(X)$  may be identified with a subsheaf of  $\mathcal{H}_{W, W_I, W_J}(X)$ , compatibly with composition. Moreover, the corresponding operators take  $W_I$ -invariant sections of the line bundle  $\mathcal{O}(\sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_I)} T_\alpha)$  to the  $W_J$ -invariant sections of the line bundle  $\mathcal{O}(\sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_J)} T_\alpha)$ . Conversely, if there are no roots such that  $T_\alpha$  and  $T_{-\alpha}$  have a common component, then  $\mathcal{H}_{W, W_I, W_J; \vec{T}}(X)$  is precisely the subsheaf of  $\mathcal{H}_{W, W_I, W_J}(X)$  cut out by this condition.*

*Proof.* Suppose first that  $\mathcal{H}_{W_I; \vec{T}}(X)$  and  $\mathcal{H}_{W_J; \vec{T}}(X)$  are covered by symmetric idempotents. This allows us to (locally) embed  $\mathcal{H}_{W, W_I, W_J; \vec{T}}(X)$  in  $\mathcal{H}_{W; \vec{T}}(X)$  as in the parameter-free case. It follows that  $\mathcal{H}_{W, W_I, W_J; \vec{T}}(X)$  acts on the  $W_I$ -invariant sections of  $k(X)$  in such a way as to take  $W_I$ -invariant sections of  $\mathcal{O}_X$  to  $W_J$ -invariant sections of  $\mathcal{O}_X$  and  $W_I$ -invariant sections of  $\mathcal{O}(\sum_{\alpha \in \Phi^-(W)} T_\alpha)$  to  $W_J$ -invariant sections of  $\mathcal{O}(\sum_{\alpha \in \Phi^-(W)} T_\alpha)$ .

Since  $\sum_{\alpha \in \Phi^-(W)} T_\alpha$  is not  $W_I$ -invariant, a  $W_I$ -invariant section of  $\mathcal{O}(\sum_{\alpha \in \Phi^-(W)} T_\alpha)$  must lie in the intersection of the images of this bundle under  $W_I$ , so in particular (taking the intersection with the image under  $w_I$ )

$$\mathcal{O}\left(\sum_{\alpha \in \Phi^-(W)} T_\alpha\right) \cap \mathcal{O}\left(\sum_{\alpha \in \Phi^+(W_I) \cup \Phi^-(W) \setminus \Phi^-(W_I)} T_\alpha\right), \quad (4.56)$$

which by hypothesis is  $\mathcal{O}(\sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_I)} T_\alpha)$ . The same calculation for  $J$  tells us that the elements of  $\mathcal{H}_{W, W_I, W_J; \vec{T}}(X)$  act as required.

For a  $W_I$ -invariant section  $\sum_{w \in W_I} c_w w W_I$  of  $\mathcal{H}_{W, W_I, W_J; \vec{T}}(X)$  to be contained in  $\mathcal{H}_{W, W_I, W_J}(X)$  is a closed condition, and thus holds in general (extending to the family with universal parameters as necessary). That it respects the given supersheaves is also a closed condition, and thus the first claim follows for general parameters.

To show equality under the conditions on  $T_\alpha$ , it suffices to compare subquotients in the respective Bruhat filtrations, and thus to compute the intersection

$$\mathcal{O}_X \cap \mathcal{O}\left(\sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_J)} T_\alpha - \sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_I)} T_{w\alpha}\right). \quad (4.57)$$

We may write

$$\sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_J)} T_\alpha - \sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_I)} T_{w\alpha} = \sum_{\alpha \in \Phi^-(W)} (T_\alpha - T_{w\alpha}) - \sum_{\alpha \in \Phi^-(W_J)} T_\alpha + \sum_{\alpha \in \Phi^-(W_I)} T_{w\alpha}. \quad (4.58)$$

Here

$$\sum_{\alpha \in \Phi^-(W)} (T_\alpha - T_{w\alpha}) = \sum_{\alpha \in \Phi^+(W) \cap w\phi^-(W)} (T_{-\alpha} - T_\alpha), \quad (4.59)$$

while

$$\sum_{\alpha \in \Phi^-(W_I)} T_{w\alpha} - \sum_{\alpha \in \Phi^-(W_J)} T_\alpha = \sum_{\alpha \in \Phi^-(W_I) \setminus \Phi^-(W_I \cap w^{-1}W_J w)} T_{w\alpha} - \sum_{\alpha \in \Phi^-(W_J) \setminus \Phi^-(W_J \cap wW_I w^{-1})} T_\alpha \quad (4.60)$$

The hypotheses ensure that there is no further cancellation, so the intersection is

$$\mathcal{O}(- \sum_{\alpha \in \Phi^+(W) \cap w\phi^-(W)} T_\alpha - \sum_{\alpha \in \Phi^-(W_J) \setminus \Phi^-(W_J \cap wW_I w^{-1})} T_\alpha), \quad (4.61)$$

agreeing with the line bundle arising in the Bruhat filtration.  $\square$

**Corollary 4.43.** *There is an isomorphism*

$$\mathcal{H}_{W, W_I, W_J; \vec{T}}(X) \cong \mathcal{O}(D_{w_0}(-\vec{T}) - D_{w_J}(-\vec{T})) \otimes \mathcal{H}_{W, W_J, W_I; \vec{T}}(X) \otimes \mathcal{O}(D_{w_I}(-\vec{T}) - D_{w_0}(-\vec{T})) \quad (4.62)$$

which is contravariant for the natural composition.

*Proof.* Extend to universal parameters, write the left-hand side as an intersection, and take the adjoint of both twists of  $\mathcal{H}_{W, W_I, W_J; \vec{T}}(X)$ . The resulting equality extends as usual to the full parameter space.  $\square$

When  $I = J$ , we denote this by  $\mathcal{H}_{W, W_I; \vec{T}}(X)$ , and call the resulting sheaf of algebras a *spherical algebra* of the Hecke algebra, which is in general a subalgebra of the algebra  $\mathcal{E}nd((\pi_* \mathcal{L}_I)^{W_I})$  corresponding to  $\vec{T} = 0$ .

## 5 Infinite groups

Most of our arguments above regarding the structure of the Hecke algebra boiled down to the combinatorics of (double) cosets in Coxeter groups and the associated Bruhat order. Indeed, virtually everything in the above discussion carries over immediately to the case of infinite Coxeter group, with one glaring exception: the Hecke algebra was defined as a sheaf of algebras on the quotient  $X/W$ , and there is no such quotient scheme when  $W$  is infinite!

Thus the primary (and to first approximation only) issue in generalizing the above construction is simply to determine what manner of object we will be constructing. Luckily, a suitable generalization of sheaves of algebras has already appeared in the literature on noncommutative geometry, namely the notion of a “sheaf algebra”.

We recall the definition from [31, §2], generalizing an earlier definition of [2, §2]. We first need the notion of a sheaf bimodule: Let  $X, Y$  be Noetherian  $S$ -schemes of finite type: An  $\mathcal{O}_S$ -central  $(\mathcal{O}_X, \mathcal{O}_Y)$ -bimodule is a quasicohherent  $\mathcal{O}_{X \times_S Y}$ -module  $M$  such that the support of any coherent subsheaf of  $M$  is finite over both  $X$  and  $Y$  (relative to the projections). We will sometimes shorthand this by saying that  $M$  is a sheaf bimodule on  $X \times_S Y$ . Note that if  $X = \text{Spec}(R_X)$ ,  $Y = \text{Spec}(R_Y)$ , then a sheaf bimodule on  $X \times_S Y$  is an  $(R_X, R_Y)$ -bimodule such that  $\Gamma(S; \mathcal{O}_S)$  is central and such that any finitely generated subbimodule is finitely generated both as a left module and as a right module.

As with ordinary bimodules, there is a notion of tensor product for sheaf bimodules. If  $M$  is a sheaf bimodule on  $X \times_S Y$  and  $N$  is a sheaf bimodule on  $Y \times_S Z$ , then we can construct

a sheaf bimodule on  $X \times_S Z$  by pulling back  $M$  and  $N$  to  $X \times_S Y \times_S Z$ , tensoring, and then projecting to  $X \times_S Z$  to obtain a sheaf  $M \otimes_Y N$ . Note that if  $\Delta_{X/S}$  is the diagonal in  $X \times_S X$ , then  $\mathcal{O}_{\Delta_{X/S}}$  is a sheaf bimodule on  $X \times_S X$ , and for any sheaf bimodule  $M$  on  $X \times_S Y$ , there is a natural isomorphism  $\mathcal{O}_{\Delta_{X/S}} \otimes_X M \cong M$ . Furthermore, this tensor product operation is naturally associative and agrees with the usual tensor product when the schemes are affine, see [31].

The tensor product provides the category of  $(\mathcal{O}_X, \mathcal{O}_X)$ -bimodules with a natural monoidal structure, thus allowing one to define a *sheaf algebra* on  $X/S$  to be a monoid object in that category; that is, a sheaf bimodule  $A$  equipped with morphisms  $\mathcal{O}_{\Delta_X} \rightarrow A$  and  $A \otimes_X A \rightarrow A$  satisfying the obvious axioms. More generally, one may also consider *sheaf categories*, in which every object of the category has an associated scheme and the Hom sets are replaced by sheaf bimodules.

One difficulty in dealing with the above construction is that it is not always easy to work with local sections of the tensor product of sheaf bimodules. Indeed, since the tensor product is a direct image, we in general need to choose an affine open covering of  $Y$  and look for compatible systems of elements of the corresponding naïve tensor products. It turns out that for *coherent* sheaf bimodules, there is a cleaner approach.

**Proposition 5.1.** *Let  $M$  be a sheaf bimodule on  $X \times_S Y$  and let  $N$  be a sheaf bimodule on  $Y \times_S Z$ . Let  $\text{Spec}(R) \cong V \subset Y$  be an affine open subset and let  $U \subset X$ ,  $W \subset Z$  be open subsets. If  $M$  is coherent and the preimage of  $V$  in the support of  $M$  contains the preimage of  $U$ , or if  $N$  is coherent and the preimage of  $V$  in the support of  $N$  contains the preimage of  $W$ , then there is a natural isomorphism*

$$\Gamma(U \times W; M \otimes_Y N) \cong \Gamma(U \times V; M) \otimes_R \Gamma(V \times W; N). \quad (5.1)$$

*Proof.* By symmetry, we may suppose that the constraint on  $M$  holds. The sections of a coherent sheaf bimodule on a product of open subsets depends only on the intersection of those open subsets on the support of the sheaf bimodule. It follows, therefore, that there is a natural isomorphism

$$\Gamma(U \times V'; M) \cong \Gamma(U \times (V \cap V'); M) \quad (5.2)$$

for any open subset  $V'$ . Computing the tensor product via an affine open covering of  $Y$  containing  $V$  gives a natural morphism

$$\Gamma(U \times W; M \otimes_Y N) \rightarrow \Gamma(U \times V; M) \otimes_R \Gamma(V \times W; N), \quad (5.3)$$

and the compatibility conditions ensure that this is an isomorphism as required.  $\square$

*Remark.* For coherent  $M$ , there is a maximal  $U$  satisfying the hypothesis: take  $X \setminus U$  to be the image of  $X$  of the preimage of  $Y \setminus V$ , and observe that finiteness implies that this image is closed, so  $U$  is open. For quasicohherent  $M$ , it is tempting to consider the intersection of the  $U$ 's corresponding to the coherent subsheaves of  $M$ , but of course this will rarely be open. Of course, if it *is* open, then taking the limit tells us that the conclusion of the Proposition continues to hold.

Of course, we would like to know that this incorporates the usual notion of a sheaf of algebras on the quotient.

**Proposition 5.2.** *Let  $f : X \rightarrow Y$  be a finite morphism of Noetherian  $S$ -schemes of finite type, and suppose that  $\mathcal{A}$  is a quasicohherent sheaf of  $\mathcal{O}_Y$ -algebras on  $Y$  equipped with an algebra morphism  $f_*\mathcal{O}_X \rightarrow \mathcal{A}$ . Then  $\mathcal{A}$  induces a sheaf algebra  $\mathcal{A}_X$  on  $X/S$  such that for any open subsets  $U, V \subset Y$ ,  $\Gamma(f^{-1}(U) \times f^{-1}(V); \mathcal{A}_X) \cong \Gamma(U \cap V; \mathcal{A})$ .*

*Proof.* As usual, we may assume that  $S$  is affine. For any affine open subset  $U \subset Y$ , the sheaf-of-algebras morphism  $f_*\mathcal{O}_X \rightarrow \mathcal{A}$  induces an algebra morphism  $\Gamma(U; f_*\mathcal{O}_X) \rightarrow \Gamma(U; \mathcal{A})$ , and thus makes  $\Gamma(U; \mathcal{A})$  a bimodule over  $\Gamma(U; f_*\mathcal{O}_X) \cong \Gamma(f^{-1}(U); \mathcal{O}_X)$ . More generally, if  $U, V \subset Y$  are two affine open subsets, then we may use the morphisms  $\Gamma(U; f_*\mathcal{O}_X) \rightarrow \Gamma(U \cap V; f_*\mathcal{O}_X)$  and  $\Gamma(V; f_*\mathcal{O}_X) \rightarrow \Gamma(U \cap V; f_*\mathcal{O}_X)$  to make  $\Gamma(U \cap V; \mathcal{A})$  a  $(\Gamma(f^{-1}(U); \mathcal{O}_X), \Gamma(f^{-1}(V); \mathcal{O}_X))$ -bimodule.

This in particular gives a family of  $(\Gamma(f^{-1}(U_i); \mathcal{O}_X), \Gamma(f^{-1}(U_j); \mathcal{O}_X))$ -bimodules associated to any affine open covering of  $Y$ . Since the coefficient rings are commutative, we may reinterpret this as a  $\Gamma(f^{-1}(U_i); \mathcal{O}_X) \otimes_{\mathcal{O}_S} \Gamma(f^{-1}(U_j); \mathcal{O}_X)$ -module structure, and then observe that

$$\Gamma(f^{-1}(U_i); \mathcal{O}_X) \otimes_{\mathcal{O}_S} \Gamma(f^{-1}(U_j); \mathcal{O}_X) \cong \Gamma(f^{-1}(U_i) \times_S f^{-1}(U_j); \mathcal{O}_{X \times_S X}). \quad (5.4)$$

The open subsets  $f^{-1}(U_i) \times_S f^{-1}(U_j)$  cover  $X \times_S X$ , and their intersections have the same form, making it easy to see that one has natural and compatible restriction maps. It follows that these module structures glue together to give a sheaf on  $X \times_S X$ . Moreover, the sheaf is supported on the preimage in  $X \times_S X$  of the diagonal in  $Y \times_S Y$ , and thus satisfies the requisite finiteness condition to be a sheaf bimodule.

It remains to see that this is a sheaf algebra. We note that for any open subset of  $Y$ , the pair of open subsets  $(f^{-1}(U), f^{-1}(U))$  satisfies the hypotheses of Proposition 5.1 for any coherent subsheaf of  $\mathcal{A}_X$  on either side, from which it easily follows that the morphisms  $f_*\mathcal{O}_X \rightarrow \mathcal{A}$  and  $\mathcal{A} \otimes_{f_*\mathcal{O}_X} \mathcal{A} \rightarrow \mathcal{A}$  induce a sheaf algebra structure on  $\mathcal{A}_X$ .  $\square$

The construction of a sheaf algebra from a sheaf of algebras suggests defining an “invariant open subset” of a sheaf algebra, as an open subset  $U$  such that the two preimages of  $U$  in the support of any coherent subsheaf of the sheaf algebra agree. If  $U$  is an affine open subset which is invariant for a given sheaf algebra  $\mathcal{A}$ , we immediately conclude that the sections on  $U \times U$  of  $\mathcal{A}$  form an algebra equipped with a morphism from  $\Gamma(U; \mathcal{O}_U)$ . The difficulty, of course, is that a typical sheaf algebra will have no invariant affine opens. Luckily, the usual construction of a sheaf by gluing depends far more on the subsets being affine than that they be open. Define an *affine localization* of  $X$  to be a nonempty affine scheme which is the directed limit of a (possibly infinite) family of open subschemes of  $X$ . As with affine opens, an affine localization is determined by its image in the underlying topological space of  $X$ , and if given a family of such subsets covering  $X$  (in a locally finite way: every open subset of  $X$  needs to be contained in a finite union of localizations), the corresponding family of morphisms will be faithfully flat. It then follows by fpqc descent that we may specify a sheaf (or morphisms of sheaves) using a covering by affine localizations in place of a covering by affine opens. One similarly finds that there is a well-behaved notion of “invariant” affine localization, and the restriction of a sheaf algebra to an invariant affine localization is an algebra.

There is still a difficulty here, in that there is no guarantee that there will always be a locally finite covering by invariant affine localizations. For instance, the only invariant affine localization of the sheaf algebra  $\overline{k}(\mathbb{P}^1)[\mathrm{PGL}_2(\overline{k})]$  on  $\mathbb{P}^1_{\overline{k}}$  is the field  $\overline{k}(\mathbb{P}^1)$  itself. If there were an fpqc base change  $S' \rightarrow S$  such that the pullback to  $(X \times_S S') \otimes_{S'} (X \times_S S')$  was covered by invariant localizations, then we could work with those localizations to understand the algebra structure and then use a further application of fpqc descent to recover the morphisms on  $X \times_S X$ . Of course, rather than make two separate applications of fpqc descent, we could simply observe that  $U_i \otimes_{S'} U_j \rightarrow X \times_S X$  give an fpqc covering and do the descent directly. But this tells us that there was no need for the  $U_i$  to cover  $X \times_S S'$ ; all we need is for them to cover  $X$ !

Given a scheme  $S$ , let  $\hat{\mathbb{A}}_S^1$  denote the localization of  $\mathbb{A}_S^1$  obtained as the filtered limit of those open embeddings such that the image is dense in every geometric fiber (equivalently, such that the

image contains the generic point of every fiber). Although the map  $\hat{\mathbb{A}}_S^1 \rightarrow \mathbb{A}_S^1$  is not an fpqc cover, the composition  $\hat{\mathbb{A}}_S^1 \rightarrow S$  is both fpqc and surjective, and thus an fpqc cover.

This construction is functorial in  $S$  and if  $S \rightarrow T$  is a finite morphism with  $T$  Noetherian, then  $\hat{\mathbb{A}}_S^1 \cong \hat{\mathbb{A}}_T^1 \times_T S$ . Note, however, that this construction does not respect open embeddings, so the obvious way to associate a sheaf to this construction does not produce a quasicoherent sheaf.

We observe that if  $X$  is projective over a Noetherian ring  $R$ , then  $\hat{\mathbb{A}}_X^1$  is affine over  $R$ . Since the construction respects closed embeddings, it suffices to consider the case  $X = \mathbb{P}_R^n$ . In that case, we note that the section  $\sum_i t^i x_i$  of the pullback of  $\mathcal{O}_X(1)$  is invertible, and thus the trivial line bundle is very ample on  $\hat{\mathbb{A}}_X^1$ , making it affine. Since this holds for any embedding of  $X$  in projective space, it follows that any very ample line bundle on  $X$  becomes trivial on  $\hat{\mathbb{A}}_X^1$ , and thus (since very ample bundles generate the Picard group) that any line bundle on  $X$  becomes trivial on  $\hat{\mathbb{A}}_X^1$ .

**Proposition 5.3.** *Suppose  $X$  and  $Y$  are projective over the Noetherian affine scheme  $S$ , and let  $M$  be a quasicoherent sheaf bimodule on  $X \times_S Y$ . Let  $M'$  be the base change of  $M$  to  $\hat{\mathbb{A}}_S^1$ . Then the fiber products of  $\hat{\mathbb{A}}_X^1$  and  $\hat{\mathbb{A}}_Y^1$  with the support of any coherent subsheaf of  $M'$  are canonically isomorphic, and the affine localization  $\hat{\mathbb{A}}_X^1 \times_{\hat{\mathbb{A}}_S^1} \hat{\mathbb{A}}_Y^1$  of the base change is an fpqc covering of  $X \times_S Y$ .*

*Proof.* The only thing to observe is that the support of any coherent subsheaf of  $M'$  is contained in the base change of the support of a coherent subsheaf of  $M$ , and thus the claim reduces to the fact that the construction  $\hat{\mathbb{A}}^1$  respects finite morphisms.  $\square$

*Remark.* In particular, given bimodules on  $X \times_S Y$  and  $Y \times_S Z$ , we can use the corresponding bimodules on  $\hat{\mathbb{A}}_X^1 \times_{\hat{\mathbb{A}}_S^1} \hat{\mathbb{A}}_Y^1$  and  $\hat{\mathbb{A}}_Y^1 \times_{\hat{\mathbb{A}}_S^1} \hat{\mathbb{A}}_Z^1$  to control the tensor product.

Thus when  $X$  is projective, or more generally when  $\hat{\mathbb{A}}_X^1$  is affine, we always have the option to replace the sheaf algebra with the actual algebra of global sections of  $M$  on  $\hat{\mathbb{A}}_X^1 \times_{\hat{\mathbb{A}}_S^1} \hat{\mathbb{A}}_X^1$ , and descent essentially reduces to checking that the algebra has a description which is independent of the auxiliary coordinate.

The reader should note that we will only be using this construction to give some alternate descriptions

According to Proposition 5.2, the algebras  $\mathcal{H}_G(X)$ ,  $\mathcal{H}_W(X)$ ,  $\mathcal{H}_{W;T}(X)$  from the previous section may all be interpreted as sheaf algebras on  $X/S$ , as can the twisted group algebras  $k(X)[G]$ ,  $\mathcal{O}_X[G]$ . The latter are quite easy to generalize to the infinite case.

**Proposition 5.4.** *[31, Lem. 2.8] Let  $g$  be an automorphism of  $X$ . Then for any quasicoherent sheaf  $M$  on  $X$ ,  $(1, g^{-1})_* M$  is a sheaf bimodule, and for  $h \in \text{Aut}(X)$  and  $N \in \text{coh}(X)$ , we have a natural isomorphism*

$$(1, g^{-1})_* M \otimes_X (1, h^{-1})_* N \rightarrow (1, (gh)^{-1})_* (M \otimes^g N). \quad (5.5)$$

*Proof.* Since  $(1, g^{-1})_* M$  is supported on the graph of an automorphism, the same applies to any coherent subsheaf, and thus it satisfies the requisite finiteness condition to be a sheaf bimodule.

Now, let  $U$  be any affine open subset of  $X$ . Then  $(U, g^{-1}(U))$  satisfy the hypotheses of Proposition 5.1, and thus we have

$$\begin{aligned} & \Gamma(U \times X; (1, g^{-1})_* M \otimes_X (1, h^{-1})_* N) \\ & \cong \Gamma(U \times g^{-1}(U); (1, g^{-1})_* M) \otimes_{\Gamma(g^{-1}(U); \mathcal{O}_X)} \Gamma(g^{-1}(U) \times X; (1, h^{-1})_* N) \\ & \cong \Gamma(U; M) \otimes_{\Gamma(g^{-1}(U); \mathcal{O}_X)} \Gamma(g^{-1}(U); N), \end{aligned}$$

where  $f \in \Gamma(g^{-1}(U); \mathcal{O}_X)$  acts on  $\Gamma(U; M)$  as multiplication by  ${}^g f$ . With this action, there is a natural isomorphism

$$\Gamma(U; M) \otimes_{\Gamma(g^{-1}(U); \mathcal{O}_X)} \Gamma(g^{-1}(U); N) \cong \Gamma(U; M) \otimes_{\Gamma(U; \mathcal{O}_X)} \Gamma(U; {}^g N) \quad (5.6)$$

given by  $m \otimes n \mapsto m \otimes {}^g n$ , and thus the result follows.  $\square$

**Definition 5.1.** Let  $X/S$  be an Noetherian  $S$ -scheme of finite type with integral geometric fibers, and let  $G$  be a finitely generated group of automorphisms of  $X$ . Then the “twisted group sheaf algebra”  $k(X)[G]$  is the sheaf

$$\bigoplus_{g \in G} (1, g^{-1})_* k(X) \quad (5.7)$$

on  $X \times X$  (with  $k(X)$  denoting the sheaf of meromorphic functions on  $X$  which are defined on the generic point of every geometric fiber of  $X$  over  $S$ ) with sheaf algebra structure induced by the natural morphisms

$$(1, g^{-1})_* k(X) \otimes_X (1, h^{-1})_* k(X) \rightarrow (1, (gh)^{-1})_* k(X) \quad (5.8)$$

coming from the Proposition.

*Remark.* We could also define this using an affine localization. The affine scheme  $\hat{\mathbb{A}}_X^1$  constructed in Proposition 5.3 is functorial for  $\text{Aut}_S(X)$ , and thus has an induced action of  $G$ . We may thus define a twisted group algebra  $k(X) \otimes_S \mathcal{O}_{\hat{\mathbb{A}}_X^1}[G]$ , and this has a natural associated descent datum. The only nontrivial thing to verify is the fact that cyclic bimodules are finitely generated on both sides, but this follows easily from the fact that  $\hat{\mathbb{A}}_X^1$  is Noetherian: the bimodule  $\mathcal{O}_{\hat{\mathbb{A}}_X^1} c_g g \mathcal{O}_{\hat{\mathbb{A}}_X^1}$  is cyclic on both sides, and any other cyclic bimodule is contained in a finite sum of such bimodules, so is finitely generated on both sides.

We may similarly define  $\mathcal{O}_X[G]$  to be the sheaf subalgebra  $\bigoplus_{g \in G} (1, g^{-1})_* \mathcal{O}_X$ , which we readily verify to contain the image of the identity and be preserved by the multiplication map.

Moreover, we have the following.

**Proposition 5.5.** *Let  $g_1, \dots, g_n, h_1, \dots, h_m$  be two finite sets of automorphisms of  $X$ , and let  $\mathcal{M}_{g_1, \dots, g_n}, \mathcal{M}_{h_1, \dots, h_m}, \mathcal{M}_{g_1 h_1, \dots, g_n h_m}$  be as defined in Lemma 4.2 and interpreted as sheaf subbimodules of  $k(X)[G]$ . Then the multiplication on  $k(X)[G]$  restricts to a morphism*

$$\mathcal{M}_{g_1, \dots, g_n} \otimes \mathcal{M}_{h_1, \dots, h_m} \rightarrow \mathcal{M}_{g_1 h_1, \dots, g_n h_m}. \quad (5.9)$$

*Proof.* Given an open subset  $V \subset X$ , we may associate an open subset  $U_V = \bigcap_i g_i(V)$ , and we claim that there is an affine open covering  $V_i$  of  $X$  such that  $U_{V_i}$  also covers  $X$ . Indeed,  $x \in U_V$  iff  $g_1^{-1}(x), \dots, g_n^{-1}(x) \in V$ , and thus if  $V_x$  is an affine open neighborhood of this set of points, we have  $x \in U_{V_x}$ . It thus suffices to specify how the above morphism acts on local sections on sets of the form  $U_V \times X$ , for which we note

$$\Gamma(U_V \times X; \mathcal{M}_{g_1, \dots, g_n} \otimes_X \mathcal{M}_{h_1, \dots, h_m}) \cong \Gamma(U_V \times X; \mathcal{M}_{g_1, \dots, g_n}) \otimes_{\Gamma(V; \mathcal{O}_V)} \Gamma(V \times X; \mathcal{M}_{h_1, \dots, h_m}). \quad (5.10)$$

But the result then follows immediately from the definition of  $\mathcal{M}_{\vec{g}}$  as the space of operators preserving holomorphy.  $\square$

*Remark.* When  $X/S$  is projective, or more generally when we have nice affine localizations as constructed above, then we could define  $\mathcal{M}_{\vec{g}}$  much more simply as the subsheaf of  $k(X)[G]$  that preserves the subring  $\mathcal{O}_{\hat{\mathbb{A}}_X^1}$ .

In particular, if  $G$  is any subgroup of the group of automorphisms of  $X/S$ , we may define a sheaf algebra  $\mathcal{H}_G^+(X)$  as the union in  $k(X)[G]$  of the sheaves  $\mathcal{M}_{\bar{g}}$  associated to finite subsets of  $G$ . More generally, we will wish to only allow poles on a proper (but  $G$ -invariant) subset of the reflection hypersurfaces, as otherwise in the affine Weyl group case we could acquire poles along the fibers where the “ $q$ ” parameter is torsion.

Suppose now that  $X/S$  is a family of abelian varieties and that  $(W, S)$  is a Coxeter group (of finite rank, but possibly infinite) equipped with an action on  $X$  of coroot type. Then we define  $\mathcal{H}_W(X)$  to be the sheaf subalgebra of  $\mathcal{H}_W^+(X)$  consisting of operators which are holomorphic away from the reflection hypersurfaces corresponding to conjugates of the simple reflections. Clearly, this agrees with our previous notation, in that when  $W$  is finite,  $\mathcal{H}_W(X)$  is the sheaf algebra on  $X/S$  associated to the sheaf of algebras on  $X/W$  we previously denoted by  $\mathcal{H}_W(X)$ .

