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# THE CORE OF A COALITIONAL PRODUCTION ECONOMY WITHOUT ORDERED PREFERENCES

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## ABSTRACT

It is shown that the core of a coalitional production economy with a balanced technology (Bohm [1974]) is nonempty, even if the consumers have preferences which are intransitive, provided the preferences are convex and continuous. Since such preferences cannot be represented by utility functions, this result does not follow from the nonemptiness of the core of a characteristic function game. Rather, the approach is closer to that of Ichiishi's [1981] social coalitional equilibrium.

# THE CORE OF A COALITIONAL PRODUCTION ECONOMY WITHOUT ORDERED PREFERENCES

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## 1. INTRODUCTION

The object of this paper is to present a set of sufficient conditions under which the core of a coalitional production economy is nonempty, even though individuals may have preferences which are not orders, i.e., not complete or transitive. Concern among economists over the realism of this postulate can be traced back to the debates over whether integrability was a necessary condition for well behaved demand curves, and the question of intransitivity was directly addressed by Georgescu-Roegen [1936]. There is a fair body of evidence amassed by psychologists (e.g., Tversky [1969]) that individual preferences may be intransitive. It is not surprising then that the past decade has seen a significant amount of work devoted to dispensing with this assumption. In addition to considerations of realism, parsimony demands that unnecessary hypotheses be discarded.

The question of whether transitivity of preferences is necessary for a consumer to have well behaved demand correspondences has been addressed by Sonnenschein [1971], Shafer [1974], and Fountain [1981]. They show that it is not necessary to assume that consumers have transitive preferences in order to derive many of the properties attributed to the demand correspondences of transitve consumers. The question of the existence of equilibrium when consumers have nonordered preferences has been examined by Schmeidler [1969], Mas-Colell [1974], Gale and Mas-Collel [1975], Shafer and Sonnenschein [1975], and Shafer [1976]. Again the conclusion is that transitivity of preferences is irrelevant to the problem.

The model of the coalitional production economy used here is that of Bohm [1974]. The method of analysis is necessarily different from Bohm's, as he make use of the nonemptiness of the core of a balanced game in characteristic function form. The problem is that to pass from a market economy to a game in characteristic function form requires that the traders have utility functions, and this in turn implies that preferences are complete and transitive. Corresponding to the core of a characteristic function form game is the set of strong equilibria of a strategic form game, that is, the set of strategy vectors that no coalition has any incentive to deviate from. The definition of strong equilibrium does not require that players have utility functions, only that they have preference relations. By modeling the market as a game in strategic form rather than in characteristic function form, the case of non-ordered preferences may be handled.

Section 2 states a lemma on the existence of strong equilibria for a certain class of games. Section 3 contains a description of a variation of Bohm's model of a coalitional production economy and a theorem on the nonemptiness of the core. The theorem is proved by using the lemma on existence of strong equilibria. Section 4 proves the lemma, although some of the more tedious details have been relegated to an appendix. Section 5 discusses the relation of this paper to the rest of the literature on intransitive preferences and possible extensions.

## 2. A LEMMA ON STRONG EQUILIBRIUM

A game G is a tuple $(N, (S_i), (F^B), F, (P_i))$ where $N = \{1,\ldots,n\}$ denotes the set of players. Each player i has a set of $S_i$ of strategies potentially available to him. Set $S = \prod_{i\varepsilon N} S_i$. For each coalition $B \subset N$ let $S^B = \prod_{i\varepsilon B} S_i$. The set of strategies which are jointly feasible for members of coalition B is denoted $F^B \subset S^B$. The set of all feasible strategy vectors is denoted by F.

Preferences are represented by a correspondence $P_i : S_i \to\to S_i$. The interpretation is that $y_i \varepsilon P_i(x_i)$ means that $y_i$ is preferred to $x_i$. Note that i's preferences depend only on his own strategy. This is highly restrictive in a game theoretic model, but adequate for dealing with the economic model considered.

A <u>strong equilibrium</u> of $G = (N, (S_i), (F_B), F, (P_i))$ is an $x \varepsilon S$ satisfying

(i) $x \varepsilon F$,

(ii) there is no $B \subset N$ and $y^B \varepsilon F^B$ satisfying $y_i^B \varepsilon P_i(x_i)$ for all $i \varepsilon B$.

In order to prove the existence of strong equilibrium the following condition will be used.



