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ELECTORAL POLITICS IN THE ZERO-SUM SOCIETY

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SOCIAL SCIENCE WORKING PAPER 472

March 1983

ABSTRACT

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In most recent work on the theory of elections, parties are assumed to compete over a multidimensional space of issues or policy variables. Distributional considerations arise only indirectly in this structure, and candidates cannot appeal directly to particular constituents or groups by offering them specific targeted benefits or services. This theory of pure "issue" politics thus ignores the prevalent constituent-service aspects of contemporary electoral politics. The present paper develops a theory of electoral competition under an alternative structure, in which candidates compete by directly offering particular benefits and services to voters. The analysis presumes a symmetry in the roles of incumbent and challenger, in that the former necessarily commits himself to an allocation first, by his actions in office, thereby presenting the challenger with a fixed target to optimize against. Voters tend to discount the challenger's promises to some degree in comparing them to the benefits currently being received under the incumbent, and cast their votes so as to maximize the level of benefits received. The main results are as follows:

1. Optimal candidate strategies in this regime turn out to be rather different from those in the classical spatial modeling framework. Challengers pursue a "divide and conquer" strategy of bidding for a minimum winning coalition of voters. Incumbents, by contrast, pursue a more even-handed strategy, attempting to appeal to all their constituents. The model thus predicts distinctive differences in the behavior of challengers and incumbents, with no tendency for the candidates to converge on a common strategy or position, as in the classical Downsian case.

2. The discount factors voters use in assessing the challenger's promises--the "incumbency premia"--can be interpreted as a set of constituent demands. If these are treated as endogenous strategic variables which voters vary so as to maximize their long-run level of the benefits, there exists an equilibrium. In equilibrium, voters capture all the benefits from the parties. The degree of inequality in the equilibrium allocation is related to the degree of risk aversion with which the electorate views candidate behavior.

3. An <u>issue</u> is a measure or proposal which, if enacted, would generate a fixed distribution of benefits and costs, and on which each candidate must take a position. We obtain simple classification of issues according to their electoral consequences, and show that one important category of issues--which we label the "controversial" issues--is strategically important. The existence of a controversial issue invariably work to the disadvantage of the incumbent; hence he

always has an incentive to suppress or remove it from the electoral arena altogether, if he can. If he cannot, it will then be optimal for the incumbent to favor the issue if and only if it is one which produces a (positive) net social benefit. Even with this optimal position, however, under general conditions the incumbent will nevertheless be defeated, by a challenger who opposed the issue and who will therefore not enact it, even though it would be socially optimal to do so. These results thus support the doubts expressed by Thurow and others, concerning the inability of a competitive democratic systems to deal effectively with major issues when distributional considerations become politically important. They also imply, however, that Thurow's proposed reforms, to strengthen party responsibility, would not help, since the problem lies in the nature of the competitive process itself.

ELECTORAL POLITICS IN THE ZERO-SUM SOCIETY

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February 1983

Theoretical work on electoral competition[1] has concentrated almost exclusively on a structure in which the candidates or parties compete over a space of issues or policy positions. Distributional considerations can arise only indirectly in such a setting, and candidates cannot appeal directly to particular constituents or groups by offering them specific targeted benefits or services. This theory of pure "issue" politics thus ignores the prevalent constituent-service aspects of contemporary electoral politics. In the present paper we investigate the nature of a competitive electoral process in an alternative, purely allocational, regime, in which candidates compete by directly offering particular benefits and services to voters.

0. Summary and Overview

The basic structure is quite simple: there are two candidates, a challenger and an incumbent, who compete for votes by promising specific benefits or services to some or all of the n voters or groups who comprise the electorate. These benefits and services, which are indexed by a single, composite private good, are positively valued by all voters, and also by the candidates themselves (or their parties and supporters). Each candidate offers an allocation $z \in \mathbb{R}^n$

of benefits to the electorate, where $z_i \geq 0$ is the amount offered to voter i, and the offers collectively must satisfy a budget constraint $\sum_i z_i \leq A$. If a candidate offering an allocation z receives a majority (of m or more votes, where $m = \frac{n+1}{2}$) he wins; each voter i then receives the promised amount z_i, while the candidate himself receives the residual $A - \sum_i z_i \geq 0$, his <u>surplus</u>. The candidate wishes to maximize his surplus,[2] so will offer voters only the minimum necessary to secure this majority. The losing candidate does not gain control of the pool of benefits, and therefore receives no surplus himself; we assume that he nevertheless attempts to minimize his opponent's surplus, by making his victory as expensive as possible.

An incumbent, being already in office and having control over the pool of benefits in the period preceeding the election, must act and actually provide benefits to his constituents during this period. He therefore commits himself to a <u>de facto</u> allocation first, before the challenger does. On the other hand voters are assumed to discount the challenger's promises to some degree in weighing them against the actual performance of the incumbent. In particular, if voter i is currently receiving a benefit of x_i from the incumbent, and is offered y_i by the challenger, we assume he votes for the challenger only if $y_i > x_i + p_i$, where p_i is the discount factor, or <u>incumbency premium</u>, of the ith voter.

These discount factors play an important role in the analysis. We can think of them as measures of voters' loyalties to the incumbent, since the larger p_i is, the more difficult it is for the challenger to obtain i's vote. (The p_i may therefore be negative as well as positive, since some voters' loyalties may be to the "out" party rather than to the incumbent.) More generally still, however, we shall interpret p_i as a kind of "price," which signals the voter's (or group's) availability to bids from either party. The p_i are fixed in the short run, and both candidates must act as "price-takers" and obtain votes by impersonally bidding for them, rather than directly negotiating or bargaining with individual voters or groups. In the longer run, however, these prices are endogenous, and can be altered by the groups themselves if they find it in their interests to do so. Thus, a group for whom $p_i = 0$ essentially pursues a policy of short-run maximization in each election, voting for whichever candidate offers it more, with no discounting. If the expected level of benefits to the group could actually be increased in the long run by applying a positive discount, however, then we would expect p_i to eventually rise. The incumbency premia thus provide a mechanism for constituents to impose demands on the political system, and to oblige the parties to cater to these demands. One important question, clearly, is whether the p_i would keep changing forever, or would ever stabilize -- i.e., whether there exists an <u>equilibrium</u> set of incumbency premia. We address the issue of equilibrium and its ramifications in the final section of the paper.

Initially, however, we take the p_i as fixed, and concentrate on characterizing the short-run behavior of candidates and voters in a single election. Since the challenger can wait until after the incumbent has committed himself to an allocation, his optimal allocation is easily determined, and is formally characterized in Theorems 1.1 and 1.2. In particular, given the vector $p \in \mathbb{R}^n$ of incumbency premia, and the incumbent's allocation $x \in \mathbb{R}^n$, the challenger must offer at least $x_i + p_i$ for i's vote; this quantity (or zero, if it is negative) is thus the "cost" of securing i's vote. The task facing the challenger is to secure a majority at the lowest possible cost, so his optimal strategy will be to offer slightly more than this amount to the m least costly voters, and nothing to the others. Challengers thus pursue a "minimal winning coalition" type of electoral strategy.

The problem facing the incumbent is rather different. In essence, his task is to make victory impossible, or failing that as expensive as possible, for the challenger, by driving up the cost of the least-cost coalition. The incumbent therefore generally cannot afford to favor some voters and ignore others, for if he did, the neglected voters would become easy, "low-cost" targets for the challenger. The strategic situation confronting the incumbent thus leads him to pursue a more broad-based electoral strategy. Optimal allocations for the imcumbent for arbitrary $p \in \mathbb{R}^n$ are partially characterized by Theorem 1.3 and Lemmas 1.1-1.4, and are fully characterized for the case of most interest, when the incumbency premia are in or near equilibrium, by Theorem 1.4. To get some sense of the nature of these allocations, suppose the incumbent chooses an allocation $x > 0$. The challenger will then bid for some least costly majority coalition C of voters. Let $x_m + p_m$ be the cost of the most costly voter in C. If it were true that $x_i + p_i > x_m + p_m$ for any voter i, the allocation x would not be optimal, for in that case the incumbent could either increase his own surplus by offering somewhat less to i, or alternatively decrease the challenger's surplus by reallocating some benefits from i to C. By similar reasoning, x would also not be optimal if $x_j + p_j < x_m + p_m$ for any voter j. Thus, if the allocation x is optimal, it must be true that $x_i + p_i = x_j + p_j$ for all i and j. Theorem 1.4 shows that when the underlying incumbency premia are near equilibrium (in the sense of Definition (3.4)), the incumbent's optimal allocation is of this form, and is given by $\hat{x}_i = a - p_i$ for all i, (where the quantity $a > 0$ is defined by $a = \min (A/m, 1/n[A + \sum_i p_i]))$. Moreover, under the premises of the theorem, it also will be true that $p_i < a$, and hence that $\hat{x}_i > 0$, for every voter i. Thus the incumbent, unlike the challenger, offers benefits to all voters.

These results, though straightforward analytically, nevertheless contrast considerably with those of the issue-oriented Dowhsian or spatial models of electoral competition, and suggest that candidates behave quite differently in an allocational setting. In the issue-oriented models the competitive process drives both candidates to adopt similar positions or strategies. In the allocational structure considered here, on the other hand, the candidates pursue distinctively different strategies, and show no tendency to converge. The nature of the differences are distinctive, and in principle empirically testable: challengers tend to pursue a divisive, minimal-winning-coalition type of strategy, while incumbents pursue a broad-based strategy, and try to appeal to all their constituents.[3] These results also imply that a successful challenger will change his electoral strategy after taking office, by trying to broaden his electoral base beyond his original core of supporters.[4]

In section 2 we turn to a different question, and consider the role of issues in this structure. By an issue we mean a measure or proposal which, if enacted, would generate a fixed distribution $b \in \mathbb{R}^n$ of benefits (or costs, if $b_i < 0$) to voters. We assume such issues are relatively "sparse," and arise only occasionally, and only one per election; moreover the benefits generated by the issue are assumed to be small relative to the pool of allocatable benefits (so that, in particular, it is always possible to fully compensate the "losers" (i.e. those for whom $b_i < 0$) if the proposal is adopted). With issues as with allocations, the incumbent must commit himself first, before the challenger does. (Because of this it is clear an issue cannot help an incumbent, since the challenger can always adopt the incumbent's position, and effectively neutralize the issue in the electoral contest (Comment 3.1).)

Issues are of various kinds. For example a socially beneficial (or disadvantageous, respectively) issue is one for which $\sum_i b_i > 0$ (or < 0, respectively). An issue is majority-preferred if there exists some majority coalition C of voters for whom $b_i > 0$ for all $i \in C$; or is a Pareto-improvement if $b_i > 0$ for all i. A typical

"special-interest" issue would be one which conveys large benefits on a small minority, while imposing costs on the rest of society, while what we might call a "Thurow"-type issue would be one which yields significant net social benefit $\sum_i b_i > 0$), yet imposes severe costs on some small minority (which, he argues, constitute an effective veto group in a democracy). (Thurow (1980).) The questions of interest are to see how such issues affect the fortunes of the candidates, and how the issues themselves ultimately fare in this electoral setting.

From a strategic point of view, the relevant classification of issues turns out to be somewhat different from any of the above, and can be described as follows: if b is an issue such that for every majority coalition C the quantity $\sum_{i \in C} b_i$ (the sum of benefits over the members of C) is non-negative, we shall say the issue is a positive one; conversely, if $\sum_{i \in C} b_i \leq 0$ for all such C, the issue is negative. We define a controversial issue as one which is neither positive nor negative. If voters are indexed in order of their b_i, i.e. so that $b_1 \leq b_2 \leq \dots \leq b_n$, evidently the least-favored majority consists of voter 1 through m. If we define $B^- = \sum_{i=1}^{m} b_i$ as the sum of benefits over this coalition, and similarly $B^+ = \sum_{i=m}^{n} b_i$ as the sum over the most-favored majority, then evidently b is controversial if and only if $B^- < 0$ and $B^+ > 0$.

Theorem 2.1 shows that it is optimal for both candidates to favor positive issues, and to oppose negative issues. Such issues therefore play no real role in the electoral contest, and do not affect the outcome. Positive issues will be ultimately adopted no matter which candidate wins (in particular, Pareto improvements will always be enacted), while negative ones will always be rejected by the winning candidate. To this extent, therefore, the electoral process copes with issues in a sensible manner.

With controversial issues things are more complex, however. To see how they affect the candidates, we first consider (in Comment 2.2) the simpler situation which results if the issue arises after the incumbent has committed himself to an allocation, but before the challenger has. The incumbent, having previously adopted his optimal allocation $\hat{x}$, must now take a position on the issue. If he favors the issue, the challenger can either match the incumbent's position (in which case he would have to bid a for any vote) or alternatively oppose it (in which case he would have to bid $y_i = \hat{x}_i + p_i + b_i = a + b_i$ for i's vote). Since the issue is controversial, there exists a majority coalition C for which the sum $\sum_{i \in C} b_i$ is negative; hence the challenger, by opposing the issue, can obtain this majority at a cost of $am + \sum_{i \in C} b_i < am$, so his surplus will be greater than if he had favored it. In particular, the challenger can always increase his surplus by $-B^-$. Alternatively, if the incumbent opposed the issue, the challenger could increase his surplus by B^+, by favoring it. From the incumbent's point of view, it is optimal for him to take whichever position minimizes his opponent's surplus, and hence to favor the issue if $-B^- \leq B^+$, or to oppose it if this inequality is reversed. The incumbent thus favors an issue if

$B^+ + B^- \geq 0$, or equivalently if $\sum b_i + b_m \geq 0$ (here b_m is the benefit of the mth or median voter, when voters are indexed so that $b_1 \leq b_2 \leq \ldots \leq b_n$). If the median voter's benefit b_m is negligibly small relative to the total social benefit $\sum_i b_i$, the issues the incumbent favors and opposes are essentially the socially beneficial and disadvantageous ones, respectively. It is always optimal for the challenger to take the opposite stand; moreover, under the conditions of Comment 2.2, the challenger will prevail in the election, and his victory will lead to rejection of the issue if it is socially beneficial, or its enactment if it was not.