Since  $W$  still has a Bruhat order, we may consider the subsheaf  $\mathcal{H}_W(X)[I]$  for any order ideal  $I \subset W$ , and if the order ideal is finite, the result will be of the form  $\mathcal{M}_I(X)$  and thus coherent. In fact, we have the following, by precisely the same argument as Lemma 4.9 and its corollaries.

**Proposition 5.6.** *If  $I$  is a finite order ideal in  $W$  and  $w \in I$  is a maximal element, then there is a short exact sequence*

$$0 \rightarrow \mathcal{H}_W(X)[I \setminus \{w\}] \rightarrow \mathcal{H}_W(X)[I] \rightarrow (1, w^{-1})_* \mathcal{O}(D_w) \rightarrow 0 \quad (5.11)$$

of sheaf bimodules, where  $D_w = \sum_{r \in R(W), rw < w} [X^r]$ .

**Corollary 5.7.** *For any reduced word  $w = s_1 \cdots s_n$ , the multiplication map*

$$\mathcal{H}_{\langle s_1 \rangle}(X) \otimes_X \cdots \otimes_X \mathcal{H}_{\langle s_n \rangle}(X) \rightarrow \mathcal{H}_W(X)[\leq w] \quad (5.12)$$

is surjective.

**Corollary 5.8.** *The construction  $\mathcal{H}_W(X)$  respects base change.*

Since any finite subset of  $W$  is contained in a finite order ideal, we also obtain the following.

**Corollary 5.9.** *The sheaf algebra  $\mathcal{H}_W(X)$  is the sheaf subalgebra of  $k(X)[W]$  generated by the sheaf subalgebras  $\mathcal{H}_{\langle s \rangle}(X)$  for  $s \in S$ .*

Since the action of  $W$  on  $X$  is of coroot type, we still have a well-defined association of coroot morphisms to the roots of  $X$ , respecting positivity, and thus the notion of a system of parameters carries over.

**Definition 5.2.** The (untwisted) Hecke algebra  $\mathcal{H}_{W; \bar{T}}(X)$  is the sheaf subalgebra of  $\mathcal{H}_W(X)$  generated by the rank 1 sheaf algebras  $\mathcal{H}_{\langle s_i \rangle, T_i}(X)$ .

Lemma 4.18 and Corollary 4.19 again carry over immediately, as does the fact that this construction respects base change. However, the description of the adjoint and the description of Proposition 4.22 both founder on the fact that they involve a sum over all positive roots. To deal with this, we will need to generalize the construction further.

We first note that if we are given an equivariant gerbe  $\mathcal{Z}$  (including, of course, the explicit isomorphisms  $\zeta_{gh} : \mathcal{Z}_g \otimes^g \mathcal{Z}_h \cong \mathcal{Z}_{gh}$ ), then as in the finite case, there is a corresponding crossed product algebra: take the sheaf bimodule

$$\bigoplus_g (1, g^{-1})_* \mathcal{Z}_g \quad (5.13)$$

with multiplication induced by  $\phi$ . Of course, this also gives a twisted version of  $k(X)[G]$  by replacing  $\mathcal{Z}_g$  by its sheaf of meromorphic sections. Note that if  $\mathcal{Z}_g$  and  $\mathcal{Z}'_g$  are meromorphically equivalent (i.e., there is a system of nonzero meromorphic maps between the line bundles which are compatible with the maps  $\zeta$ ), then this induces an isomorphism between the corresponding crossed product algebras. In particular, the meromorphic crossed product algebra associated to a given equivariant gerbe is isomorphic to the usual twisted group algebra iff there is a consistent family of meromorphic sections of  $\mathcal{Z}_g$ , iff the equivariant gerbe has the form  $\mathcal{Z}_g = \mathcal{O}(Z_g)$  where  $Z_g$  is a cocycle valued in Cartier divisors. (More generally, if one chooses arbitrary meromorphic sections, one obtains a meromorphic equivariant gerbe in which the line bundles are trivial but the maps  $\zeta_{gh}$  are only meromorphic, giving a class in  $Z^2(G; k(X)^*)$  and making the algebra a crossed product algebra in the usual sense.)

As we have already mentioned, this sort of twisting is too general to allow us to define analogues of  $\mathcal{H}_G^+(X)$ ; for instance, given any homomorphism  $G \rightarrow \Gamma(S; \mathcal{O}_S^*)$ , there is an automorphism of  $k(X)[G]$  that takes  $g$  to  $\chi(g)g$ , and this automorphism will not preserve  $\mathcal{H}_G^+(X)$  if  $\ker \chi$  does not contain every reflection. We avoid the question of determining precisely what data is needed to specify a twist (apart from noting that it should consist of consistent trivializations of the equivariant gerbe data over the complete local rings associated to reflection hypersurfaces), and instead concentrate solely on a particularly important class of twists in the Hecke algebra setting.

For Hecke algebras, in principle the only thing we need in addition to an equivariant gerbe structure is data specifying the rank 1 subalgebras. In general, of course, there would be no reason to expect that the algebras so generated will be well-behaved. Since the relations in the Coxeter group only come from finite rank 2 subgroups, and we understand the corresponding algebras, this suggests that what we need is a collection of twists in rank 1 that are compatible in (finite) rank 2.

With this in mind, fix a line bundle  $\gamma_i$  for each  $i$ , subject to the condition that if  $\langle s_i, s_j \rangle$  is finite, then there is a line bundle  $\gamma_{ij}$  such that  $\gamma_i \otimes \gamma_{ij}^{-1}$  is the pullback of a line bundle on  $X/\langle s_i \rangle$  and  $\gamma_j \otimes \gamma_{ij}^{-1}$  is the pullback of a line bundle on  $X/\langle s_j \rangle$ . This gives rise to an equivariant gerbe structure as follows. First, we take  $\mathcal{Z}_{s_i} := \gamma_i \otimes^{s_i} \gamma_i^{-1}$ , at which point every reduced word gives rise to a corresponding line bundle:

$$\mathcal{Z}_{s_1} \otimes^{s_1} \mathcal{Z}_{s_2} \otimes^{s_1 s_2} \mathcal{Z}_{s_3} \otimes \dots \otimes^{s_1 \dots s_{m-1}} \mathcal{Z}_{s_m}. \quad (5.14)$$

Since  $\gamma_i \otimes \gamma_{ij}^{-1}$  has a unique  $\langle s_i \rangle$ -equivariant structure that descends to the quotient, we have a canonical isomorphism  $\mathcal{Z}_{s_i} \cong \gamma_{ij} \otimes^{s_i} \gamma_{ij}^{-1}$ , which in turn induces an isomorphism between the line bundles corresponding to the two sides of the braid relation. Moreover, the restriction of  $\mathcal{Z}_{s_i}$  to  $X^{(s_i)}$  is naturally isomorphic to the trivial bundle, and thus the same applies to its restriction to the (nonempty and proper over  $S$ ) subscheme  $X^W$ , and the isomorphism corresponding to the braid relation is compatible with these trivializations. Thus any composition of such isomorphisms is also compatible with the trivializations, implying compatibility in general.

The resulting equivariant gerbe  $\mathcal{Z}_\gamma$  induces a twisted version of  $k(X)[W]$  which we denote by  $k(X)[W]_\gamma$ . This contains natural sheaf subalgebras corresponding to the simple reflections, namely

$$\gamma_i \otimes \mathcal{H}_{\langle s_i, T_i \rangle}(X) \otimes \gamma_i^{-1}, \quad (5.15)$$

and we may thus define  $\mathcal{H}_{W; \vec{T}; \gamma}(X)$  to be the sheaf subalgebra generated by these rank 1 subalgebras. The usual arguments do not *quite* carry over to the twisted case, as we no longer have the description as holomorphy preserving maps. This turns out not to be a terribly significant issue, however.

**Lemma 5.10.** *Let  $w \in W$  be given by the reduced word  $w = s_1 \cdots s_l$ . Then the multiplication map*

$$\mathcal{H}_{\langle s_1 \rangle, \vec{T}; \gamma}(X) \otimes_X \cdots \otimes_X \mathcal{H}_{\langle s_l \rangle, \vec{T}; \gamma}(X) \rightarrow \mathcal{H}_{W; \vec{T}; \gamma}(X)[\leq w] \quad (5.16)$$

*is surjective.*

*Proof.* We first note that if  $W$  is finite, then this follows from the fact that we may describe  $\mathcal{H}_{W; \vec{T}; \gamma}(X)$  as a twist by a line bundle of  $\mathcal{H}_{W; \vec{T}}(X)$ . Now, consider a general product map

$$\mathcal{H}_{\langle s'_1 \rangle, \vec{T}; \gamma}(X) \otimes_X \cdots \otimes_X \mathcal{H}_{\langle s'_l \rangle, \vec{T}; \gamma}(X) \rightarrow \mathcal{H}_{W; \vec{T}; \gamma}(X), \quad (5.17)$$

where  $s'_1 \cdots s'_l$  may not even be a reduced word. If  $s'_i = s'_{i+1}$  for some  $i$ , then we may use the fact that the rank 1 subbimodules are subalgebras to obtain a product map with the same image and fewer reflections. Next, suppose there is a subword of the form  $s'_i s'_j \cdots$  giving one side of the standard braid relation. The image of the corresponding tensor product of rank 1 algebras is contained in the corresponding rank 2 algebra, and finiteness of the rank 2 subalgebra ensures that the map is surjective. The same applies to the other side of the braid relation, and thus we may apply any braid relation to our word without changing the image of the tensor product.

We may thus conclude first that the product map corresponding to a nonreduced word has the same image as the product map corresponding to a reduced word, and second that the image of the product map corresponding to a reduced word depends only on the corresponding element of  $W$ . In particular, it follows that  $\mathcal{H}_{W; \vec{T}; \gamma}(X)$  is spanned by the images of product maps corresponding to reduced words.

Now, given a finite Bruhat order ideal  $I$ , choose reduced words for the maximal elements of  $I$  and consider the submodule  $M_I$  of  $\mathcal{H}_{W; \vec{T}; \gamma}(X)[I]$  spanned by the corresponding products; by the previous paragraph, this is independent of the choices of reduced words. To show that this is all of  $\mathcal{H}_{W; \vec{T}; \gamma}(X)[I]$  for all  $I$ , it will suffice to show that

$$M_I \cap \mathcal{H}_{W; \vec{T}; \gamma}(X)[I'] = M_{I'} \quad (5.18)$$

for any  $I' \subset I$ ; indeed, any local section  $\sum_w c_w w \in \mathcal{H}_{W; \vec{T}; \gamma}(X)$  is contained in *some* module of the form  $M_{I'}$ , and the claim would show that it was contained in  $M_{I'}$  where  $I'$  is the minimal order ideal containing the support of  $\sum_w c_w w$ .

Of course, we may as well assume that  $I' = I \setminus \{w\}$  for some maximal element  $w \in I$ , and since then only one of the products can produce nonzero coefficients of  $w$ , we reduce to the case  $I = [\leq w]$ . If  $w = s_1 \cdots s_l$  is a reduced word for  $w$ , then the sheaf bimodule

$$\mathcal{H}_{\langle s_1 \rangle, \vec{T}; \gamma}(X) \otimes_X \cdots \otimes_X \mathcal{H}_{\langle s_l \rangle, \vec{T}; \gamma}(X) \quad (5.19)$$

has a natural subbimodule induced by the Bruhat filtrations of the rank 1 algebras, namely the subbimodule generated by the tensor products in which one of the factors is omitted. The image of the subbimodule is supported on the order ideal  $[< w] := [\leq w] \setminus \{w\}$ , and taking the leading coefficients of the product gives an isomorphism on the quotient bimodule, and thus an element of the image has vanishing leading coefficient iff it is in the image of the subbimodule. Every maximal element of  $[< w]$  has a reduced word obtained by omitting one reflection from  $s_1 \cdots s_l$ , and thus the resulting span is precisely  $M_{[< w]}$ , as required.  $\square$

**Corollary 5.11.** *If  $I$  is a finite order ideal in  $W$  and  $w \in I$  is a maximal element, then there is a short exact sequence*

$$0 \rightarrow \mathcal{H}_{W; \vec{T}; \gamma}(X)[I \setminus \{w\}] \rightarrow \mathcal{H}_{W; \vec{T}; \gamma}(X)[I] \rightarrow (1, w^{-1})_* (\mathcal{Z}_{\gamma, w} \otimes \mathcal{O}(D_w(\vec{T}))) \rightarrow 0 \quad (5.20)$$

*of sheaf bimodules.*

**Corollary 5.12.** *The construction of the sheaf algebra  $\mathcal{H}_{W;\vec{T};\gamma}(X)$  respects base change.*

We should note, of course, that the data  $\gamma_i$  is more than we really need to specify the algebras. The first thing to note is that the construction is functorial under isomorphisms of  $\gamma$  in a very strong way: if  $\gamma_i \cong \gamma'_i$  for each  $i$ , then not only is there an induced isomorphism  $\mathcal{H}_{W;\vec{T};\gamma}(X) \cong \mathcal{H}_{W;\vec{T};\gamma'}(X)$ , but this isomorphism is independent of the choice of isomorphisms  $\gamma_i \cong \gamma'_i$ . In particular, this means that we could take the  $\gamma_i$  to be sections of the relative Picard scheme instead of actual line bundles. Tensoring  $\gamma_i$  by the pullback of a line bundle on  $X/\langle s_i \rangle$  also has no effect on the algebra. Thus what we truly are specifying is a point of the Picard scheme of the generic fiber of the corresponding (geometrically) hyperelliptic curve, modulo the class in  $\text{Pic}^2$  induced by the hyperelliptic involution.

Next, suppose that  $D_1, \dots, D_n$  are Cartier divisors such that  $Z_{s_i} = D_i - s_i D_i$  extends to a cocycle valued in Cartier divisors, and consider the line bundles  $\gamma_i = \mathcal{O}(D_i)$ . Since the action of  $W$  on the group of Cartier divisors is a permutation module, its restriction to any finite subgroup is induced from a trivial module, so has trivial  $H^1$ . In particular  $Z|_{\langle s_i, s_j \rangle}$  is a coboundary of some  $D_{ij}$  for any finite rank 2 parabolic subgroup. If  $D_i - D_{ij}$  has even valuation along any (separable) component of  $[X^{s_i}]$ , and similarly for  $D_j - D_{ij}$ , then we may take  $\gamma_{ij} = \mathcal{O}(D_{ij})$ . Note that since  $D_{ij}$  is only determined up to  $\langle s_i, s_j \rangle$ -invariant divisors, we can change its parity along each orbit of  $\langle s_i, s_j \rangle$ -reflection hypersurfaces independently, and thus if  $s_i$  and  $s_j$  are not conjugate, this condition can always be satisfied, and otherwise reduces to a single parity constraint.

Since the conditions on twisting data are preserved under tensor product, we can always twist by such a family, to obtain a new twisting datum denoted  $\gamma(\vec{D})$ . Since the  $Z_{s_i}$  extends to a cocycle valued in Cartier divisors, the resulting equivariant gerbe comes with a natural meromorphic equivalence to the original equivariant gerbe, and thus we have an induced isomorphism  $k(X)[W]_\gamma \cong k(X)_{\gamma(\vec{D})}$  for any  $\gamma$ .

To understand such isomorphisms more generally, we will need to understand cocycles valued in Cartier divisors. The fact that  $\text{Hom}(W, \mathbb{Z}) = 0$  implies that any *coinduced* module for  $W$  has trivial  $H^1$ . Since Cartier divisors are a sum of *induced* modules, there can be (and are) cocycles valued in Cartier divisors which are not coboundaries. However, since the induced modules are contained in the corresponding coinduced modules, we can always express such a cocycle as a coboundary in the larger module (of integer-valued functions on the set of irreducible Cartier divisors). Note that since the typical element of a coinduced module will not have coboundary in the induced submodule, we need to add the condition that any element of  $w$  only changes finitely many values of the function; naturally, it suffices to verify the condition for the simple reflections.

For instance, if we interpret  $\sum_{\alpha \in \Phi^+(W)} T_\alpha$  as giving an integer-valued function on irreducible Cartier divisors (i.e., the sum over  $\alpha \in \Phi^+(W)$  of the valuation of  $T_\alpha$  along the given divisor), then any element of  $W$  only changes finitely many values of the function, and thus we obtain a well-defined coboundary  $\sum_{\alpha \in \Phi^+(W) \cap w\Phi^-(W)} (T_\alpha - T_{w\alpha})$ .

We may also use such formal sums to define (meromorphically trivial) twisting data; if  $\Gamma$  is an integer-valued function on irreducible Cartier divisors such that  $\Gamma - s_i \Gamma$  has finite support, then we may obtain a divisor  $D_i$  with the same coboundary on  $\langle s_i \rangle$  by restricting  $\Gamma$  to the union of the support of  $\Gamma - s_i \Gamma$  and the components of the reflection hyperplanes. Similarly, if  $\langle s_i, s_j \rangle$  is finite, then we may obtain a divisor  $D_{ij}$  by restricting  $\Gamma$  to the union of the supports of  $\Gamma - w\Gamma$  for  $w \in \langle s_i, s_j \rangle$ , and find that  $D_{ij} - D_i$  and  $D_{ij} - D_j$  are pullbacks, so that  $\gamma_i = \mathcal{O}(D_i)$  gives a well-defined twisting datum. We denote the twist of some other  $\gamma$  by this meromorphically trivial datum by  $\gamma(\Gamma)$ .

We then introduce the notation

$$\mathcal{O}(\Gamma) \otimes \mathcal{H}_{W;\vec{T};\gamma}(X) \otimes \mathcal{O}(-\Gamma) \tag{5.21}$$

for  $\mathcal{H}_{W;\vec{T};\gamma}(X)$  viewed as a subalgebra of  $k(X)[W]_\gamma$ . Note that if  $\Gamma' - \Gamma$  is a Cartier divisor, then

$$\mathcal{O}(\Gamma') \otimes \mathcal{H}_{W;\vec{T};\gamma}(X) \otimes \mathcal{O}(-\Gamma') \cong \mathcal{O}(\Gamma' - \Gamma) \otimes (\mathcal{O}(\Gamma) \otimes \mathcal{H}_{W;\vec{T};\gamma}(X) \otimes \mathcal{O}(-\Gamma)) \otimes \mathcal{O}(\Gamma - \Gamma') \quad (5.22)$$

where the outer twist on the right is the usual twist by a line bundle. We may also define a sheaf  $\mathcal{O}(\Gamma') \otimes \mathcal{H}_{W;\vec{T};\gamma}(X) \otimes \mathcal{O}(-\Gamma)$  in this case by  $\mathcal{O}(\Gamma' - \Gamma) \otimes (\mathcal{O}(\Gamma) \otimes \mathcal{H}_{W;\vec{T};\gamma}(X) \otimes \mathcal{O}(-\Gamma))$ .

**Proposition 5.13.** *Let  $\vec{T}, \vec{T}'$  be two systems of parameters for  $W$  on  $X$ . Then*

$$\mathcal{H}_{W;\vec{T}+\vec{T}';\gamma}(X) = \mathcal{O}\left(-\sum_{\alpha \in \Phi^+(W)} T'_\alpha\right) \otimes \mathcal{H}_{W;\vec{T}+\vec{T}';\gamma}(X) \otimes \mathcal{O}\left(\sum_{\alpha \in \Phi^+(W)} T'_\alpha\right) \quad (5.23)$$

as subalgebras of  $k(X)[W]_\gamma$ .

*Proof.* This reduces immediately to the corresponding claim in the rank 1 case, where (after removing the common twist by  $\gamma$ ) it reads

$$\mathcal{H}_{A_1, T+T'}(C) = \mathcal{O}(-T') \otimes \mathcal{H}_{A_1, T+{}^sT'}(C) \otimes \mathcal{O}(T'). \quad (5.24)$$

For general parameters (such that no two of  $T, {}^sT, T', {}^sT'$  have a common component), this is straightforward: it is easy to see that  $\mathcal{H}_{A_1, T+T'}(C)$  preserves the subsheaf  $\mathcal{O}(-T')$ , and  $\mathcal{H}_{A_1, T+{}^sT'}(C)$  preserves the subsheaf  $\mathcal{O}(-{}^sT')$ , and this gives both inclusions.  $\square$

The proof of Proposition 4.22 carries over to give the following.

**Proposition 5.14.** *Suppose that  $T_\alpha$  and  $T_{-\alpha}$  have no common component for any  $\alpha \in \Phi(W)$ . Then*

$$\mathcal{H}_{W;\vec{T};\gamma}(X) = \mathcal{H}_{W;0;\gamma}(X) \cap \mathcal{O}\left(-\sum_{\alpha \in \Phi^+(W)} T_\alpha\right) \otimes \mathcal{H}_{W;0;\gamma}(X) \otimes \mathcal{O}\left(\sum_{\alpha \in \Phi^+(W)} T_\alpha\right). \quad (5.25)$$

We also note the following fact, which allows us to decouple the conditions associated to different parameters.

**Proposition 5.15.** *Suppose that  $\vec{T}$  and  $\vec{T}'$  are such that  $T_\alpha$  and  $T'_\alpha$  have no common component for any  $\alpha$ . Then*

$$\mathcal{H}_{W;\vec{T}+\vec{T}';\gamma}(X) = \mathcal{H}_{W;\vec{T};\gamma}(X) \cap \mathcal{H}_{W;\vec{T}';\gamma}(X). \quad (5.26)$$

*Proof.* The rank 1 subalgebras on the left are contained in the corresponding subalgebras on the right, so algebra on the left is certainly contained in the intersection on the right. To see equality, we use the Bruhat filtration and observe that each subquotient on the left is the intersection of the corresponding subquotients on the right.  $\square$

*Remark.* This easily gives a version of Proposition 5.14 in which  $\mathcal{H}_{W;\vec{T}+\vec{T}';\gamma}(X)$  is given as an intersection of two versions of  $\mathcal{H}_{W;\vec{T};\gamma}(X)$ .

The construction of the adjoint in the finite case carries over. Note that the naïve adjoint  $\sum_w c_w w \mapsto \sum_w w c_w$  induces a natural isomorphism  $k(X)[W]_\gamma^{\text{op}} \cong k(X)[W]_{\gamma^{-1}}$ . (In terms of the sheaf algebra itself, all we are doing is swapping the two factors of  $X \times_S X$ .) To describe how this acts on the Hecke algebras, it will be helpful to denote the formal sum  $\sum_{\alpha \in \Phi^+(W)} ([X^{r_\alpha}] - \vec{T})$  by  $D_{w_0}(\vec{T})$ , and similarly for  $D_{w_0}$ . This of course agrees with the usual notation whenever the longest element  $w_0 \in W$  actually exists.

**Proposition 5.16.** *The naïve adjoint on  $k(X)[W]_\gamma$  induces an isomorphism*

$$\mathcal{H}_{W;\vec{T};\gamma}(X)^{op} \cong \mathcal{O}(D_{w_0}(\vec{T})) \otimes \mathcal{H}_{W;\vec{T};\gamma^{-1}}(X) \otimes \mathcal{O}(-D_{w_0}(\vec{T})) \cong \mathcal{O}(D_{w_0}) \otimes \mathcal{H}_{W;-\vec{T};\gamma^{-1}}(X) \otimes \mathcal{O}(-D_{w_0}). \quad (5.27)$$

*Proof.* Again, this reduces immediately to the rank 1 case.  $\square$

Diagram automorphisms of course work as well in the infinite case; the only caveat is that unlike in the finite case, the parameters (and twisting) need not be preserved by the diagram automorphism. More generally, if  $H$  is a group of automorphisms of  $X$  acting as diagram automorphisms of  $W$  and preserving the parameters, and there is an equivariant gerbe  $\mathcal{Z}_h$  on  $H$  such that  ${}^h\gamma_i \sim \gamma_i \otimes \mathcal{Z}_h$  for each  $i$ , then the corresponding holomorphic crossed product algebra normalizes the Hecke algebra, and we can combine them into a larger algebra associated to the extended Coxeter Group  $W \rtimes H$ . (In the  $C_n$  case we consider in detail below, we will see that even the requirement that the parameters be invariant can be finessed.)

Suppose  $A$  and  $B$  are sheaf algebras, on  $X/S$  and  $Y/S$  respectively. An  $(A, B)$ -bimodule is then simply a sheaf bimodule  $M$  on  $X \times_S Y$  equipped with multiplication maps  $A \otimes_X M \rightarrow M$ ,  $M \otimes_Y B \rightarrow M$  making the obvious diagrams commute. (Note that the restriction of  $M$  to a compatible pair of localizations is a bimodule over the corresponding restrictions of  $A$  and  $B$ .) The tensor product is then defined in the obvious way, so that we may define induced modules. Restriction is of course also easy to define, though Frobenius reciprocity is somewhat tricky, as there are difficulties with defining Hom on sheaf bimodules in general.

This is not a problem for the analogue of Proposition 4.27; the only change is that  $M$  should be replaced by a suitable bimodule. In the finite case, this is no difficulty: when  $W$  is finite, any  $\mathcal{H}_{W;\vec{T}}(X)$ -module in the usual sense determines a corresponding  $(\mathcal{H}_{W;\vec{T}}(X), \mathcal{O}_{X/W})$ -bimodule structure.

**Proposition 5.17.** *Suppose  $I, J \subset S$  are such that the parabolic subgroups  $W_I, W_J$  are finite. Then for any  $(\mathcal{H}_{W_J;\vec{T};\gamma}(X), \mathcal{O}_Y)$ -bimodule  $M$  and any maximal chain in the Bruhat order on  ${}^I W^J$ , the subquotient corresponding to  $w \in {}^I W^J$  in the resulting filtration of  $\text{Res}_{W_I}^{W;\vec{T};\gamma} \text{Ind}_{W_J}^{W;\vec{T};\gamma} M$  is the  $(\mathcal{H}_{W_I;\vec{T};\gamma}(X), \mathcal{O}_Y)$ -bimodule*

$$\text{Ind}_{W_I(w)}^{W_I;\vec{T};\gamma} \mathcal{Z}_{\gamma,w} \otimes \mathcal{O}(D_w(\vec{T})) \otimes w \text{Res}_{W_J(w)}^{W_J;\vec{T};\gamma} M. \quad (5.28)$$

If  $W_I$  is a finite parabolic subgroup, then the module  $\mathcal{O}_X$  we considered above becomes a  $(\mathcal{H}_{W_I;\vec{T}}(X), \mathcal{O}_{X/W_I})$ -bimodule. To be precise, the corresponding sheaf bimodule the direct image in  $X \times_S X/W_I$  of the structure sheaf of the diagonal in  $X \times_S X$ . Its global sections on  $U \times V$  may then be identified with  $\Gamma(U \cap \pi_I^{-1}(V); \mathcal{O}_X)$ , and if  $U$  is  $W_I$ -invariant, the action is given by the usual action of the algebra  $\Gamma(U \times U; \mathcal{H}_{W_I;\vec{T}}(X))$ .

Since there are difficulties in general with defining Hom functors on sheaf bimodules, we cannot easily define  $M^{W_I}$  as the Hom from the appropriate induced module. Instead, we define  $M^{W_I}$  (for  $\gamma$  trivial on  $W_I$ ) as follows: take the direct image of  $\text{Res}_{W_I}^{W;\vec{T};\gamma} M$  in  $(X/W_I) \times_S Y$ , view this as a family of sheaves on  $X/W_I$ , and apply  $-^{W_I}$  as defined in the sheaf-of-algebras setting. If we also define  $M/W_I$  for a right module by  $M \otimes_{\mathcal{H}_{W_I;\vec{T}}(X)} \mathcal{O}_X$ , then we have the following.

**Lemma 5.18.** *If  $M$  and  $N$  are right-, respectively left-modules over  $\mathcal{H}_{W_I;\vec{T}}(X)$ , then there is a natural morphism*

$$(M/W_I) \otimes_{X/W_I} N^{W_I} \rightarrow M \otimes_{\mathcal{H}_{W_I;\vec{T}}(X)} N. \quad (5.29)$$

In particular, if the twisting datum  $\gamma$  is trivial on  $W_I$ , then there is an evaluation morphism

$$\mathrm{Ind}_{W_I}^{W; \vec{T}; \gamma} \mathcal{O}_X \otimes_{X/W_I} M^{W_I} \rightarrow M. \quad (5.30)$$

*Proof.* By the associativity of tensor product, the first claim reduces to showing that there is a natural morphism  $\mathcal{O}_X \otimes_{X/W_I} N^{W_I} \rightarrow N$ , which in turn comes from the fact  $N^{W_I}$  is a subbimodule of the direct image of  $N$  in  $(X/W_I) \times_S Y$  together with an adjunction between said direct image functor and  $\mathcal{O}_X \otimes_{X/W_I} -$ .