<u>Balancedness</u>: A family $\beta$ of subsets of N is <u>balanced</u> if for each $B \varepsilon \beta$ there is a $\lambda_B \geq 0$ (called a <u>balancing weight</u>) such that for each $i \varepsilon N$

$$\sum_{i\varepsilon B\varepsilon\beta} \lambda_B = 1$$

The game G is <u>balanced</u> if whenever $\beta$ is a balanced family with balancing weights $\{\lambda_B\}$, and $x^B \varepsilon F^B$ for each $B \varepsilon \beta$, then $x \varepsilon F$ where $x_i = \sum_{i\varepsilon B\varepsilon\beta} \lambda_i x_i^B$.

<u>LEMMA</u>. Let $G = (N, (S_i), (F^B), (P_i))$ be a game satisfying

G1. For each i, $S_i$ is a nonempty compact convex subset of $\mathbb{R}^{k_i}$.

G2. For each $B \subset N$, $B \neq \emptyset$, $F^B$ is a nonempty compact subset of $S^B$.

G3. F is convex and compact.

G4. For each i,

(a) $P_i$ has open graph in $S_i \times S_i$,

(b) $x_i \notin P_i(x_i)$.

(c) $P_i(x_i)$ is convex (but possibly empty).

G5. G is balanced.

Then G has a strong equilibrium.

The proof of the lemma is postponed until after a discussion of the economic model which serves to motivate the definitions above.

## 3. THE MODEL OF THE ECONOMY

The model is basically that of Bohm [1974], who presents several examples of technologies satisfying the assumptions.

The commodity space is $\mathbb{R}^\ell$. There is a finite set $N = \{1,\dots,n\}$ of consumers who are characterized by their consumption sets $\tilde{S}_i \subset \mathbb{R}^\ell$, their preference correspondences $P_i : \tilde{S}_i \to\to \tilde{S}_i$, and their endowments $w_i \in \mathbb{R}^\ell$. With each coalition $B \subset N$ is associated a nonempty production possibility set $Y^B \subset \mathbb{R}^\ell$. The total production possibility set is $Y \subset \mathbb{R}^\ell$.

An <u>allocation</u> is a list of commodity vectors $(x_i)_{i = 1,\dots,n}$ such that $x_i \in \tilde{S}_i$ for each $i \in N$ and $\sum_{i\in N} x_i - \sum_{i\in N} w_i \in Y$. The set of all allocations is denoted $F$.

For each $B \subset N$, $B \neq \emptyset$, define $F^B$ by $x^B \in F^B$ if and only if $x_i^B \in \tilde{S}_i$ for all $i \in B$ and

$$\sum_{i\in B} x_i - \sum_{i\in B} w_i \in Y^B.$$

We say that coalition $B$ can <u>improve upon</u> allocation $x = (x_i)$ if there is $z^B \in F^B$ satisfying $z_i^B \in P_i(x_i)$ for all $i \in S$.

The <u>core</u> of the economy $E = (N, (\tilde{S}_i), (w_i), (Y^B), Y, (P_i))$ is the set of all allocations which cannot be improved upon by any nonempty coalition.

The technology $((Y^B), Y)$ is <u>balanced</u> if for every balanced family $\beta$ of subsets of $N$ with balancing weights $\lambda_B$,

$$\sum_{B\in\beta} \lambda_B Y^B \subset Y.$$



In particular since $\{N\}$ is a balanced family with balancing weight $\lambda_N = 1$, $Y^N \subset Y$. Similarly, $\sum_{i\in N} Y^{\{i\}} \subset Y$.

For any set $X \subset \mathbb{R}^\ell$ let $AX$ denote the asymptotic cone of $X$.

<u>THEOREM.</u> Let the economy $E = (N, (\tilde{S}_i), (w_i), (Y^B), Y, (P_i))$ satisfy the following assumptions.

E1. For each $i$, $S_i \subset \mathbb{R}^\ell_+$ is closed, convex and bounded from below and $w_i \in S_i$.

E2. For each $i$,

(a) $P_i = \tilde{S}_i \to \tilde{S}_i$ has open graph in $\tilde{S}_i \times \tilde{S}_i$.

(b) $x_i \notin P_i(x_i)$.

(c) $P_i(x_i)$ is convex (but possibly empty.)

E3. For each $i$, $0 \in Y^{\{i\}}$.