A rather perverse outcome thus occurs, at least when the incumbent cannot readjust his allocation to try to compensate for the vulnerabilities created by the issue. The more complex case, in which he can optimize over his issue position and allocation simultaneously, is analyzed in Lemmas 2.1-2.5, summarized in Theorem 2.2; qualitatively, the results are rather similar. The incumbent favors a controversial issue if and only if $\sum_i b_i \geq 0$, i.e. it is socially beneficial (Lemma 2.5). In this case it will be optimal for him to allocate more to the "losers" who are disadvantaged by the issue (Lemma 2.4); with this allocation either position becomes optimal for the challenger (Lemma 2.3). The incumbent's surplus is strictly less than it would have been in the absence of the issue (Theorem 1.4, (2) and (3) of Lemma 2.3), and if the issue is divisive enough (i.e. if $-B^-$ is large enough), the challenger will win ((1) of Lemma 2.3).

Some implications of these results are as follows: A rational incumbent favors the public interest (i.e. favors issues which are socially beneficial, and opposes those which are not), simply because this is the most profitable position for him electorally. For a controversial issue, however, this is only a "second-best" strategy, since such an issue always works to his disadvantage, no matter what position he takes on it.[5] An incumbent thus has an even stronger incentive to suppress controversial issues altogether. To the extent that incumbent officeholders can control and manipulate the political agenda, therefore, we should expect them to try to keep such issues off the agenda; or, failing that, to at least keep them out of the electoral arena, for example by referring them to other jurisdictions, or the bureaucracy or courts, for resolution. Challengers, on the other hand, have the opposite incentive, and at least in the short run stand to benefit from having elections fought over controversial issues.[6]

With controversial issues and against a rational incumbent, it is optimal for the challenger to oppose the incumbent's position, and hence to oppose the public intent. Moreover such issues work to the advantage of the challenger, so if the incumbent's margin was small or nonexistent to begin with, and/or the issue divisive enough, the challenger will prevail. The public interest thus fares poorly in this electoral process: elections will often be won by candidates who oppose measures which would improve the social welfare, or who advocated undesirable special interest causes. These findings thus support many of Thurow's (1980) conclusions.[7]

We turn finally to a more fundamental question, concerning the equilibrium of the underlying incumbency premia. In the argument so far the p_i have been taken as fixed. As suggested at the outset, however, these are actually policy variables, and in the long run may be altered by the various voters or groups themselves, if they find it in their interests to do so. Until now the candidates have been the only strategically active agents, with voters playing an essentially passive role: once confronted by the candidates' offers, voters can only cast their ballots and accept whatever benefits have been promised by the winning candidate; if his surplus is large, most of the benefits will accrue to the candidate and his party, and few to the citizens for whom they were presumably originally intended. In the longer run, however, the incumbency premia provide a means for voters to influence outcomes, and to induce candidates to become more responsive to their demands. Thus, for example, in an era in which the incumbent is dominant and regularly wins with a large surplus, a voter or group which finds itself taken for granted and inadequately provided for may seek redress by gradually weakening its loyalties to the incumbent. This may encourage the challenger to bid more energetically for i's vote, and possibly even lower the challenger's cost sufficiently to enable him to win; even if not, the mere threat of defection may induce the incumbent to increase i's benefit, to retain his vote. To the extent that such influences increase the level of i's expected benefit, it is clearly in i's interest to change his incumbency premium accordingly; and we should expect him to eventually do so. The original n-tuple p of premia is therefore unstable, since at least one group i has both the incentive and the ability to change it, by altering its own component p_i. We define an <u>equilibrium</u> as an n-tuple of incumbency premia which is not unstable in this sense (Definition 3.1).

It is certainly conceivable that there is no such equilibrium, and that any $p \in \mathbb{R}^n$ is subject to continual change by some or all voters (for example, by perpetually striving to make their p_i ever smaller, thus driving them towards $-\infty$). As it turns out, however, equilibria can be shown to exist in this structure, and to be rather plausible in nature. The equilibria of interest — the "non-degenerate" ones -- are characterized by Theorem 3.2. Some implications of this result are as follows:

First, in equilibrium the surplus to winning candidate is zero ((4) of Theorem 3.2): all benefits are thus distributed to the electorate, and none retained by the parties. In the long run, therefore, the ability of voters to shift loyalties does serve as an effective control on the behavior of the political elites, and ultimately forces them to use the benefits to increase the welfare of citizens rather than simply enrich themselves. (This result is reminiscent of the zero-profit condition of a competitive economic equilibrium.)

Second, in equilibrium every voter receives a strictly positive level of benefit ((3) of Theorem 3.2). This may at first glance appear to be a somewhat "egalitarian" outcome. In the more

relevant welfare sense, however, it is not, since it takes no account of underlying income or social inequalities. Indeed, since many of the benefits in question arise from programs or policies intended to redress these underlying inequalities, social equity is promoted by providing them to the needy or disadvantaged. Theorem 3.2 implies, however, that when the disposition of such benefits enters the political arena and becomes subject to manipulation by politicians seeking electoral gain, they will be distributed more widely, and offered to all voters irrespective of need. The equilibrium allocation of benefits is thus not one likely to promote social equality.[8] (This tendency may also give some insight into the often-noted "reciprocity norm" in Congressional public works spending, whereby projects are allocated to all districts irrespective of economic justification or partisanship.)

Part (3) of Theorem 3.2 states that any equilibrium p must satisfy $\sum_i p_i = (\frac{m}{m-1})A$ (or in effect, that the average incumbency premium must equal the per capita level of benefits available). When this equality does not hold p is not in equilibrium, and there will be voters or groups who can benefit by changing their premiums accordingly. Until the equilibrium is restored, however, the electoral process will be temporarily biased in favor of one or the other of the candidates -- the incumbent if $\sum p_i$ is too large, or the challenger if too low (Theorem 1.4). Thus, consider the situation in which the system is initially in equilibrium, when the pool of benefits is suddenly and exogenously decreased. The p_i would then be too high relative to A. In the long run voters will eventually adjust them downwards and restore equilibrium, but in the interim, before this occurs, there will be a series of elections in which $\sum p_i$ exceeds its equilibrium level, and in these elections the incumbent will win, and will earn a positive surplus ((1) of Theorem 1.4). It thus makes perfect sense for an incumbent to support a balanced-budget amendment, or other proposal which imposes an exogenous cap on public benefits, since he stands to profit handsomely in the transitional period. (The converse is also true, that the challenger would profit from an exogenous increase in A; the effect here is weaker, however, since after the initial election the victorious challenger becomes the incumbent and is thus disadvantaged, and his opponent over time will eventually capture a sizeable share of the windfall.)

An equilibrium is not unique, and in general there will be many $p \in \mathbb{R}^n$ which are potential equilibria. In a two-party electoral system, in particular, we might well expect two different equilibria to be present, which depend on the identity of the incumbent party. Denote the two parties by α and β (until now we have distinguished between parties or candidates only on the basis of their incumbency status), and by p^α and p^β the two corresponding equilibria. Whenever party α is incumbent each voter i discounts the challenger's bid by p_i^α (and thus votes for him only if $y_i > p_i^\alpha + x_i$), and conversely applies the discount factor p_i^β whenever β is incumbent. A voter for whom $p_i^\alpha = p_i^\beta$ behaves the same no matter which party is incumbent. If p_i^α and p_i^β differ, however, i in effect has partisan preferences; in this

case $p_i^\alpha > p_i^\beta$ reflects a preference for party α, while if $p^\beta > p^\alpha$ i is more favorably disposed toward incumbents of the other party. Let us rewrite the incumbency premia in a slightly different and more convenient form by defining two components, $\alpha_i = 1/2\,[p_i^\alpha - p_i^\beta]$, the pure <u>partisanship</u> component, and $\pi_i = 1/2\,[p_i^\alpha + p_i^\beta]$, the pure <u>incumbency</u> component. Then, clearly, $p_i^\alpha = \pi_i + \alpha_i$ and $p_i^\beta = \pi_i - \alpha_i$, so i's premium is the sum of the incumbency effect plus or minus (depending on the identity of the incumbent) the partisan effect. The partisan component α_i is thus essentially a measure of i's party identification, since it reflects his loyalty toward or intrinsic preference for α (or β, if $\alpha_i < 0$).

The concept of party identification has played a major role in empirical work on voting and elections. The basic rationalistic premise in the theoretical literature, however, has been that voters view parties and elections instrumentally, as means toward their ultimate ends of attaining better policies or more benefits, and it has proven difficult to reconcile (or even incorporate) a notion of intrinsic party loyalties into this instrumentalist framework. For this reason the concept has played little or no role in the theoretical literature. In the present structure, however, there is a natural way of defining long-run partisan preferences. We may thus inquire into the extent to which such intrinsic loyalties are compatible with individual rationality, and more generally into the nature of the equilibrium distribution of partisan preferences.

(1) of Theorem 3.4 implies that each α_i must satisfy

$-A/m < \alpha_i < A/m$, so may be either positive or negative, and certainly need not be zero. Thus individual partisan preferences for either party are perfectly compatible with individual rationality. At the aggregate level, however, it follows from (2) of Theorem 3.4 that

$$\sum \alpha_i = 1/2\ [\sum p_i^\alpha - \sum p_i^\beta] = 1/2\ [(\frac{m-1}{m})A - \frac{m-1}{m})A] = 0$$

Thus, in equilibrium, the electorate as a whole is not biased in favor of either party. A distribution of partisan preferences which favored one party would be unstable, and there would be voters or groups who stand to gain, in the long run, by weakening their loyalties: over time, such adjustments continue until the electorate reaches a partisan balance and a new equilibrium is established. This result thus provides a theoretical explanation of the historical tendency, noted by Sellers (1965) and others, for two-party systems to return to partisan balance over time.

1. <u>Candidate Behavior</u>

We begin with some preliminary definitions and notation. There is a finite set $N = \{1,2,...,n\}$ of <u>voters</u> (or groups), indexed by i. The number n of voters is odd, with $m = (n+1)/2$. A <u>coalition</u> C is a subset of N, and #C is the cardinality of, or number of voters belonging to, the coalition C. We denote by $[j,k]$, $(j,k]$, $[j,k)$, etc. the coalitions $\{i:i \geq j,\ i \leq k\}$, $\{i:i > j,\ i \leq k\}$, $\{i:i \geq j,\ i < k\}$, etc. C is a <u>majority</u> coalition if $\#C \geq m$, and $M^* = \{C \subset N: \#C \geq m\}$ is the set of such majority coalitions. For any vector $z \in \mathbb{R}^n$ and coalition $C \subset N$, $z(C)$ denotes the quantity $z(C) = \sum_{i \in C} z_i$.

An <u>allocation</u> by a party is a vector $z \in \mathbb{R}^n$ which is nonnegative, $z \geq 0$, and which satisfies the budget constraint $z(N) \leq A$, where $A > 0$ is the fixed budget of benefits or services available for allocation to voters or for consumption by the political parties. There are two political parties (or candidates), the <u>incumbent</u>, party 1, and the <u>challenger</u>, party 2. We denote by $X = \{x,x',...\}$ the possible allocations by the incumbent, and by $Y = \{y,y',...\}$ the possible allocations by the challenger.

The <u>incumbency premium</u> for voter i is a real number p_i, which may be positive or negative (or zero). Given the allocations x and y, voter i votes for the challenger if $y_i > x_i + p_i$; otherwise, if $y_i \leq x_i + p_i$, he votes for the incumbent. The n-tuple of incumbency premiums is a vector $p \in \mathbb{R}^n$.

Given x, y and p, let C be the set of voters who vote for the incumbent: thus $C = \{i: x_i + p_i \geq y_i\}$. If C constitutes a majority, i.e. $C \in M^*$, the incumbent <u>wins</u> the election. We then define: $v_i(x,y;p) = x_i$, the <u>payoff</u> to each voter i; $s_1(x,y;p) = A - x(N) \geq 0$, the <u>surplus</u> to the incumbent; and $s_2(x,y;p) = - s_1(x,y;p)$, the surplus to the challenger. Otherwise, if $C \notin M^*$, the <u>challenger</u> wins, and the payoffs and surpluses are $v_i(x,y;p) = y_i$ for any i, $s_2(x,y;p) = A - y(N)$, and $s_1(x,y;p) = - s_2(x,y;p)$.