The claim for  $\mathrm{Ind}_{W_I}^{W; \vec{T}; \gamma} \mathcal{O}_X$  follows by

$$\mathrm{Ind}_{W_I}^{W; \vec{T}; \gamma} \mathcal{O}_X \cong \mathcal{H}_{W; \vec{T}; \gamma}(X) \otimes_{\mathcal{H}_{W_I; \vec{T}}} \mathcal{O}_X = \mathcal{H}_{W; \vec{T}; \gamma}(X)/W_I. \quad (5.31)$$

□

*Remark.* In particular, if  $M$  is a sheaf algebra on  $X$  containing  $\mathcal{H}_{W_I; \vec{T}}$ , then the subquotient  $(M/W_I)^{W_I}$  is naturally a sheaf algebra on  $X/W_I$ . (And, of course, this extends to sheaf categories in which each endomorphism sheaf algebra contains a Hecke algebra.)

Since we are defining  $-^{W_I}$  by reduction to the finite case, most of the calculations there carry over. We find that (assuming  $\gamma$  is trivial on  $W_I$  and  $W_J$ ) the submodule of  $\mathrm{Res}_{W_I}^{W; \vec{T}; \gamma} \mathrm{Ind}_{W_J}^{W; \vec{T}; \gamma} \mathcal{O}_X$  corresponding to any finite Bruhat order ideal has strongly flat invariants for  $W_I$ , and thus

$$\mathcal{H}_{W, W_J, W_I; \vec{T}; \gamma}(X) := (\mathrm{Ind}_{W_J}^{W; \vec{T}; \gamma} \mathcal{O}_X)^{W_I} \quad (5.32)$$

is an  $S$ -flat sheaf bimodule on  $X/W_I \times_S X/W_J$ , and this construction commutes with base change. The subquotients in the corresponding Bruhat filtration may all be described in the following way. For each  $w \in {}^I W^J$ , there is a corresponding line bundle  $\mathcal{L}_w$  on  $X/W_{I(w)}$  (constructed from  $\vec{T}$  and  $\gamma$ ) such that the subquotient is the direct image in  $X/W_I \times_S X/W_J$  of the  $(X/W_{I(w)}, X/W_{J(w)})$ -bimodule  $(1, w^{-1})_* \mathcal{L}_w$ . More precisely, the line bundle  $\mathcal{L}_w$  is the descent to  $X/W_{I(w)}$  of the  $(W_{I(w)})$ -equivariant(!) line bundle

$$\mathcal{Z}_{\gamma, w} \otimes \mathcal{O}(D_w(\vec{T})) \otimes \mathcal{O}(D_{w_I}(-\vec{T}) - D_{w_{I(w)}}(-\vec{T})). \quad (5.33)$$

Note that the evaluation map from the lemma induces a morphism

$$\mathrm{Ind}_{W_J}^{W; \vec{T}; \gamma} \mathcal{O}_X \otimes \mathcal{H}_{W, W_I, W_J; \vec{T}; \gamma}(X) \rightarrow \mathrm{Ind}_{W_I}^{W; \vec{T}; \gamma} \mathcal{O}_X, \quad (5.34)$$

and this restricts to a morphism

$$\mathcal{H}_{W, W_J, W_K; \vec{T}; \gamma}(X) \otimes \mathcal{H}_{W, W_I, W_J; \vec{T}; \gamma}(X) \rightarrow \mathcal{H}_{W, W_I, W_K; \vec{T}; \gamma}(X), \quad (5.35)$$

making this a sheaf category as in the finite case. In particular, this makes  $\mathcal{H}_{W, W_I; \vec{T}; \gamma}(X)$  a sheaf algebra on  $(X/W_I)/S$  as expected.

As in previous cases, it will be helpful to have an alternate description of these algebras that lets us more easily verify that a given operator is a local section. We first consider the case without parameters. Given a coset  $wW_I$ , there is a corresponding sheaf bimodule  $k(X)wW_I$  on  $X \times X/W_I$ , with the obvious left and right module structures.

**Proposition 5.19.** *If the root kernels of  $W_I$  and  $W_J$  on  $X$  are diagonalizable, then we may identify  $\mathcal{H}_{W, W_I, W_J}(X)$  with the sheaf subbimodule of  $\bigoplus_{w \in W_I} k(X)wW_I$  that takes (locally) holomorphic  $W_I$ -invariant functions to locally holomorphic  $W_J$ -invariant functions.*

*Proof.* This follows as in the finite case.  $\square$

If  $\Gamma_I, \Gamma_J$  are  $W_I, W_J$ -invariant functions which are even on reflection hypersurfaces and have finitely supported difference, then for any twisting datum  $\gamma$  which is trivial on  $W_I$  and  $W_J$ ,

$$\mathcal{O}(\Gamma') \otimes \mathcal{H}_{W; \vec{T}; \gamma} \otimes \mathcal{O}(-\Gamma) \quad (5.36)$$

becomes a left  $\mathcal{H}_{W_J; \vec{T}}$ -module and a right  $\mathcal{H}_{W_I; \vec{T}}$ -module, and thus has a corresponding spherical module which we may denote by

$$\mathcal{O}(\Gamma') \otimes \mathcal{H}_{W, W_I, W_J; \vec{T}; \gamma} \otimes \mathcal{O}(-\Gamma). \quad (5.37)$$

Let us abusively denote the infinite sum  $\sum_{\alpha \in \Phi^+(W)} [X^{r_\alpha}]$  by  $D_{w_0}$ . Then the adjoint takes the following form.

**Proposition 5.20.** *If the root kernels of  $W_I$  and  $W_J$  on  $X$  are diagonalizable, then there is an isomorphism*

$$\mathcal{H}_{W, W_I, W_J}(X) \cong \mathcal{O}(D_{w_0} - D_{w_I}) \otimes \mathcal{H}_{W, W_J, W_I}(X) \otimes \mathcal{O}(D_{w_J} - D_{w_0}) \quad (5.38)$$

which is contravariant with respect to composition.

*Proof.* Embedding the left-hand side in  $\mathcal{H}_W(X)$  using the idempotents and taking the adjoint gives

$$e_I^* \mathcal{O}(D_{w_0}) \otimes \mathcal{H}_W(X) \otimes \mathcal{O}(-D_{w_0}) e_J^* \quad (5.39)$$

We have

$$\mathcal{O}(D_{w_0}) \otimes \mathcal{H}_{W_J}(X) \otimes \mathcal{O}(-D_{w_0}) = \mathcal{O}(D_{w_J}) \otimes \mathcal{H}_{W_J}(X) \otimes \mathcal{O}(-D_{w_J}) \quad (5.40)$$

so that the calculation reduces to identifying

$$e_I^* \mathcal{O}(D_{w_I}) \otimes \mathcal{H}_W(X) \otimes \mathcal{O}(-D_{w_J}) e_J^* \cong \mathcal{H}_{W, W_J, W_I}(X). \quad (5.41)$$

But this follows from the calculation in the finite case.  $\square$

*Remark.* Note that an analogous statement applies in the twisted case as long as we can express the twisting data in terms of some function  $\Gamma$ , as we can then break the various twists into a  $W_I$  or  $W_J$ -invariant piece and a finite piece, and thus reduce to the untwisted case.

**Proposition 5.21.** *If the root kernels of  $W_I$  and  $W_J$  on  $X$  are diagonalizable, then*

$$\begin{aligned} & \mathcal{H}_{W, W_I, W_J; \vec{T}}(X) \\ & \subset \mathcal{H}_{W, W_I, W_J}(X) \cap \mathcal{O}\left(\sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_J)} T_\alpha\right) \otimes \mathcal{H}_{W, W_I, W_J}(X) \otimes \mathcal{O}\left(-\sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_I)} T_\alpha\right), \end{aligned} \quad (5.42)$$

with equality unless there is a root  $\alpha$  such that  $T_\alpha$  and  $T_{-\alpha}$  have a common component.

*Proof.* Over the locus of  $S$  covered by symmetric idempotents, we may use those idempotents to locally identify  $\mathcal{H}_{W, W_I, W_J; \vec{T}}(X)$  with a submodule of  $\mathcal{H}_{W, W_I, W_J}(X)$ . This identification is compatible with the identification of meromorphic fibers, so extends to a global identification on each fiber covered by symmetric idempotents, and from there to the closure of the symmetric idempotent locus.

Similarly, local idempotents embed  $\mathcal{H}_{W,W_I,W_J;\vec{T}}(X)$  in

$$\mathcal{O}\left(-\sum_{\alpha \in \Phi^+(W)} T_\alpha\right) \otimes \mathcal{H}_W(X) \otimes \mathcal{O}\left(\sum_{\alpha \in \Phi^+(W)} T_\alpha\right) = \mathcal{O}\left(\sum_{\alpha \in \Phi^-(W)} T_\alpha\right) \otimes \mathcal{H}_W(X) \otimes \mathcal{O}\left(-\sum_{\alpha \in \Phi^-(W)} T_\alpha\right), \quad (5.43)$$

and the idempotents eliminate the contributions of  $T_\alpha$  for  $\alpha \in W_I, W_J$  respectively.

To see that the inclusion is tight, we need merely verify that both sides have the same Bruhat subquotients, which reduces to verifying that the negative part of

$$\sum_{\alpha \in \Phi^+(W) \cap w\Phi^-(W)} (T_{-\alpha} - T_\alpha) + \sum_{\alpha \in \Phi^-(W_J) \setminus \Phi^-(W_J \cap wW_I w^{-1})} (T_{w\alpha} - T_\alpha) \quad (5.44)$$

has no further cancellation.  $\square$

**Corollary 5.22.** *If the root kernels of  $W_I$  and  $W_J$  on  $X$  are diagonalizable, then there is an isomorphism*

$$\mathcal{H}_{W,W_I,W_J;\vec{T}}(X) \cong \mathcal{O}(D_{w_0}(-\vec{T}) - D_{w_I}(-\vec{T})) \otimes \mathcal{H}_{W,W_J,W_I;\vec{T}}(X) \otimes \mathcal{O}(D_{w_J}(-\vec{T}) - D_{w_0}(-\vec{T})) \quad (5.45)$$

which is contravariant with respect to composition.

## 6 The (double) affine case

The most interesting case for our purposes is when the Coxeter group is an affine Weyl group  $\tilde{W}$ . We actually want to modify the construction (very) slightly in that case, as the abelian variety being acted on is slightly larger than we would like. That is, rather than have an  $n+1$ -dimensional variety with an invariant map to an elliptic curve, we would rather act on the fibers of that map. If we pull back the sheaf bimodule  $\mathcal{H}_{\tilde{W};\vec{T};\gamma}(X)$  from  $X \times_S X$  to  $X \times_{X/A_{\tilde{W}}} X$ , then we find (by considering what happens on invariant localizations, say) that the result is still naturally a sheaf algebra. The group no longer acts faithfully on every fiber, but the various calculations involving the Bruhat filtration carry over without difficulty, so that we still obtain a flat family of sheaf algebras generated by the rank 1 subalgebras. One caveat is that  $T_\alpha$  and  $T_\beta$  need not be transverse for  $\alpha \neq \pm\beta$ ; if they correspond to the same root of the finite root system, then their divisors differ only by a translation.

Still, we have the following definition. First, if  $X$  is a torsor over the abelian scheme  $A$ , an action of  $\tilde{W}$  on  $X$  by affine reflections is simply an action such that every simple reflection fixes a hypersurface and the action on  $A$  factors through a faithful action of the corresponding finite Weyl group. Any such action arises from an action of coroot type by specializing the parameter  $q$ . In addition, every finite parabolic subgroup still acts by reflections, and thus in particular we still have good notions of systems of parameters and twisting data. The one caution is then when expressing twisting data as a coboundary of some formal sum of divisors, one needs to assume  $q$  non-torsion. This is not truly an issue, however, as one can simply take the limit of the twisting data from the non-torsion case (which simply adds an extra level of formality to the formal sum).

**Definition 6.1.** Let  $\tilde{W}$  act on  $X$  by affine reflections, and let  $\vec{T}$  be a system of parameters and  $\gamma$  a twisting datum. The corresponding *elliptic double affine Hecke algebra*  $\mathcal{H}_{\tilde{W};\vec{T};\gamma}(X)$  is the sheaf subalgebra of  $k(X)[\tilde{W}]$  generated by the rank 1 subalgebras  $\mathcal{H}_{\langle s \rangle;\vec{T};\gamma}(X)$  for  $s \in S$ .

**Proposition 6.1.** *The subsheaf of  $\mathcal{H}_{\tilde{W};\vec{T};\gamma}(X)$  corresponding to any finite Bruhat order ideal is an  $S$ -flat coherent sheaf on  $X \times_S X$ .*

**Proposition 6.2.** *We have*

$$\mathcal{H}_{\tilde{W};\vec{T};\gamma}(X) \subset \mathcal{H}_{W;0;\gamma}(X) \cap \mathcal{O}\left(-\sum_{\alpha \in \Phi^+(W)} T_\alpha\right) \otimes \mathcal{H}_{W;0;\gamma}(X) \otimes \mathcal{O}\left(\sum_{\alpha \in \Phi^+(W)} T_\alpha\right), \quad (6.1)$$

with equality unless there are  $\alpha \in \Phi^+(W)$ ,  $\beta \in \Phi^-(W)$  such that  $T_\alpha$  and  $T_\beta$  have a common component.

*Remark.* When  $q$  is torsion, then in fact each  $\alpha$  has infinitely many  $\beta$  (both positive and negative) such that  $T_\alpha = T_\beta$ , and thus not only is the hypothesis for equality never satisfied, but the formal sum  $\sum_{\alpha \in \Phi^+(W)} T_\alpha$  does not actually specify an integer-valued function on irreducible Cartier divisors. As we mentioned, this is more of a notational issue than an actual obstruction to the inclusion, as the resulting family of meromorphically trivial twisting data extends naturally to the torsion  $q$  case.

The description of  $\mathcal{H}_{W;0}(X)$  as holomorphy-preserving operators continues to hold, as long as  $W$  acts faithfully, or more generally for any Bruhat order ideal that injects in  $\text{Aut}(X)$ . In that case, the result of Lemma 4.2 gives conditions along the reflection hyperplanes which are precisely analogues of the residue conditions of [9].

One new phenomenon that arises in the affine case is that the group can fail to act faithfully. Since the finite Weyl group acts faithfully on  $A$ , the kernel is necessarily contained in the translation subgroup, and we see that there is a kernel precisely when  $q$  is torsion. In that case, the action of  $\tilde{W} \cong \Lambda \rtimes W$  on  $X$  factors through a semidirect product of the form  $\tilde{W}_r := (\Lambda/r\Lambda) \rtimes W$  where  $r$  is the order of  $q$ , and one finds that  $\mathcal{H}_{\tilde{W};\vec{T};\gamma}(X)$  is the sheaf algebra associated to a sheaf of algebras on the quotient  $X/\tilde{W}_r$ . The centralizer of  $k(X)$  in the generic fiber has the form  $k(X)[r\Lambda]$ , and thus the center of the generic fiber is  $k(X/\tilde{W}_r)[r\Lambda]^W$ . This agrees with the generic fiber of the center, and thus the center of  $\mathcal{H}_{\tilde{W};\vec{T};\gamma}(X)$  for  $q$  torsion is the coordinate sheaf of an integral  $X$ -scheme of relative dimension  $n$ . We conjecture that this center is Noetherian, and further that  $\mathcal{H}_{\tilde{W};\vec{T};\gamma}(X)$  is coherent over the center; together, this would make  $\mathcal{H}_{\tilde{W};\vec{T};\gamma}(X)$  Noetherian.

More generally, we conjecture that  $\mathcal{H}_{\tilde{W};\vec{T};\gamma}(X)$  is always Noetherian (in some appropriate sense). Assuming the technique of [1] could be adapted to the case of sheaf algebras, this would reduce to the case that everything is defined over a finite field, and thus to the case that  $q$  is torsion.

There is, of course, a different limit one might consider as  $q \rightarrow 0$  (or as  $q$  becomes torsion): rather than take the limit to a twisted group algebra in which the group does not act faithfully, one might instead take a limit to an algebra of differential-reflection operators; i.e., take the limit in such a way as to consider how the operators themselves actually act. This would presumably be the correct way to interpret the algebra of holomorphy-preserving operators if one does not suppress the poles for  $q$  torsion, but is not directly accessible via our techniques. In particular, the resulting algebra would not be generated by the rank 1 subalgebras, and the corresponding spherical algebra is almost certainly not a domain (in the sense given below) when  $q$  is a nontrivial torsion element.

The most important case for the spherical algebra construction is when  $W_I = W$  is the corresponding finite Weyl group. In that case, we note that each coset  $\tilde{W}/W$  contains a unique translation, and thus we may interpret elements of the spherical algebra as (elliptic) difference operators, with  $\mathcal{H}_{\tilde{W},W;0}(X)$  for non-torsion  $q$  corresponding to difference operators that (locally) preserve  $W$ -invariant holomorphic functions. Note that the Bruhat order on  $\tilde{W}/W$  is simply the usual dominance order on weights.

This has the following consequence. We say that a sheaf algebra is a domain if the product of a nonzero section on  $U \times V$  and a nonzero section on  $V \times W$  is always a nonzero section on  $U \times W$ .

**Proposition 6.3.** *Suppose that the root kernel for  $W$  on  $X$  is diagonalizable. Then for any twisting datum  $\gamma$  which is trivial on  $W$ , every fiber of the spherical algebra  $\mathcal{H}_{\tilde{W}, W; \vec{T}; \gamma}(X)$  is a domain.*

*Proof.* We first note that the inclusion of  $\mathcal{H}_{\tilde{W}, W; \vec{T}; \gamma}(X)$  in  $k(X)[\Lambda]_\gamma$  is injective on fibers. This follows from the fact that we can compute  $\mathcal{H}_{\tilde{W}, W; \vec{T}; \gamma}(X)$  as the  $W$ -invariant submodule of an  $S$ -flat module with strongly flat invariants, and thus the inclusion

$$\mathcal{H}_{\tilde{W}, W; \vec{T}; \gamma}(X) \subset \text{Ind}_{\tilde{W}}^{\tilde{W}; \vec{T}; \gamma} \mathcal{O}_X \quad (6.2)$$

is injective on fibers; as the induced module injects in the induced module of  $k(X)$  and this equals  $k(X)[\Lambda]_\gamma$ , the desired injectivity follows.

In particular, any local section of a fiber on a product of  $W$ -invariant open subsets can be identified with a  $W$ -invariant element of  $k(X)[\Lambda]_\gamma$ . Since this identification is compatible with the multiplication, it will suffice to show that  $k(X)[\Lambda]_\gamma$  is a domain. Since  $\Lambda$  is a finitely generated free abelian group, there exist injective homomorphisms  $\Lambda \rightarrow \mathbb{R}$ , allowing us to define a total ordering on  $\Lambda$  compatible with the group law. In particular, for any nonzero element  $\sum_{\lambda \in \Lambda} c_\lambda[\lambda]$  of  $k(X)[\Lambda]_\gamma$ , there is a corresponding notion of “leading monomial”, defined as  $c_\lambda[\lambda]$  where  $\lambda$  is the largest element of the support. If  $f$  has leading monomial  $f_\lambda[\lambda]$  and  $g$  has leading monomial  $g_\mu[\mu]$ , then  $fg$  has leading monomial  $\zeta_{\lambda\mu}(f_\lambda \otimes g_\mu)$ , and is therefore nonzero as required.  $\square$

*Remark.* Again, we conjecture that this domain is Noetherian in an appropriate sense.

In the affine case, the spherical algebra has an additional symmetry. For general Coxeter groups, the usual symmetry replacing  $\vec{T}$  by  $-\vec{T}$  has an issue in the spherical algebra case. The proof of that symmetry relied on the fact that  $\sum_{\alpha \in \Phi(W)} T_\alpha$  has no effect on twisting, so that twisting by  $\sum_{\alpha \in \Phi^+(W)} T_\alpha$  and  $-\sum_{\alpha \in \Phi^-(W)} T_\alpha$  have the same effect, letting one move the twist to the other half of the intersection. For the spherical algebra, this operation instead turns  $-\sum_{\alpha \in \Phi^-(W) \setminus \Phi^-(W_I)} T_\alpha$  into  $\sum_{\alpha \in \Phi^+(W) \cup \Phi^-(W_I)} T_\alpha$ , and thus does not give an algebra of the same form. However, in the *affine* case, it turns out that there actually *is* a system of parameters  $\vec{T}'$  such that

$$\sum_{\alpha \in \Phi^+(\tilde{W}) \cup \Phi^-(W)} T_\alpha = \sum_{\alpha \in \Phi^-(\tilde{W}) \setminus \Phi^-(W)} T'_\alpha. \quad (6.3)$$

In both cases, we can break up the sum as a sum over roots of  $W$ , and find that on the left-hand side we have a sum over translates of  $T_\alpha$  by nonnegative multiples of some  $q_\alpha$ , while on the right-hand side we have a sum over translates of  $T'_\alpha$  by negative multiples of  $q_\alpha$ . We may thus simply take  $T'_\alpha$  to be the translate of  $T_{-\alpha}$  by  $q_\alpha$ .

Another feature of the double affine case is that the inverse map acts on the poset  ${}^I W^I$  as a diagram automorphism (which is often trivial). In particular, if we can arrange for the pullback of the adjoint through the diagram automorphism to have isomorphic twist datum, then this gives an actual involution of the Hecke algebra, which allows us to consider self-adjoint operators (and even have a reasonable chance of proving commutativity). For a specific example of this phenomenon in the  $C_n$  case, see Theorem 7.22 below.

Note that although we only stated many of the results above for the spherical algebra in the untwisted case, there is certainly no difficulty in extending them to the case that the twisting datum is the coboundary of a line bundle. Since we want the restriction to  $W$  to be trivial, that line bundle should be  $W$ -invariant, so that it is a multiple of the standard  $W$ -invariant line bundle. The resulting value for  $\gamma_0$  is a certain degree 0 line bundle depending on  $q$  and that multiple; by varying that multiple, we find (at least when  $q$  is non-torsion) that we obtain a Zariski dense subset of a 1-parameter family of such twisting data, and thus the formula for the adjoint and the gauging symmetry for  $\vec{T}$  both work for any twisting datum in the family.

## 7 The $C^\vee C_n$ case

We now restrict our attention to the case that the affine Weyl group is of type  $C$ . This has a natural action on the family  $\mathcal{E}^{n+1}$  given as follows:

$$\begin{aligned} s_0(z_1, \dots, z_n, q/2) &= (q - z_1, z_2, \dots, z_n, q/2) \\ s_n(z_1, \dots, z_n, q/2) &= (z_1, \dots, z_{n-1}, -z_n, q/2), \end{aligned}$$

and for  $1 \leq i \leq n-1$ ,  $s_i$  swaps  $z_i$  and  $z_{i+1}$ . Here we denote the last coordinate in  $\mathcal{E}^{n+1}$  by  $q/2$ , so that  $q$  is twice that coordinate. We use this notation since the corresponding family of actions of  $\tilde{C}_n$  on  $\mathcal{E}^n$  (and thus the resulting Hecke algebras) depends only on  $q$ , but it will be convenient (and more symmetric) to be able to divide  $q$  by 2. We find for this action that the simple coroot morphisms are  $q/2 - z_1, z_1 - z_2, \dots, z_{n-1} - z_n, z_n$ , and thus that this is in fact an action of coroot type, as required for our theory.

We will want to enlarge this to include an action of the diagram automorphism:

$$\omega(z_1, \dots, z_n) = (q/2 - z_n, \dots, q/2 - z_1), \quad (7.1)$$

which clearly permutes the simple coroot morphisms as expected.

The root curves are all isomorphic to  $\mathcal{E}$ , and the simple root morphisms are given by

$$(-1, 0, \dots, 0), (1, -1, 0, \dots, 0), (0, 1, -1, 0, \dots, 0), \dots, (0, \dots, 0, 1, -1, 0), (0, \dots, 0, 1, 0). \quad (7.2)$$

It follows that the root kernel for any finite parabolic subgroup is trivial, so that there will be no difficulties with invariants and (local) idempotents.

In fact, we have the following.

**Proposition 7.1.** *For any finite parabolic subgroup  $W_I \subset \tilde{C}_n$ , the quotient  $\mathcal{E}^n/W_I$  is smooth over  $\mathcal{M}_{1,1}$ .*

*Proof.* Any quotient  $\mathcal{E}^n/W_I$  is a product of symmetric powers of  $\mathcal{E}$  and a quotient or two of the form  $\mathcal{E}^m/C_m$  (using the diagram automorphism to identify parabolic subgroups involving  $s_0$  with the usual hyperoctahedral group). Symmetric powers of a curve are smooth, so there is no difficulty. For the quotient by the full hyperoctahedral group, we first observe that the quotient by the normal subgroup of order  $2^m$  is just the product of  $n$  copies of the quotient of  $\mathcal{E}$  by  $[-1]$ , a.k.a.  $\mathbb{P}^1$ . We thus find that  $\mathcal{E}^m/C_m \cong (\mathbb{P}^1)^m/S_m \cong \mathbb{P}^m$ , giving smoothness as required.  $\square$

“Miracle flatness” immediately gives the following, which in particular ensures that the spherical algebras we consider will be locally free in a suitable sense (i.e., that their direct images in either copy of  $\mathcal{E}^n/W$  are locally free).

**Corollary 7.2.** *For any parabolic subgroup  $W_I \subset C_n$ , the quotient maps  $\mathcal{E}^n \rightarrow \mathcal{E}^n/W_I$  and  $\mathcal{E}^n/W_I \rightarrow \mathcal{E}^n/C_n$  are flat.*

Suppose for the moment that  $\vec{T}_0 = \omega \vec{T}_n$ . Then we may consider the extended double affine Hecke algebra obtained by adjoining  $\mathcal{O}_X \omega$  to  $\mathcal{H}_{\tilde{C}_n; \vec{T}}(\mathcal{E}^n)$ . This has a natural  $\mathbb{Z}/2\mathbb{Z}$ -grading, and it will be useful to think of it as corresponding to a (sheaf) category with two objects rather than a sheaf algebra. That is, we have objects 0 and 1 with endomorphisms given by the sheaf algebra  $\mathcal{H}_{\tilde{C}_n; \vec{T}}(\mathcal{E}^n)$  and the remaining morphisms given in either direction by the sheaf bimodule

$\mathcal{H}_{\tilde{C}_n; \tilde{T}}(\mathcal{E}^n)\omega$ . Doing this actually lets us remove the constraint on the parameters: we may always obtain a sheaf category with two objects by taking

$$\begin{aligned}\mathrm{Hom}(0, 0) &= \mathcal{H}_{\tilde{C}_n; \tilde{T}}(\mathcal{E}^n), \\ \mathrm{Hom}(0, 1) &= \mathcal{H}_{\tilde{C}_n; \omega \tilde{T}}(\mathcal{E}^n)\omega, \\ \mathrm{Hom}(1, 0) &= \mathcal{H}_{\tilde{C}_n; \tilde{T}}(\mathcal{E}^n)\omega, \\ \mathrm{Hom}(1, 1) &= \mathcal{H}_{\tilde{C}_n; \omega \tilde{T}}(\mathcal{E}^n),\end{aligned}$$

where  $\mathcal{H}_{\tilde{C}_n; \tilde{T}}(\mathcal{E}^n)\omega$  denotes the bimodule obtained by twisting the regular bimodule of the DAHA by the isomorphism induced by  $\omega$ .

The Bruhat order on the extended affine Weyl group induces a Bruhat filtration on this category (which agrees with the usual Bruhat filtration coming from viewing each Hom bimodule as a regular module of some DAHA). Although of course this filtration is useful in itself, we will also make great use of a filtration obtained from a much coarser order. The category is certainly generated by the rank 1 subalgebras along with  $\omega$ , but since  $\omega$  permutes the rank 1 subalgebras, we can actually omit the subalgebras corresponding to  $s_0$  from the generators. As a result, we find that the category is generated by the two morphisms corresponding to  $\omega$  along with the Hecke algebras in each degree corresponding to the finite Weyl group. We may then define a filtration on each Hom bimodule by the number of times  $\omega$  was used; e.g., the degree  $\leq d$  piece of  $\mathrm{Hom}(0, d \bmod 2)$  is the image of

$$\mathcal{H}_{C_n; \tilde{T}}(X)\omega\mathcal{H}_{C_n; \omega \tilde{T}}(X)\omega\mathcal{H}_{C_n; \tilde{T}}(X)\dots \quad (7.3)$$

(with  $d$  copies of  $\omega$  and  $d + 1$  finite Hecke algebras as tensor factors). We can express this as the image of a product of rank 1 algebras and  $d \bmod 2$  copies of  $\omega$  and then move all of the copies of  $\omega$  to the end, so that the proof of Lemma 5.10 tells us that the result is a Bruhat interval. Moreover, that Bruhat interval is clearly a union of  $(W, W)$  double cosets, so is determined by the corresponding set of dominant weights; we find that the condition to be degree  $\leq d$  is simply that the first coefficient of the dominant weight is  $\leq d/2$ .