E4. For each $B \subset N$, $Y^B$ is closed and there is an $x^B \in \tilde{S}^B$ satisfying $\sum_{i\in B} x_i - \sum_{i\in B} w_i \in Y^B$.

E5. $Y$ is closed and convex and $AY \cap \mathbb{R}^\ell_+ = \{0\}$.

E6. $((Y^B), Y)$ is balanced.

Then $E$ has a nonempty core.

Note that if $0 \in Y^B$ for every coalition $B$, then assumption E4. will be satisfied.

Proof of Theorem. Bohm [1974] has already shown that under these hypotheses each $F^B$ is compact and that F is compact. Each $F^B$ is nonempty by assumption E4.

Letting $S_i$ be a compact convex subset of $\tilde{S}_i$ large enough so that for all $B \subset N$, $F^B \subset \prod_{i\varepsilon B} S_i$ and $F \varepsilon \prod_{i\varepsilon N} S_i$.

Then the game $(N, (S_i), (F^B), F, (P_i|_{S_i}))$ satisfies the hypotheses of the lemma. The only hypothesis that needs checking is that of balancedness: Let $\beta$ be a balanced family of coalitions with balancing weights $\lambda_B$ and let $x^B \varepsilon F^B$ for each $B \varepsilon \beta$. Then there is a $y^B \varepsilon Y^B$ such that $\sum_{i\varepsilon B} x_i^B = \sum_{i\varepsilon B} w_i + y^B$. Then setting $x_i = \sum_{i\varepsilon B\varepsilon\beta} \lambda_B x_i^B$ it follows that

$$\sum_{i\varepsilon N} x_i = \sum_{i\varepsilon N} \sum_{i\varepsilon B\varepsilon\beta} \lambda_B w_i + \sum_{B\varepsilon\beta} \lambda_B y_B$$

$$= \sum_{i\varepsilon N} w_i + \sum_{B\varepsilon\beta} \lambda_B y^B.$$

Since $((Y^B), Y)$ is balanced, $\sum_{i\varepsilon N} x_i - \sum_{i\varepsilon N} w_i \varepsilon Y$ and so $x \varepsilon F$. Thus by the lemma the game $(N, (S_i), (F^B), (P_i|_{S_i}))$ has a strong equilibrium which is clearly an allocation in the core of the economy E. |||

## 4. PROOF OF LEMMA

The following is a well-known alternate characterization of balanced families. Let $e^1,\ldots,e^n$ be the unit coordinate vectors in $\mathbb{R}^N$ and set $m_B = \frac{1}{|B|}\sum_{i\varepsilon B} e^i$. Then $\beta$ is a balanced family if and only if $m_N \varepsilon \text{co}\{m_B : B \varepsilon \beta\}$, where co denotes the convex hull.

The proof relies on the following result which is a special



case of a theorem of Fan [1969] and Browder [1967]. Let $K \subset \mathbb{R}^n$ be compact and convex, and let $\gamma, \mu : K \rightarrow\!\!\rightarrow \mathbb{R}^n$ be upper hemicontinuous with compact and convex values. Suppose that for each $x \varepsilon K$ there exist three points $y \varepsilon K$, $u \varepsilon \gamma(x)$, $v \varepsilon \mu(x)$ and a real number $\lambda > 0$ such that $y = x + \lambda(u - v)$. Then there is a $z \varepsilon K$ such that $\gamma(z) \cap \mu(z) \neq \emptyset$.

Begin the proof of the lemma by defining $v_i = S_i \times S_i \rightarrow \mathbb{R}_+$ by

$$v_i(y_i,x_i) = \text{distance}\,[(x_i,y_i),(\text{Gr } P_i)^c],$$

where $\text{Gr } P_i = \{(x_i,z_i) = z_i \varepsilon P_i(x_i)\}$, the graph of $P_i$. (This construction is due to Shafer and Sonnenschein [1975].) Each $v_i$ is continuous (as $\text{Gr } P_i$ is open) and $v_i(y_i,x_i) > 0$ if and only if $y_i \varepsilon P_i(x_i)$. The function $v_i$ acts as a "pseudo-utility" for $P_i$, and possesses the following important property. Suppose $v_i(z_i^k,x_i) \geq w$ for $k = 1,\ldots,p$. Let $z_i$ be a convex combination of $z_i^1,\ldots,z_i^k$. Then $v_i(z_i,x_i) \geq w$. The proof of this may be found in the appendix.