Each party is assumed to be interested in obtaining as large a surplus as possible. Given p and the incumbent's allocation x, let $\hat{s}_2(x;p) = \sup_{y \in Y} s_2(x,y;p)$. It would be natural to say an allocation $\hat{y}$ is "optimal" for the challenger if $s_2(x,\hat{y};p) = \hat{s}_2(x;p)$. However if

$\hat{s}_2(x;p) > 0$, strictly optimal allocation of this kind need not exist: for in that case the allocations y yielding a positive surplus $s_2(x,y;p) = A - y(N)$ are those such that $y_i > x_i + p_i$, all $i \in C$, for some $C \in M^*$, so if $x_i + p_i \geq 0$, attempting to maximize the quantity $A - y(N)$ would lead to a corner solution such that $y_i = x_i + p_i$, at which the challenger would lose i's vote and thus his majority (and hence his positive surplus). Rather than use such an allocation, the challenger would instead bid slightly more for the voters $i \in C$, thereby ensuring a majority and a slightly less-than-optimal, but positive, surplus. Thus, for any $\varepsilon > 0$, we shall say an allocation y is _ε-optimal_ for the challenger if $s_2(x,y;p) \geq \hat{s}_2(x;p) - \varepsilon$, and is winning if it is possible for the challenger to win. The ε-optimal allocations for the challenger are characterized as follows:

Theorem 1.1. Given p and the incumbent's allocation x, define $q_i^x = \max(0, x_i+p_i)$, all i, and w(x) as the minimum of $q^x(C)$ over the set of majority coalitions C, i.e. $w(x) = \min_{C \in M^*} q^x(C)$. Then:

(1) If $w(x) \geq A$ the challenger cannot win against x. His surplus is $\hat{s}_2(x;p) = s_2(x,y;p) = -[A - x(N)] < 0$ no matter what allocation y he uses, i.e. every $y \in Y$ is ε-optimal.

(2) If $w(x) < A$ the challenger can win, and $\hat{s}_2(x;p) = A - w(x) > 0$. An allocation y is ε-optimal if and only if $y(N) \leq w(x) + \varepsilon$, and $y_i > x_i + p_i$ all $i \in C$, for some $C \in M^*$.

Proof (1): In order for the challenger to win there must exist an allocation y and majority coalition $C \in M^*$ such that $y_i > x_i + p_i$, all

$i \in C$. Since $y_i \geq 0$, this would imply either $x_i + p_i < 0 = q_i^x \leq y_i$ or $0 \leq x_i + p_i = q_i^x < y_i$ for each $i \in C$; moreover the latter must hold for at least one $i \in C$ (for otherwise we would have $0 = q^x(C) \geq w(x)$, a contradiction of the hypothesis that $w(x) \geq A$). Hence $y(C) > q^x(C) \geq w(x) \geq A$. This is impossible, however, for y would then violate the budget constraint $A \geq y(N)$, since clearly $y(N) \geq y(C)$. Therefore the challenger cannot win, implying $s_2(x,y;p) = -s_1(x,y;p) = -[A - x(N)]$ for any allocation $y \in Y$, and hence that any such allocation is ε-optimal for all $\varepsilon > 0$.

(2) Let $\hat{C}$ be a majority coalition which minimizes $q^x(C)$, i.e. $q^x(\hat{C}) = w(x)$. For sufficiently small $\varepsilon > 0$, the allocation y^ε given by

$$y_i^\varepsilon = \begin{cases} q_i^x + \frac{\varepsilon}{n} & \text{for } i \in \hat{C} \\ \frac{\varepsilon}{n} & \text{otherwise} \end{cases}$$

is winning, and yields a surplus of $s_2(x,y^\varepsilon;p) = A - y^\varepsilon(N) = A - q^x(\hat{C}) - \varepsilon$. Hence $\hat{s}_2(x;p) \geq A - q^x(\hat{C}) = A - w(x) > 0$. Since any ε-optimal allocation y must also be winning, it must satisfy $y_i \geq q_i^x$, all $i \in C$, for some majority coalition $C \in M^*$. But then $y(N) \geq y(C) \geq q^x(C) \geq w(x)$, implying $\hat{s}_2(x;p) - \varepsilon \leq s_2(x,y;p) = A - y(N) \leq A - w(x)$. Hence, letting $\varepsilon \to 0$, it follows that $\hat{s}_2(x;p) = A - w(x)$, from which the rest of (2) follows immediately. QED

The allocations of interest are those which are ε-optimal for small ε; thus (with some abuse of terminology) we define:

Definition 1.1. An allocation $\hat{y}$ is optimal for the challenger against x if for any sequence $\varepsilon^j \to 0$, $\varepsilon^j > 0$ there exists a sequence of allocations y^j such that $y^j \to \hat{y}$ and y^j is ε^j-optimal against x for all j. We denote by $\hat{Y}(x;p)$ the set of allocations which are optimal against x.

In practice, of course, a rational challenger would not use an optimal allocation, but would instead choose an ε-optimal one. However, the limiting allocations provide a complete characterization of the ε-optimal allocations (for small ε), and we shall henceforth confine attention to them. An explicit characterization is as follows:

Theorem 1.2 Given p and x, define q_i^x as in Theorem 1.1, and let voters be indexed so that $q_i^x \leq q_{i+1}^x$, all i. Then:

(1) If $q^x[1,m] \geq A$ the challenger loses. His surplus is $\hat{s}_2(x;p) = -[A - x(N)]$, and every allocation y is optimal.

(2) If $q^x[1,m] < A$ the challenger can win. His optimal surplus is $\hat{s}_2(x;p) = A - q^x[1,m] > 0$. An allocation $\hat{y}$ is optimal if and only if

$$\hat{y}_i = \begin{cases} q_i^x & \text{for } i \leq m \\ 0 & \text{otherwise} \end{cases}$$

for some such indexing of voters.

Proof Since $q_i^x \leq q_{i+1}^x$, all i, it follows that $[1,m] = \{1,2, \ldots , m\}$ minimizes $q^x(C)$ over $C \in M^*$. (1) and all but the "only if" part of (2) then follow immediately from Theorem 1.1. To show the rest, note that if C minimizes $q^x(C)$ over $C \in M^*$, then $q_i^x \leq q_{i'}^x$, for all $i \in C$, $i' \notin C$, (for otherwise substituting i' for i would yield a coalition $C' = C \cup \{i'\} - \{i\} \in M^*$ for which $q^x(C') < w(x)$, a contradiction), so there exists some indexing such that $C = [1,\#C]$. If $\#C > m$, it must be true that $q_i^x = 0$ for all $i \in (m,\#C]$ (for otherwise $q^x[1,m] < q^x(C) = w(x)$, again a contradiction), which implies the result. QED

Turning now to the other party, we shall say an allocation is optimal for the incumbent if it guarantees him as large a surplus as possible: thus, given p, let $\hat{s}_1(p) = \max_{x \in X} \inf_{y \in Y} s_1(x,y;p)$. Then $\hat{x}$ is optimal for the incumbent if $\inf_{y \in Y} s_1(\hat{x},y;p) = \hat{s}_1(p)$. We denote by $\hat{X}(p)$ the set of such allocations. They are characterized by the following series of results:

Theorem 1.3 Given p, for any allocation x define w(x) as in Theorem 1.1. An allocation $\hat{x}$ is then optimal for the incumbent iff either

(1) $w(\hat{x}) \geq A$ and $\hat{x}(N) = \min_{\{x: w(x) \geq A\}} x(N)$, i.e. $\hat{x}$ is winning, and does so at minimum cost. The incumbent's surplus is then $\hat{s}_1(p) = A - \hat{x}(N) \geq 0$.

(2) $w(\hat{x}) < A$ and $w(\hat{x}) = \max_{x \in X} w(x)$, i.e. $\hat{x}$ is losing, but maximizes the

challenger's cost of winning. The incumbent's surplus is then $\hat{s}_1(p) = -[A - w(\hat{x})] < 0$.

<u>Proof</u> Follows directly from Theorem 1.1 and the fact that the incumbent wishes to maximize the quantity

$\inf_y \hat{s}_1(x,y;p) = \inf_y [-s_2(x,y;p)] = -\sup_y s_2(x,y;p) = -\hat{s}_2(x;p)$ over $x \in X$.

QED

We shall say an allocation x is a <u>trivial</u> optimum if every $x' \in X$ is optimal for the incumbent. Then:

<u>Lemma 1.1</u> Let voters be indexed so that $p_i \leq p_{i+1}$, all i, and define $\underline{p}_i = \min(0,p_i)$, all i. The following statements are then equivalent:

(1) There exists a trivial optimum for the incumbent, or equivalently $\hat{y} = 0$ is uniquely optimal for the challenger, against any x.

(2) The challenger wins with surplus $\hat{s}_2(p) = A = -\hat{s}_1(p)$, against any x.

(3) $\underline{p}[m,n] \leq -A$, or equivalently $\underline{p}(C) \leq A$ for every $C \in M^*$.

<u>Proof</u> (3) ⇒ (2): Let r be the largest integer such that $p_i < 0$ for all $i \leq r$. Suppose $\underline{p}[m,n] \leq -A$. Then clearly $r \geq m$, and $\underline{p}[m,n] = p[m,r]$. For any allocation x, let $J^+ = \{i \leq r:\ p_i + x_i > 0\}$. If $\#J^+ \geq (r - m + 1) = \#[m,r]$, then evidently $p(J^+) \leq p[m,r]$ from the indexing and the fact that $p_i < 0$ for $i \leq r$. Hence $q^x(J^+) = (x + p)(J^+) =$ $x(J^+) + p(J^+) \leq A + p[m,r] \leq A - A = 0$, since $x(J^+) \leq x(N) \leq A$ and

$p[m,r] = \underline{p}[m,n] \leq -A$. This is a contradiction, however, since $q_i^x > 0$ for an $i \in J^+$. Hence it must be that $\#J^+ \leq r - m$, and hence that $\#\{i \leq r:\ x_i + p_i \leq 0\} = r - \#J^+ \geq m$. Thus there exists a coalition C, $\#C = m$, such that $x_i + p_i \leq 0$ for all $i \in C$, so for any $\varepsilon > 0$ the allocation

$$y_i^1 = \begin{cases} \varepsilon & \text{for } i \in C \\ 0 & \text{otherwise} \end{cases}$$

would win for the challenger and yield a surplus of $s_2(x,y^1;p) = A - m\varepsilon$. Since $s_2(x,y;p) \leq A$ for any x, y, it follows that $A \geq \sup_y s_2(x,y;p) \geq s_2(x,y^1;p) = A - m\varepsilon$, so letting $\varepsilon \to 0$ we have $\hat{s}_2(x;p) = \sup_y s_2(s,y;p) = A$. Since this holds for all $x \in X$ it follows that $\hat{s}_1(x) = \max_x -\hat{s}_2(s;p) = -A$.

(2) ⇒ (1): Clearly $s_2(x,y;p) \leq A$ for any x, y, so for any $x' \in X$, it must be true that $\hat{s}_1(p) = \max_x [-\hat{s}_2(x;p)] \geq -\hat{s}_2(x';p) =$ $-\sup_y [s_2(x',y;p)] \geq -A$. Hence $\hat{s}_1(p) = -A$ implies $-\hat{s}_2(x';p) = -A$, for all $x' \in X$, i.e. that every such allocation is a trivial optimum for the incumbent.

(1) ⇒ (3): Suppose a trivial optimum exists but, contrary to (3), that $\underline{p}[m,n] > -A$. A trivial optimum exists iff $\hat{s}_2(x;p) = -\hat{s}_1(p)$ for all $x \in X$. This could not be true if $\hat{s}_1(p) \geq 0$, for then the incumbent would always win (with any x), so his surplus would be $s_1(x,y;p) = A - x(N)$, which clearly depends on the choice of x. Hence the incumbent must lose, and $0 > \hat{s}_1(p) = -\hat{s}_2(x;p)$ for all x. As

before, let r be the largest integer such that $p_i < 0$ for all $i \leq r$. If $r \leq m$ then under any allocation of the form $x_i = \delta \geq 0$, all $i \geq r$, $[1,m]$ would be a least-cost coalition, so $\hat{s}_2(x;p) = A - q^x[1,m] = A - p[r,m] - (m - r + 1)\delta$, which clearly depends on the choice of δ, a contradiction; hence $r > m$. Now consider the vector x defined by

$$x_i = \begin{cases} 0 & \text{for } i < m \\ -p_i + \delta & \text{for } i \in [m,r] \\ \delta & \text{for } i > r \end{cases}$$

If $\delta \geq 0$ clearly $x_i \geq 0$ for all i. Since $p[m,r] = \underline{p}[m,n]$ and $\underline{p}[m,n] > -A$ by hypothesis, evidently $x(N) = 0 + (\delta - p)[m,r] + \delta(r,n] = -p[m,r] + m\delta = -\underline{p}[m,n] + m\delta < A + m\delta$. Hence for sufficiently small $\delta > 0$, x will be a feasible allocation. Evidently $[1,m]$ will still be a least-cost coalition for the challenger, and $q^x[1,m] = q^x[1,m) + q^x_m = 0 + \delta$. Hence his surplus is $\hat{s}_2(x;p) = A - q^x[1,m] = A - \delta$, which, again, depends on δ, a contradiction which proves the result. QED

Next, we have:

<u>Lemma 1.2</u> Let voters be indexed so that $p_i \leq p_{i+1}$, all i, and define $\bar{p}_i = \max(0,p_i)$, all i. The following statements are then equivalent:

(1) $\hat{x} = 0$ is uniquely optimal for the incumbent.

(2) The incumbent wins with surplus $\hat{s}_1(p) = A = -\hat{s}_2(p)$, against any y.

(3) $\bar{p}[1,m] \geq A$, or equivalently $\bar{p}(C) \geq A$ for all $C \in M^*$.

<u>Proof</u> (3) ⇒ (2): If (3) holds then the allocation $\hat{x} = 0$ satisfies $q^{\hat{x}}_i = \bar{p}_i \leq \bar{p}_{i+1} = q^{\hat{x}}_{i+1}$, all i. Hence $[1,m]$ is a least-cost coalition and its cost to the challenger is $q^{\hat{x}}[1,m] = \bar{p}[1,m] \geq A$, from (3). Hence from Theorem 1.1 the incumbent wins with $\hat{x}$, and his surplus is $\hat{s}_1(p) = A - \hat{x}(N) = A$.

(2) ⇒ (1): If (2) holds then any optimal allocation x must satisfy $A = \hat{s}_2(p) = A - x(N)$, implying $x(N) = 0$, and hence that $\hat{x} = 0$ is the unique optimal allocation.