This leads to a “graded” (or “compactified”) version of the extended DAHA: take the sheaf category with objects  $\mathbb{Z}$  generated by elements  $\omega \in \mathrm{Hom}(k, k + 1)$  and algebras  $\mathcal{H}_{C_n; \tilde{T}}(X) \subset \mathrm{Hom}(2k, 2k)$ ,  $\mathcal{H}_{C_n; \omega \tilde{T}}(X) \subset \mathrm{Hom}(2k + 1, 2k + 1)$ . (We may think of this as a sort of Rees algebra corresponding to the filtration, taking into account parity.)

Note that when asking whether two (small) sheaf categories are isomorphic, there are actually two natural notions. The issue here is that for any automorphism of an object of a category, there is a corresponding inner automorphism of the category as a whole (and more generally for any assignment of an automorphism to each object). In our case, any unit on parameter space gives such an inner automorphism, and since we are primarily interested in the individual fibers, we should extend that to allow *local* units. This leads to a notion of twisting objects by line bundles; note that the resulting sheaf category will still be locally isomorphic to the original sheaf category. In general, we will often only state that given sheaf categories are isomorphic locally on the base; this is mainly to save the bookkeeping effort of determining precisely which line bundles one needs to twist by to make the isomorphism global. In particular, when specifying line bundles and equivariant gerbes, the  $z$ -independent terms of the polarization and weight will be largely irrelevant, so we can make the simplest consistent choice without having to worry too much about which choice would make later polarizations simpler.

Of course, we have yet to incorporate a twisting datum. We first note that the underlying equivariant gerbe induces a cocycle valued in pairs of polarizations and weights. We can embed

the  $\tilde{C}_n$ -module of polarizations in the degree 2 subspace of  $\mathbb{Q}[\vec{z}, q, \vec{\pi}][1/q]$  (where  $\vec{\pi}$  corresponds to additional factors of  $\mathcal{E}$  which we include to allow some room for further parameters), and similarly for the  $\tilde{C}_n$ -module of weights. Since  $q$  is  $\tilde{C}_n$ -invariant, both rational  $\tilde{C}_n$ -modules are isomorphic to the corresponding module  $\mathbb{Q}[\vec{z}, \vec{\pi}]$  obtained by specializing  $q = 1$ . We can compute  $H^1$  of this module by restriction to the translation subgroup, where the filtration by degree makes it easy to verify that  $H^1$  is trivial. It follows that the given cocycle is the coboundary of a pair  $(p_3(\vec{z}, q, \vec{\pi})/q, p_1(\vec{z}, q, \vec{\pi})/q)$  where  $p_3$  and  $p_1$  are homogeneous polynomials of degree 3 and 1 respectively. Of course,  $p_3$  and  $p_1$  are only determined modulo 0-cocycles, but those are again easily seen to be just the polynomials independent of  $\vec{z}$ .

In our case, since our primary interest is in the spherical algebras, we want the twisting datum to be trivial on  $C_n$ . The same must in particular hold for the equivariant gerbe, so that  $p_3$  and  $p_1$  must be  $C_n$ -invariant polynomials. (More precisely, they must be  $C_n$ -invariant modulo 0-cocycles, but we can then average over  $C_n$  without changing the coboundary.) Since we are allowed to ignore terms independent of  $\vec{z}$ , we see that we may as well take  $p_1 = 0$  and  $p_3 = \frac{\lambda(q/2, \vec{\pi})}{q} \sum_i \frac{z_i^2}{2}$  for some linear functional  $\lambda$  with rational coefficients. Imposing the condition that the coboundary consist of actual polarizations then forces  $\lambda$  to have integer coefficients. That is, the value of the coboundary at  $s_0$  is  $p_3(\vec{z}) - {}^{s_0}p_3(\vec{z}) = \lambda(q/2, \vec{\pi})z_1 - \lambda(q/2, \vec{\pi})\frac{q}{2}$ , which is integral iff  $\lambda$  is integral since  $q/2$  is a variable.

To extend this to a twisting datum, it suffices to choose a meromorphic section of the restriction to  $\langle s_0 \rangle$  and represent the corresponding Cartier divisor as  $D_0 - {}^{s_0}D_0$  with  $D_0$  transverse to  $[X^{s_0}]$ . Since  $D_0$  is only determined modulo  $s_0$ -invariant divisors and linear equivalence, it is equivalent to specify the polarization

$$k \frac{z_1(z_1 - q)}{2} + \frac{\lambda(q/2, \vec{\pi})z_1}{2}, \quad (7.4)$$

with  $k \in \{0, 1\}$ . (Again, the constant term is irrelevant for our purposes.) Note that this imposes a stronger integrality constraint on  $\lambda$ , which must now have even coefficients.

Of course, we saw above when discussing the elliptic Gamma “function” that not every suitably integral cocycle has meromorphic sections, even for the translation subgroup. Luckily, in our case, we can easily write down explicit products of elliptic Gamma functions that do the trick. To be precise, consider the product

$$\prod_{1 \leq i \leq n} \Gamma_q(a \pm z_i) \quad (7.5)$$

where  $a$  is linear in  $q/2$  and  $\pi$ . Here and below, we have used the shorthand notation that multiple arguments to  $\Gamma_q$  or  $\vartheta$  represent a product and the appearance of  $\pm$  in the argument means that *both* signs should be used; thus  $\Gamma_q(a \pm z_i) = \Gamma_q(a + z_i)\Gamma_q(a - z_i)$ .

This is  $C_n$ -invariant, and has polarization (ignoring the  $z$ -independent term)

$$(2a - q) \sum_i \frac{z_i^2}{2}. \quad (7.6)$$

Moreover, we find

$$\frac{{}^{s_0} \prod_{1 \leq i \leq n} \Gamma_q(a \pm z_i)}{\prod_{1 \leq i \leq n} \Gamma_q(a \pm z_i)} = \frac{\vartheta(a - z_1)}{\vartheta(a - q + z_1)}, \quad (7.7)$$

and thus this corresponds to the twisting datum with  $D_0 = [a]$ . This gives the general degree 1 case, and we may obtain the general degree 0 case by taking a ratio of two such products.

Such a choice of product induces an embedding of the twisted Hecke algebra in the algebra of meromorphic difference-reflection operators; equivalently, meromorphic sections of the twisted

Hecke algebra act on formal functions of the form  $f\gamma$  where  $\gamma$  is the product of  $\Gamma_q$  symbols. This lets us extend the holomorphy-preserving property to the twisted case: if  $f$  is locally holomorphic away from the poles of the sections of the cocycle corresponding to  $\gamma$ , then so its image. This leads to issues where  $\gamma$  has poles, but we have enough choice in how we represent things that the corresponding invariant localizations give a finite covering. (I.e., in the degree 0 case, we may take  $\gamma = \prod_{1 \leq i \leq n} \Gamma_q(v + a \pm z_i) / \Gamma_q(v \pm z_i)$  with  $v$  varying; in degree 1, we take  $\gamma = \prod_{1 \leq i \leq n} \Gamma_q(v \pm z_i, w \pm z_i) / \Gamma_q(v + w - a \pm z_i)$  with  $v, w$  varying.)

To include the diagram automorphism, we note that because we extended to a category above, we may choose a different product of  $\Gamma_q$  symbols for each object, and need only have a line bundle  $\mathcal{L}_i$  in each degree such that sections of  $\mathcal{L}_i\omega$  take  $k(X)\gamma_i$  to  $k(X)\gamma_{i+1}$ . This is easy enough, since

$$\frac{\omega \prod_{1 \leq i \leq n} \Gamma_q(a \pm z_i)}{\prod_{1 \leq i \leq n} \Gamma_q(a - q/2 \pm z_i)} = \prod_{1 \leq i \leq n} \vartheta(a - q/2 - z_i)$$

$$\frac{\omega \prod_{1 \leq i \leq n} \Gamma_q(a \pm z_i)}{\prod_{1 \leq i \leq n} \Gamma_q(a + q/2 \pm z_i)} = \prod_{1 \leq i \leq n} \frac{1}{\vartheta(a - q/2 + z_i)}.$$

In the degree 0 case, this ends up not changing the twisting datum, but in degree 1, the polarization of  $\mathcal{L}_i$  ends up alternating between positive and negative. This would increase the number of cases we would need to consider, so it will be convenient to enlarge the category even further by replacing each object by a sequence of objects, one for each (isomorphism class of) line bundle on  $\mathbb{P}^n$ . The Hom bimodule between two objects in the enlarged category is then just the twist of the original Hom bimodule by the pair of line bundles (inverting the one on the domain side). The benefit of this is that we can move between the two degree 1 cases by twisting by  $\mathcal{O}_{\mathbb{P}^n}(\pm 1)$ , and thus in the enlarged category, there is only one case.

It turns out that even the above category is not quite general enough to include everything we want to do for the spherical algebras. As a result, we will first focus our attention on the case in which the endpoints do not have any parameters assigned. We also specialize to the usual Macdonald-ish case, in which  $T_i$  for  $1 \leq i \leq n-1$  is the divisor  $t + x_i - x_{i+1} = 0$ . It also turns out that most of the symmetries of the algebra do not preserve the untwisted case, but *do* preserve a particular twist; this leads us to make a somewhat odd-appearing choice in parametrizing twists. The specific basis we use for the  $\mathbb{Z}^2$  of objects is inspired by [20] (where it in turn came from the geometry of rational surfaces).

**Definition 7.1.** The even elliptic DAHA  $\mathcal{H}_{\eta';q,t}^{(n)}$  (of type  $C$ ) is the smallest sheaf category on  $\mathcal{E}^{n+3}$  with objects  $\mathbb{Z}\langle s, f \rangle$  such that

$$\mathcal{H}_{\eta';q,t}^{(n)}(ds + d'_1 f, ds + d'_2 f) = \mathcal{O}_{\mathbb{P}^n}(d'_2) \otimes \mathcal{H}_{C_n; \vec{t}}(\mathcal{E}^{n+3}) \otimes \mathcal{O}_{\mathbb{P}^n}(-d'_1) \quad (7.8)$$

and  $\mathcal{H}_{\eta';q,t}^{(n)}(ds + d'_1 f, (d+1)s + d'_2 f) \supset \mathcal{L}\omega$  where  $\mathcal{L}$  is the line bundle with polarization

$$-(d'_1 - d'_2 + 1) \sum_i z_i^2 - ((n-1)t + \eta' + (d - d'_1 + 1)q) \sum_i z_i + (d - d'_1)q^2/4. \quad (7.9)$$

The odd elliptic DAHA  $\mathcal{H}_{x_0;q,t}^{(n)}$  is defined similarly, except that  $\mathcal{L}$  has polarization

$$-(2d'_1 - 2d'_2 + 3) \sum_i \frac{z_i^2}{2} - ((n-1)t + x_0 + (3d - 2d'_1 + 2)q/2) \sum_i z_i + (3d - 2d'_1)q^2/8. \quad (7.10)$$

*Remark.* In this case, we were able to choose the  $z$ -independent part of the polarization to make everything globally consistent, where  $\mathcal{O}_{\mathbb{P}^n}(1)$  is chosen so as to pull back to the line bundle with polarization  $\sum_i z_i^2$ . We can recover the usual (compactified) elliptic DAHA by restricting the algebra  $\mathcal{H}_{-(n-1)t-q;q,t}^{(n)}$  to the subset  $\mathbb{Z}(s+f)$  of objects; if we restrict to even multiples and then invert the sections  $1 \in \text{Hom}(d(s+f), (d+2)(s+f))$ , we recover the uncompactified elliptic DAHA. We should note that while  $\mathcal{H}_{\eta';q,t}^{(n)}(0, 2s+2f)$  always contains 1, this is only true locally on the base for  $\mathcal{H}'_{x_0;q,t}(0, 2s+3f)$ , and it is not possible to fix this without also changing the particular representative of  $\mathcal{O}_{\mathbb{P}^n}(1)$ .

We similarly let  $\mathcal{S}_{\eta';q,t}^{(n)}$  and  $\mathcal{S}_{x_0;q,t}^{(n)}$  denote the corresponding spherical categories (i.e., replacing each Hom bimodule by the appropriate subquotient). Each Hom bimodule in one of these categories is a sheaf bimodule on the quotient  $\mathbb{P}^n = \mathcal{E}^n/C_n$ , every local section of which is a meromorphic difference operator on  $\mathcal{E}^n$ .

**Proposition 7.3.** *The subsheaf corresponding to any Bruhat order ideal in either  $\mathcal{S}_{\eta';q,t}^{(n)}$  or  $\mathcal{S}_{x_0;q,t}^{(n)}$  is a coherent sheaf bimodule on  $\mathbb{P}^n \times_{\mathcal{E}^3} \mathbb{P}^n$ , and the direct image in either  $\mathbb{P}^n$  is locally free.*

*Proof.* This reduces to the corresponding statement for the subquotients in the Bruhat filtration, each of which comes from a line bundle on the quotient by a parabolic subgroup of  $C_n$ , and is therefore flat on the quotient  $\mathbb{P}^n$ .  $\square$

By comparison with the univariate case (which we will discuss in more detail shortly), we are led to define generalizations (“blowups”) of these algebras with even more parameters. Recall that local sections of the spherical algebras are difference operators. We let  $T_i$  denote the operator that pulls back through  $z_i \mapsto z_i + q$ ; note that in terms of our convention for how group elements act,  $T_i$  is the same as the action of the translation  $z_i \mapsto z_i - q$ . We extend this to half-integer powers of  $T_i$  by using the chosen  $q/2$ . Every local section is then a left linear combination of monomials  $T^{\vec{k}} := \prod_i T_i^{k_i}$  in which all  $k_i$  are half-integers in the same coset of  $\mathbb{Z}$  (determined by the coefficient of  $s$  in the degree of the operator in the category).

**Definition 7.2.** The sheaf category  $\mathcal{S}_{\eta',x_1,\dots,x_m;q,t}^{(n)}$  is the sheaf category on  $\mathbb{P}^n/\mathcal{E}^{m+3}$  with objects  $\mathbb{Z}\langle s, f, e_1, \dots, e_m \rangle$  defined by taking  $\mathcal{S}'_{\eta',x_1,\dots,x_m;q,t}^{(n)}(d_1s + d'_1f - r_{11}e_1 - \dots - r_{1m}e_m, d_2s + d'_2f - r_{21}e_1 - \dots - r_{2m}e_m)$  to be the subsheaf of  $\mathcal{S}'_{\eta';q,t}^{(n)}(d_1s + d'_1f, d_2s + d'_2f)$  consisting (locally) of operators  $\mathcal{D}$  such that the left coefficient of  $\prod_{1 \leq i \leq n} T_i^{k_i}$  vanishes on the divisors  $z_i = x_j - (2l - d_2 + 1)q/2$  for  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ ,  $k_i + (d_2 - d_1)/2 + r_{1j} \leq l < r_{2j}$  and the divisors  $z_i = -x_j + (2l - d_2 + 1)q/2$  for  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ ,  $-k_i + (d_2 - d_1)/2 + r_{1j} \leq l < r_{2j}$ .

*Remark.* Note that by symmetry, it would have sufficed to impose the first set of vanishing conditions.

We also construct a sheaf category  $\mathcal{S}'_{x_0,x_1,\dots,x_m;q,t}^{(n)}$  by imposing the same conditions on  $\mathcal{S}'_{x_0;q,t}^{(n)}$ .

Note that in this definition, we need to impose the vanishing conditions on the family as a whole; on individual fibers, the condition along individual divisors may be too strong (when the divisors are reflection hypersurfaces) or too weak (when the divisors are not distinct). As a result, it is a nontrivial question whether the family is flat, and even when it is flat, it is conceivable that the individual fibers might fail to inject in the category of meromorphic operators (which could in turn allow the fibers to acquire zero divisors).

Luckily, since we defined these using the spherical algebra of an elliptic DAHA, there is an obvious approach to studying these categories: construct them as spherical subquotients of a suitable

subcategory of  $\mathcal{H}_{\eta';q,t}^{(n)}$  or  $\mathcal{H}_{x_0;q,t}^{(n)}$ . There are actually multiple choices one might make here, as one can view a divisor  $z_i = x_j - kq/2$  as a pullback from the coroot curve associated to either endpoint. Although it might seem natural to choose the endpoint that matches the parity of  $k$ , it will be easier for present purposes to consistently use  $s_0$ . Other choices may lead to more natural Hecke algebras, however; for instance the classical double affine Hecke algebra of type  $C^\vee C_n$  corresponds to assigning two parameters to  $s_n$  and two parameters to  $s_0$ .

Since we are keeping the parameters away from the finite Weyl group, the vanishing conditions should be appropriately  $C_n$ -equivariant, and thus the vanishing condition for the left coefficient of  $w$  should depend only on  $wC_n$  and be equivariant on the left. Each coset of  $wC_n$  has a unique representative  $\prod_i T_i^{k_i}$ , and we readily verify that the vanishing conditions we imposed above transform well under  $C_n$ . We may thus define  $\mathcal{H}_{\eta',x_1,\dots,x_m;q,t}^{(n)}$  by imposing the resulting vanishing conditions, and similarly for  $\mathcal{H}_{x_0,x_1,\dots,x_m;q,t}^{(n)}$ .

We note that in addition to the functoriality on fibers implied by the fact that this is defined over the moduli stack, we also have functoriality with respect to translation by 2-torsion.

**Proposition 7.4.** *If  $\tau$  is an fppf-local section of  $\mathcal{E}[2]$ , then there are isomorphisms*

$$\begin{aligned} \mathcal{H}_{\eta',x_1,\dots,x_m;q,t}^{(n)} &\cong \mathcal{H}_{\eta',x_1+\tau,\dots,x_m+\tau;q,t}^{(n)} \\ \mathcal{H}_{x_0,x_1,\dots,x_m;q,t}^{(n)} &\cong \mathcal{H}_{x_0+\tau,x_1+\tau,\dots,x_m+\tau;q,t}^{(n)} \end{aligned} \quad (7.11)$$

*Proof.* Conjugating by the involution  $(z_1, \dots, z_n) \mapsto (z_1 + \tau, \dots, z_n + \tau)$  induces such an isomorphism on the generators for  $m = 0$  and acts as described on the additional vanishing conditions for  $m > 0$ .  $\square$

*Remark 1.* In fact, we could have defined these categories over the moduli stack of hyperelliptic curves of genus 1, at the cost of making the action of  $C_n$  slightly more complicated (with  $s_n$  acting by  $z_n \mapsto s(z_n)$ , where  $s$  is the hyperelliptic involution), in which case these isomorphisms follow by functoriality. This would have made a number of later formulas more complicated, as well as making it more difficult to discuss line bundles and  $\Gamma_q$  symbols.

*Remark 2.* The above isomorphism involves translation by  $\tau$  in every degree. If one instead only translates in the odd degrees, none of the parameters visible in the notation change, but  $q/2$  is replaced by  $q/2 + \tau$ . In particular, the algebra is indeed independent of the choice of  $q/2$ .

We have the following useful ‘‘elementary transformation’’ symmetry. Here and below, we simplify things by observing that our sheaf categories satisfy a natural translation symmetry in which translation in the group of objects corresponds to translations of the parameters by multiples of  $q/2$ , and thus it suffices to consider Hom bimodules starting at the 0 object.

**Proposition 7.5.** *There are (locally on the base) natural isomorphisms*

$$\begin{aligned} &\mathcal{H}_{x_0,x_1,x_2,\dots,x_m;q,t}^{(n)}(0, ds + d'f - r_1e_1 - \dots - r_me_m) \\ &\cong \prod_{1 \leq i \leq n} \Gamma_q((r_1 + (1-d)/2)q - x_1 \pm z_i) \\ &\mathcal{H}_{x_0-x_1,-x_1,x_2,\dots,x_m;q,t}^{(n)}(0, ds + (d' - r_1)f - (d - r_1)e_1 - r_2e_2 - \dots - r_me_m) \\ &\prod_{1 \leq i \leq n} \Gamma_q(q/2 - x_1 \pm z_i)^{-1} \end{aligned} \quad (7.12)$$

and

$$\begin{aligned}
& \mathcal{H}_{\eta', x_1, x_2, \dots, x_m; q, t}^{(n)}(0, ds + d'f - r_1 e_1 - \dots - r_m e_m) \\
& \cong \prod_{1 \leq i \leq n} \Gamma_q((r_1 + (1-d)/2)q - x_1 \pm z_i) \\
& \quad \mathcal{H}'_{\eta' - x_1, -x_1, x_2, \dots, x_m; q, t}^{(n)}(0, ds + (d + d' - r_1)f - (d - r_1)e_1 - r_2 e_2 - \dots - r_m e_m) \\
& \quad \prod_{1 \leq i \leq n} \Gamma_q(q/2 - x_1 \pm z_i)^{-1}. \tag{7.13}
\end{aligned}$$

*Proof.* It is easy to see that the corresponding categories of meromorphic operators are isomorphic (locally on the base), and the pseudo-conjugation by  $\Gamma_q$  symbols respects the twisting data, so the image of the right-hand side in the meromorphic category corresponding to the left-hand side satisfies the same conditions on the reflection hypersurfaces. The conditions along the divisors corresponding to  $x_i$  for  $2 \leq i \leq m$  are clearly the same, and it is straightforward the check that the same holds for  $i = 1$ .  $\square$

*Remark.* Since the definition is also clearly invariant under permutations of  $x_1, \dots, x_m$ , we may apply this symmetry in any  $x_i$ , and then by composition in any subset of the  $x_i$ . If the subset has odd size, then the isomorphism switches between  $\mathcal{H}^{(n)}$  and  $\mathcal{H}'^{(n)}$ , while even subsets induce isomorphisms between extended DAHAs of the same parity. In particular, for each of the two families, there is an action of  $W(D_m)$  on the parameter space that extends to an action on the family of sheaf categories.

If we can show that these sheaf categories have well-behaved Bruhat filtrations (i.e., in which the subquotients are obtained from the  $m = 0$  case by imposing the vanishing conditions), then the same will immediately hold for their spherical subquotients. Unfortunately, we cannot simply copy the arguments we used in the usual elliptic Hecke algebra case; although the upper bound on the Bruhat subquotients works the same way, the category structure means there is no longer a canonical way to associate a multiplication map to a reduced word.

There are some special cases which are easy, however. As before, we may restrict our attention to Hom bimodules starting from the 0 object.

The following is a trivial consequence of the definition.

**Proposition 7.6.** *If  $r_m \leq 0$ , then*

$$\begin{aligned}
& \mathcal{H}_{\eta', x_1, \dots, x_m; q, t}^{(n)}(0, ds + d'f - r_1 e_1 - \dots - r_m e_m) \\
& \quad = \mathcal{H}'_{\eta', x_1, \dots, x_{m-1}; q, t}^{(n)}(0, ds + d'f - r_1 e_1 - \dots - r_{m-1} e_{m-1}) \\
& \mathcal{H}_{x_0, x_1, \dots, x_m; q, t}^{(n)}(0, ds + d'f - r_1 e_1 - \dots - r_m e_m) \\
& \quad = \mathcal{H}'_{x_0, x_1, \dots, x_{m-1}; q, t}^{(n)}(0, ds + d'f - r_1 e_1 - \dots - r_{m-1} e_m)
\end{aligned}$$

Applying the elementary transformation symmetry in  $x_m$  means that the case  $r_m \geq d$  is also straightforward to deal with. There is one more case which is nice.

**Proposition 7.7.** *There is a system of parameters  $\vec{T}$  and a twisting datum  $\gamma$  such that each Hom bimodule*

$$\mathcal{H}_{\eta', x_1, \dots, x_m; q, t}^{(n)}(0, d(2s + 2f - e_1 - \dots - e_m)) \tag{7.14}$$

*is equal to the corresponding Bruhat interval in  $\mathcal{H}_{\vec{C}_n, \vec{T}; \gamma}(\mathcal{E}^{n+m+3})$ , and similarly for*

$$\mathcal{H}'_{x_0, x_1, \dots, x_m; q, t}^{(n)}(0, d(2s + 3f - e_1 - \dots - e_m)). \tag{7.15}$$

*Proof.* We can rephrase the vanishing conditions for generic parameters as stating that

$$\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \Gamma_q(q/2 - x_j \pm z_i) \mathcal{D} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \Gamma_q(q/2 - x_j \pm z_i)^{-1} \quad (7.16)$$

has holomorphic coefficients. The cocycle in Cartier divisors associated to this product of  $\Gamma_q$  symbols is precisely the right form to come from a system of parameters (associated to the orbit of  $s_0$ ).  $\square$

It turns out that it suffices to understand these cases.

**Theorem 7.8.** *For any vector  $v = ds + d'f - r_1e_1 - \cdots - r_me_m$ , the bounds on the Bruhat subquotients of  $\mathcal{H}_{\eta', x_1, \dots, x_m; q, t}^{(n)}(0, v)$  and  $\mathcal{H}'_{x_0, x_1, \dots, x_m; q, t}(0, v)$  coming from the vanishing conditions are saturated. In particular, both sheaf categories are locally free, and the map from any Hom bimodule to the sheaf bimodule of meromorphic operators is injective on fibers.*

*Proof.* We first use the elementary transformation symmetry to replace all the cases with  $r_i \geq d$  with cases with  $r_i \leq 0$  (possibly changing the parity), and thus with  $r_i = 0$ . In particular, we observe that the cases  $d \in \{0, 1\}$  of the Theorem reduce to the subcase  $r_1 = \cdots = r_m = 0$ , and thus to the original elliptic DAHA, where we certainly have saturation. For  $d > 1$ , if some  $r_i = 0$ , we can simply omit that parameter and thus reduce to a case with smaller  $m$ . We thus find that it suffices to prove saturation when  $0 < r_1, \dots, r_m < d$ .

We consider the even case  $\mathcal{H}^{(n)}$ , with the odd case  $\mathcal{H}'^{(n)}$  being entirely analogous. Let  $C_m := 2s + 2f - e_1 - \cdots - e_m$ , and suppose by induction that we have saturation for  $v - C_m$ . It will then suffice to show that the image of the multiplication map

$$\mathcal{H}_{\eta', x_1, \dots, x_m; q, t}^{(n)}(v - C_m, v) \otimes \mathcal{H}'_{x_0, x_1, \dots, x_m; q, t}(0, v - C_m) \rightarrow \mathcal{H}'_{\eta', x_1, \dots, x_m; q, t}(0, v) \quad (7.17)$$

saturates the bound. Since the first factor is a Bruhat interval in an honest elliptic DAHA, it in particular has a global section 1, giving an inclusion

$$\mathcal{H}_{\eta', x_1, \dots, x_m; q, t}^{(n)}(0, v - C_m) \subset \mathcal{H}'_{\eta', x_1, \dots, x_m; q, t}(0, v), \quad (7.18)$$

identifying the former with the Bruhat interval of the latter corresponding to the dominant weight  $(d/2 - 1, d/2 - 1, \dots, d/2 - 1)$ . The bounds on the subquotients are clearly the same, and thus we have saturation for any  $w$  in this interval.

For the rest of the module, we note that both sides are bimodules over the finite Hecke algebra, and since the  $x_i$  parameters have no effect on this algebra, we may reduce to the case of a Bruhat interval  $[\leq w]$  with  $w$  a minimal representative of  $C_n \setminus \tilde{C}_n$ . Let  $\lambda(w)$  be the corresponding dominant weight. We have already shown that the leading coefficient map is saturated when  $\lambda(w)_1 \leq d/2 - 1$ , so without loss of generality may assume  $\lambda(w)_1 = d/2$ . It then follows from the structure of minimal coset representatives that not only is  $s_0w < w$ , but  $\lambda(s_0w)$  is obtained from  $\lambda(w)$  by reducing some coefficient from  $d/2$  to  $d/2 - 1$ . This changes the bound on the leading coefficient bundle by precisely the leading coefficient divisor of  $s_0$  in the relevant DAHA, and thus gives saturation as required.  $\square$

Passing to the spherical subquotient gives us the following.

**Corollary 7.9.** *For any vector  $v = ds + d'f - r_1e_1 - \cdots - r_me_m$ , the bounds on the Bruhat subquotients of  $\mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(n)}(0, v)$  and  $\mathcal{S}'_{x_0, x_1, \dots, x_m; q, t}(0, v)$  coming from the vanishing conditions are saturated. In particular, both sheaf categories are locally free, and the map from any Hom bimodule to the sheaf bimodule of meromorphic operators is injective on fibers.*

To understand the significance of this result, we will need to understand some special cases. The case  $t = 0$  is of particular interest, due to the following. The tensor and symmetric power constructions on modules extend to sheaf bimodules so carry over to analogous constructions for algebras and categories. (Of course, in the latter cases, one should take the symmetric subobject, not the quotient object.) The following is an immediate consequence of the fact that the corresponding  $A_{n-1}$  Hecke algebra is just the usual twisted group algebra.