(1) ⇒ (3): Suppose $\hat{x} = 0$ is nontrivially optimal. Evidently $q^{\hat{x}}_i = \max(0, 0+p_i) = \bar{p}_i$ for all i, and $[1,m]$ is a least-cost coalition for the challenger. If the challenger won then $p_m \leq 0$ would imply $p_i \leq 0$, whence $q^x_i = 0$, all $i \leq m$, whence $\hat{s}_2(p) = A - q^x[1,m] = A$, and hence (from (2) of Lemma 1.1) that $\hat{x}$ is a trivial optimum, a contradiction of the initial hypothesis. The remaining possibility is $p_m < 0$. In that case, under the allocation $x_i' = \delta > 0$, all i, $[1,m]$ is still a least-cost coalition, and $q^{x'}[1,m] \geq q^{\hat{x}}[1,m] + \delta > q^{\hat{x}}[1,m]$, so $\hat{s}_2(x',p) = A - q^{x'}[1,m] < A - q^{\hat{x}}[1,m] = s_2(\hat{x};p)$, so $\hat{x}$ would not be optimal, again a contradiction. Thus the challenger cannot win against $\hat{x}$, so (1) of Theorem 1.1 must hold, implying $A \leq q^{\hat{x}}[1,m] = \bar{p}[1,m]$, i.e. that (3) above holds. QED

The interesting case is when $x \neq 0$ is a nontrivial optimum. As an immediate consequence of Lemmas 1.1 and 1.2, we note:

Comment 1.1 Let voters be indexed so that $p_i \leq p_{i+1}$, all i, and define $\underline{p}_i$ and $\overline{p}_i$ as in Lemmas 1.1 and 1.2. The following statements are then equivalent:

(1) There exists a nontrivial optimum $\hat{x} \neq 0$ for the incumbent.

(2) $-A < \hat{s}_1(p) < A$ and $-A < \hat{s}_2(p) < A$.

(3) $\overline{p}[1,m] < A$ and $\underline{p}[m,n] > -A$.

Next, we note

Lemma 1.3 Suppose there exists a nontrivial optimum $\hat{x} \neq 0$ for the incumbent, and let w(x) be defined as in Theorem 1.1. Then either

(1) $w(\hat{x}) = A$ and $\hat{x}(N) = \min_{\{x:w(x)\geq A\}} x(N) \leq A$, in which case the incumbent wins and $A > \hat{s}_1(p) \geq 0$, or

(2) $w(\hat{x}) = \max_x w(x) < A$ and $\hat{x}(N) = A$, in which case the challenger can win and $A > \hat{s}_2(p) > 0$.

Proof (1): If (1) of Theorem 1.3 holds and $w(\hat{x}) > A$ then an allocation

$$x'_i = \begin{cases} \hat{x}_i - \varepsilon & \text{if } \hat{x}_i > 0 \\ \hat{x}_i & \text{otherwise} \end{cases}$$

would satisfy $x'(C) \geq \hat{x}(C) - n\varepsilon$ for any $C \in M^*$, so for sufficiently small $\varepsilon > 0$, $w(x') \geq A$, i.e. x' would also be winning. But since $\hat{x} \neq 0$, $x'(N) < \hat{x}(N)$ so x' would increase the incumbent's surplus, i.e. $\hat{x}$ would not be optimal.

(2) If (2) of Theorem 1.3 holds but $x(N) < A$ then there exists an allocation $x_i' = \hat{x}_i + \varepsilon$ for some $\varepsilon > 0$. From Comment 1.1 $A > \hat{s}_2(p) = A - w(\hat{x})$, whence $0 < w(\hat{x}) \leq q^{\hat{x}}(C)$ for any $C \in M^*$. But for any such C, $q^{x'}(C) > q^{x}(C)$ (since $0 < q_i^{\hat{x}} = \hat{x}_i + p_i < x'_i + p_i = q_i^{x'}$ for at least one $i \in C$), so $w(x') > w(\hat{x})$ and $\hat{x}$ would not be optimal, again. Hence $x(N) = A$. QED

The following provides a more explicit, though partial, characterization of the nontrivially optimal allocations:

Lemma 1.4 Suppose $\hat{x} \neq 0$ is a nontrivial optimum for the incumbent. Let voters be indexed so that $q_i^{\hat{x}} \leq q_{i+1}^{\hat{x}}$, and $p_i \leq p_{i+1}$ if $q_i^{\hat{x}} = q_{x+1}^{\hat{x}}$, for all i, and let ℓ be the smallest integer for which $q_i^{\hat{x}} > 0$ and r the largest for which $x_i > 0$. Then:

(1) $1 \leq \ell \leq r \leq n$ and $\ell \leq m$

(2) $p_i \leq p_{i+1}$ for all $i \geq r$ and $i < \ell$

(3) $\hat{x}_i > 0$ implies $q_i^{\hat{x}} > 0$ for all i

(4) If $r > m$ then $i < \ell$ implies $p_i < 0$, and $\hat{x}$ is of the form

$$\hat{x}_i = \begin{cases} \alpha - p_i > 0 & \text{for } i \in [\ell, r] \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$. (In this case $p_i \leq p_{i+1}$ for all i).

<u>Proof</u> (3): Suppose (3) did not hold, i.e. that $\hat{x}_{i*} < 0$, $q^{\hat{x}}_{i*} = 0$ for some i*. By the reasoning used in proving Lemm a 1.3, if the incumbent wins, then the allocation $x_i' = 0$ for $i = i*$, $= \hat{x}_i$ otherwise would also win, yet would increase the incumbent's surplus. Alternatively, if the incumbent lost then the allocation $x_i' = 0$ for $i = i*$, $= \hat{x}_i + \frac{x_{i*}}{n-1}$ otherwise would decrease the challenger's surplus. Hence $\hat{x}$ would not be optimal, a contradiction.

(1): $\hat{x} \neq 0$ implies $1 \leq r \leq n$, and with (3) above implies $1 \leq \ell \leq r$. If $\ell > m$ then $q^{\hat{x}}[1,m] = 0$ so $\hat{s}_2(p) = A$, which from Lemma 1.1 would contradict the hypothesis that $\hat{x}$ is nontrivially optimal. Hence $\ell \leq m$.

(2): From the indexing and definition of r, clearly $p_r \leq q^{\hat{x}}_r \leq q^{\hat{x}}_i = p_i \leq q^{\hat{x}}_{i+1} = p_{i+1}$ for $i > r$, so $p_{i+1} \geq p_i$ for $i \geq r$. Similarly $i < \ell$ implies $q^{\hat{x}}_i = \max(0,p_i) = 0$, so from the indexing $p_i \leq p_{i+1}$ for $i < \ell - 1$.

To extend this to $i < \ell$, suppose the contrary, i.e. $p_{\ell-1} > p_\ell$. Since $p_\ell < p_{\ell-1} \leq q^{\hat{x}}_{\ell-1} = 0 < q^{\hat{x}}_\ell = p_\ell + \hat{x}_\ell$, it follows that $p_\ell < 0$ and $\hat{x}_\ell > 0$. If the incumbent wins then the allocation

$$x'_i = \begin{cases} \hat{x}_\ell - (p_{\ell-1} - p_\ell) & \text{for } i = \ell - 1 \\ 0 & \text{for } i = \ell \\ \hat{x}_i & \text{otherwise} \end{cases}$$

would satisfy $q^{x'}_i = q^{\hat{x}}_i$ for $i \neq \ell, \ell - 1$, $q^{x'}_\ell = 0 = q^{\hat{x}}_{\ell-1}$, $0 < q^{x'}_{\ell-1} = q^{\hat{x}}_\ell \leq q^{\hat{x}}_i$ for $i \geq \ell$, so since $\ell \leq m$, [1,m] would still be a least-cost coalition for the challenger, and its cost would still be $q^{x'}[1,m] = q^{\hat{x}}[1,m]$. However $x'(N) = x'(N - \{\ell-1,\ell\}) + x'_{\ell-1} + x'_\ell = \hat{x}(N - \{\ell-1,\ell\}) + \hat{x}_\ell - (p_{\ell-1} - p_\ell) + 0 = \hat{x}(N) - (p_{\ell-1} - p_\ell) < \hat{x}(N)$ (since $\hat{x}_{\ell-1} = 0$, from (3) above), so x' would still win and would increase the incumbent's surplus, a contradiction. Alternatively, if the challenger could win against $\hat{x}$, then the allocation

$$x'_i = \begin{cases} 0 & \text{for } i < \ell - 1 \\ \hat{x}_\ell - (p_{\ell-1} - p_\ell) + \varepsilon & \text{for } i = \ell - 1 \\ 0 & \text{for } i = \ell \\ \hat{x}_i + \varepsilon & \text{for } i > \ell \end{cases}$$

where $\varepsilon = \frac{(p_{\ell-1} - p_\ell)}{(n - \ell + 1)} > 0$, would decrease his surplus, again a contradiction of the hypothesis that $\hat{x}$ is optimal.

(4): Suppose $q^{\hat{x}}_r > q^{\hat{x}}_\ell$. Let $T = \{i: \hat{x}_i > 0 \text{ and } q^{\hat{x}}_i = q^{\hat{x}}_r\}$, $t = \#T$, and consider an allocation of the form

$$x'_i = \begin{cases} \hat{x}_i - \varepsilon & \text{for } i \in T \\ \hat{x}_i = 0 & \text{for } i \notin T \cup \{\ell\} \end{cases}$$

for some small $\varepsilon > 0$. If the incumbent wins let $x'_\ell = x_\ell + (t - 1)\varepsilon$. For sufficiently small ε, [1,m] would still be a least-cost coalition and $q^{x'}[1,m] - q^{\hat{x}}[1,m] = (x' - \hat{x})(T \cap [1,m]) + x'_\ell - \hat{x}_\ell =$

$-\varepsilon\ \#(T \cap [1,m]) + (t-1)\varepsilon \geq 0$ (since $r \in T$ by definition, and $r > m$ by assumption, implying $\#(T \cap [1,m]) \leq t - 1$). But since $x'(N) - \hat{x}[N] = -\varepsilon t + (t-1)\varepsilon = -\varepsilon < 0$, x' would increase the incumbent's surplus, which is impossible. Similarly, if the challenger wins, setting $x'_\ell = t\varepsilon$ would decrease his surplus, again a contradiction. Hence it cannot be that $q_r^{\hat{x}} > q_\ell^{\hat{x}}$, which from the indexing implies that $q_r^{\hat{x}} = q_\ell^{\hat{x}} = q_i^{\hat{x}}$ all $i \in [\ell,r]$. Hence, taking $\alpha \equiv q_\ell^{\hat{x}} > 0$ it follows that $\hat{x}$ is of the stated form.

Moreover if $p_{i*} \geq 0$ for some $i* < \ell$, we could construct an allocation $x*$ (where $x*_{i*} = (r - \ell + 1)\varepsilon$ if the challenger wins or $(r - \ell)\varepsilon$ if the incumbent does, $x*_i = \hat{x}_i - \varepsilon$ for $i \in [\ell,r]$, $x*_i = 0$ otherwise) which increase the incumbent's surplus, by the same reasoning as above. QED

Rather than attempt a complete characterization of $\hat{X}(p)$ for all possible p, we shall confine attention to incumbency premiums which are in equilibrium, or nearly so (in a sense which is defined precisely in the section below). Optimal candidate strategies for these p are given by the following important result:

<u>Theorem 1.4.</u> Let $a = \min(A/m, (1/n)[A + p(N)])$. If $|p_i| \leq a$ ($< a$, respectively) is optimal (uniquely optimal, respectively) for the incumbent. The election outcome and surplus to the winning candidate is then as follows:

(1) If $p(N) \geq \frac{m-1}{m}A$ then $a = A/m \leq (1/n)[A + p(N)]$ and the incumbent wins, with surplus $\hat{s}_1(p) = [A - \hat{x}(N)] = [p(N) - (\frac{m-1}{m})A] \geq 0$

(2) If $p(N) < \frac{m-1}{m}A$ then $a = (1/n)[A + p(N)] < A/m$ and the challenger wins, with surplus
$\hat{s}_2(\hat{x};p) = [A - a \cdot m] = \frac{m}{n}[(\frac{m-1}{m})A - p(N)] > 0$.

<u>Proof</u> Note that $A/m \leq (1/n)[A + p(N))]$ iff $\frac{nA}{m} \leq A + p(N)$ iff $\frac{m-1}{m}A \leq p(N)$, (since $n - m = m - 1$), and conversely. Let voters be indexed and ℓ and r defined as in Lemma 1.4.

(1): If $a = A/m$ then $x(N) = (A/m - p)(N) = \frac{nA}{m} - p(N) \leq A$ (by the second inequality above), so $\hat{x}$ is a feasible allocation. For any $C \in M*$, $q^{\hat{x}}(C) = (A/m)\#C \geq (A/m)m = A$ since $\#C \geq m$, so the incumbent wins, and his surplus is $A - \hat{x}(N) = A - \frac{nA}{m} + p(N) = p(N) - \frac{m-1}{m}A \geq 0$.

Now consider some optimal allocation x', and let voters be indexed and ℓ and r defined as in Lemma 1.4, with respect to this allocation. The incumbent must still win, so $q^{x'}(C) \geq A$ for any $C \in M*$. If $r \leq m$ this would imply $q_i^{x'} = p_i \leq A/m$ for all $i > r$, and hence (from the indexing) that $q_i^{x'} \leq A/m$ for all i. If this inequality were strict for any i, however, then $q^{x'}(C) < A$ for any coalition C which contains i and $\#C = m$, so the incumbent would lose, which is impossible. Hence $q_i^{x'} = A/m$, all i, implying $\hat{x} = x'$ and hence that $\hat{x}$ is uniquely optimal.

The remaining possibility is that $r > m$. In this case, from (4) of Lemma 1.4, there exists $\alpha > 0$ such that $x_i' = \alpha - p_i > 0$ for

$i \in [\ell, r]$, $x_i' = 0 > p_i$ for $i < \ell$, while $p_i \geq \alpha$ and $x_i' = 0$ for $i > r$. Clearly the incumbent must win with x', so $q^{x'}[1,m] \geq A$, implying $m(A/m) = A \leq q^{x'}[1,m] = q^{x'}[1,\ell] + q^{x'}[\ell,m] = 0 + \alpha(m - \ell + 1)$. Hence $\alpha \geq A/m$. Equality would imply $\ell = 1$, in which case $\hat{x} = x'$ so $\hat{x}$ is uniquely optimal. Otherwise, suppose $\alpha > A/m$ and $\ell > 1$. Then $r = n$ (since $p_i \leq A/m < \alpha$ for all i), and we can rewrite the above inequality as $(\ell - 1)(A/m) \leq (\alpha - A/m)(m - \ell + 1) =$ $1/2[2(\alpha - A/m)(m - \ell + 1)] = 1/2(\alpha - A/m)([n - \ell + 1] - [\ell - 2]) \leq$ $1/2(\alpha - A/m)(n - \ell + 1)$ (using the facts that $2m = n + 1$ and $\ell \geq 2$). x' must yield at least as large a surplus as $\hat{x}$, so $x'(N) \leq \hat{x}(N)$, i.e. $x'(N) = x'[1,\ell) + '[\ell,n] =$ $0 + (x' - \hat{x})[\ell,n] + \hat{x}[\ell,n] \leq \hat{x}(N) = \hat{x}[1,\ell) + \hat{x}[\ell,n]$, implying $(x' - \hat{x})[\ell,n] = (\alpha - A/m)(n - \ell + 1) \leq \hat{x}[1,\ell) =$ $(A/m - p)[1,\ell) \leq 2(A/m)(\ell - 1)$, since $p_i \geq -A/m$, all i. Clearly this and the earlier inequalities will be consistent only if all are actually equalities, implying $\hat{x}(N) = x(N)$ and hence that $\hat{x}$ is also optimal. Moreover if $p_i > -A/m$ for all i the above inequality would be strict, a contradiction of the earlier inequality, so the hypothesis $\alpha > A/m$, $\ell > 1$ cannot hold, i.e. $x' = \hat{x}$ and $\hat{x}$ is the unique optimal allocation for the incumbent.

(2): If $a = (1/n)[A + p(N)] < A/m$ then $\hat{x}(N) = (1/n[A + p(N) - p](N)$ $= A + p(N) - p(N) = A$, so the allocation is feasible, again. For any C, $\#C = m$, evidently $q^{\hat{x}}(C) = (1/n[A + p(N)])m < (A/m)m = A$ so the challenger can win, and his optimal surplus is
$A - q^{\hat{x}}(C) = nA/n - m/n[A + p(N)] = \frac{m - 1}{n} A - (m/n)p(N) =$

$m/n\ [\frac{m - 1}{m} A - p(N)] > 0$. The optimality and uniqueness of $\hat{x}$ are argued as above. QED

2. Issues

An issue is a proposal or measure which, if enacted, would result in a specific distribution of gains or losses to voters, i.e. a fixed vector $b \in \mathbb{R}^n$ of net benefits. Each party must adopt a position on the issue, i.e. favor or oppose it. If a party favors it and then wins the election using an allocation z, each voter i subsequently receives a benefit of $z_i + b_i$; alternatively, if the winning party opposes the issue, the subsequent benefit to i is only z_i. The incumbent must commit himself to a position and an allocation before the challenger does. Voters use the same decision rule as before, except that then they now include the benefits arising from the candidates' issue positions in their calculations (e.g. if the incumbent opposes the issue and the challenger favors it, i votes for the incumbent if and only if $x_i + p_i \geq y_i + b_i$, and so forth).

A strategy for a party consists of a position and an allocation, e.g. (favor, x). Optimal strategies are defined in the obvious way. We shall say a position (or allocation) is optimal for a party if there is an optimal strategy involving that position (or allocation); if all optimal strategies involve that position (or allocation), it is uniquely optimal. A position is conditionally optimal with respect to an arbitrary (not necessarily optimal) allocation if it maximizes the party's surplus (minimum surplus, for an incumbent) over the set of strategies containing that allocation;

an allocation conditionally optimal with respect to a position is similarly defined. (Clearly a strategy (π, z) is optimal if and only if π and z are conditionally optimal with respect to z and π, respectively.) We denote by $\hat{s}_j(b,p)$ the optimal surplus for candidate j, given p and the issue b, and by $\hat{s}_2(x,\pi;b,p)$ the challenger's optimal surplus against the strategy (x,π).

If both candidates adopt the same position on an issue, the outcome and surpluses will depend only on their allocations. Thus the challenger can always guarantee himself at least $\hat{s}_2(p)$ by matching the incumbent's position and using the optimal allocation of Theorem 1.1; hence

Comment 2.1. For any issue b, $\hat{s}_1(b,p) \leq \hat{s}_1(p)$ and $\hat{s}_2(b,p) \geq \hat{s}_2(p)$; no issue can help the incumbent.

Issues can be classified in various ways. For example an issue is a weak Pareto-improvement over the status quo if $b_i > 0$, all i (or conversely is Pareto-inferior if $b_i < 0$, all i); socially beneficial (or disadvantageous, respectively) if $b(N) > 0$ ($b(N) < 0$, respectively); or is majority preferred if there is some majority coalition $C \in M^*$ for which $b_i > 0$, all $i \in C$. Strategically, however, the following classification turns out to be the fundamental one:

Definition 2.1. Given an issue b, let voters be indexed so that $b_i \leq b_{i+1}$, all i. Then the issue is

(1) positive if $b[1,m] \geq 0$, or equivalently $b(C) \geq 0$, all $C \in M^*$.

(2) negative if $b[m,n] \leq 0$, or equivalently $b(C) \leq 0$, all $C \in M^*$; or

(3) controversial if it is neither positive or negative, i.e. $b[1,m] < 0$ and $b[m,n] > 0$.

If the inequalities in (1) or (2) are strict, the issue is strictly positive or negative, respectively.

In analyzing the electoral impact of such issues, we shall confine attention to situations in which p is near equilibrium, and in which the distribution of benefits and costs generated by the issue itself is small relative to the allocated benefits. In particular, we shall henceforth assume the issue b to be "small" in the following sense:

Definition 2.2 Given p and b, let voters be indexed so that $b_i \leq b_{i+1}$, all i, and define $a = \min(A/m, (1/n)[A + p(N)])$. The issue b is "small" (relative to p) if $|p_i \pm (b_n - b_1)| < a$ for all i.

We now have:

Theorem 2.1. Let b be a positive issue which is "small." It is optimal for both candidates to favor it; uniquely so for the incumbent, if the issue is strictly positive, and for the challenger as well if he can win. The outcome and surplus to the winning candidate are unaffected, i.e. $\hat{s}_j(b,p) = \hat{s}_j(p)$ for either candidate j.

(Analogous results for a negative issue b are obtained by applying the assertions above to a new issue b* = −b.)

<u>Proof</u> Let $x_i = a - p_i$, where as usual $a = \min[A/m, 1/n\ A + p(N)]$. It follows from Definition 2.3 that $a > 0$, and $x_i > 0$, all i. Let voters be indexed so that $b_1 \leq b_2 \leq \ldots \leq b_m$.

Suppose first that the incumbent favors the issue, and uses the allocation x. If the challenger opposed the issue, he would have to bid $q_i' = \max(0, a + b_i)$ for i's vote, so $C' = [1,m]$ will be a least-cost coalition for him, and its cost will be $q'(C') \geq (a + b)(C') = am + b(C') \geq am$ (since $b(C') \geq 0$ for a positive issue). If instead he favored the issue he would have to bid only a for any vote, so the cost of C′ would be am; hence, given this strategy by the incumbent, it is optimal for the challenger to favor the issue, uniquely so if the issue is strictly positive and the challenger can win (if not any position would be optimal for him).

Now suppose the incumbent opposed the issue. If the challenger favors it, he would have bid $q_i^2 = \max(0, a - b_i)$ for i's vote, so $C^2 = [m,n]$ is a least-cost coalition, and its cost is $q^2(C^2) = \max(0, a - b)(C^2) < am$ (since $b_i > 0$ for all $i \in [m,n] = C^2$, for a positive issue). On the other hand if the challenger also opposed the issue he would have to bid a for any voter i's vote, so the ost of securing a majority would be am; hence it is optimal (uniquely so, if the challenger can win) for him to favor it. Moreover since the minimum cost to the challenger would be less in this case, it is uniquely optimal (conditional on x) for the incumbent

to favor the issue.

It remains to show that x is optimal. But if the incumbent favors the issue and uses x, then since the challenger will also favor it, the issue will play no role, and the outcome and surpluses will be as given in Theorem 1.4 (x is identical to the allocation $\hat{x}$ of Theorem 1.4). Hence, in view of Comment 2.1, the strategy (favor, x) is optimal for the incumbent.

To establish uniqueness, if the incumbent used an allocation $x' \neq x$, then whatever his position π, the challenger could adopt the same position and then obtain a surplus of $\hat{s}_2(x';p) > \hat{s}_2(x;p) = \hat{s}_2(x;\pi;b,p)$, from the uniqueness part of Theorem 1.4; hence x is uniquely optimal for the incumbent, from which it follows that it is also uniquely optimal for him to favor the issue. QED

Hence positive issues will be supported and negative ones opposed by both candidates, and such issues will not affect the nature of the allocational contest between the candidates.

Controversial issues are another matter. To get some insight into the impact of such issues, it will be useful to first consider the simpler situation in which the issue arises after the incumbent has already committed himself to an allocation (but before the challenger has). The incumbent, not anticipating the emergence of the issue, would then employ the allocation $\hat{x}$ of Theorem 1.4; subsequently, when the issue arises, he can take whatever position is best for him, but cannot readjust his allocation $\hat{x}$. The challenger, on the other hand, can decide upon his own position and allocation simultaneously, with full knowledge of his opponent's strategy.

Let voters be indexed so that $b_i \leq b_{i+1}$, all i, and define $B^- = b[1,m]$ and $B^+ = b[m,n]$ as the sum of benefits over the least- and most-favored majority coalitions, respectively; since the issue is controversial, $B^- < 0 < B^+$. We then have:

Comment 2.2 Let b be a controversial issue which is "small," and suppose the incumbent uses the allocation $\hat{x}$ of Theorem 1.4. No matter what position the incumbent takes, it is uniquely optimal for the challenger to take the opposite position. It is conditionally optimal for the incumbent to favor the issue if $B^+ \geq |B^-|$ (uniquely so, if the inequality is strict). In this case the challenger can win with surplus

$$\hat{s}_2(\hat{x}, \text{favor}; b, p) = \begin{cases} |B^-| & \text{if } \hat{s}_1(p) \geq 0 \\ \hat{s}_2(p) + |B^-| & \text{otherwise} \end{cases}$$

(Analogous results for the case $B^+ \leq |B^-|$ are obtained by applying the above assertions to the issue $b' = -b$.)

Proof Let voters be indexed so that $b_i \leq b_{i+1}$, all i. If both candidates take the same position, the challenger must bid a for any voter's vote, or am to obtain a majority. If the incumbent favored and challenger opposed the issue the challenger must bid $\max(0, a + b_i)$ for i's vote, so [1,m] is a least-cost coalition. Since the issue is controversial, $b_i < b_n - b_1$, and since it is small, $|b_n - b_1| < a$. Hence $a + b_i > 0$ and the cost of the coalition is

$(a + b)[1,m] = a \cdot m + B^- < a \cdot m$ (since $B^- < 0$) so it is optimal for the challenger to oppose it. By the same reasoning, if the incumbent opposed and the challenger favored the issue, the challenger would have to bid $a - b_i$ for i's vote, [m,n] would be a least-cost coalition, and its cost would be $(a - b)[m,n] = a \cdot m - B^+ > a \cdot m$. Hence, again, it is optimal for the challenger to take the opposite position (uniquely so, if he can win). It is conditionally optimal for the incumbent to take whichever position is most costly for the challenger, i.e. to favor the issue if $|B^-| \leq -B^+$ or to oppose it if the inequality is reversed.

In the former case, the cost to the challenger of securing a majority is $|B^-|$ less than it would have been in the absence of the issue, i.e. in the pure allocation game of Theorem 1.4. If $a = A/m$ the incumbent would have won originally, so now the challenger does, with surplus $\hat{s}_2(b,p) = |B^-|$; otherwise, if $a = 1/n\ [A + p(N)]$ the challenger wins in both cases, with surplus $\hat{s}_2(b,p) = \hat{s}_1(p) + |B^-|$. QED

If the incumbent opposes the issue the challenger, by favoring it, can obtain the votes of those who would benefit from it more cheaply than otherwise, while if the incumbent favors it those who bear its costs become more vulnerable to the challenger. Hence irrespective of what stand the incumbent takes on it, a controversial issue creates opportunities for the challenger. The incumbent minimizes this vulnerability by favoring the issue if $B^+ \geq |B^-|$, or equivalently $0 \leq B^+ + B^- = b[m,n] + b[1,m] = b[1,n] + b_m = b(N) + b_m$;

thus

if the benefit b_m to the median voter is negligibly small, the incumbent favors a controversial issue if it is socially beneficial, and opposes it if not. The challenger, however, takes the opposite position, and prevails. Thus the socially inoptimal position is ultimately victorious. All this is conditional on $\hat{x}$, and assumes the incumbent cannot readjust his allocation to compensate for the vulnerabilities created by his stand on the issue.