**Proposition 7.10.** *One has the following isomorphisms (locally on the base):*

$$\begin{aligned}\mathcal{S}_{\eta', x_1, \dots, x_m; q, 0}^{(n)} &\cong \text{Sym}^n(\mathcal{S}_{\eta', x_1, \dots, x_m; q, 0}^{(1)}) \\ \mathcal{S}'_{x_0, x_1, \dots, x_m; q, 0}^{(n)} &\cong \text{Sym}^n(\mathcal{S}'_{x_0, x_1, \dots, x_m; q, 0}^{(1)}).\end{aligned}$$

*Remark.* Note that the tensor product of  $n$  univariate difference operators is described as follows. The  $i$ th operator acts on  $k(\mathcal{E}^n)$  by pulling back the coefficients from the  $i$ th factor and only translating the  $i$ th coordinate. These actions commute, and thus we may compose them to obtain the tensor product difference operator on  $k(\mathcal{E}^n)$ . The tensor product of the algebras is then the image of the tensor product of sheaves under this operation, and the symmetric power consists of those operators in the tensor product that commute with  $S_n$ .

There is a similar description for  $t = q$  coming from the following symmetry (essentially Corollary 5.22, combined with the fact that the conditions associated to  $x_1, \dots, x_m$  are unaffected).

**Proposition 7.11.** *One has the following isomorphisms (locally on the base):*

$$\begin{aligned}\mathcal{S}_{\eta', x_1, \dots, x_m; q, q-t}^{(n)} &\cong \prod_{1 \leq i < j \leq n} \Gamma_{p,q}(t \pm z_i \pm z_j) \mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(n)} \prod_{1 \leq i < j \leq n} \Gamma_{p,q}(t \pm z_i \pm z_j)^{-1} \\ \mathcal{S}'_{x_0, x_1, \dots, x_m; q, q-t}^{(n)} &\cong \prod_{1 \leq i < j \leq n} \Gamma_{p,q}(t \pm z_i \pm z_j) \mathcal{S}'_{x_0, x_1, \dots, x_m; q, t}^{(n)} \prod_{1 \leq i < j \leq n} \Gamma_{p,q}(t \pm z_i \pm z_j)^{-1}.\end{aligned}$$

We also note the following version of the adjoint symmetry. It will be convenient to express the adjoint in terms of a formal density; in particular,  $dT$  simply represents a formal  $\tilde{C}_n$ -invariant measure.

**Proposition 7.12.** *The adjoint with respect to the formal density*

$$\prod_{1 \leq i < j \leq n} \frac{\Gamma_q(t \pm z_i \pm z_j)}{\Gamma_q(\pm z_i \pm z_j)} \prod_{1 \leq i \leq n} \frac{1}{\Gamma_q(\pm 2z_i)} dT \quad (7.19)$$

*induces (locally on the base) contravariant isomorphisms*

$$\begin{aligned}\mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(n)} &\cong \mathcal{S}_{-\eta', -x_1, \dots, -x_m; q, t}^{(n)} \\ \mathcal{S}'_{x_0, x_1, \dots, x_m; q, t}^{(n)} &\cong \mathcal{S}'_{-x_0, -x_1, \dots, -x_m; q, t}^{(n)}\end{aligned}$$

*acting on objects as  $v \mapsto -v$ .*

*Remark.* We will refer to this formal adjoint as the ‘‘Selberg’’ adjoint, as the formal density consists of the interaction terms in the elliptic Selberg integral. More generally, composing the Selberg adjoint with a sequence of elementary transformations gives an adjunction involving densities of the form

$$\prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq k} \Gamma_q(u_i \pm z_i)}{\Gamma_q(\pm 2z_i)} \prod_{1 \leq i < j \leq n} \frac{\Gamma_q(t \pm z_i \pm z_j)}{\Gamma_q(\pm z_i \pm z_j)} dT \quad (7.20)$$

where the  $u_i$  depend on the  $x_i$  and the specific domain and codomain objects. Here we should think of the operators as mapping between two different inner product spaces, so that the two formal integrals are against (slightly) different densities.

We were somewhat vague in our descriptions of the subquotients of the Bruhat filtration above, as the specific divisors that are forced into a given coefficient are somewhat complicated to describe in general. For the most part, though, the important information about the subquotients is not how they are built up out of divisors, but simply which line bundle one ends up with in the end. (Recall that the subquotients are obtained from line bundles which are invariant under some parabolic subgroup  $W_I$  by descending to  $X/W_I$  then taking the direct image to  $X/C_n$ .)

Propositions 7.10 and 7.11 make this information relatively straightforward to determine. Any filtered isomorphism preserves the Bruhat subquotients, and thus the associated polarizations must be invariant under  $t \mapsto q - t$  (modulo line bundles on the base, that is). Modulo line bundles on the base, the  $t$ -dependent contribution to the polarization is linear, and thus the  $t \mapsto q - t$  symmetry forces it to be trivial. In other words, the *subquotients* are (locally on the base) independent of  $t$  and thus by the first Proposition are induced by the subquotients for  $n = 1$ . More precisely, for  $t = 0$ , the line bundle on  $\mathcal{E}^n$  associated to a given dominant weight is the outer tensor product of the univariate subquotients associated to the parts of the weight; one then descends to the quotient by the stabilizer in  $C_n$  of the weight.

It turns out that the univariate case has already been studied. Let  $\Gamma\mathcal{S}^{(n)}$ ,  $\Gamma\mathcal{S}'^{(n)}$  denote the associated “global section” categories; more precisely, these are sheaves of categories on  $\mathcal{E}^{m+3}$  in which each Hom sheaf is the direct image of the corresponding Hom bimodule. Since we included twisting by  $\mathcal{O}_{\mathbb{P}^n}(1)$  in the definition of the category, one can recover  $\mathcal{S}^{(n)}$  and  $\mathcal{S}'^{(n)}$  from their global section categories: for each Hom bimodule  $M$  in the sheaf category, the corresponding graded module (relative to the Segre embedding of  $\mathbb{P}^n \times \mathbb{P}^n$ ) can be extracted from the global section category.

In [20], two families of categories  $\mathcal{S}_{\eta, \eta', x_1, \dots, x_m; q, p}$  and  $\mathcal{S}'_{\eta, x_0, x_1, \dots, x_m; q, p}$  were constructed on an analytic curve  $\mathbb{C}^*/\langle p \rangle$ . These have the same group of objects as our categories, and an interpretation of the local sections of the Hom sheaves as difference operators. Moreover, the categories for general  $m$  are cut out from the categories for  $m = 0$  by suitable vanishing conditions, while the categories for  $m = 0$  are described via explicit generators given in terms of theta functions. Switching from multiplicative to additive notation and replacing  $\theta$  by  $\vartheta$  then extends this to arbitrary curves. (In fact, [20] gave such an extension by specifying an explicit gauging by products of Gamma functions that makes everything elliptic, and observing that the elliptic functions extend. But of course one could do the same gauging in terms of  $\vartheta$  and  $\Gamma_q$  symbols, so the resulting categories are the same.)

Although those operators are not quite  $C_1$ -symmetric in our sense, they are close: indeed, each operator formally takes functions invariant under  $z \mapsto (1 - d_1)q + \eta - z$  to functions invariant under  $z \mapsto (1 - d_2)q + \eta - z$ . This, of course, is easy enough to fix: if we base change to have an element  $\eta/2$  (and recall that we already have an element  $q/2$ ), then we can compose on both sides by a suitable translation to make the operator honestly  $C_1$ -symmetric.

**Proposition 7.13.** *Locally on the base, the global section categories are isomorphic to the  $C_1$ -symmetric versions of the categories  $\mathcal{S}$ ,  $\mathcal{S}'$  constructed in [20]. More precisely, if  $v, w$  are arbitrary elements of the object group  $\mathbb{Z}\langle s, f, e_1, \dots, e_m \rangle$ , then*

$$\Gamma\mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(1)}(v, w) \cong \mathcal{S}_{2c, 2c + \eta', c + x_1, \dots, c + x_m; q; \mathcal{E}}(v, w) \quad (7.21)$$

$$\Gamma\mathcal{S}'_{x_0, x_1, \dots, x_m; q, t}^{(1)}(v, w) \cong \mathcal{S}'_{2c, c + x_0, c + x_1, \dots, c + x_m; q; \mathcal{E}}(v, w). \quad (7.22)$$

*Proof.* For  $m = 0$ ,  $\mathcal{S}_{2c, 2c + \eta'; q; \mathcal{E}}$  and  $\mathcal{S}'_{2c, c + x_0; q; \mathcal{E}}$  are generated in degrees  $f$ ,  $s$ ,  $s + f$ . The degree  $f$  operators are clearly elements of the corresponding global section category, and the  $C_1$ -symmetry

along with the fact that the only poles are  $[X^{s_1}]$  implies the same for the degree  $s$  and  $s+f$  operators. Since the subcategory generated in this way saturates the Bruhat filtration, the categories are actually isomorphic.

For each of the four categories, every Hom sheaf for  $m > 0$  is contained in the appropriate Hom sheaf for  $m = 0$ , and the image of a local section of  $\mathcal{S}$  or  $\mathcal{S}'$  satisfies the correct vanishing conditions to be a local section of  $\Gamma\mathcal{S}^{(1)}$  or  $\Gamma\mathcal{S}'^{(1)}$ . Moreover, the Bruhat filtration and the analogous filtration by order tells us that both Hom sheaves are direct images of vector bundles on  $\mathbb{P}^1$  with the same Hilbert polynomials, and must therefore be identified by the isomorphism.  $\square$

This leads to a particularly nice interpretation of our categories. If the rational surface  $X_m$  is obtained from a Hirzebruch surface  $X_0$  by blowing up  $m$  points of a smooth anticanonical curve, then the line bundles on  $X_m$  are parametrized by the group  $\mathbb{Z}\langle s, f, e_1, \dots, e_m \rangle$ . This then gives rise to a category on this group of objects by taking the full subcategory of  $\text{coh}(X_m)$  in which the objects consist of one line bundle of each isomorphism class. We can, of course, do this over the entire moduli stack of such surfaces, which turns out (at least for  $m > 0$ ) to be isomorphic to  $\mathcal{E}^{m+1}$ . We then obtain a sheaf of categories on this base by taking the appropriate sheaf version of Hom between line bundles. It was shown in [20] that this sheaf of categories is precisely the specialization to  $q = 0$  of  $\mathcal{S}$  or  $\mathcal{S}'$ , depending on whether  $X_0$  comes from a vector bundle of even or odd degree. (One caveat here is that the fibers in this category can be slightly different from the categories associated to individual surfaces; Hom spaces in the latter may jump in the presence of  $-2$ -curves, while the global category is flat.)

The same, therefore, applies to our categories, and thus for general  $n$ ,  $\mathcal{S}_{\eta', x_1, \dots, x_m; 0, 0}^{(n)}$  and  $\mathcal{S}'_{x_0, \dots, x_m; 0, 0}^{(n)}$  can be interpreted as symmetric powers of rational surfaces. (To be precise, each fiber is equivalent to a subcategory of the subcategory of line bundles on such a symmetric power, which is full whenever the ratio of line bundles is acyclic.) The categories with  $q = 0$  and general  $t$  are thus commutative deformations of such powers (some sort of compactified discrete elliptic Calogero-Moser spaces), while the categories with general  $q, t$  are further noncommutative deformations.

We can also obtain analogous deformations for  $\mathbb{P}^2$ , though in that case only the global section category makes sense. If we restrict  $\Gamma\mathcal{S}'_{x_0; q, t}^{(n)}$  to the objects in  $\mathbb{Z}\langle s + f \rangle$ , then for  $n = 1, q = t = 0$ , we can identify consecutive Hom spaces in such a way as to obtain the polynomial algebra in three generators. (For  $n = 1, q \neq 0$ , we instead get the three-generator Sklyanin algebra of [1, 3], see [20].) Thus for general  $n, q = 0$ , we again obtain a family of commutative deformations of  $\text{Sym}^n(\mathbb{P}^2)$ , and further noncommutative deformations for general parameters.

There are some caveats to the above discussion. One is that since we are including *all* line bundles in the construction, there is no canonical way to associate a projective variety for  $q = 0$  or a noncommutative analogue in general: in general, we would need to make an explicit choice of ample divisor, or make some other choice of what it means for a module over the category to be torsion (i.e., map to the 0 sheaf). For  $n = 1$ , it was shown in [20] that any reasonable choice of ample divisor (in particular, any divisor which is ample on every  $X_m$ ) gives rise to the same quotient category, and thus there is no difficulty. Unfortunately, the argument there relied heavily on showing that various product maps are surjective, and the analogous surjectivity fails for  $n > 1$  even for  $q = t = 0$ . We therefore leave this as an open question.

There is also an issue here that, due to some difficulties in applying the Hecke algebra ideas to the  $\mathbb{P}^2$  case, we cannot always prove flatness for general ample divisors. For any surface other than  $\mathbb{P}^2$ , this is not a significant issue, as there will always be a nonempty subcone of the ample cone for which everything does work as expected. For  $\mathbb{P}^2$ , or in general outside this subcone, we will at

least be able to show that each Hom space is flat outside some finite (and most likely empty) set of bad pairs  $(q, t)$ .

We should also note that since the data in each case includes an explicit morphism to a Hirzebruch surface, that a priori the category might depend on this map (and, when  $X_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ , on the choice of ruling) and not just on the surface  $X_m$ . We will show in the following section that (just as for  $n = 1$ ) this is not an issue, but the argument is decidedly nontrivial.

Before proceeding to studying flatness for the global section category, we should note that there are also interpretations of  $\mathcal{H}^{(1)}$  and  $\mathcal{H}'^{(1)}$  in terms of the categories constructed in [20]. The point is that for  $n = 1$ , we are taking a spherical algebra relative to a master Hecke algebra. Since this is the endomorphism algebra of a vector bundle on  $\mathbb{P}^1$ , we immediately find that the spherical algebra and the DAHA are Morita equivalent. We find the following.

**Proposition 7.14.** *Let  $v, w \in \mathbb{Z}\langle s, f, e_1, \dots, e_m \rangle$ . Then*

$$\Gamma\mathcal{H}_{\eta', x_1, \dots, x_m; q}^{(1)}(v, w) \cong \begin{pmatrix} \Gamma\mathcal{S}_{\eta', x_1, \dots, x_m; q}^{(1)}(v, w) & \Gamma\mathcal{S}_{\eta', x_1, \dots, x_m; q}^{(1)}(v - 2f, w) \\ \Gamma\mathcal{S}_{\eta', x_1, \dots, x_m; q}^{(1)}(v, w - 2f) & \Gamma\mathcal{S}_{\eta', x_1, \dots, x_m; q}^{(1)}(v - 2f, w - 2f) \end{pmatrix}, \quad (7.23)$$

and similarly for  $\Gamma\mathcal{H}'^{(1)}$ .

*Remark.* One can also apply this at the level of sheaf categories on  $\mathbb{P}^1$ . There one finds (per the analogous statement of [20]) that the spherical sheaf category is the sheaf “ $\mathbb{Z}$ -algebra” associated to a noncommutative  $\mathbb{P}^1$ -bundle on  $\mathbb{P}^1$  [32]. There is, of course, no reason why we could not apply the Morita equivalence associated to  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  to any such noncommutative  $\mathbb{P}^1$ -bundle on  $\mathbb{P}^1$  and thus obtain an associated DAHA (which will always be a degeneration of the elliptic DAHA).

In [20], it was shown that the algebras  $\mathcal{S}$  satisfy a “Fourier transform” symmetry swapping  $\eta$  and  $\eta'$  and swapping  $s$  and  $f$ , which in turn induces a symmetry

$$\Gamma\mathcal{S}_{\eta', x_1, \dots, x_m; q, 0}^{(1)} \cong \Gamma\mathcal{S}_{-\eta', x_1 - \eta'/2, \dots, x_m - \eta'/2; q, 0}^{(1)} \quad (7.24)$$

again swapping  $s$  and  $f$ . (We will show in the next section how to extend this to general  $n$ .) Since the description of  $\Gamma\mathcal{H}^{(1)}$  in terms  $\Gamma\mathcal{S}^{(1)}$  is not invariant under swapping  $s$  and  $f$ , this symmetry does not actually extend to the DAHA itself. It turns out that, at least for  $m = 0$ , this is a consequence of the compactification we performed. Each Hom space of degree  $2s + 2f$  contains an element 1 (in a Fourier-invariant way!); if we localize with respect to those elements, then the objects  $v$  and  $v + 2s + 2f$  of the category become isomorphic, and thus we may replace  $v - 2f$  by  $v + 2s$  in the above description. Using the translation symmetry, we can then subtract  $2s$  from  $v$  and  $v + 2s$  at the cost of changing the parameters slightly. But then swapping  $s$  and  $f$  recovers the above description of  $\Gamma\mathcal{H}^{(1)}$ . In other words, the localized DAHAs actually *do* satisfy a Fourier transformation symmetry (though it is not clear how to describe it in terms of explicit operators). It is likely that something similar holds in general (including  $n > 1$ ), but this will require a better understanding of the relevant Morita equivalences.

In the univariate setting, one can gain some insight from the results of [15] on the traditional  $C^\vee C_1$  Hecke algebra. This suggests in general that if one replaces  $-2f$  above by  $-2f + e_1 + \dots + e_k$ , that this will have the effect of moving the parameters  $x_1, \dots, x_k$  from  $s_0$  to  $s_n$ . If so, then the Fourier transform would continue to extend to the noncompact elliptic DAHA in the presence of non- $t$  parameters, but would effectively swap the roles of the two roots vis-à-vis the parameters.

This degeneration also gives strong evidence that the full  $\mathrm{SL}_2(\mathbb{Z})$  action will *not* extend to the elliptic DAHA (compact or not). Indeed, if one looks at the action of  $\mathrm{SL}_2(\mathbb{Z})$  on the corresponding surfaces in the case that the spherical algebra is abelian, one finds that it relies on the fact that

the anticanonical curve at infinity is singular. Each generator of  $\mathrm{SL}_2(\mathbb{Z})$  blows up a singular point and then blows down a different component of the anticanonical curve, so that the anticanonical curve has the same shape and its complement has not changed, but the actual projective surface has. Blowing up a smooth point of the anticanonical curve in general *does* change the complement of the anticanonical curve, and thus we cannot expect this operation to survive to the elliptic level.

As with sheaves in general, when we take global sections in a family of sheaf categories, the fibers of the global sections can differ considerably from the global sections of the fibers. That is, there is a natural morphism from each fiber of the global section category to the global section category of the corresponding fiber, but this morphism can fail to be either injective or surjective. The failure of injectivity is particularly bad when we consider that the kernel of the map does not inherit an interpretation in terms of difference operators. In particular, if we have such a failure of injectivity, then we can no longer be confident that the fiber is a domain.

Each Hom sheaf in the global section category is the direct image of the corresponding Hom bimodule, and we can factor the direct image through one of the projections  $\mathbb{P}^n \times \mathbb{P}^n \rightarrow \mathbb{P}^n$  to find that the Hom sheaf is the direct image of a vector bundle on  $\mathbb{P}^n$ . If every fiber of that vector bundle is acyclic, then Grauert tells us that taking the direct image actually *does* commute with passing to fibers. It turns out that this holds (modulo some genericity assumptions in some cases) for a sufficiently large class of degrees to allow us to prove in general that the map is always injective and that the global section category is flat.

There are two ways to show acyclicity. One is to show that every subquotient of the Bruhat filtration is acyclic; the other is to use the symmetric power description for  $t = 0$  to deduce acyclicity for  $t = 0$  and thus in a neighborhood of  $t = 0$  by semicontinuity. In either case, the bundle is either itself a symmetric power or is built up out of symmetric powers, and thus we need to understand when such a bundle is acyclic.

Given a sheaf  $M$  on a scheme  $X$ , we may define a sheaf  $\mathrm{Sym}^n(M)$  on the symmetric power  $\mathrm{Sym}^n(X)$  by descending  $M^{\boxtimes n}$  through the quotient by  $S_n$ .

**Lemma 7.15.** *Let  $X$  be a projective scheme over a field  $k$ , and let  $M$  be an acyclic sheaf on  $X$ . Then  $\mathrm{Sym}^n(M)$  is an acyclic sheaf on  $\mathrm{Sym}^n(X)$  for all  $n \geq 1$ .*

*Proof.* First, note that if  $k$  has characteristic  $p > n$  or 0, then this is immediate, since  $\mathrm{Sym}^n(M)$  is a direct summand of an acyclic sheaf, namely the direct image under a finite morphism of the acyclic sheaf  $M^{\boxtimes n}$ .

In general, we proceed by induction on the pair  $(n, \dim X)$  relative to the product partial order. Let  $\mathcal{O}_X(1)$  be a very ample divisor on  $X$ , and note that  $\mathrm{Sym}^n(\mathcal{O}_X(1))$  is at least ample on  $\mathrm{Sym}^n(X)$ . (It can fail to be very ample!) In particular, there exists  $l > 0$  so that

$$\mathrm{Sym}^n(M) \otimes \mathrm{Sym}^n(\mathcal{O}_X(1))^l \cong \mathrm{Sym}^n(M(l)) \tag{7.25}$$

is acyclic. Choose a nonzero section of  $\mathcal{O}_X(l)$ , and use it to embed  $M$  as a subsheaf of  $M(l)$ . This one-step filtration of  $M(l)$  induces a symmetric power filtration  $F$  of  $\mathrm{Sym}^n(M(l))$  such that  $F_{m+1}/F_m$  is the direct image on  $\mathrm{Sym}^n(X)$  of the sheaf  $\mathrm{Sym}^m(M(l)/M) \boxtimes \mathrm{Sym}^{n-m}(M)$  on  $\mathrm{Sym}^m(X) \times \mathrm{Sym}^{n-m}(X)$ . By induction each subquotient  $F_{m+1}/F_m$  for  $m > 0$  is acyclic, as each factor is either a symmetric power of lower degree or supported on a lower-dimensional projective scheme. Since  $F_1 = \mathrm{Sym}^n(M)$ , it follows that  $\mathrm{Sym}^n(M(l))/\mathrm{Sym}^n(M)$  is acyclic, and thus that  $H^p(\mathrm{Sym}^n(M)) = 0$  for  $p > 1$ .

For all  $m \geq 0$ , we have  $H^0(\mathrm{Sym}^n(M(m))) \cong \mathrm{Sym}^n(H^0(M(m)))$ , and thus  $h^0(\mathrm{Sym}^n(M(m))) = \binom{h^0(M(m)) + n - 1}{n}$ . Applying this to  $m \gg 0$  lets us compute the Hilbert polynomial of  $\mathrm{Sym}^n(M)$ , and

then setting  $m = 0$  gives

$$\chi(\mathrm{Sym}^n(M)) = \binom{\chi(M) + n - 1}{n} = \binom{h^0(M) + n - 1}{n} = h^0(\mathrm{Sym}^n(M)). \quad (7.26)$$

Since we have already shown that the higher cohomology spaces vanish, this implies that  $h^1$  also vanishes, and the claim follows.  $\square$

By considering subquotients for the Bruhat filtration, we obtain the following.

**Lemma 7.16.** *Suppose  $d' \geq d - 1$ . Then every fiber of  $\mathcal{S}_{\eta';q,t}^{(n)}(0, ds + d'f)$  is acyclic for the morphism to parameter space.*

**Lemma 7.17.** *Suppose  $2d'/3 \geq d - 1$ . Then every fiber of  $\mathcal{S}'_{x_0;q,t}{}^{(n)}(0, ds + d'f)$  is acyclic for the morphism to parameter space.*

**Lemma 7.18.** *Suppose  $d' \geq \max(d, d/2 + r_1)$  and  $0 \leq r_1 \leq d$ . Then every fiber of  $\mathcal{S}_{\eta',x_1;q,t}^{(n)}(0, ds + d'f - r_1e_1)$  is acyclic for the morphism to parameter space.*

*Proof.* The first two lemmas are straightforward. For the third, note that if  $r_1 \leq d/2$ , then imposing the vanishing conditions subtracts  $r_1$  from the degree of the top subquotient,  $r_1 - 1$  from the next, etc., until we reach 0, and in each case there is sufficient degree to do this without becoming negative (or 0, apart from the subquotient supported on  $\mathbb{P}^1$ ). For  $r_1 > d/2$ , we apply the elementary transformation symmetry to reduce to a  $r_1 \leq d/2$  case with the opposite parity. Each case gives a convex cone in which we are guaranteed acyclicity, and combining the cones gives the desired result.  $\square$

**Proposition 7.19.** *Suppose  $v = ds + d'f - r_1e_1 - \cdots - r_me_m$  satisfies the inequalities  $d' \geq \max(d, d/2 + r_1)$ ,  $d \geq r_1 + r_2$  and  $r_1 \geq r_2 \geq \cdots \geq r_m \geq 0$ , and either  $v = 0$  or  $2d + 2d' - r_1 - r_2 - \cdots - r_m > 0$ . Then every fiber of  $\mathcal{S}_{\eta',x_1,\dots,x_m;q,t}^{(n)}(0, v)$  is acyclic for the morphism to parameter space.*

*Proof.* Every subquotient is the direct image under a finite morphism of an outer tensor product of symmetric powers of subquotients of the  $n = 1$  case. The constraints on  $v$  ensure that every univariate subquotient is acyclic: either (the direct image of) a line bundle of positive degree on  $\mathcal{E}$  or a line bundle of nonnegative degree on  $\mathbb{P}^1$ . It follows that every Bruhat subquotient for general  $n$  is acyclic, and thus the same holds for the full Hom space.

To check the univariate assertion, note that for  $d = 0$ , the vector bundle is simply  $\mathcal{O}_{\mathbb{P}^1}(d')$ , so there is no problem. For  $d > 0$ , the leading subquotient in the filtration comes from a line bundle on  $\mathcal{E}$  of degree  $2d + 2d' - r_1 - \cdots - r_m$ , so is positive, and the corresponding subsheaf is the same as the Hom sheaf obtained by subtracting  $2s + 2f - e_1 - \cdots - e_m$  from  $v$  (unless  $d = 1$ , when the subsheaf is trivial and there is nothing further to discuss). If  $m > 1$  (the  $m = 1$  case already having been dealt with), then this subtraction preserves all of the inequalities except possibly  $r_m \geq 0$  and  $2d + 2d' - r_1 - \cdots - r_m > 0$ . The first inequality could only be violated if we had  $r_m = 0$ , in which case we might as well have omitted that parameter. For the other inequality, we are adding  $m - 8$  to the left-hand side, so there is no problem if  $m \geq 8$ . But if  $m < 8$ , then the inequality is implied by the other inequalities.  $\square$

Note that this Proposition is already stronger than it seems, as we can always arrange to have the inequalities  $d \geq r_1 + r_2$ ,  $r_1 \geq r_2 \geq \cdots \geq r_m \geq 0$  by applying a suitable combination of elementary transformations and setting negative  $r_i$  to 0. Indeed, with the exception of the final

inequality, this is just stating that the vector  $v$  is in the fundamental chamber for the corresponding action of  $W(D_m)$ . In particular, for any vector  $v$ , we can use this Proposition to find an explicit  $d'$  such that  $v + d'f$  satisfies acyclicity. (The existence of such a  $d'$  was of course already guaranteed by Serre vanishing.) If  $r_1 \leq d/2$ , then this bound is pretty close to tight (based on what we know about the  $n = 1$  case), but for  $r_1 \geq d/2$ , the following result suggests that there is considerable room for improvement.

**Proposition 7.20.** *Suppose  $v = ds + d'f - r_1e_1 - \cdots - r_me_m$  satisfies the inequalities  $d' \geq d \geq r_1 + r_2$ ;  $r_1 \geq r_2 \geq \cdots \geq r_m \geq 0$ ; and either  $v = 0$  or  $2d + 2d' - r_1 - r_2 - \cdots - r_m > 0$ . Then there is a codimension  $\geq 2$  subscheme of parameter space such that every fiber of  $\mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(n)}(0, v)$  on the complement is acyclic for the morphism to parameter space.*

*Proof.* If  $n = 1$ , then these inequalities are enough to guarantee acyclicity, per [20]. It follows that any fiber with  $t = 0$  satisfies acyclicity, and thus the open subscheme on which acyclicity holds contains the divisor  $t = 0$ . By the  $t \mapsto q - t$  symmetry, the acyclic locus also contains the divisor  $t = q$ . This pair of divisors is relatively ample over  $\mathcal{M}_{1,1}$  for the  $\mathcal{E}^2$  parametrizing  $q$  and  $t$ , and thus their complement contains no point of codimension  $\leq 1$ .  $\square$

We of course conjecture that the codimension  $\geq 2$  subscheme is always empty.