To analyze the more general case when the incumbent can optimize over his allocation and position simultaneously, we must first define some additional quantities. As before, let voters be indexed so that $b_i \leq b_{i+1}$, all i, and again define $B^- \equiv b[1,m]$ and $B^+ \equiv b[m,n]$. For any number h, define $I^+(h) \equiv \{i \geq m : b_i > h\}$, $I^-(h) \equiv \{i \leq m : b_i > h\}$, and $f(h) \equiv b(I^+(h)) - b(I^-(h))$. Evidently $f(h) = 0 - 0 = 0$ for $h \geq b_n$, $f(h) = B^+ - B^- > -B^- > 0$ for $h \leq b_1$, and is continuous and strictly decreasing in h for $h \in [b_1, b_n]$. Hence there exists a unique $h^* \in (b_1, b_n)$ such that $f(h^*) = -B^-$. We now define: $I^+ \equiv I^+(h^*)$, $I^- \equiv I^-(h^*)$,

$$g_i \equiv \begin{cases} b_i - h^* & \text{for } i \in I^- \cup I^+ \\ 0 & \text{otherwise} \end{cases}$$

$\beta^+ \equiv g(I^+)$, $\beta^- \equiv g(I^-)$. (Note that $f(h^*) = \beta^+ - \beta^- = -B^-$, $\beta^+ > 0$, $\beta^- \geq 0$, and $g(N) > 0$.)

<u>Lemma 2.1</u> Let b be a controversial issue, and let voters be indexed and the quantities h^*, β^-, β^+, B^-, B^+, g defined as above. The

following statements are then equivalent:

(1) $\beta^- > 0$.

(2) $h^* < b_m$.

(3) $\sum_{i \geq m} (b_i - b_m) < -B^-$.

(4) $b_m > \frac{b(N)}{m - 1}$.

<u>Proof</u> (1) $\Rightarrow$ (2): Since $\beta^- = g(I^-) = (b - h^*)(I^-)$, $\beta^- > 0$ implies $I^- \neq 0$ and hence that $b_i > h^*$ for some $i \in I^- \subset [1,m]$, whence from the indexing $b_m \geq b_i > h^*$.

(2) $\Rightarrow$ (3): $h^* < b_m$ implies $I^+ = [m,n]$ and $m \in I^-$, whence $\beta^- = g(I^-) \geq b_m - h^* > 0$ and $\sum_{i \geq m} (b_i - b_m) > \sum_{i \geq m} (b_i - h^*) = g(I^+) = \beta^+ = -B^- + \beta^- > -B^-$.

(3) $\Rightarrow$ (1): If we set $h = b_m$ evidently $I^-(h) = 0$, $I^+(h) = [m,n]$ and $f(h) = \sum_{i \geq m} [b_i - b_m]$, so if (3) holds $f(h) > -B^-$, which implies that $h^* < b_m$, whence $I^- \neq 0$, whence $\beta^- = (b - h^*)(I^-) \geq b_m - h^* > 0$.

(3) $\Leftrightarrow$ (4): Evidently $\sum_{i \geq m} (b_i - b_m) + B^- = b[m,n] - mb_m + b[1,m] = b(N) - (m - 1)b_m$, so (3) holds iff $b_m > \frac{b(N)}{m - 1}$. QED

Next, we have:

<u>Lemma 2.2</u> Let b be a controversial issue which is "small," with

voters indexed and β^+, g, etc. as defined in Lemma 2.1. Define: $\hat{x}_i \equiv \alpha - p_i - g_i$, all i, where $\alpha \equiv \min\left((1/m)[A + \beta^+], 1/n[A + p(N) + g(N)]\right)$. Then $\hat{x}$ is a feasible allocation for the incumbent, and $\hat{x}_i > 0$, $\hat{x}_i + p_i > 0$ and $\hat{x}_i + p_i + b_i > 0$ for all i.

<u>Proof</u> Since $\alpha \leq (1/n)[A + p(N) + g(N)]$ it follows that $x(N) = (\alpha - p - g)(N) = \alpha \cdot n - p(N) - g(N) \leq n(1/n)[A + p(N) + g(N)] - p(N) - g(N) = A$, so the budget constraint is satisfied. Moreover since $\beta^+ > 0$ and $g(N) > 0$ it follows that $\alpha = \min\left((1/m)[A + \beta^+], (1/n)[A + p(N) + g(N)]\right) > \min\left(A/m, (1/n)[A + p(N)]\right) = a > 0$, while $g_i < b_n - b_i$ by construction. Since the issue is small, $p_i + (b_n - b_1) < a$, whence $\hat{x}_i = \alpha - p_i - g_i > a - p_i - (b_n - b_1) > 0$. Similarly b small implies $(b_n - b_1) < a$, whence $\hat{x}_i + p_i = \alpha - g_i > a - (b_n - b_1) > 0$. It is readily verified that

$$\hat{x}_i + p_i + b_i = \begin{cases} \alpha + h^* & \text{for } i \in I^+ \cup I^- \\ \alpha + b_i & \text{otherwise} \end{cases}$$

For a controversial issue $|b_i| < (b_n - b_1)$, and $(b_n - b_1) < a$ since the issue is small, so $-b_i \leq |b_i| < a < \alpha$ whence $\alpha + b_i > 0$. The same reasoning applies to h^*. QED

<u>Lemma 2.3</u> Let b and $\hat{x}$ be as in Lemma 2.2. If the incumbent favors the issue and uses the allocation $\hat{x}$, either position will be optimal for the challenger. If the challenger uses an optimal strategy, the outcome and surplus to the incumbent will be as follows:

The incumbent wins if and only if

1(a,b) $$p(N) - \left(\frac{m-1}{m}\right)A \geq \begin{cases} -[B^- - \frac{B^-}{m}] & \text{if } b_m \leq \frac{b(N)}{m-1}, \text{ or} \\ -[B^- + (\frac{B^+}{m} - b_m)] & \text{if } b_m > \frac{b(N)}{m-1}. \end{cases}$$

In this case his surplus is

2(a,b) $$\hat{s}_1(1,\hat{x}) = \begin{cases} p(N) - \left(\frac{m-1}{m}\right)A + [B^- - \frac{B^-}{m}] & \text{if } b_m \leq \frac{b(N)}{m-1} \\ p(N) - \left(\frac{m-1}{m}\right)A + [B^- + (\frac{B^+}{m} - b_m)] & \text{if } b_m > \frac{b(N)}{m-1}. \end{cases}$$

Otherwise, if he loses, his surplus is

3(a,b) $$\hat{s}_1(1,\hat{x}) = \begin{cases} (\frac{m}{n})p(N) - \left(\frac{m-1}{n}\right)A + \frac{m}{n}[B^- - \frac{B^-}{m}] & \text{if } b_m \leq \frac{b(N)}{m-1} \\ (\frac{m}{n})p(N) - \left(\frac{m-1}{n}\right)A + \frac{m}{n}[B^- + (\frac{B^+}{m} - b_m)] & \text{if } b_m > \frac{b(N)}{m-1} \end{cases}$$

<u>Proof</u> If the challenger also favors the issue, he must bid a strictly positive (by Lemma 2.2) amount $q_i \equiv \hat{x}_i + p_i = \alpha - g_i$ for voter i's vote. If $i \in I^- \cup I^+$ then $g_i = b_i - h^* \leq b_{i+1} - h^* = g_{i+1}$ from the indexing, so [m,n] is a least-cost coalition to the challenger, and its cost is $q[m,n] = \alpha m - g[m,n] = \alpha m - \beta^+$.

Alternatively, if the challenger opposes the issue, he must bid $0 < q_i' \equiv \hat{x}_i + p_i + b_i = \alpha - g_i + b_i$ for i's vote, where

$$q_i' = \begin{cases} \alpha - (b_i - h^*) + b_i = \alpha + h^* & \text{for } i \in I^- \cup I^+ \\ \alpha + b_i & \text{otherwise} \end{cases}$$

Since $h^* \geq b_i \geq b_{i-1}$ for all $i \in I^- \cup I^+$, it follows that [1,m] is a least-cost coalition. Its cost to the challenger is $q'[1,m] = (\alpha - g + b)[1,m] = \alpha m - g[1,m] + b[1,m] = \alpha m - \beta^- + B^- = \alpha m - \beta^+$, since $\beta^+ - \beta^- = -B^-$ by construction. Hence either strategy is optimal.

To prove the remainder, note that if $\alpha = 1/m[A + \beta^+]$ the cost to the challenger is $q[m,n] = \alpha m - \beta^+ = m(1/m)[A + \beta^+] - \beta^+ = A$; thus the incumbent wins if and only if $1/m[A + \beta^+] \leq 1/n[A + p(N) + g(N)]$.

Consider first the case $b_m \leq \frac{b(N)}{m-1}$. Then, from Lemma 2.1, $\beta^- = 0$, so $g(N) = \beta^+ = -B^-$, and the incumbent wins iff $1/m[A - B^-] \leq 1/n[A + p(N) - B^-]$, or $(n - m)A - mp(N) \leq (n - m)B^-$ which since $n - m = m - 1$ is equivalent to (1a). If this inequality holds and the incumbent wins, his surplus is $\hat{s}_1(1,\hat{x}) = A - \hat{x}(N) = A - [\alpha n - p(N) - q(N)] = A - n(1/m[A + \beta^+]) + p(N) + \beta^+ = p(N) - (\frac{m-1}{m})A - (\frac{m-1}{m})(-B^-)$, which is equivalent to (2a). Alternatively, if the inequality fails then the incumbent loses, $\alpha = 1/n[A + p(N) + g(N)]$, and $\hat{s}_1(1,\hat{x}) = -(A - q[m,n]) = -A + \alpha m - \beta^+ = -A + m(1/n[A + p(N) + g(N)]) - \beta^+$

$= (\frac{m-n}{n})A + \frac{m}{n}p(N) + (\frac{m-n}{n})\beta^+ = \frac{m}{n}p(N) - (\frac{m-1}{n})A + (\frac{m-1}{n})B^-$, implying (3a).

Now consider the case $b_m > \frac{b(N)}{m-1}$. From Lemma 2.1, $\beta^- > 0$, whence $I^+ = [m,n]$ and $g(N) = (b - h^*)(I^+ \cup I^-) = (b - h^*)(I^+) + (b - h^*)(I^-) - (b - h^*)(I^+ \cap I^-) = \beta^+ + \beta^- - (b_m - h^*)$, since $I^+ \cap I^- = \{m\}$. As before, the incumbent wins iff $(1/m)[A + \beta^+] \leq (1/n)[A + p(N) + g(N)] = (1/n)[A + p(N) + \beta^+ + \beta^- - (b_m - h^*)]$, or equivalently, after some manipulation, $p(N) - (\frac{m-1}{m})A \geq (\frac{m-1}{m})\beta^+ - \beta^- + (b_m - h^*) = \beta^+ - \beta^- - \frac{\beta^+}{m} + (b_m - h^*)$. Since $\beta^+ - \beta^- = -B^-$ and $\beta^+ = (b - h^*)(I^+) =$

$(b - h^*)[m,n] = B^+ - mh^*$, this inequality becomes

$p(N) - (\frac{m-1}{m})A \geq -B^- - \frac{B^+ - mh^*}{m} + b_m - h^* = -B^- - \frac{B^+}{m} + b_m$, i.e. (1b). It is straightforward to verify that the surpluses are as given in (2b) and (3b). QED

<u>Lemma 2.4</u> Let b and $\hat{x}$ be as in Lemma 2.2. If the incumbent favors the issue, the allocation $\hat{x}$ is conditionally optimal for him.

<u>Proof</u> We must show that no other allocation can increase the incumbent's surplus. Suppose that such an allocation, z, did exist, and consider the vectors (potential allocations for the challenger) $y_i = z_i + p_i$ for $i \in [m,n]$, = 0 otherwise, and $y_i' = z_i + p_i + b_i$ for $i \in [1,m]$, = 0 otherwise.

If (1) of Lemma 2.3 holds these vectors must satisfy $y[m,n] \geq A$, $y'[1,m] \geq A$, since otherwise the challenger would win; hence, from the definitions of y and q (from the proof of Lemma 3.3) it follows that $(z + p)[m,n] = y[m,n] \geq A = q[m,n] = (\hat{x} + p)[m,n]$, whence $z[m,n] \geq \hat{x}[m,n]$; similarly $z[1,m] \geq \hat{x}[1,m]$, from the definitions of y' and q'. Moreover, since by hypothesis z increases the incumbent's surplus, $z(N) < \hat{x}(N)$. These inequalities together imply $z[1,m) < \hat{x}[1,m)$, $z(m,n] < \hat{x}(m,n]$, $z_m > x_m$. From the second of these, there must exist a voter $i^* > m$ for which $z_{i^*} < \hat{x}_{i^*}$, and since $\hat{x}_{i^*} + p_{i^*} \leq \hat{x}_m + p_m$ by the construction of $\hat{x}$, it follows that $z_{i^*} + p_{i^*} < \hat{x}_m + p_m$. Hence, taking C as the majority coalition $C = [1,m) \cup \{i^*\}$, we have $(z + p)(C) = (z + p)[1,m) + z_{i^*} + p_{i^*} < (\hat{x} + p)[1,m) + \hat{x}_m + p_m = (\hat{x} + p)[1,m]$. Hence, by opposing the issue and bidding $y_i' = z_i + p_i + b_i$ for $i \in [1,m]$, the challenger's cost would be $y'[1,m] = (z + p + b)[1,m] < (\hat{x} + p + b)[1,m] = q'[1,m] = A$, i.e. he would win, a contradiction of the hypothesis that z increases the incumbent's surplus.

If the inequality (1) does not hold, then an allocation z which increases the incumbent's surplus would have to satisfy $y[m,n] > q[m,n]$, $y'[1,m] > q'[1,m]$, and z feasible would imply $z(N) \leq A = \hat{x}(N)$. By analogous reasoning, these inequalities imply $y[m,n] = (z + p)[m,n] < q[m,n]$, and hence that the challenger could increase his surplus, and therefore decrease the incumbent's surplus, by also favoring the issue and bidding y_i for $i \in [1,m]$. Hence, no allocation z can increase the incumbent's surplus, so long as he favors the issue. QED

<u>Lemma 2.5</u>. Let b be a controversial issue which is "small." It is optimal (uniquely optimal, respectively) for the incumbent to favor the issue if and only if $b(N) \geq 0$ ($b(N) > 0$, respectively).

<u>Proof</u> It the incumbent favors the issue his conditionally optimal surplus is given by Lemma 2.3, in view of Lemma 2.4. Conversely, opposing the issue b is equivalent to favoring the issue $b^* \equiv -b$; hence the conditionally optimal allocation x^* and surpluses can be obtained by applying Lemma 2.3 to the issue b^*.

Denote various quantities appearing in Lemma 2.3 by $P \equiv p(N) - (\frac{m-1}{m})A$, $Q \equiv [B^- - \frac{B^-}{m}]$, $R \equiv [B^- + \frac{B^+}{m} - b_m]$. (Thus if $b_m \leq \frac{b(N)}{m-1}$ the incumbent wins iff $P \geq -Q$, etc.) Let Q^* and R^* be the

corresponding quantities for the issue $b^* = -b$. Evidently $b_m^* = -b_m$ and $b^*(N) = -b(N)$. Moreover since B^{*+} is $b^*[m^*,n^*]$ when voters are arranged in order of increasing b_{i*}^*, or equivalently (since $b_i^* = -b_i$), in order of decreasing b_i, evidently $[m^*,n^*]$ consists of the voters $[1,m]$ when voters are reordered so that $b_{i+1} \geq b_i$, i.e. $B^{*+} = b^*[m^*,n^*] = -b[m^*,n^*] = -b[1,m] = -B^-$. By the same reasoning, $B^{*-} = -B^+$. Hence $Q^* \equiv [B^{*-} - \frac{B^{*-}}{m}] = [-B^+ + \frac{B^+}{m}]$ and $R^* = [B^{*-} + \frac{B^{*+}}{m} - b^*_m] = [-B^+ - \frac{B^-}{m} + b_m]$.

Consider first the case $b_m < \frac{b(N)}{m-1}$. If the incumbent favors the issue he wins iff $P \geq Q$, and his surplus is $P + Q$ or $\frac{m}{n}(P + Q)$ if he wins or loses, respectively. Alternatively, if he opposes it (i.e. favors $b^* \equiv -b$) then since $b^*_m = -b_m > \frac{-b(N)}{m-1} = \frac{b^*(N)}{m-1}$, parts (b) of Lemma 2.3 applying, i.e. the incumbent wins iff $P \geq -R^*$, and his surplus is $P + R^*$ or $\frac{m}{n}(P + R^*)$, respectively.

If $P \geq \max(-Q,-R^*)$ the incumbent wins in either case, so it is optimal to favor iff the surplus by favoring is at least as large as that when opposing, i.e. $P + Q \geq P + R^*$, i.e. $Q \geq R^*$. If $P < \min(-Q,-R^*)$ he loses in both cases, and the same condition follows. If $-Q \leq P < R^*$ the incumbent can win only by favoring, so that position is uniquely optimal, while if $-R^* \leq P < Q$ he wins only by opposing, so it cannot be optimal to favor the issue. These assertions together imply that it is optimal for the incumbent to favor the issue if and only if $Q \geq R^*$, i.e. $R^* \equiv -B^+ - \frac{B^-}{m} + b_m \leq B^- - \frac{B^-}{m} \equiv Q$, or equivalently $b_m \leq B^- + B^+ = b(N) + b_m$, i.e. $b(N) \geq 0$.

If $b_m > \frac{b(N)}{m-1}$ then by the same reasoning it is optimal for the incumbent to favor b if and only if $R \geq Q^*$, i.e.

$B^- + \frac{B^+}{m} - b_m \geq -B^+ + \frac{B^+}{m}$, or equivalently $b(N) \geq 0$, again.

The final possibility is $b_m = \frac{b(N)}{m-1}$, in which case $b_m{}^* = \frac{b^*(N)}{m-1}$ so parts (a) of Lemma 2.3 apply to both b and b*. It will then be optimal for the incumbent to favor b* iff $Q \geq Q^*$, i.e.

$B^- - \frac{B^-}{m} \geq -B^+ + \frac{B^+}{m}$, or equivalently $0 \leq (\frac{m-1}{m})[B^- + B^+] =$
$(\frac{m-1}{m})[b(N) + b_m] = (\frac{m-1}{m})[b(N) + \frac{b(N)}{m-1}]$
$= (\frac{m-1}{m})(\frac{m-1+1}{m})b(N)$, i.e. $b(N) \geq 0$. QED

We may summarize these various results as follows:

Theorem 2.2 Let b be a controversial issue which is "small." It is optimal for the incumbent to favor the issue if and only if $b(N) \geq 0$, i.e. the issue is socially beneficial. If the incumbent uses an optimal strategy, either position is optimal for the challenger. The outcome and surplus to the winning candidate will be as given in Lemma 2.3. (Analogous results for the case $b(N) \leq 0$ are obtained by applying these assertions to a new issue $b' = -b$.)

3. Equilibrium

Let us now think of the incumbency premiums as prices or constituent demands, under the control of the individual voters or groups. Voter i, by raising or lowering his price p_i, makes himself less or more available to the challenger, which may in turn affect the outcome of the election, and the payoff he subsequently receives; he will attempt to set his price to ensure as large a payoff as possible. All prices are assumed fixed in advance of the election. The two parties then choose optimal allocations $\hat{x}$, $\hat{y}$, and after the election each voter will receive a payoff of $v_i = \hat{x}_i$ or $v_i = \hat{y}$, depending on which party wins. The optimal allocations are not necessarily unique, so for each p there will be a set $V(p) \subset \mathbb{R}^n$ of possible n-tuples of payoffs to voters, one for each possible winning optimal allocation. Each voter or group i is assumed to have well-defined (though not necessarily complete, or transitive) preference over the sets of possible payoffs or, equivalently, over the n-tuples p. These preferences are representable by a binary preference relation $\underset{i}{\succsim}$, with strict preference $\underset{i}{\succ}$ and indifference $\underset{i}{\sim}$ being defined in the usual way.

We can now define an equilibrium, in the obvious fashion. As a matter of notation, to focus on variations in some voter i's premium p_i, we denote by p_{-i} the (n-1)-tuple $(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$, and by (p_i, p_{-i}) the full n-tuple $(p_1, \ldots, p_i, \ldots, p_n)$.

Definition 3.1. An n-tuple $p \in \mathbb{R}^n$ is an equilibrium if for no voter i is there a price p_i' such that $(p_i', p_{-i}) \underset{i}{\succ} (p_i, p_{-i})$.

Some price vectors satisfying this definition are of limited interest. For example, if all prices were set so high that the incumbent could win with the allocation $x = 0$, no voter would receive a positive payoff. If no voter could affect this by changing his own price alone, then we have a sort of "equilibrium by default," despite the fact that the payoff to every voter is zero. We shall say such an equilibrium is __degenerate__: to be more precise, let $\bar{v}_i(p)$ be the maximum possible payoff to i at p, i.e. $\bar{v}_i(p) = \max_{v \in \bar{V}(p)} v_i$. We then define

__Definition 3.2.__ An equilibrium p is __degenerate__ if $\bar{v}_i(p) = 0 = \bar{v}_i(p_i', p_{-i})$ for all i and all p_i'.

The following property will be useful:

__Lemma 3.1__ Suppose there exists a nontrivial optimum $\hat{x} \neq 0$ for the incumbent. If $p_{i*} = 0$ for any $i* \in N$ then there exist optimal allocations $x' \in \hat{X}(p)$, $y' \in \hat{Y}(x';p)$ such that $v_{i*}(x',y';p) > 0$.

__Proof__ Let voters be indexed and r and ℓ defined as in Lemma 1.4, with respect to some optimal allocation $\hat{x}$. If $r > m$ then (4) of Lemma 1.4 applies, so $i < \ell$ implies $p_i \geq \alpha > 0$. Hence, since $p_{i*} = 0$, it follows that $i* \in [\ell, r]$, which implies the result.

Otherwise, if $r \leq m$, since $q_r^x > 0$ (from (3) of Lemma 1.4 and the definition of r), it must be true that $i* \leq r$ (for $i* > r$ would imply $\hat{x}_{i*} = 0$ and hence that $q_{i*}^{\hat{x}} = 0 < q_r^{\hat{x}}$ which is inconsistent with the indexing of votes). If $\hat{x}_{i*} > 0$ then $0 < q_{i*}^{\hat{x}} \leq q_m^{\hat{x}}$ (from (3) of Lemma 1.4 and the fact that $i* \leq r \leq m$), again implying the conclusion.

The remaining possibility is $0 = \hat{x}_{i*} = q_{i*}^{\hat{x}}$. Consider the allocation

$$x'_i = \begin{cases} \hat{x}_i - \varepsilon & \text{for } i \in T \\ t\varepsilon & \text{for } i = i*, \\ 0 & \text{otherwise} \end{cases}$$

where $T = \{i: \hat{x}_i > 0\} \subset [1,m]$ and $t = \#T$. Since $q_{i*}^{\hat{x}} = 0 < q_r^{\hat{x}} \leq q_{m+1}^{\hat{x}} = p_{m+1}$ using (3) of Lemma 1.4 and the fact that $r < m + 1$, for sufficiently small $\varepsilon > 0$ it will still be true that $q_{i*}^{x'} = t\varepsilon < p_{m+1}$ and $q_i^{x'} \leq q_i^{\hat{x}} \leq p_{m+1} = q_{i+1}^{x'}$, for all $i < m$ so [1,m] is still a least-cost coalition, and evidently $x'(N) = \hat{x}(N)$, $q^{x'}[1,m] = q^{\hat{x}}[1,m]$, so x' is also optimal. But then $v_i(x',y) = x'_i = t\varepsilon > 0$ for any y if the incumbent wins, while if the challenger wins there exists an optimal $\hat{y}$ such that $v_i(x',\hat{y}) = \hat{y}_i = q_i^{x'} = t\varepsilon > 0$. QED

The degenerate equilibria can now be completely characterized by the following result:

__Lemma 3.2.__ For any p let voters indexed so that $p_i \leq p_{i+1}$, for all i. Then p is a degenerate equilibrium if and only if either

(1) $\bar{p}[1,m-1] \geq A$, or alternatively

(2) $\underline{p}[m+1,n] \leq -A$.

__Proof__ If: For any i and p'_i let $p' = (p'_i, p_i)$. For any $C \in M*$ evidently $\bar{p}'(C) \geq \bar{p}'(C - \{i\}) = \bar{p}(C - \{i\}) \geq \bar{p}[1,m-1]$,

from the indexing and the fact that $\bar{p}_i' \geq 0$. Hence if (1) above holds then (3) of Lemma 1.2 also holds, so the incumbent wins and his uniquely optimal allocation is $\hat{x} = 0$, whence $V(\hat{x},y;p') = 0$ all $\hat{x} \in \hat{X}$, whence $\bar{V}(p') = 0$ for any such p'. By an analogous argument $p'(C) \leq \bar{p}[m+1,n]$ for all $C \in M*$, so (2) above implies (3) of Lemma 1.1 and hence that the challenger wins and $\hat{y} = 0$ is uniquely optimal, whence $\overline{V}(p') = 0$.

Only If: Suppose neither (1) nor (2) were true. Choose $p'_m = 0$, and designate by $[1',m']$ and $[m',n']$ coalitions which minimize $\overline{p}'(C)$ and maximize $\underline{p}'(C)$ over $C \in M^*$, respectively (identified by reindexing voters in order of increasing p'_i). Since (1) fails, $A > \overline{p}[1,m-1] = \overline{p}[1,m-1] + \overline{p}_m' = \overline{p}'[1,m] \geq \overline{p}'[1',m']$. Similarly, since (2) fails, $-A < \underline{p}[m+1,n] = \underline{p}[m+1,n] + \underline{p}_m' = \underline{p}'[m,n] \leq p'[m',n']$. Hence (2) of Comment 1.1 holds, so there exists a nontrivial optimum for the incumbent. Since $p_m' = 0$ Lemma 3.1 implies $v_m(x,y;p') > 0$ for some $x \in \hat{X}$, $y \in \hat{Y}(x;p')$, i.e. $0 < \overline{v}_m(p') = \overline{v}_m(p_m',p_{-m})$, so p is not a degenerate equilibrium. QED

The non-degenerate equilibria are the ones of interest. To obtain a more explicit characterization of them, we introduce some slight additional structure on voter preferences. Since each voter i is ultimately interested in maximizing his own payoff v_i, his preferences over sets of payoff n-tuples are assumed to reflect this. In particular, if V and V' are two such sets such that $v_i \geq v_i'$ for all $v \in V$, $v' \in V'$, then we shall say that V <u>dominates</u> V' <u>for</u> <u>i</u>.