**Corollary 7.21.** *Suppose  $v = ds + d'f - r_1e_1 - \cdots - r_me_m$  satisfies the inequalities  $d' \geq d \geq r_1 + r_2$  and  $r_1 \geq r_2 \geq \cdots \geq r_m \geq 0$ . Then there is a codimension  $\geq 2$  subscheme of parameter space (empty if  $d' \geq d/2 + r_1$ ) on the complement of which  $\Gamma\mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(n)}(0, v)$  is flat and the map to meromorphic difference operators is injective on fibers.*

*Proof.* If  $v = 0$  or  $2d + 2d' - r_1 - \cdots - r_m > 0$ , then this follows from acyclicity, so suppose  $2d + 2d' - r_1 - \cdots - r_m \leq 0$ . This is the degree of the leading subquotient of the univariate filtration; if it is negative, then this leading subquotient never has a global section, while if it is 0, the line bundle depends nontrivially on the parameters, and thus *generically* does not have a global section. Either way, the direct image of a nontrivial symmetric power of the leading univariate subquotient will always be 0, and the same holds for an outer tensor product with such a power.

Consider a Bruhat order ideal (i.e., an order ideal in the poset of dominant weights) contained in the interval  $[\leq (d/2, \dots, d/2)]$ . If this order ideal contains a dominant weight with  $\lambda_1 = d/2$ , then there is such a weight which is a maximal element of the order ideal. Since the subquotient corresponding to that maximal element has no direct image, removing it has no effect on the direct image. We thus find that the direct image of the interval  $[\leq (d/2, \dots, d/2)]$  is the same as the direct image of the interval  $[\leq (d/2 - 1, \dots, d/2 - 1)]$ , and thus we reduce to  $v - (2s + 2f - e_1 - \cdots - e_m)$  as before.  $\square$

For  $0 \leq m \leq 7$  (or for  $m = -1$ , i.e.,  $\mathbb{P}^2$ ), the corresponding commutative surface is a (possibly singular) del Pezzo surface with a choice of smooth anticanonical curve and a sequence of blowdowns to a Hirzebruch surface. The anticanonical embedding of this surface is given by the graded algebra

$$\bigoplus_{d \geq 0} \Gamma\mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(1)}(0, d(2s + 2f - e_1 - \cdots - e_m)); \quad (7.27)$$

we can interpret this as a graded algebra by using the fact that 1 is in

$$\Gamma\mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(1)}(d(2s + 2f - e_1 - \cdots - e_m), (d + 1)(2s + 2f - e_1 - \cdots - e_m)) \quad (7.28)$$

for any  $d$ . This is the Rees algebra of the natural filtration on the spherical algebra

$$\bigcup_{d \geq 0} \Gamma \mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(1)}(0, d(2s + 2f - e_1 - \dots - e_m)), \quad (7.29)$$

the coordinate ring of the complement of the chosen smooth anticanonical curve. Taking the multivariate versions thus gives deformations of the (anticanonically embedded) symmetric powers of  $X_m$  and  $X_m \setminus E$ , and, apart from possible codimension  $\geq 2$  exceptions for  $\mathbb{P}^2$ , these deformations are always flat, and every fiber is a domain.

If  $X_m$  has  $-2$ -curves or  $m > 7$ , then the anticanonical divisor is no longer ample, and thus one can no longer expect to obtain a deformation of  $\text{Sym}^n(X_m)$  as a graded algebra, or of  $\text{Sym}^n(X_m \setminus E)$  as a filtered algebra. This is why we generalized the spherical algebra construction to the above categories: one needs to work with *some* non-(pluri)anticanonical divisor, and it is then easier to include all divisors.

There is an interesting phenomenon that arises for the spherical algebra in the  $m = 8$  case. The global section algebra in this case is trivial (consisting only of the global section 1), but this merely reflects the fact that the generic fiber has no nontrivial global sections. The univariate subquotients in this case are all multiples of  $x_1 + \dots + x_8 - 2\eta'$ , and thus if this value is  $r$ -torsion, then any subquotient of weight a multiple of  $r$  will be trivial. (In terms of surfaces, this corresponds to the case that  $X_8$  is an elliptic surface, in which one fiber consists of  $r$  copies of the chosen anticanonical curve.) As a result, the dimension of global sections of such a fiber in a given Bruhat interval can in principle be as large as the number of such weights contained in the interval, or (a priori) as small as 1.

It turns out that at least for  $r = 1$  (i.e., when the elliptic surface has a section), this upper bound is attained (i.e., the dimension of global sections in a Bruhat interval is equal to the size of the Bruhat interval), and furthermore those global sections satisfy a surprising property. Note that it suffices to find  $n + 1$  global sections of degree  $2s + 2f - e_1 - \dots - e_8$ , as we can then obtain global sections with arbitrary dominant weight by taking products.

**Theorem 7.22.** *On any fiber such that  $2\eta' = x_1 + \dots + x_8$ , the space of global sections of  $\mathcal{S}_{\eta', x_1, \dots, x_8; q, t}^{(n)}(0, 2s + 2f - e_1 - \dots - e_8)$  is  $n + 1$ -dimensional, and any two global sections commute.*

*Proof.* Certainly,  $n + 1$  is an upper bound on the number of global sections, since there are  $n + 1$  subquotients, each of which has a unique global section. The given Hom bimodule is contained in the Hom bimodule  $\mathcal{S}_{\eta', x_1, \dots, x_7; q, t}^{(n)}(0, 2s + 2f - e_1 - \dots - e_7)$ , and the latter Hom bimodule satisfies acyclicity. Since the  $m = 7$  bimodule has 2 global sections when  $n = 1$ , it has  $n + 1 = \binom{2+n-1}{n}$  global sections when  $t = 0$  and thus (by flatness) in general. We thus need to show that those global sections are actually global sections of the subsheaf we want.

Let  $\mathcal{D}$  be such a global section. This is determined by the left coefficients  $c_m$  of  $\prod_{1 \leq i \leq m} T_i^{-1}$  for  $0 \leq m \leq n + 1$ , where  $c_m$  is  $S_m \times C_{n-m}$ -invariant. Each  $c_m$  is a section of a line bundle  $\mathcal{L}_m \otimes \mathcal{O}(D_m)$ , where  $\mathcal{L}_m$  comes from the equivariant gerbe and  $D_m$  comes from the allowed poles and forced zeros. The allowed poles are somewhat complicated, since we are not assuming that  $c_m$  is a leading coefficient, but the forced zeros are the same as they would have been if it were a leading coefficient. The symmetric power property then tells us that when  $t = 0$ , the forced zeros associated to  $t$  must all cancel allowed poles, and any allowed pole associated to a root of type  $D_n$  must be cancelled in this way.

The remaining zeros and poles can be deduced from the univariate case, and we thus find that

$c_m$  is a multiple of

$$\begin{aligned} & \prod_{1 \leq i < j \leq m} \frac{\vartheta(t - z_i - z_j, q + t - z_i - z_j)}{\vartheta(-z_i - z_j, q - z_i - z_j)} \prod_{\substack{1 \leq i \leq m \\ m < j \leq n}} \frac{\vartheta(t - z_i \pm z_j)}{\vartheta(-z_i \pm z_j)} \\ & \times \prod_{1 \leq i \leq m} \frac{\prod_{1 \leq j \leq 7} \vartheta(q/2 + x_j - z_i)}{\vartheta(-2z_i, q - 2z_i)} \prod_{m < j \leq n} \frac{1}{\vartheta(-q - 2z_j, q - 2z_j)}, \end{aligned} \quad (7.30)$$

in the sense that the ratio is a holomorphic section of the line bundle with polarization

$$\sum_{1 \leq i \leq m} (z_i^2/2 - (q/2 + x_8)z_i) + \sum_{m < j \leq n} 4z_j^2, \quad (7.31)$$

modulo line bundles on the base. (The only difference between this and the leading coefficient of the corresponding Bruhat interval are the factors  $\vartheta(-q - 2z_i, q - 2z_i)$  for  $m < j \leq n$ .) This line bundle has degree 1 in each  $z_i$  for  $1 \leq i \leq m$ , so every holomorphic section has the same dependence on those variables, which we can read off from the polarization.

We thus conclude that  $c_m / \prod_{1 \leq i \leq m} \vartheta(q/2 + x_8 - z_i)$  is independent of  $z_1$  through  $z_m$ , so is still holomorphic. As a result, we find that every global section of the  $m = 7$  bimodule is also a global section for  $m = 8$ ; more precisely, the holomorphy gives it generically, but the condition is closed, so it holds in general.

It remains to show commutativity. We first note as a sanity check that the  $n + 1$  leading term operators

$$\prod_{1 \leq i < j \leq m} \frac{\vartheta(t - z_i - z_j, q + t - z_i - z_j)}{\vartheta(-z_i - z_j, q - z_i - z_j)} \prod_{\substack{1 \leq i \leq m \\ m < j \leq n}} \frac{\vartheta(t - z_i \pm z_j)}{\vartheta(-z_i \pm z_j)} \prod_{1 \leq i \leq m} \frac{\prod_{1 \leq j \leq 8} \vartheta(q/2 + x_j - z_i)}{\vartheta(-2z_i, q - 2z_i)} T_i^{-1} \quad (7.32)$$

commute. It follows that on any fiber with  $x_1 + \cdots + x_8 = 2\eta'$ , the global section algebra of the spherical algebra

$$\bigcup_{d \geq 0} \mathcal{S}_{\eta', x_1, \dots, x_8; q, t}^{(n)}(0, d(2s + 2f - e_1 - \cdots - e_8)) \quad (7.33)$$

has abelian associated graded; the above leading term operators give one element for each dominant weight, so generate the associated graded.

This global section algebra has a particularly nice symmetry: it is preserved by the formal adjoint with respect to the density

$$\prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq 8} \Gamma_q(q/2 + x_j \pm z_i)}{\Gamma_q(\pm 2z_i)} \prod_{1 \leq i < j \leq n} \frac{\Gamma_q(t \pm z_i \pm z_j)}{\Gamma_q(\pm z_i \pm z_j)} dT. \quad (7.34)$$

Indeed, this is the composition of the Selberg adjoint and all 8 elementary transformations, with the total effect on the parameters being  $\eta' \mapsto x_1 + \cdots + x_8 - \eta' = \eta'$ . Note that although such isomorphisms are usually only defined up to a unit, we can eliminate that freedom by insisting that the adjoint of 1 be 1. This is triangular with respect to Bruhat order, and is trivial on the associated graded, since it fixes the generators and the associated graded is abelian. Since a triangular involution which is 1 on the diagonal is 1, we find that this formal adjoint acts trivially on the entire global section algebra. Since an algebra consisting entirely of self-adjoint operators is abelian, we conclude that the generators commute as required.  $\square$

Since the  $n + 1$  operators are filtered by Bruhat order, it is natural from an integrable systems perspective to designate the first nontrivial operator (with leading term  $\propto T_1^{-1}$ ) as the Hamiltonian. This has leading coefficient

$$\frac{\prod_{1 \leq j \leq 8} \vartheta(q/2 + x_j - z_1)}{\vartheta(-2z_1, q - 2z_1)} \frac{\prod_{2 \leq j \leq n} \vartheta(t - z_1 \pm z_j)}{\prod_{2 \leq j \leq n} \vartheta(-z_1 \pm z_j)}, \quad (7.35)$$

which turns out to be a mild reparametrization of the leading coefficient of the Hamiltonian proposed by van Diejen in [6] (see also [33, (3.12-3.14)], with the caveat that one must gauge the operator), and later shown to be integrable in [12]. In fact, one can verify (we omit the details) that van Diejen's operator satisfies the appropriate residue conditions to be a global section of  $\mathcal{S}_{\eta', x_1, \dots, x_8; q, t}^{(n)}(0, 2s + 2f - e_1 - \dots - e_8)$ , and thus we have given a new proof that van Diejen's Hamiltonian is integrable.

*Remark.* Since our Hecke algebra methods gave a new proof of the existence of the commuting operators which were constructed in [12], it is natural to wonder whether there might be applications in the other direction; that is, using their R-matrix based approach to construct global sections of other Hom sheaves in our spherical DAHA categories. Such a construction might make it possible to prove flatness in general without having to exclude a codimension  $\geq 2$  subscheme; if a given Hom bimodule generically has  $N$  global sections, then to prove flatness and injectivity in a neighborhood of a given fiber, it suffices to construct  $N$  local sections on a neighborhood of the fiber such that the restrictions to the fiber are linearly independent.

The connection to elliptic surfaces suggests a possible generalization of this integrable system. If  $x_1 + \dots + x_8 - 2\eta'$ , instead of being 0, is a torsion point of order  $r$ , then we again find that there are many trivial Bruhat subquotients, and thus it becomes nontrivial to determine how many global sections the spherical algebra has. We cannot answer this in general, but we can, at least, show that the  $r$ -torsion condition forces there to be *some* nontrivial global sections.

**Proposition 7.23.** *Let  $E$  be an elliptic curve and  $\eta', x_1, \dots, x_8, q, t$  be points of  $E$  such that  $x_1 + \dots + x_8 - 2\eta'$  is a torsion point of order  $r$ . Then the corresponding fiber of the spherical algebra  $\bigcup_d \mathcal{S}_{\eta', x_1, \dots, x_8; q, t}^{(n)}(0, d(2s + 2f - e_1 - \dots - e_8))$  has a global section of dominant weight  $r$  with nonzero leading term.*

*Proof.* Indeed, every subquotient in the Bruhat filtration for the order ideal  $[< r]$  is acyclic: the bottom subquotient is  $\mathcal{O}_{P^n}$ , while the remaining subquotients are nontrivial elements of  $E^n[r]$ .  $\square$

It is then natural to conjecture that the resulting Hamiltonian is integrable, or more precisely the following.

**Conjecture 1.** *Under the same hypotheses, the fiber of*

$$\mathcal{S}_{\eta', x_1, \dots, x_8; q, t}^{(n)}(0, r(2s + 2f - e_1 - \dots - e_8)) \quad (7.36)$$

*has  $n + 1$  global sections, all of which commute.*

Both parts of the proof for  $r = 1$  fail here: the  $m = 7$  surface has too many global sections, and the adjoint no longer gives an element of the same spherical algebra. There is some experimental evidence for this Conjecture, however: for  $n = r = 2$ , the analogous statement for a suitable degeneration to a nodal curve holds by a computer calculation. (We will briefly discuss how to construct such degenerations at the end of the next section.) This statement is, of course, trivial for  $n = 1$ , but it is worth noting there that the global sections  $\mathcal{S}_{\eta', x_1, \dots, x_8; 0, 0}^{(1)}(0, r(2s + 2f - e_1 - \dots - e_8))$  are just the pullback of  $\mathcal{O}(1)$  from the base of the elliptic fibration.

## 8 The (spherical) $C^\vee C_n$ Fourier transform

Our objective in the present section is to prove the following result.

**Theorem 8.1.** *There is, locally on the base, an isomorphism*

$$\Gamma\mathcal{S}_{2c, x_1, \dots, x_m; q, t}^{(n)} \cong \Gamma\mathcal{S}_{-2c, x_1 - c, \dots, x_m - c; q, t}^{(n)} \quad (8.1)$$

acting on objects as  $ds + d'f - r_1e_1 - \dots - r_me_m \mapsto d's + df - r_1e_1 - \dots - r_me_m$  and triangular with respect to the Bruhat filtration. Moreover, this isomorphism commutes (up to local units) with the Selberg adjoint.

We refer to this isomorphism as the ‘‘Fourier transform’’: in particular, note that it takes multiplication operators (of degree  $d'f$ ) to difference operators (of  $d's$ ) and (at least on the parameters) is an involution. (In addition, though we will not be using this fact, the Fourier transform can be represented in the analytic setting by a formal integral operator [19].)

Before constructing the Fourier transform, we give some consequences. The simplest is that we can conjugate the symmetry by an elementary transformation.

**Corollary 8.2.** *There is, locally on the base, an isomorphism*

$$\Gamma\mathcal{S}'_{x_1 + 2c, x_1, x_2, \dots, x_m; q, t}^{(n)} \cong \Gamma\mathcal{S}'_{x_1 - c, x_1 + c, x_2 - c, \dots, x_m - c; q, t}^{(n)} \quad (8.2)$$

acting on objects as  $ds + d'f - r_1e_1 - r_2e_2 - \dots - r_me_m \mapsto (d' - r_1)s + d'f - (d' - d)e_1 - r_2e_2 - \dots - r_me_m$ .

This also tells us that the deformations of  $\mathbb{P}^2$  we constructed are independent of  $x_0$  (as one would expect).

**Corollary 8.3.** *The restriction to  $\mathbb{Z}(s + f)$  of  $\Gamma\mathcal{S}'_{x_0; q, t}^{(n)}$  is (fpf locally) independent of  $x_0$ .*

*Proof.* The previous Corollary gives (locally) an isomorphism

$$\Gamma\mathcal{S}'_{x_0, x_0 - 2c; q, t}^{(n)} \cong \Gamma\mathcal{S}'_{x_0 - 3c, x_0 - c; q, t}^{(n)} \quad (8.3)$$

The action on objects takes  $d(s + f)$  to  $d(s + f)$ , so this local isomorphism induces a local isomorphism

$$\Gamma\mathcal{S}'_{x_0; q, t}^{(n)}|_{\mathbb{Z}(s + f)} \cong \Gamma\mathcal{S}'_{x_0 - 3c; q, t}^{(n)}|_{\mathbb{Z}(s + f)} \quad (8.4)$$

for any  $x_0$  and  $c$ . It follows that any two geometric fibers with the same values of  $q, t$  are isomorphic.  $\square$

*Remark 1.* More generally, if  $(E, x_0, x_1, q, t)$  is a point of  $\mathcal{E}^4$  over some scheme  $S$ , then we have an isomorphism  $\Gamma\mathcal{S}'_{x_0; q, t}^{(n)}|_{\mathbb{Z}(s + f)} \cong \Gamma\mathcal{S}'_{x_1; q, t}^{(n)}|_{\mathbb{Z}(s + f)}$  defined Zariski locally on  $S$  as long as  $x_1 - x_0 \in 3E(S)$ . Without this assumption, there may very well be no such isomorphism; indeed for  $n = 1, q = 0$ , these are essentially the homogeneous coordinate rings of the embeddings of  $C$  via  $[x_0] + 2[0]$  and  $[x_1] + 2[0]$ .

*Remark 2.* When  $c \in E[3]$ , this isomorphism becomes an automorphism, but is quite nontrivial.

The most significant consequence is the following. Note here that we are, as usual, taking global sections *before* passing to fibers.

**Theorem 8.4.** *For any  $v \in \mathbb{Z}\langle s, f, e_1, \dots, e_m \rangle$ , there is a codimension  $\geq 2$  subscheme of parameter space on the complement of which  $\Gamma\mathcal{S}'_{\eta', x_1, \dots, x_m; q, t}^{(n)}(0, v)$  is flat and the map to meromorphic difference operators is injective on fibers.*

*Proof.* Applying the Fourier transform has no effect on flatness (since it is an isomorphism), and the Fourier transform will be constructed via an action on meromorphic difference operators, and thus injectivity on fibers is also preserved. This allows us to reduce to Corollary 7.21, as in [20]. To be precise, let  $v = ds + d'f - r_1e_1 - \cdots - r_me_m$ . We may apply a permutation and an even number of elementary transformations to put  $v$  into the fundamental chamber for  $W(D_m)$ . Moreover, if  $r_m < 0$ , then we may set it to 0 without changing the sheaf of global sections, and in this way may arrange to have  $d \geq r_1 + r_2$  and  $r_1 \geq \cdots \geq r_m \geq 0$ . If  $d' \geq d$ , then we may apply Corollary 7.21. If  $d' < 0$ , then we observe that  $\Gamma \mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(n)}(0, v) = 0$ , so the result again follows. Otherwise, we apply the Fourier transform. Since this strictly decreases  $d$  but keeps it nonnegative, the claim follows by induction.  $\square$

*Remark.* Of course, the codimension  $\geq 2$  subscheme is the same as that of the appropriate special case of Corollary 7.21.

Just as elliptic pencils gave rise to integrable systems above, there is something analogous (if slightly weaker) for rational pencils. Call a small category with object set  $\mathbb{Z}$  “quasi-abelian” if there is a commutative graded algebra  $A$  such that  $\text{Hom}(j, k) \cong A[k - j]$  for all  $j, k$ , with composition given by multiplication.

**Corollary 8.5.** *Suppose  $v \in \mathbb{Z}\langle s, f, e_1, \dots, e_m \rangle$  is the class of a rational pencil on the rational surface  $X_m$ . Then*

$$\Gamma \mathcal{S}_{\eta', x_1, \dots, x_m; q, t}^{(n)}|_{\mathbb{Z}v} \quad (8.5)$$

*is quasi-abelian, and the corresponding graded algebra is a free polynomial algebra in  $n+1$  generators.*

*Proof.* If  $v = f$ , this is easy: the global sections of degree  $df$  are just multiplication by  $C_n$ -invariant sections of the bundle with polarization  $d \sum_i z_i^2$ , and this is precisely the pullback of  $\mathcal{O}_{\mathbb{P}^n}(d)$ . More generally, it follows from the theory of rational surfaces (see [18]) that  $v$  represents a rational pencil iff it is in the orbit of  $f$  under the group  $W(E_{m+1})$  generated by the Fourier transform and  $W(D_m)$ .  $\square$

Just as integrable systems lead to natural eigenvalue equations, such “quasi-integrable” systems lead to generalized eigenvalue problems. A *generalized eigenfunction* of a space  $\mathcal{D}$  of operators is a function  $f$  such that the image  $\mathcal{D}f$  is 1-dimensional. (We then obtain an associated “generalized eigenvalue”, namely the point in  $\mathbb{P}(\mathcal{D})$  associated to the kernel of the map  $D \mapsto Df$  on  $\mathcal{D}$ .)

Given a quasi-integrable system associated to a rational pencil, we have for each  $d$  a map  $\phi_d$  from  $A[1]$  to the space of operators, such that  $\phi_{d+1}(y)\phi_d(x) = \phi_{d+1}(x)\phi_d(y)$ . We may then consider for each  $d$  the generalized eigenvalue problem associated to  $\phi_d(A[1])$ . For any generalized eigenfunction  $f_d$  for  $\phi_d(A[1])$ , let  $f_{d+1}$  be a nonzero representative of  $\phi_d(A[1])f_d$ . Then for suitable  $y \in A[1]$ , we have

$$\phi_{d+1}(x)f_{d+1} = \phi_{d+1}(x)\phi_d(y)f_d = \phi_{d+1}(y)\phi_d(x)f_d = \lambda_d(x)\phi_{d+1}(y)f_{d+1} \quad (8.6)$$

for all  $x \in A[1]$ , and thus  $f_{d+1}$  is a generalized eigenfunction for  $\phi_{d+1}(A[1])$ . More generally, if  $V \subset A[1]$  is such that the corresponding generalized eigenvalue problem for  $\phi_{d+1}(V)$  is nondegenerate (i.e., for each point of projective space, the corresponding problem has at most 1-dimensional solution space), then any generalized eigenfunction  $f_d$  for  $\phi_d(V)$  is a generalized eigenfunction for  $\phi_d(A[1])$ , since then  $\phi_d(y)f_d$  is a generalized eigenfunction for  $\phi_{d+1}(V)$ .

There are two cases of particular interest. In the case  $v = s + f - e_1 - e_2$ ,  $\eta' = -(n-1)t - q$ , the generators of the quasi-integrable system are operators of the form considered in [21], and the quasi-abelian property turns into the quasi-commutation relation used there. The corresponding generalized eigenvalue problem is precisely the difference equation [21, Prop. 3.9] satisfied by the

elliptic interpolation functions. (The interpolation kernel of [19] is also a generalized eigenfunction for the same space of operators, [19, Prop. 3.12].)

The biorthogonal functions of [22, 21] are also generalized eigenfunctions of such an integrable system, corresponding to  $v = 2s + 2f - e_1 - e_2 - e_3 - e_4 - 2e_5$  and  $\eta' = -(n-1)t - q$ . Indeed, the first-order difference operators considered in [22] correspond to products of operators of degrees  $s - e_5$ ,  $s + 2f - e_1 - e_2 - e_3 - e_4 - e_5$  and  $s + f - e_i - e_j - e_5$ ,  $1 \leq i < j \leq 4$ , giving rise to 8 operators of degree  $2s + 2f - e_1 - e_2 - e_3 - e_4 - 2e_5$ . The biorthogonal functions are generalized eigenfunctions of the span of these 8 operators, and the generalized eigenvalues are all distinct points of  $\mathbb{P}^7$ . It thus follows that any biorthogonal function is a generalized eigenfunction for the full space of operators. (With more effort, one can in fact verify that the generalized eigenvalues are given by suitable specializations of the leading coefficients; this is a consequence in general of the fact that the Fourier transform respects leading coefficients.)

We now turn to constructing the Fourier transform. The traditional approach would be to construct a Fourier transform on the DAHA and then observe that it restricts to a transform on the spherical algebra. One significant issue that arises here is that although we have a reasonable facsimile of a presentation, it is at the level of sheaves, not at the level of global sections, while the Fourier transform does not make sense in terms of sheaves (since it does not preserve multiplication). Furthermore, most of the rank 1 subalgebras we used to generate the DAHA do not have *any* nontrivial global sections (the leading Bruhat subquotient is a generically nontrivial line bundle of degree 0 in every variable). As a result, it seems unlikely that the Fourier transform on the DAHA (assuming it exists) would have a construction that was significantly simpler than the construction we give in the spherical case. Beyond that, there is another issue: as we discussed above, the description of the rank 1 DAHA via a Morita equivalence to the spherical algebra strongly suggests that the Fourier transform only exists for the noncompact version of the DAHA. In other words, the Fourier transform on the DAHA would not respect the filtration by degree; since this filtration comes from the Bruhat filtration, the latter also could not be preserved. As a result, even having a Fourier transform for the DAHA would not be enough to prove the Theorem; one also needs to understand why the spherical version is triangular!

We thus wish an approach that works directly with the spherical algebra. Note that since the action of the Fourier transform on objects preserves  $\mathbb{Z}\langle s, f \rangle$ , the Fourier transform for  $m > 0$  restricts to a transform of the same sort for  $m = 0$ . Moreover, since every Hom sheaf is contained in one of degree in  $\mathbb{Z}\langle s, f \rangle$ , it suffices to specify how the transform acts on such sheaves and show that it preserves the various subsheaves of interest. We thus focus our initial attention on the case  $m = 0$ .

In the univariate setting, the Fourier transform was easy to construct: for generic parameters, one can give an explicit presentation for the category (with generators of degrees  $s$  and  $f$ ) and this presentation has an obvious symmetry. Moreover, a slightly larger set of elements generates the category even without the genericity condition, and one can determine how the transform must act on those elements by taking a suitable limit.

Although we have analogues of those generators (and will indeed be able to describe their Fourier transforms explicitly), this approach founders in the multivariate setting for two reasons. The first is that the operators of degree  $s$  and  $f$  do not even come close to generating the category for  $n > 1$  in general: for  $q = 0$ ,  $t = 0$ , the full category is the bihomogeneous coordinate ring of  $\text{Sym}^n(\mathbb{P}^1 \times \mathbb{P}^1)$ , while the elements of degrees  $s$  and  $f$  lie in the subring corresponding to the quotient  $\mathbb{P}^n \times \mathbb{P}^n$ . If we include elements of degree  $s + f$ , the situation is somewhat better (we will see that these come close enough to generating to be useful), but this only forces us to confront the fact that we have absolutely no understanding of the *relations* satisfied by these elements.

As a result, we will need some way to construct the Fourier transform which is explicitly a homomorphism. We will do this by constructing a transform on a much larger algebra of operators, and then show that it preserves the particular subspace we care about. The simplest way to construct a homomorphism on a category of operators is to apply a gauge transformation: assign an operator to each object and apply the associated quasi-conjugation.

The first step in constructing such operators is to determine on what spaces they act, and thus we need to think a bit about where our existing operators act. Define a family of (gerbe) polarizations

$$P_d(\eta'; q, t) := -((n-1)t - (d-1)q + \eta') \sum_i z_i^2 / q. \quad (8.7)$$

If  $F$  is the product of a  $\Gamma_q$  symbol with polarization  $P_{d'_1-d_1}(\eta'; q, t)$  and a rational function on  $\mathcal{E}^n$ , then we can apply any global section of a fiber of  $\mathcal{S}_{\eta'; q, t}^{(n)}(d_1 s + d'_1 f, d_2 s + d'_2 f)$  and the result will be a rational function times a  $\Gamma_q$  symbol with polarization  $P_{d'_2-d_2}(\eta'; q, t)$ .

Thus the Fourier transform should be given by operators that take functions with polarization  $P_d(\eta'; q, t)$  to functions with polarization  $P_{-d}(-\eta'; q, t)$ , or equivalently take  $P_0(\eta' + dq; q, t)$  to  $P_0(-\eta' - dq; q, t)$ . There are issues in general, but there is one important case in which operators of this form do indeed exist. Indeed, the simplest way to obtain an operator mapping  $P_d(0; q, t)$  to  $P_{-d}(0; q, t)$  would be to take a global section of  $\mathcal{S}_{0; q, t}^{(n)}(df, ds)$ , assuming such a global section exists.