<u>Definition 3.3.</u> Voter preferences are said to <u>respect dominance</u> if $p \underset{i}{\succsim} p'$ whenever V(p) dominates V(p') for i.

We then have:

<u>Lemma 3.3.</u> Suppose voter preferences respect dominance. If p is a nondegenerate equilibrium, then $\overline{v}_i(p) > 0$ for every voter i.

<u>Proof</u> Let voters be indexed so that $p_i \leq p_{i+1}$ for all i. Suppose that $\overline{p}[1,m] < A$ and $\underline{p}[m,n] > -A$ but that $\overline{v}_j(p) = 0$ for some voter j. Let $p'_j = 0$, $p' = (p_j',\underline{p}_j)$. Evidently $\overline{p}_j' = 0 \leq \overline{p}_j$ and $\underline{p}_j' = 0 \geq \underline{p}_j$, so $\overline{p}'[1,m] \leq \overline{p}[1,m] < A$ and $\underline{p}'[m,n] \geq \overline{p}[m,n] > -A$. Hence Lemma 3.2 applies, implying $\overline{v}_j(p) > 0$, i.e. $\overline{v}_j(p_j',\underline{p}_j) > 0 = \overline{v}_j(p_j,\underline{p}_j)$, a contradiction of the hypothesis that p is an equilibrium.

There are two remaining possibilities to consider:

(1): $\overline{p}[1,m] \geq A$: In this case, from Lemma 1.2, x = 0 is uniquely optimal for the incumbent, and the incumbent wins, so $\overline{v}_i(p) = 0$ for all i. Since p is a nondegenerate equilibrium, from Lemma 3.2 it must be true that $\overline{p}[1,m-1] < A$, and hence that $0 < \overline{p}_m = p_m$. But if we choose $p_m' = 0$ it then follows from the 'only if' argument of Lemma 3.2 that $\overline{V}_m(p_m',p_{-m}) > 0 = \overline{V}_m(p_m,p_{-m})$, so p could not be an equilibrium.

(2): $\underline{p}[m,n] \leq -A$ implies that the challenger wins with y = 0, whence V(p) = 0 which by an analogous argument leads to a contradiction of the hypothesis that p is an equilibrium. QED

The set of price n-tuples which satisfy this necessary condition will be of interest later; they are in a sense "closer" to being in equilibrium than those for which $\overline{v}_i(p) = 0$ for some voters. More precisely,

Definition 3.4. A price vector p is "near equilibrium" if it is contained in an open set S on which $\bar{v}_i(p') > 0$ for every i, at all $p' \in S$.

We now have

Theorem 3.1. A necessary and sufficient condition for p to be "near equilibrium" is that $|p_i| < a \equiv \min (A/m,\ 1/n[A+p(N)])$ for every i.

Proof If: If the inequality holds then Theorem 1.4 applies. It follows that $a > 0$ and $x_i = a - p_i > 0$, all i is the incumbent's unique optimal allocation. Moreover if $a = A/m$ the incumbent wins, so $\bar{v}_i(p) = x_i > 0$, all i. If $a < A/m$ the challenger wins. Since $q_i^x = a > 0$ for all i, any coalition C such that $\#C = m$ is a minimum-cost coalition, and any allocation of the form

$$y_i = \begin{matrix} a & \text{for } i \in C \\ 0 & \text{otherwise} \end{matrix}$$

for any such C, is optimal. Since every i belongs to some such C it follows that $\bar{v}_i(p) = a > 0$ for all i. Clearly the inequality, and hence the conclusion $V(p') > 0$ also holds on some neighborhood of p.

Only If: To prove the converse suppose p is near equilibrium and that the incumbent wins, but that $p_1 < -A/m$. Without loss of generality we can suppose that $p_1 < p_2 < \cdots < p_n$ (replacing the original p by a neighboring one, if necessary). p near equilibrium implies $v_1(p) > 0$ and hence that $x_1 > 0$ for some optimal allocation x, which again without loss of generality we can suppose to be of the form $x_i > 0$ iff $x_i + p_i = \alpha$ for some $\alpha > 0$ (From Lemma 1.4 this follows immediately if $r > m$, while if $r \leq m$ a reallocation of x(N) among $\{i: x_i > 0\}$ will create a new optimal allocation of this form). Since the incumbent wins and $\ell = 1$, either $\alpha \leq A/m$ (if $r \leq m$) or $\alpha = A/m$ (if $r > m$). Let $\varepsilon = \frac{x_1}{(n-1)}$ and define a new allocation x' by $x_1 = 0$, $x'_i = x_i + \varepsilon$ otherwise. Clearly $x'(N) = x(N)$, and evidently [1,m] is still a least-cost coalition, so x optimal implies $0 \geq q^{x'}[1,m] - q^x[1,m] = -\alpha + (m-1)\varepsilon$, which (using $\varepsilon = \frac{x_1}{(n-1)} = \frac{(\alpha - p_1)}{(n-1)}$) in turn implies $p_1 \geq -\alpha \geq -A/m$, a contradiction of the original hypothesis that $p_i < -A/m$. The alternative hypothesis, that $p_n > A/m$ would lead to a similar contradiction. Hence, from the indexing, $-A/m \leq p_i \leq A/m$, all i. Since the incumbent wins it follows from Theorem 1.4 that $1/n[A + p(N)] \geq A/m$ and hence that $|p_i| \leq \min (A/m,\ 1/n[A + p(N)]) = a$, all i.

The other case, p near equilibrium, challenger wins, is argued analogously, leading to the conclusion that $|p_i| \leq 1/n[A + p(N)] = \min (A/m, 1/n[A + p(N)]) = a$. Finally, if $|p_i| = a$ for any i, clearly the inequality would not hold on any open neighborhood of p; hence p near equilibrium implies the inequality is strict. QED

Note that when p is near equilibrium the optimal allocations will be as given in Theorem 1.3. Returning to the equilibria themselves, we have

Lemma 3.4. Suppose voter preferences respect dominance. If p is a nondegenerate equilibrium, then

(1) $|p_i| < a \equiv \min(A/m, (1/n)[A + p(N)])$ for all i,

(2) $p(N) \leq (\frac{m-1}{m})A$, and

(3) $\hat{s}_1(p) \leq 0$

Proof (1): From Lemma 3.3 it must be true that $\bar{v}_i(p) > 0$ all i, which from the necessity argument of Theorem 3.2 implies (1) above.

(2): From (1) above Theorem 1.4 applies. Hence $p(N) > \frac{m-1}{m} A$ would imply $a = A/m$, the incumbent wins, with a strictly postive surplus $\hat{s}_1(p) = A - x(N) > 0$, and his unique optimal allocation is $x_i = A/m - p_i$, all i. For some i* let $p_{i*}' = p_{i*} - \varepsilon$ for some $\varepsilon > 0$, and define $p' = (p_{i*}', p_{i*})$. Clearly for sufficiently small ε (1) will still hold, so the incumbent will still win, and his unique optimal allocation x' will be $x_{i*}' = x_{i*} + \varepsilon$, $x_i' = x_i$ for $i \neq i*$. Hence $v_i(x,y;p') = x_i' > x_i = v_i(x',y;p)$ for all optimal x,y,x',y', so V(p') dominates V(p) for i and p could not be an equilibrium. Hence $p(N) \leq \frac{m-1}{m} A$.

(3): From (2) either $p(N) = \frac{m-1}{m} A$ in which case (1) and Theorem 1.4 imply $\hat{s}_1(p) = 0$, or else $p(N) < \frac{m-1}{m} A$, implying $\hat{s}_1(p) < 0$ QED

Note that (3) implies, in particular, that either the challenger wins, or if the incumbent does his surplus is zero. In view of (1) above and Theorem 1.3, the incumbent's optimal allocation will make every minimal winning coalition a least cost-coalition to the challenger, since #C = m implies $q^x(C) = a \cdot m$. Thus, if the

challenger can win, he can do so with many optimal allocations (one for each such C), and voter i's payoff win be either $a > 0$ or 0, depending on whether he happens to belong to the chosen coalition or not. If p and p' are two vectors satisfying (1) then i can receive either zero or a positive payoff in either case, so neither dominates the other. If $a > a'$, however, he receives a higher payoff whenever he belongs to the chosen coalition, so there is a sense in which his payoff under p is <u>conditionally</u> better than that under p'. To put things more generally, let us define for any payoff vector $v \in \mathbb{R}^n$ the set of voters who receive positive payoffs, $C(N) = \{i: v_i > 0\}$, and for any set V of such payoffs let (V) = {C(N) : v ∈ V}. Then we shall say that V <u>conditionally</u> dominates V' for i if (V) = (V') and $i \in C(v) = C(v')$ implies $v_i \geq v_i'$ for all $v \in V$, $v' \in V$.

Definition 3.5. Voter preferences <u>respect conditional dominance</u> if $p \underset{i}{\gtrsim} p'$ whenever V(p) dominates, or conditionally dominates, V(p'), for i.

Theorem 3.2. Suppose voter preferences respect conditional dominance. If p is a nondegenerate equilibrium, then:

(1) $|p_i| < a = A/m = 1/n[A + p(N)]$, all i.

(2) $p(N) = \frac{m-1}{m} A$,

(3) $v_i(x,y;p) > 0$ all i and all $x \in \hat{X}(p)$, $y \in \hat{Y}(\hat{x};p)$, and

(4) $\hat{s}_1(p) = 0 = \hat{s}_2(p)$.

Proof (2): From (1) and (2) of Lemma 3.4, $|p_i| < a$, all i, and $p(N) \leq \frac{m-1}{m} A$, which is equivalent to $A/m \geq 1/n[A + p(N)] = a$. Suppose the inequality were strict. Then Theorem 1.4 would imply: the challenger wins; his optimal allocations are $\{y{:}y_i = a$ for $i \in C$, $= 0$ otherwise, for some $C \subset N$, $\#C = m\}$; and hence that the conditional payoff to any voter i^* is a if $i^* \in C$, 0 otherwise, for any such C. If $p_{i*}' = p_{i*} + \varepsilon$, $p' = p_i', \underline{p}_i)$, then for sufficiently small $\varepsilon > 0$ it will still be true that $p'(N) < \frac{m-1}{m} A$ and $|p_{i*}| < a$, so by the same reasoning i^*'s conditional payoff will be $a' = 1/n[A + p'(N)] = 1/n[A + p(N) + \varepsilon] = a + \varepsilon/n > a$ if $i^* \in C$, 0 otherwise, where again C ranges over the set of coalitions for which $\#C = m$. Hence $V(p')$ conditionally dominates $V(p)$, so p would not be an equilibrium. Hence if p is an equilibrium the inequality cannot be strict, i.e. $p(N) = \frac{m-1}{m} A$.

(1): Follows from Lemma 3.4 and (2) above.

(3): From (1) and (2) above and Theorem 1.4 it follows that the incumbent wins, and that his unique optimal allocation is $\hat{x}_i = a - p_i > 0$, all i. Hence $v_i(\hat{x},y;p) = \hat{x}_i > 0$.

(4): Follows from Theorem 1.4 and (2) above. QED

FOOTNOTES

1. For example, among many others, Downs (1957), Davis and Hinich (1966), Davis, Hinich, and Ordeshook (1970), McKelvey and Ordeshook (1970), Kramer (1977), (1978).

2. The more conventional assumption, originating in Downs (1957), is that each candidate strives to maximize his probability of victory, or perhaps his vote share. Pursuit of such goals is purely instrumental in Downs, however, being only a necessary step towards the candidate's ultimate objective of enjoying the spoils of office. Our assumption that candidates are surplus maximizers thus more directly incorporates this ultimate goal into the analysis.

3. Some indirect but nevertheless suggestive evidence on challengers versus incumbents is reported by Hershey (197). She interviewed campaign managers and candidates for congressional and statewide races in Wisconsin during the 1970 election, and found that most challengers (13 of 16 candidates, 17 of 18 campaign managers) would be satisfied with a bare, minimal winning coalition victory, whereas most incumbents (11 of 12 candidates, 10 of 11 campaign managers) would not, and strive for larger margins.

4. Compare, for example, Fenno's description of the early, "expansionist" stage of a congressman's constituency career: "In the expansionist stage . . . before [his] first election . . .

the first step is to solidify a primary constituency, a core of strongest supporters who will carry a primary campaign, if necessary, and who will, in any case, provide the backbone for a general election campaign. The second step is to cultivate the broader re-election constituency by reaching out for additional elements of support." (Fenno (1978), pp. 172).

5. As Murray Kempton puts it, in commenting on LBJ's subsequent deemphasis of the populist issues on which he campaigned and won in his first election, "To get elected is to become an incumbent, and to be an incumbent is to view with more alarm than hope any attempt to acquaint society's victims with their greivances." (Kempton (1983)).

6. There are two important qualifications to this: first, the challenger's advantage arises from the fact that the incumbent must commit himself first. A challenger who attempted to directly interject an issue into the campaign himself would presumably find it difficult to do so without at least implicitly taking a position on it himself, in which case he would in effect be making the first commitment, so the incumbent would gain the advantage. Moreover, a victorious challenger becomes an incumbent in the next election, and the controversial issue which helps him now may return to haunt him in the future.

7. Thurow, however, attributes the problem to the lack of party

discipline in the United States. In our analysis both parties are in effect perfectly disciplined, however; thus our analysis suggests the problem lies deeper, in the nature of the competitive electoral process itself.

8. One exception to this, of course, would be the pure "rent-seeking society" (Kruger ( )): if there is no private sector there are presumably no underlying economic inequalities, so equality in the provision of public benefits will indeed result in social equality. Expansion of the public sector is thus one way of reconciling the otherwise conflicting ends of social equality and the equalization of benefits.

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