For  $d = 1$ , this is not too hard, and indeed we can understand global sections of order 1 in general.

**Lemma 8.6.** *For any point of  $\mathcal{E}^3$ , the corresponding fiber of  $\mathcal{S}_{\eta'; q, t}^{(n)}(0, s + d'f)$  is spanned by operators of the form*

$$\begin{aligned} D_q^{(n)}(u_0, u_1, \dots, u_{2d'+1}; t) \\ = \sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r < 2d'+2} \vartheta(u_r + \sigma_i z_i)}{\vartheta(2\sigma_i z_i)} \prod_{1 \leq i < j \leq n} \frac{\vartheta(t + \sigma_i z_i + \sigma_j z_j)}{\vartheta(\sigma_i z_i + \sigma_j z_j)} \prod_{1 \leq i \leq n} T_i^{\sigma_i/2} \end{aligned} \quad (8.8)$$

with  $u_0 + \dots + u_{2d'+1} = q + \eta'$ .

*Proof.* The interval  $[\leq (1/2, \dots, 1/2)]$  in the Bruhat order consists of a single double coset, and thus the space of global sections is (up to multiplication by an explicit product of  $\vartheta$  functions) the space of  $S_n$ -invariant sections of the appropriate line bundle. That space is spanned by products of the form  $\prod_{1 \leq i \leq n} f(z_i)$  where  $f$  is a section of the corresponding line bundle on  $\mathcal{E}$ , and any such section can be factored into  $\vartheta$  functions.  $\square$

**Proposition 8.7.** *For any  $d \geq 0$ , the space of global sections of  $\mathcal{S}_{0; q, t}^{(n)}(df, ds)$  is 1-dimensional.*

*Proof.* We proceed by induction in  $d$ , with the case  $d = 0$  being obvious. Suppose we are given a nonzero global section  $D_d^{(n)}(q, t) \in \mathcal{S}_{0; q, t}^{(n)}(df, ds)$ . Then for any  $u, v$ , the operator

$$D_q^{(n)}((d+1)q/2 \pm u; t) D_d^{(n)}(q, t) \prod_{1 \leq i \leq n} \vartheta(z_i \pm v) - D_q^{(n)}((d+1)q/2 \pm v; t) D_d^{(n)}(q, t) \prod_{1 \leq i \leq n} \vartheta(z_i \pm u) \quad (8.9)$$

is a section of  $\mathcal{S}_{0; q, t}^{(n)}((d-1)f, (d+1)s)$ . Moreover, we know the leading coefficient of  $D_d^{(n)}(q, t)$  up to a scalar multiple, and may therefore verify that both operators have the same leading coefficient. Since every subquotient below the top of the corresponding univariate vector bundle has negative degree, none of the multivariate subquotients below the top have polarizations represented by

positive semidefinite matrices. Thus none of those subquotients have any global sections, let alone symmetric ones. It follows that a section of  $\mathcal{S}_{0;q,t}^{(n)}((d-1)f, (d+1)s)$  with vanishing leading coefficient must in fact be 0, and thus

$$D_q^{(n)}((d+1)q/2 \pm u; t) D_d^{(n)}(q, t) \prod_{1 \leq i \leq n} \vartheta(z_i \pm v) = D_q^{(n)}((d+1)q/2 \pm v; t) D_d^{(n)}(q, t) \prod_{1 \leq i \leq n} \vartheta(z_i \pm u). \quad (8.10)$$

Equivalently,

$$D_q^{(n)}((d+1)q/2 \pm u; t) D_d^{(n)}(q, t) \prod_{1 \leq i \leq n} \vartheta(z_i \pm u)^{-1} \quad (8.11)$$

is independent of  $u$ . In particular, the apparent  $u$ -dependent poles of this product of operators are not, in fact, singularities, and thus this gives a section of  $\mathcal{S}_{0;q,t}^{(n)}((d+1)f, (d+1)s)$  as required. That this is the only global section up to scalar multiples follows by observing that again all Bruhat subquotients below the top have indefinite polarizations, while the top subquotient is trivial.  $\square$

Following the above proof, we define  $D_d^{(n)}(q, t)$  by the recurrence

$$D_{d+1}^{(n)}(q, t) = D_q^{(n)}((d+1)q/2 \pm u; t) D_d^{(n)}(q, t) \prod_{1 \leq i \leq n} \vartheta(z_i \pm u)^{-1}, \quad (8.12)$$

with base case  $D_0^{(n)}(q, t) = 1$ . (Note that  $D_1^{(n)}(q, t) = D_q^{(n)}(; t)$ .) Equivalently,  $D_d^{(n)}(q, t)$  is the unique global section of  $\mathcal{S}_{0;q,t}^{(n)}(df, ds)$  with leading term

$$\prod_{1 \leq i < j \leq n} \frac{\Gamma_q(dq + t - z_i - z_j)}{\Gamma_q(t - z_i - z_j)} \prod_{1 \leq i \leq j \leq n} \frac{\Gamma_q(-z_i - z_j)}{\Gamma_q(dq - z_i - z_j)} \prod_{1 \leq i \leq n} T_i^{-d/2}. \quad (8.13)$$

Since translation by  $s + f$  does not change the parameters, this also gives a global section of  $\mathcal{S}_{0;q,t}^{(n)}(d_0(s + f) + df, d_0(s + f) + ds)$  for any  $d_0$ .

The following result shows that these operators indeed behave like Fourier transforms.

**Proposition 8.8.** *We have the operator relations*

$$D_q^{(n)}((d+1)q/2 \pm u; t) D_d^{(n)}(q, t) = D_{d+1}^{(n)}(q, t) \prod_{1 \leq i \leq n} \vartheta(z_i \pm u) \quad (8.14)$$

$$D_d^{(n)}(q, t) D_q^{(n)}(-dq/2 \pm u; t) = \prod_{1 \leq i \leq n} \vartheta(z_i \pm u) D_{d+1}^{(n)}(q, t), \quad (8.15)$$

and, if  $u_0 + u_1 + u_2 + u_3 = (d+1)q$ ,

$$D_d^{(n)}(q, t) D_q^{(n)}(u_0, u_1, u_2, u_3; t) = D_q^{(n)}(u_0 + dq/2, u_1 + dq/2, u_2 + dq/2, u_3 + dq/2; t) D_d^{(n)}(q, t). \quad (8.16)$$

*Proof.* In each case, both sides are sections of the same Hom sheaf of  $\mathcal{S}_{0;q,t}^{(n)}$  with the same leading coefficient, and only the top subquotient has positive semidefinite polarization.  $\square$

It turns out that if we adjoined formal inverses of the operators  $D_d^{(n)}(q, t)$  and declared them to be  $D_{-d}^{(n)}(q, t)$ , then the result would indeed define a Fourier transform on a certain subcategory of the category with  $\eta' = 0$  (in which the Hom sheaves of degree  $ds + d'f$  for  $d > d'$  are replaced by the images under the Fourier transform of the Hom sheaves of degree  $d's + df$ ). Proving this directly is somewhat tricky, however, as unlike in the univariate setting, there does not appear

to be a readily accessible test for right divisibility by  $D_d^{(n)}(q, t)$ . And, of course, even using the translation symmetry, this would at best give us a transform for  $\eta' \in \mathbb{Z}q$ , which is especially weak when  $q$  is torsion.

The key idea for proceeding further is that the relation

$$D_d^{(n)}(q, t)D_q^{(n)}(-dq/2 \pm u; t) = \prod_{1 \leq i \leq n} \vartheta(z_i \pm u)D_{d+1}^{(n)}(q, t) \quad (8.17)$$

gives us a system of recurrences that we can use to solve for coefficients of  $D_d^{(n)}(q, t)$ . Indeed, it follows from this relation that

$$D_d^{(n)}(q, t)D_q^{(n)}(-dq/2 \pm u; t)|_{u=z_i} = 0 \quad (8.18)$$

for  $1 \leq i \leq n$ . Since the operators are symmetric, let us consider the specialization  $u = z_n$ . The coefficient of  $\prod_i T_i^{k_i - (d+1)/2}$  in this specialized operator is a linear combination of the left coefficients of  $\prod_i T_i^{l_i - d/2}$  in  $D_d^{(n)}(q, t)$  for  $\max(k_i - 1, 0) \leq l_i \leq k_i$ . The coefficient in this linear combination for  $\vec{l} = \vec{k}$  is

$$\frac{\prod_{1 \leq i \leq n} \vartheta(-k_i q + z_n - z_i, k_i q + z_i + z_n) \prod_{1 \leq i < j \leq n} \vartheta(t + dq - (k_i + k_j)q - z_i - z_j)}{\prod_{1 \leq i < j \leq n} \vartheta(dq - (k_i + k_j)q - z_i - z_j)}, \quad (8.19)$$

and thus we can solve for the coefficient of  $\prod_i T_i^{k_i - d/2}$  in  $D_d^{(n)}(q, t)$ , at least generically. In fact, we find the only difficulty arises when  $\vartheta(-k_n q) = 0$ , so if  $q$  is not torsion and  $\vec{k} \neq 0$ , there will always be one of the  $n$  specializations that allows us to solve for the coefficient of  $\prod_i T_i^{k_i - d/2}$  in terms of coefficients of terms which are smaller in dominance order. In other words,  $D_d^{(n)}(q, t)$  is determined by the given relation along with the choice of leading coefficient.

The fact that we can control coefficients near the leading coefficients suggests a way to proceed further: take an appropriate completion! Define a nonarchimedean metric on the  $\mathbb{Z}/2\mathbb{Z}$ -graded algebra  $k(X)[T_1, \dots, T_n, \prod_i T_i^{-1/2}]$  by

$$\left| \sum_{\vec{k}} c_{\vec{k}} \prod_i T_i^{k_i} \right| := \max_{\vec{k}: c_{\vec{k}} \neq 0} \exp\left(-\sum_i k_i\right). \quad (8.20)$$

We call an element of the corresponding completion a *formal difference operator*. We in particular denote the completion of the subalgebra  $k(X)[T_1, \dots, T_n]$  by  $k(X)[[T_1, \dots, T_n]]$ . This construction of course applies equally well to the case of twisted difference operators, or even to the corresponding category in which the objects are polarizations  $P_d(\eta'; q, t)$  with fixed  $\eta'$ . We will mostly suppress the twisting from the notation.

The major advantage of formal difference operators is that the ring has a large number of units. Indeed, the usual argument for inverting a commutative formal power series with invertible constant term applies equally well in the noncommutative setting to give the following.

**Proposition 8.9.** *If  $D \in k(X)[[T_1, \dots, T_n]]$  has nonzero constant term, then  $D$  is a unit.*

Since  $\prod_i T_i^{-1/2}$  is also clearly invertible, we find that any of the operators  $D_d^{(n)}(q, t)$  are invertible as formal operators. In fact, in the ring of formal operators we can solve for  $D_d^{(n)}(q, t)$  in terms of  $D_{d+1}^{(n)}(q, t)$  and in this way define  $D_d^{(n)}(q, t)$  for  $d < 0$ . We then find by an easy induction that  $D_{-d}^{(n)}(q, t) = D_d^{(n)}(q, t)^{-1}$ .

In addition to these inner automorphisms, we also have automorphisms coming from gauging by  $\Gamma_q$  symbols and translations on  $E$ . Let  $T_\omega(c)$  denote the translation of all variables by  $c$ , so that  $T_\omega(q/2) = \prod_{1 \leq i \leq n} T_i^{1/2}$ . Then for any  $\Gamma_q$  symbol  $\Gamma$  of polarization

$$(\eta' - \eta'') \sum_i z_i^2/q + 2((n-1)t + q + \eta')c \sum_i z_i/q, \quad (8.21)$$

there is an induced isomorphism

$$D \mapsto \Gamma T_\omega(c) D T_\omega(-c) \Gamma^{-1} \quad (8.22)$$

from  $\text{End}(P_0(\eta'; q, t))^0$  (the subspace involving only integer powers of  $T_i$ ) to  $\text{End}(P_0(\eta''; q, t))^0$ . With this in mind, we define a “formal gauging operator” from  $P_0(\eta'; q, t)$  to  $P_0(\eta''; q, t)$  to be an object of the form

$$\Gamma T_\omega(c) D \quad (8.23)$$

where  $D$  is a unit in the endomorphism ring. The “leading term” of such an operator is the formal symbol  $\Gamma T_\omega(c) f$  where  $f$  is the constant term of  $D$ . The formal gauging operators form a group, with a natural subgroup consisting of elements of the form  $\Theta T_\omega(kq/2) D$  where  $\Theta$  is a product of  $\vartheta$  symbols. If  $G_1, G_2$  are formal gauging operators such that  $G_1 G_2^{-1}$  lies in the subgroup of formal difference operators, then for any formal difference operator  $D$  with only integer shifts (and with coefficients having appropriate polarizations),  $G_1 D G_2^{-1} := (G_1 G_2^{-1}) G_2 D G_2^{-1}$  will again be a formal difference operator. This extends to the half-integer case by writing  $D = T_\omega(q/2) D'$  and  $G_1 D G_2^{-1} := (G_1 T_\omega(q/2) G_2^{-1}) G_2 D' G_2^{-1}$ . In either case, the operation clearly respects multiplication as long as the gauging operators match up.

**Proposition 8.10.** *There is a unique family of formal gauging operators  $\mathcal{D}_{q,t}^{(n)}(c)$  from  $P_0(2c; q, t)$  to  $P_0(-2c; q, t)$  with leading term*

$$\prod_{1 \leq i \leq j \leq n} \frac{\Gamma_q(-z_i - z_j)}{\Gamma_q(-2c - z_i - z_j)} \prod_{1 \leq i < j \leq n} \frac{\Gamma_q(t - 2c - z_i - z_j)}{\Gamma_q(t - z_i - z_j)} T_\omega(c) \quad (8.24)$$

such that  $\mathcal{D}_{q,t}^{(n)}(-dq/2) = D_d^{(n)}(q, t)$  for all  $d \in \mathbb{Z}$ . Moreover, if one divides any coefficient of  $\mathcal{D}_{q,t}^{(n)}(c)$  by the leading term, then the only  $z$ -independent poles of the resulting meromorphic function on  $\mathcal{E}^{n+3}$  are along hypersurfaces for which  $q$  is torsion.

*Proof.* If such a family of operators exists, then it must satisfy

$$\mathcal{D}_{q,t}^{(n)}(c) D_q^{(n)}(c \pm u; t)|_{u=z_i} = 0 \quad (8.25)$$

for  $1 \leq i \leq n$ . This gives an algebraic (and triangular) system of equations for the coefficients of  $\mathcal{D}_{q,t}^{(n)}(c)$  which we have already seen has at most one solution (and if it has a solution, the only  $z$ -independent poles are where  $q$  is torsion). Since it has a solution on the Zariski-dense set of divisors  $c \in \mathbb{Z}q$ , it must have a solution in general.  $\square$

*Remark.* For an analytic approach to constructing such operators, see [19].

Since we understand a Zariski dense subset of these operators, we can immediately deduce some relations.

**Proposition 8.11.** *The operators  $\mathcal{D}_{q,t}^{(n)}(c)$  satisfy the operator identities*

$$D_q^{(n)}(-c \pm u; t) \mathcal{D}_{q,t}^{(n)}(c + q/2) = \mathcal{D}_{q,t}^{(n)}(c) \prod_{1 \leq i \leq n} \vartheta(z_i \pm u) \quad (8.26)$$

$$\mathcal{D}_{q,t}^{(n)}(c) D_q^{(n)}(c \pm u; t) = \prod_{1 \leq i \leq n} \vartheta(z_i \pm u) \mathcal{D}_{q,t}^{(n)}(c - q/2) \quad (8.27)$$

and, if  $u_0 + u_1 + u_2 + u_3 = q + 2c$ ,

$$\mathcal{D}_{q,t}^{(n)}(c) D_q^{(n)}(u_0, u_1, u_2, u_3; t) = D_q^{(n)}(u_0 - c, u_1 - c, u_2 - c, u_3 - c; t) \mathcal{D}_{q,t}^{(n)}(c). \quad (8.28)$$

We also note the following ‘‘braid relation’’, generalizing the first two identities.

**Proposition 8.12.** *The operators  $\mathcal{D}_{q,t}^{(n)}(c)$  satisfy the operator identities*

$$\prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 - d \pm z_i)}{\Gamma_q(t_0 + d \pm z_i)} \mathcal{D}_{q,t}^{(n)}(c + d) \prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 - c \pm z_i)}{\Gamma_q(t_0 + c \pm z_i)} = \mathcal{D}_{q,t}^{(n)}(c) \prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 - c - d \pm z_i)}{\Gamma_q(t_0 + c + d \pm z_i)} \mathcal{D}_{q,t}^{(n)}(d) \quad (8.29)$$

In particular,  $\mathcal{D}_{q,t}^{(n)}(c)^{-1} = \mathcal{D}_{q,t}^{(n)}(-c)$ .

*Proof.* Consider the composition

$$\prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 + d \pm z_i)}{\Gamma_q(t_0 - d \pm z_i)} \mathcal{D}_{q,t}^{(n)}(c) \prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 - c - d \pm z_i)}{\Gamma_q(t_0 + c + d \pm z_i)} \mathcal{D}_{q,t}^{(n)}(d) \prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 + c \pm z_i)}{\Gamma_q(t_0 - c \pm z_i)}. \quad (8.30)$$

If we substitute

$$\mathcal{D}_{q,t}^{(n)}(d) = D_q^{(n)}(-d \pm (t_0 - c); t) \mathcal{D}_{q,t}^{(n)}(d + q/2) \prod_{1 \leq i \leq n} \vartheta(z_i \pm (t_0 - c))^{-1} \quad (8.31)$$

then apply the easy relation

$$\begin{aligned} & \prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 - c - d \pm z_i)}{\Gamma_q(t_0 + c + d \pm z_i)} D_q^{(n)}(c - d - t_0, t_0 - c - d; t) \\ &= D_q^{(n)}(c - d - t_0, t_0 + c + d; t) \prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 + q/2 - c - d \pm z_i)}{\Gamma_q(t_0 + q/2 + c + d \pm z_i)}, \end{aligned} \quad (8.32)$$

we can combine the two operators:

$$\mathcal{D}_{q,t}^{(n)}(c) D_q^{(n)}(c \pm (t_0 + d); t) = \prod_{1 \leq i \leq n} \vartheta(z_i \pm (t_0 + d)) \mathcal{D}_{q,t}^{(n)}(c - q/2) \quad (8.33)$$

and find that the result simplifies to the case  $(c, d, t_0) \mapsto (c - q/2, d + q/2, t_0 + q/2)$  of the above composition. In other words, the given operator is invariant under such translations, so by density is invariant under any translation  $(c, d, t_0) \mapsto (c - u, d + u, t_0 + u)$ . Taking  $u = c$  gives

$$\prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 + 2c + d \pm z_i)}{\Gamma_q(t_0 - d \pm z_i)} \mathcal{D}_{q,t}^{(n)}(0) \prod_{1 \leq i \leq n} \frac{\Gamma_q(t_0 - d \pm z_i)}{\Gamma_q(t_0 + 2c + d \pm z_i)} \mathcal{D}_{q,t}^{(n)}(c + d) = \mathcal{D}_{q,t}^{(n)}(c + d), \quad (8.34)$$

since  $\mathcal{D}_{q,t}^{(n)}(0) = 1$ . □

*Remark 1.* Compare the proof of [21, Thm. 4.1]. The similarity in arguments is not at all a coincidence: The analytic construction of  $\mathcal{D}_{q,t}^{(n)}(c)$  in terms of the interpolation kernel of [19] implies that one can obtain the elliptic binomial coefficients of [21] as specializations of the coefficients of  $\mathcal{D}_{q,t}^{(n)}(c)$ , making [21, Thm. 4.1] a (Zariski dense) special case of the above braid relation.

*Remark 2.* We refer to this as the braid relation for the following reason. The action of the Fourier transform on objects is a reflection in an appropriate inner product (the intersection form of the surface!), as are the generators of the  $W(D_m)$  action. Each generator has a certain action on operators. If one takes into account the action on parameters, the generators are involutions, and all relevant braid relations are satisfied, so that this gives an action of a Coxeter group  $W(E_{m+1})$  on the family. Only one of the braid relations is nontrivial to verify, and it reduces to the above relation.

We thus define a Fourier transform on formal difference operators in the following way. If the formal operator  $D$  maps the polarization  $P_0(2c; q, t)$  to the polarization  $P_0(2c'; q, t)$ , then its Fourier transform  $\hat{D}$  is the operator

$$\mathcal{D}_{q,t}^{(n)}(c')D\mathcal{D}_{q,t}^{(n)}(-c) \quad (8.35)$$

mapping  $P_0(-2c; q, t)$  to  $P_0(-2c'; q, t)$ . There is some choice here (since the polarizations only depend on  $2c, 2c'$ ), but luckily it is not particularly serious.

**Lemma 8.13.** *If  $\tau$  is a 2-torsion point, then*

$$\mathcal{D}_{q,t}^{(n)}(c + \tau) = T_\omega(\tau)\mathcal{D}_{q,t}^{(n)}(c) = \mathcal{D}_{q,t}^{(n)}(c)T_\omega(\tau). \quad (8.36)$$

*Proof.* Indeed, the recurrence we used to solve for the coefficients of  $\mathcal{D}_{q,t}^{(n)}(c)$  is equivariant under translation by 2-torsion.  $\square$

For our purposes, we will always be working in the subcategory with objects  $P_0(-dq + \eta'; q, t)$ , and will take  $c, c'$  in the Fourier transform to be the appropriate linear combination of  $q/2$  and some fixed  $\eta'/2$ . We have, of course, already computed some instances of the Fourier transform:

$$\prod_{1 \leq i \leq n} \vartheta(z_i \pm u) \mapsto D_q^{(n)}(q/2 - c \pm u; t) \quad (8.37)$$

$$D_q^{(n)}(c + q/2 \pm u; t) \mapsto \prod_{1 \leq i \leq n} \vartheta(z_i \pm u) \quad (8.38)$$

$$D_q^{(n)}(u_0, u_1, u_2, q + 2c - u_0 - u_1 - u_2; t) \mapsto D_q^{(n)}(u_0 - c, u_1 - c, u_2 - c, q + c - u_0 - u_1 - u_2; t), \quad (8.39)$$

where in each case the input is a (general) section of  $\mathcal{S}_{2c; q, t}^{(n)}$  starting from the 0 object, of degree  $f, s$  and  $s + f$  respectively.

**Theorem 8.14.** *If  $c' - c$  is an integer multiple of  $q/2$ , then the Fourier transform is holomorphic; that is, the Fourier transform of any holomorphic family of operators is a holomorphic family of operators.*

*Proof.* The only issue is when  $q$  is torsion, as otherwise both  $\mathcal{D}_{q,t}^{(n)}(-c)$  and  $\mathcal{D}_{q,t}^{(n)}(c')$  are holomorphic (in the sense that they have no  $z$ -independent poles other than those for  $q$  torsion).

Consider a multiplication operator  $h$ . If this is  $C_n$ -invariant, we can express it as a ratio of holomorphic  $C_n$ -invariant theta functions. The algebra of such theta functions is generated

by functions  $\prod_{1 \leq i \leq n} \vartheta(u \pm z) = (-1)^n \prod_{1 \leq i \leq n} \vartheta(z_i \pm u)$ , and thus any holomorphic family of  $C_n$ -invariant functions  $h$  has holomorphic Fourier transform. (The leading term of the Fourier transform of an operator is easy to determine, so we find that the Fourier transform of the denominator is indeed invertible.)

Now, let  $h$  be a general multiplication operator. To show that  $\hat{h}$  is holomorphic, we need to show that every coefficient is holomorphic. The coefficient of  $\prod_i T_i^{k_i}$  has denominator dividing  $\prod_{1 \leq j \leq \max(k_1, \dots, k_n)} \vartheta(jq)$ , and by Hartog's Lemma it suffices to prove that the coefficient is holomorphic at the generic point of every component of the corresponding divisor. Each coefficient is a finite linear combination of shifts of  $h$ , and we are evaluating it at a point with generic  $(z_1, \dots, z_n)$ . In particular, none of the points where we are evaluating  $h$  are in the same  $C_n$  orbit (though we may be hitting the same point multiple times). It follows that there exists a  $C_n$ -invariant function  $g$  such that the corresponding sum for  $h - g$  is holomorphic: simply take  $g$  to be a very good approximation near the points where  $h$  is being evaluated. Since  $\hat{g}$  is holomorphic and this coefficient of the Fourier transform of  $h - g$  is holomorphic, it follows that the given coefficient of  $\hat{h}$  is holomorphic as required.

Now, let  $D$  be an operator of the form  $D_q^{(n)}(c + q/2 \pm u; t)$ , which again has a holomorphic Fourier transform. If  $q \neq 0$ , then the space of operators

$$k(X)D_q^{(n)}(c + q/2 \pm u; t)k(X) \tag{8.40}$$

is a  $2^n$ -dimensional vector space on the left. Indeed, each of the  $2^n$  shifts that appear induce different automorphisms of  $k(X)$ . It follows that any element of that space has holomorphic Fourier transform (except possibly where  $q = 0$ ). Since that space contains elements  $\propto \prod_i T_i^{\pm 1/2}$  for every combination of signs, we have proved holomorphy of the Fourier transform on a set of (topological) generators of the ring of twisted formal difference operators. The Fourier transform is continuous with respect to the nonarchimedean metric, so the result follows in general.

It remains to consider the case  $q = 0$ . This splits into two components, depending on whether  $q/2 = 0$  or  $q/2$  is nontrivial 2-torsion. The latter case reduces to the first, however, since everything is invariant under translation by 2-torsion. We may thus restrict our attention to the local ring at the generic point with  $q/2 = 0$ . In that case, the special fiber of the ring of twisted formal difference operators is abelian, since all shifts are trivial. As a result, the algebra over the local ring picks up an additional operation on operators:

$$(D_1, D_2) \mapsto (D_1 D_2 - D_2 D_1) / \pi, \tag{8.41}$$

where  $\pi$  is a uniformizer. This takes any pair of holomorphic operators to a holomorphic operator, and the Fourier transform respects this operation. We may thus use this operation to construct operators with known holomorphic Fourier transform. It turns out that the usual proof of independence of automorphisms of fields can be expressed in terms of this operation, and thus we still obtain the full  $2^n$ -dimensional space of operators.  $\square$

*Remark.* The proof for  $q = 0$  is of course based on the standard fact that an automorphism of a family of noncommutative algebras preserves the induced Poisson structure on any commutative fiber.

Of course, the algebra of formal difference operators is far too large, and doesn't even have an action of  $C_n$  (as it preserves neither the metric nor the topology). So we need to show that the operators we care about map to operators which not only have finite support, but have  $C_n$  symmetry. Luckily, this is a closed condition, so it suffices to prove it generically.

**Lemma 8.15.** *On the generic fiber, the  $\mathbb{Z}/2\mathbb{Z}$ -graded algebra  $\bigcup_d \mathcal{S}_{2c;0,0}^{(n)}(0, d(s+f))$  is generated by  $\mathcal{S}_{2c;0,0}^{(n)}(0, s+f)$ .*

*Proof.* In fact, we claim that for  $d \gg 0$ ,  $\mathcal{S}_{2c;0,0}^{(n)}(0, d(s+f))$  is spanned by products of  $d$  elements of  $\mathcal{S}_{2c;0,0}^{(n)}(0, s+f)$ . Since this contains the spaces for all smaller  $d$  of the same parity, the result will immediately follow.

Since this graded algebra is the homogeneous coordinate ring of  $\text{Sym}^n(\mathbb{P}^1 \times \mathbb{P}^1)$ , what we are in fact claiming is that the ample bundle  $\text{Sym}^n(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1))$  is very ample. In general, it follows from [4, §1.3] that over any field of characteristic 0,  $\text{Sym}^n(\mathcal{O}_{\mathbb{P}^m}(1))$  is very ample on  $\text{Sym}^n(\mathbb{P}^m)$ , and thus the same holds for the symmetric power of any closed subscheme of  $\mathbb{P}^m$ .  $\square$

*Remark.* It is likely that this fails in small characteristic. It is certainly the case that  $\text{Sym}^n(\mathcal{O}_{\mathbb{P}^m}(1))$  can fail to be very ample on  $\text{Sym}^n(\mathbb{P}^m)$ ; indeed this already happens for  $\text{Sym}^3(\mathbb{P}^2)$  in characteristic 3. In addition, even in characteristic 0, the  $\mathbb{Z}$ -graded algebra is not generated in degree 1 if  $n$  is sufficiently large. Indeed, one has  $h^0(\text{Sym}^n(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1))) = \binom{n+3}{3}$ , while  $h^0(\text{Sym}^n(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2))) = \binom{n+8}{8}$ . So for  $n \gg 0$ , even if we take into account noncommutativity, there are simply not enough sections of degree 1 for their products to account for every section of degree 2!

**Corollary 8.16.** *For any  $d$ , the Fourier transform induces an isomorphism of stalks*

$$\Gamma \mathcal{S}_{2c;q,t}^{(n)}(0, d(s+f))_{q=t=0} \cong \Gamma \mathcal{S}_{-2c;q,t}^{(n)}(0, d(s+f))_{q=t=0}. \quad (8.42)$$

*Proof.* Fix a basis of the global sections of the fiber over the generic point with  $q = t = 0$ . Each such global section can be expressed as a polynomial in sections of degree 1; if we choose an extension to the stalk for each degree 1 operator that appears, then the result will be a basis of the stalk of degree  $d$  operators in which every element is a polynomial in first-order operators. It follows that every element of the basis has Fourier transform in

$$\bigcup_{e \geq 0} \Gamma \mathcal{S}_{-2c;q,t}^{(n)}(0, e(s+f))_{q=t=0}, \quad (8.43)$$

but the Fourier transform clearly preserves the space of operators  $\prod_i T_i^{-l/2} k(X)[[T_1, \dots, T_n]]$  for each  $l$ , and thus the Fourier transform is actually in  $\Gamma \mathcal{S}_{-2c;q,t}^{(n)}(0, d(s+f))_{q=t=0}$  as required. The inverse operation is of course just the Fourier transform again.  $\square$

**Corollary 8.17.** *For any  $d$ , the Fourier transform induces a (local) isomorphism of sheaves of categories*

$$\Gamma \mathcal{S}_{2c;q,t}^{(n)}|_{\mathbb{Z}(s+f)} \cong \Gamma \mathcal{S}_{-2c;q,t}^{(n)}|_{\mathbb{Z}(s+f)}. \quad (8.44)$$

*Proof.* The given Hom sheaves of the global section category are flat and the map to difference operators is injective on fibers. We may thus identify sections with holomorphic families of difference operators and apply the Fourier transform to obtain a holomorphic family of formal difference operators. The generic point of this family is a section of the other global section category, and thus the family itself is a section.  $\square$

**Corollary 8.18.** *For  $d \leq d'$ , the Fourier transform induces a morphism*

$$\Gamma \mathcal{S}_{2c;q,t}^{(n)}(0, ds + d'f) \rightarrow \Gamma \mathcal{S}_{-2c;q,t}^{(n)}(0, d's + df). \quad (8.45)$$

*Proof.* If  $d = 0$ , this is easy, as the algebra is generated in degree 1, and we know the result there. More generally, given a section  $D$  of  $\Gamma\mathcal{S}_{2c;q,t}^{(n)}(0, ds + d'f)$  and any section  $g$  of  $\Gamma\mathcal{S}_{-2c;q,t}^{(n)}(d's + df, d's + d'f)$ , consider the composition  $\hat{g}D \in \Gamma\mathcal{S}_{2c;q,t}^{(n)}(0, d's + d'f)$ , which makes sense since  $g$  has degree  $(d' - d)f$ . Since the Fourier transform is an (order-preserving) involution, we find that  $\hat{g}D$  has Fourier transform  $g\hat{D}$ , so that  $g\hat{D}$  is a section of  $\Gamma\mathcal{S}_{-2c;q,t}^{(n)}(0, d's + d'f)$  for any  $g$ . But this implies that  $\hat{D}$  is actually a section of  $\Gamma\mathcal{S}_{-2c;q,t}^{(n)}(0, d's + d'f)$  as required.  $\square$

**Corollary 8.19.** *For any  $d, d'$ , the sheaf  $\Gamma\mathcal{S}_{2c;q,t}^{(n)}(0, ds + d'f)$  is flat and the map to difference operators is injective on fibers. Moreover, the Fourier transform induces an isomorphism*

$$\Gamma\mathcal{S}_{2c;q,t}^{(n)}(0, ds + d'f) \cong \Gamma\mathcal{S}_{-2c;q,t}^{(n)}(0, d's + df) \quad (8.46)$$

for all  $d, d'$ .

*Proof.* We already know this if  $d < 0$  or  $d \leq d'$ , so suppose  $d \geq d'$ . It suffices to show injectivity on fibers, as it implies that the  $\text{Tor}_1$  of the cokernel is flat. Thus, let  $D \in \Gamma\mathcal{S}_{2c;q,t}^{(n)}(0, ds + d'f)$  be a local section such that the corresponding difference operator vanishes on some fiber. Consider the Fourier transform

$$\Gamma\mathcal{S}_{-2c;q,t}^{(n)}(0, d's + df) \rightarrow \Gamma\mathcal{S}_{2c;q,t}^{(n)}(0, ds + d'f). \quad (8.47)$$

The domain is locally free and injective on fibers, and the codomain is at least *generically* free of the same rank. Since the Fourier transform is invertible at the level of operators, this map is injective, and thus an isomorphism. It follows that there is a section  $D' \in \Gamma\mathcal{S}_{-2c;q,t}^{(n)}(0, d's + df)$  such that  $D - \hat{D}'$  is generically 0. But this, of course, implies that  $D' = \hat{D}$ . In particular, the corresponding fiber of  $D'$  vanishes, which means that in a suitable local basis we have  $D' = \sum_i c_i D_i$ , in which each  $c_i$  vanishes on a divisor passing through that fiber. We then have  $D = \sum_i c_i \hat{D}_i$  with each  $\hat{D}_i$  a local section of  $\Gamma\mathcal{S}_{2c;q,t}^{(n)}(0, ds + d'f)$ . It follows that the section corresponding to  $D$  vanishes at the fiber, so that injectivity holds.  $\square$

To finish the proof of the theorem, we need to show that the transform respects Bruhat order, that it respects the vanishing conditions associated to  $x_1, \dots, x_m$ , and that it commutes with the Selberg adjoint. Each of these have analogous statements for general formal difference operators, and in the first two cases reduce to the fact that (due to continuity) the Fourier transform affects leading coefficients in easy to control ways.

For the Bruhat order, we actually obtain a finer (inclusion) partial order in the formal setting.

**Proposition 8.20.** *Let  $D$  be a holomorphic family of formal difference operators from  $P_0(2c; q, t)$  to  $P_0(2c + lq; q, t)$ . Let  $S \subset \mathbb{Z}^n \cup (1/2, \dots, 1/2)\mathbb{Z}^n$  be the set of vectors  $\vec{v}$  such that for some  $\vec{k} \in \mathbb{N}^n$ , the left coefficient of  $\prod_i T_i^{v_i - k_i}$  is nonzero, and let  $\hat{S}$  be the corresponding set for  $\hat{D}$ . Then  $\hat{S} = (l/2, \dots, l/2) + S$ .*

*Proof.* Conjugating by  $T_\omega(c)$  or a  $\Gamma_q$  symbol has no effect on the support of an operator, and multiplication by  $T_\omega(lq/2)$  shifts the support by  $(l/2, \dots, l/2)$ . The remaining operation consists of left- and right-multiplication by units in  $k(X)[[T_1, \dots, T_n]]$ , and this clearly preserves the set  $S$ .  $\square$

For the vanishing conditions, we have the following. Note that we only consider half of the vanishing conditions, as in the formal setting it only makes sense to consider conditions on the leading few terms. Also, for convenience, we only consider the generic case.

**Proposition 8.21.** *Over the generic point  $(E, x, c, q, t) \in \mathcal{E}^4$  and for integers  $r, l$ , consider the space of formal difference operators  $D$  mapping  $P_0(2c; q, t)$  to  $P_0(2c + lq; q, t)$  such that  $D \prod_i T_i^{r/2}$  involves only integer shifts. If the left coefficients of both  $D$  and*

$$\prod_{1 \leq i \leq n} \Gamma_q(x + rq/2 - z_i)^{-1} D \prod_{1 \leq i \leq n} \Gamma_q(x - z_i) \quad (8.48)$$

*are holomorphic along all hypersurfaces of the form  $z_i \in x + rq/2 + kq$ ,  $k \in \mathbb{Z}$ , then the left coefficients of both  $\hat{D}$  and*

$$\prod_{1 \leq i \leq n} \Gamma_q(x - c + (r - l)q/2 - z_i)^{-1} \hat{D} \prod_{1 \leq i \leq n} \Gamma_q(x - c - z_i) \quad (8.49)$$

*are holomorphic along all hyperplanes of the form  $z_i \in x - c + (r - l)q/2 + kq$ ,  $k \in \mathbb{Z}$ .*

*Proof.* By definition, we have

$$\hat{D} = \mathcal{D}_{q,t}^{(n)}(c + lq/2) D \mathcal{D}_{q,t}^{(n)}(-c). \quad (8.50)$$

There are only countably many hypersurfaces of the form  $z_i = y$  on which some left coefficient of  $\mathcal{D}_{q,t}^{(n)}(-c)$  and  $\mathcal{D}_{q,t}^{(n)}(c + lq/2)$  has a pole (including poles of the meromorphic sections of equivariant gerbes corresponding to the leading coefficients). Since  $x$  is generic, it follows that all three factors on the right are holomorphic on the given orbits of hypersurfaces, and thus so is the product.

The claim for

$$\prod_{1 \leq i \leq n} \Gamma_q(x - c + (r - l)q/2 - z_i)^{-1} \hat{D} \prod_{1 \leq i \leq n} \Gamma_q(x - c - z_i) \quad (8.51)$$

analogously reduces to checking possible poles of

$$\prod_{1 \leq i \leq n} \Gamma_q(x - z_i)^{-1} \mathcal{D}_{q,t}^{(n)}(-c) \prod_{1 \leq i \leq n} \Gamma_q(x - c - z_i) \quad (8.52)$$

and

$$\prod_{1 \leq i \leq n} \Gamma_q(x - c + (r - l)q/2 - z_i)^{-1} \mathcal{D}_{q,t}^{(n)}(c + lq/2) \prod_{1 \leq i \leq n} \Gamma_q(x + rq/2 - z_i). \quad (8.53)$$

In each case, the gauging only multiplies the coefficients by holomorphic theta functions, so cannot introduce any new poles.  $\square$

To get an analogue for the Selberg adjoint, there is a mild difficulty coming from the fact that the Selberg adjoint was only defined for  $C_n$ -symmetric operators, and the obvious extension does not make sense for formal operators. Luckily, the formal adjoint with respect to the inner product

$$\int f(z_1, \dots, z_n) g(-z_1, \dots, -z_n) \prod_{1 \leq i < j \leq n} \frac{\Gamma_q(t \pm z_i \pm z_j)}{\Gamma_q(\pm z_i \pm z_j)} \prod_{1 \leq i \leq n} \frac{1}{\Gamma_q(\pm 2z_i)} dT \quad (8.54)$$

*does* make sense for formal difference operators and formal gauging operators and agrees with the Selberg adjoint in the  $C_n$ -symmetric case. Using this as the definition of the Selberg adjoint for formal operators gives the following, which immediately implies consistency of the Fourier transform with the Selberg adjoint.

**Proposition 8.22.** *The operators  $\mathcal{D}_{q,t}^{(n)}(c)$  are self-adjoint under the Selberg adjoint.*

*Proof.* The Selberg adjoint has the correct leading term, so it suffices to show that

$$\mathcal{D}_{q,t}^{(n)}(c)^{\text{ad}_t} D_q^{(n)}(c \pm u; t) = \prod_{1 \leq i \leq n} \vartheta(z_i \pm u) \mathcal{D}_{q,t}^{(n)}(c - q/2)^{\text{ad}_t}. \quad (8.55)$$

Since  $\prod_{1 \leq i \leq n} \vartheta(z_i \pm u)$  is self-adjoint, this reduces to checking that

$$D_q^{(n)}(c \pm u; t)^{\text{ad}_t} = D_q^{(n)}(q/2 - c \pm u; t), \quad (8.56)$$

an easy verification.  $\square$

*Remark.* In fact, one has in general

$$D_q^{(n)}(u_0, \dots, u_{2d'+1}; t)^{\text{ad}_t} = D_q^{(n)}(q/2 - u_0, \dots, q/2 - u_{2d'+1}; t), \quad (8.57)$$

either by a direct computation or by using the fact that both are sections of the same Hom sheaf, and with the same leading coefficient.

We mention a couple of further consequences of the proof. First, the fact that the Fourier transform is determined by its values where we know it explicitly has consequences in the analytic setting. Indeed, in [19], a kernel function  $\mathcal{K}_c^{(n)}(\vec{x}; \vec{y}; q, t)$  was constructed, with the property that for  $D$  of degree  $s$ ,  $f$ , or  $s + f$ , one had

$$D_{\vec{x}} \mathcal{K}_c^{(n)}(\vec{x}; \vec{y}; q, t) = \hat{D}_{\vec{y}}^{\text{ad}_t} \mathcal{K}_c^{(n)}(\vec{x}; \vec{y}; q, t) \quad (8.58)$$

It follows from the above proof and continuity that this holds for *all* operators which are global sections of the appropriate Hom spaces. In particular, this applies to operators of degree  $2s + 2f - e_1 - \dots - e_8$  (i.e., the van Diejen/Komori-Hikami integrable system considered in Theorem 7.22), showing that the associated formal integral operator takes eigenvalue equations of this form to eigenvalue equations of the same form.

Also, we have already mentioned the consequence that the resulting deformations of  $\text{Sym}^n(\mathbb{P}^2)$  only depend (geometrically) on  $E$ ,  $q$ , and  $t$ . It is worth mentioning the specific form that the given isomorphisms take. The isomorphism

$$\Gamma \mathcal{S}'_{x_0; q, t}{}^{(n)}|_{\mathbb{Z}(s+f)} \cong \Gamma \mathcal{S}'_{x_1; q, t}{}^{(n)}|_{\mathbb{Z}(s+f)} \quad (8.59)$$

is given (up to a choice of element  $(x_0 - x_1)/3$ ) by gauging by the operator

$$\begin{aligned} G_d(x_0, x_1) := & \prod_{1 \leq i \leq n} \Gamma_q \left( -\frac{(d-1)q}{2} - \frac{2x_0 + x_1}{3} \pm z_i \right) \\ & \mathcal{D}_{q,t}^{(n)}((x_0 - x_1)/3) \\ & \prod_{1 \leq i \leq n} \Gamma_q \left( -\frac{(d-1)q}{2} - \frac{x_0 + 2x_1}{3} \pm z_i \right)^{-1} \end{aligned} \quad (8.60)$$

in degree  $d$ ; i.e., the action on morphisms from  $d_1(s + f)$  to  $d_2(s + f)$  is given by

$$D \mapsto G_{d_2}(x_0, x_1) D G_{d_1}(x_0, x_1)^{-1}. \quad (8.61)$$

In particular, we see that when  $(x_0 - x_1)/3$  is 3-torsion, the resulting automorphism is still quite nontrivial. In addition, the translation symmetry of the category involves changing  $x_0$ , and thus

although one *can* identify it with a graded algebra at the cost of choosing an element  $q/3$ , the resulting graded algebra does not actually have a representation in (finite) difference operators. If we restrict to the “anticanonical” model, i.e., to  $\mathbb{Z}(3s + 3f)$ , then the Hom space of degree  $3s + 3f$  contains the 1-dimensional subspace of operators of degree  $s$ , spanned by

$$G_0(x_0, x_0 - 3q/2) = \prod_{1 \leq i \leq n} \Gamma_q(-x_0 \pm z_i) \mathcal{D}_{q,t}^{(n)}(-q/2) \prod_{1 \leq i \leq n} \Gamma_q(-q/2 - x_0 \pm z_i)^{-1}. \quad (8.62)$$

If we adjoin the inverse of such an operator, then the result in degree 0 may be identified with a filtered algebra of *formal* difference operators. We can include elements of degree not a multiple of 3 at the cost of choosing  $q/3$  and allowing some  $\Gamma_q$  factors and shifts by multiples of  $q/3$ . Indeed, the braid relation tells us (assuming compatible choices when dividing by 3) that  $G_d(x_1, x_2)G_d(x_0, x_2) = G_d(x_0, x_2)$ , and thus the various isomorphisms between categories with parameter  $x_0 + kq/2$  are all compatible. It follows that if we compose an operator mapping  $d_1(s + f)$  to  $d_2(s + f)$  with the formal operators giving isomorphisms  $x_0 \mapsto x_0 + d_1q/2$  and  $x_0 + d_2q/2 \mapsto x_0$ , then the result will be compatible with compositions and will be the same as if we only used the spaces with  $d_1 = 0$ . Of course, even in the univariate setting, the resulting algebra is not likely to be easy to describe in any direct fashion...

One final thing to mention is that our description of first-order operators in Lemma 8.6 as well as our description of the operators  $\mathcal{D}_{q,t}^{(n)}(c)$  are both quite well suited to considering degenerations of the tuple  $(E, c, q, t)$ . In light of the fact that operators of degree  $s + f$  are generically very ample, we can give at least indirect descriptions of the limiting algebras  $\bigcup_a \mathcal{S}_{\eta';q,t}^{(n)}(0, d(s + f))$  by specifying their elements of degree 1, and understanding the extension to the whole category simply requires keeping track of the elements of degree  $f$  as well.

Taking the limit can be somewhat tricky in general, as it may be necessary to gauge by suitable functions before the limit is well-defined. The simplest approach is to choose a suitable gauge transformation to make the operators elliptic before taking the limit; this introduces additional parameters which we can then eliminate by a further limit. Indeed, we find that for any operator  $D \in \Gamma \mathcal{S}_{\eta';q,t}^{(n)}(0, ds + d'f)$ , the gauge transformation

$$\begin{aligned} & \prod_{1 \leq i \leq n} \frac{\Gamma_q(\eta' - dq/2 - d'q + (n-1)t + v_0 + v_1 + v_2 \pm z_i)}{\Gamma_q(-dq/2 + v_0 \pm z_i, -dq/2 + v_1 \pm z_i, -dq/2 + v_2 \pm z_i)} \\ & D \\ & \prod_{1 \leq i \leq n} \frac{\Gamma_q(v_0 \pm z_i, v_1 \pm z_i, v_2 \pm z_i)}{\Gamma_q(\eta' + (n-1)t + v_0 + v_1 + v_2 \pm z_i)} \end{aligned} \quad (8.63)$$

is elliptic (for fixed  $v_0, v_1, v_2$ ). We can then take the limit as  $p \rightarrow 0$  and remove the parameters by gauging back by an appropriate product of  $q$ -Pochhammer symbols  $(x; q)_\infty := \prod_{0 \leq j} (1 - q^j x)$ . We obtain a limit  $\Gamma \mathcal{S}_{\eta';q,t;*}^{(n)}(0, s + d'f)$  (with  $q, t, \eta'$  and  $\vec{z}$  in the multiplicative group) consisting of operators of the form

$$\sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq i \leq n} \frac{z_i^{\sigma_i} f(z_i^{\sigma_i})}{1 - z_i^{2\sigma_i}} \prod_{1 \leq i < j \leq n} \frac{1 - tz_i^{\sigma_i} z_j^{\sigma_j}}{1 - z_i^{\sigma_i} z_j^{\sigma_j}} \prod_{1 \leq i \leq n} T_i^{\sigma_i/2} \quad (8.64)$$

where  $f(z)$  is a univariate Laurent polynomial with exponents ranging from  $1 - d'$  to  $d' - 1$  satisfying the condition  $[z^{d'-1}]f(z) = q\eta'[z^{1-d'}]f(z)$  on its extreme coefficients. The Fourier transformation

has a corresponding limit, represented by operators  $\mathcal{D}_{q,t:*}^{(n)}(c)$  satisfying

$$\mathcal{D}_{q,t:*}^{(n)}(c) \prod_{1 \leq i \leq n} \frac{((vcd)z_i^{\pm 1}; q)}{((v/cd)z_i^{\pm 1}; q)_\infty} \mathcal{D}_{q,t:*}^{(n)}(d) = \prod_{1 \leq i \leq n} \frac{((vd)z_i^{\pm 1}; q)_\infty}{((v/d)z_i^{\pm 1}; q)_\infty} \mathcal{D}_{q,t:*}^{(n)}(cd) \prod_{1 \leq i \leq n} \frac{((vc)z_i^{\pm 1}; q)_\infty}{((v/c)z_i^{\pm 1}; q)_\infty} \quad (8.65)$$

with leading term

$$\prod_{1 \leq i \leq n} \frac{\theta_{q^{1/2}}(-1/z_i)}{\theta_{q^{1/2}}(-1/cz_i)} \prod_{1 \leq i \leq j \leq n} \frac{(1/c^2 z_i z_j; q)_\infty}{(1/z_i z_j; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(t/c^2 z_i z_j; q)_\infty}{(t/z_i z_j; q)_\infty} T_\omega(c) \quad (8.66)$$

and special case

$$\mathcal{D}_{q,t:*}^{(n)}(q^{-1/2}) = \sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq i \leq n} \frac{z_i^{\sigma_i}}{1 - z_i^{2\sigma_i}} \prod_{1 \leq i < j \leq n} \frac{1 - tz_i^{\sigma_i} z_j^{\sigma_j}}{1 - z_i^{\sigma_i} z_j^{\sigma_j}} \prod_{1 \leq i \leq n} T_i^{\sigma_i/2}. \quad (8.67)$$

Note that taking  $v = 0$  in the limit of the braid relation gives  $\mathcal{D}_{q,t:*}^{(n)}(c)\mathcal{D}_{q,t:*}^{(n)}(d) = \mathcal{D}_{q,t:*}^{(n)}(cd)$ , so that we may interpret  $\mathcal{D}_{q,t:*}^{(n)}(c)$  as a fractional power of the operator for  $c = q^{-1/2}$ , which in turn is a lowering operator appearing in the theory of Koornwinder polynomials. We can further extend this limit to the case of sufficiently general blowups (i.e., with all  $x_i$  finite) by imposing the appropriate conditions on the leading coefficients; this is how we tested the  $n = r = 2$  case of Conjecture 1.

Another noteworthy limit involves gauging by a translation so as to break the  $z \mapsto 1/z$  symmetry, and taking a limit in the resulting parameter. This gives a Fourier transform represented by operators satisfying

$$\mathcal{D}_{q,t:**}^{(n)}(c) \prod_{1 \leq i \leq n} \frac{((vcd)z_i; q)}{((v/cd)z_i; q)_\infty} \mathcal{D}_{q,t:**}^{(n)}(d) = \prod_{1 \leq i \leq n} \frac{((vd)z_i; q)_\infty}{((v/d)z_i; q)_\infty} \mathcal{D}_{q,t:**}^{(n)}(cd) \prod_{1 \leq i \leq n} \frac{((vc)z_i; q)_\infty}{((v/c)z_i; q)_\infty} \quad (8.68)$$

with leading term

$$\prod_{1 \leq i \leq n} \frac{\theta_{q^{1/2}}(-1/z_i)}{\theta_{q^{1/2}}(-1/cz_i)} T_\omega(c) \quad (8.69)$$

and special case

$$\mathcal{D}_{q,t:**}^{(n)}(q^{-1/2}) = \sum_{I \subset \{1, \dots, n\}} (-1)^{|I|} |t|^{|I|(|I|-1)/2} \prod_{1 \leq i \leq n} z_i^{-1} \prod_{i \in I, j \notin I} \frac{z_j - tz_i}{z_j - z_i} \prod_{i \in I} T_i^{1/2} \prod_{i \notin I} T_i^{-1/2}, \quad (8.70)$$

a.k.a. the lowering operator for  $GL_n$ -type Macdonald polynomials. The Hom spaces of degree  $s + d'f$  have similar, if somewhat more complicated forms, obtained by gauging the  $q^{-1/2}$  case of the Fourier transform operator by suitable products of Pochhammer symbols. We omit the details, except to note that the results again look like operators arising in Macdonald theory.

There are some other symmetry breaking limits (e.g., the image of  $\eta' \rightarrow 0$  under the Fourier transform); we omit the details. Of course, such symmetry-breaking limits have an invidious effect on the Bruhat ordering; for instance, the “leading term” must now incorporate all  $\sim 2^n$   $S_n$ -orbits corresponding to the given  $C_n$ -orbit of weights. As a result, in more degenerate cases, it can be difficult to figure out the correct way to compactify the algebra. This can be fixed in some cases by realizing that the  $S_n$ -symmetric operator is actually a shadow of a  $C_n$ -symmetric operator acting on a power of a reducible curve (a hyperelliptic curve of *arithmetic* genus 1). Similarly, there are differential limits living on a power of the nonreduced curve  $y^2 = 0$ .

## References

- [1] M. Artin, J. Tate, and M. Van den Bergh. Some algebras associated to automorphisms of elliptic curves. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 33–85. Birkhäuser Boston, Boston, MA, 1990.
- [2] M. Artin and M. Van den Bergh. Twisted homogeneous coordinate rings. *J. Algebra*, 133(2):249–271, 1990.
- [3] A. I. Bondal and A. E. Polishchuk. Homological properties of associative algebras: the method of helices. *Izv. Ross. Akad. Nauk Ser. Mat.*, 57(2):3–50, 1993.
- [4] M. Brion. Stable properties of plethysm: on two conjectures of Foulkes. *Manuscripta Math.*, 80(4):347–371, 1993.
- [5] J. H. Conway and N. J. A. Sloane. Low-dimensional lattices. II. Subgroups of  $gl(n, F)$ . *Proc. Roy. Soc. London Ser. A*, 419:29–68, 1988.
- [6] J. F. van Diejen. Integrability of difference Calogero-Moser systems. *J. Math. Phys.*, 35(6):2983–3004, 1994.
- [7] M. Eichler and D. Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.
- [8] W. Fulton and M. Olsson. The Picard group of  $\mathcal{M}_{1,1}$ . *Algebra Number Theory*, 4(1):87–104, 2010.
- [9] V. Ginzburg, M. Kapranov, and E. Vasserot. Residue construction of Hecke algebras. *Adv. Math.*, 128(1):1–19, 1997.
- [10] V. A. Gritsenko. Fourier-Jacobi functions in  $n$  variables. *J. Soviet Math.*, 53(3):243–252, 1988.
- [11] N. M. Katz and B. Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.
- [12] Y. Komori and K. Hikami. Quantum integrability of the generalized elliptic Ruijsenaars models. *J. Phys. A*, 30(12):4341–4364, 1997.
- [13] A. Krieg. Jacobi forms of several variables and the Maaß space. *J. Number Theory*, 56(2):242–255, 1996.
- [14] E. Looijenga. Root systems and elliptic curves. *Invent. Math.*, 38(1):17–32, 1976.
- [15] A. Oblomkov. Double affine Hecke algebras of rank 1 and affine cubic surfaces. *Int. Math. Res. Not.*, (18):877–912, 2004.
- [16] A. Polishchuk. *Abelian varieties, theta functions and the Fourier transform*, volume 153 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003.
- [17] E. Rains and S. Ruijsenaars. Difference operators of Sklyanin and van Diejen type. *Comm. Math. Phys.*, 320(3):851–889, 2013.
- [18] E. M. Rains. Generalized Hitchin systems on rational surfaces. arXiv:1307.4033.

- [19] E. M. Rains. Multivariate quadratic transformations and the interpolation kernel. arXiv:1408.0305.
- [20] E. M. Rains. The noncommutative geometry of elliptic difference equations. arXiv:1607:08876.
- [21] E. M. Rains.  $BC_n$ -symmetric abelian functions. *Duke Math. J.*, 135(1):99–180, 2006.
- [22] E. M. Rains. Transformations of elliptic hypergeometric integrals. *Ann. of Math. (2)*, 171(1):169–243, 2010.
- [23] H. Rosengren. Elliptic hypergeometric functions. arXiv:1608.06161.
- [24] S. N. M. Ruijsenaars. First order analytic difference equations and integrable quantum systems. *J. Math. Phys.*, 38:1069–1146, 1997.
- [25] K. Saito. Extended affine root systems. II. Flat invariants. *Publ. Res. Inst. Math. Sci.*, 26(1):15–78, 1990.
- [26] T. Sekiguchi. On projective normality of Abelian varieties. II. *J. Math. Soc. Japan*, 29(4):709–727, 1977.
- [27] T. Shioda. On elliptic modular surfaces. *J. Math. Soc. Japan*, 24:20–59, 1972.
- [28] J. H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.
- [29] V. P. Spiridonov and S. O. Warnaar. Inversions of integral operators and elliptic beta integrals on root systems. *Adv. Math.*, 207(1):91–132, 2006.
- [30] J. R. Stembridge. Tight quotients and double quotients in the Bruhat order. *Electron. J. Combin.*, 11(2):#R14, 2005.
- [31] M. Van den Bergh. A translation principle for the four-dimensional Sklyanin algebras. *J. Algebra*, 184(2):435–490, 1996.
- [32] M. Van den Bergh. Non-commutative  $\mathbb{P}^1$ -bundles over commutative schemes. *Trans. Amer. Math. Soc.*, 364(12):6279–6313, 2012.
- [33] J. F. van Diejen. Difference Calogero-Moser systems and finite Toda chains. *J. Math. Phys.*, 36(3):1299–1323, 1995.