For each $B \subset N$ set

$$V^B(x) = \{w \varepsilon \mathbb{R}^N : \exists z^B \varepsilon F^B \ \forall i\varepsilon B \ \ w_i \leq v_i(z_i^B, x_i)\}.$$

If $i \notin B$, then $w \varepsilon V^B(x)$ places no restriction on $w_i$. Thus x is a strong equilibrium if and only if $x \varepsilon F$ and $\bigcup_{B\varepsilon N} V^B(x) \cap \mathbb{R}^N_{++} = \emptyset$.

The sets $V^B(x)$ are analogues of the characteristic function of a game without side payments and the arguments of Shapley [1973] and Ichiishi [1981] may be applied. The following line of argument is very similar to Ichiishi [1981].

Since each $v_i$ is continuous and each $F^B$ is compact, there is some $M \geq 0$ such that for all $x \in S$, and $z^B \in F^B$, $v_i(z_i^B, x_i) \leq M$ for all $i \in B$. Put $a^i = -Mne^i \in \mathbb{R}^n$ (where $e^i$ is the $i^{th}$ unit coordinate vector of $\mathbb{R}^n$) and set $\Delta = \text{co}\,\{a^i = i \in N\}$. For each $B \subset N$ set $m_B = \frac{1}{|B|}\sum_{i \in B} a^i$.

For each $y \in \Delta$ set $\tau(y,x) = \max\,\{t > 0 : y + t(1,..,1) \in \bigcup_{B \subset N} V^B(x)\}$, and put $w(y,x) = y + \tau(y,x)(1,...,1)$. Note that $\tau(y,x) < M(n+1)$ and $w(y,x) \leq M(1,...,1)$. Since $v_i$ is always nonnegative, $V^{\{k\}}(x)$ always includes $\{w : w_k \leq 0\}$. Suppose that some $w_k(y,x) < 0$, then $w(y,x) = y + \tau(y,x)(1,...,1)$ is in the interior of $V^{\{k\}}(x)$, which contradicts the definition of $\tau$. Thus $w(y,x) \geq 0$.

The next step is to show that if $x \in F$ and $w(y,x) \leq 0$, then $x$ is a strong equilibrium. Suppose not. Then for some $z^B \in F^B$, $z_i^B \in P_i(x)$ for all $i \in B$, so $v_i(z_i^B, x_i) > 0$ for all $i \in B$. Thus there is a $w \in V^B(x)$ with $w > 0$. But then $y + \tau(y,x)(1,...,1) = w(y,x) \leq 0$ is in the interior of $V^B(x)$, which contradicts the definition of $\tau$.

Thus the search for a strong equilibrium has been reduced to the following problem: Find $x \in F$ and $y \in \Delta$ such that $w(y,x) \leq 0$. To this end make the following constructions. For each $B \in N$, set $\Gamma^B(x) = \{y \in \Delta : w(y,x) \in V^B(x)\}$. Let $E(x,y)$ equal

$$\{z \in F : z \text{ minimizes distance } [v(\,\cdot\,,x), \{w : w \leq w(y,x)\}]\}$$

where the ith component of $v(x,y)$ is $v_i(x_i, y_i)$.

Define $\gamma, \mu : S \times \Delta \twoheadrightarrow S \times \Delta$ by



$$\gamma(x,y) = \{x\} \times \text{co}\,\{m_B : y \in \Gamma^B(x)\}$$

and

$$\mu(x,y) = \text{co}\,E(x,y) \times \{m_N\}.$$

The correspondences $\gamma$ and $\mu$ so defined satisfy the hypothesis of the Fan-Browder theorem. The proof of this claim is given in the appendix.

It follows from the Fan-Browder theorem that there are $\bar{x}, \bar{y}, x^*, y^*$ satisfying

$$(\bar{x}, \bar{y}) \in \mu(x^*, y^*) \cap \gamma(x^*, y^*).$$

In other words

(1) $\bar{x} \in \text{co}\,E(x^*, y^*)$.

(2) $\bar{y} = m_N$.

(3) $\bar{x} = x^*$.

(4) $\bar{y} \in \text{co}\,\{m_B : y^* \in \Gamma^B(x^*)\}$.

By (2) and (4), $\beta = \{B : y^* \in \Gamma^B(x^*)\}$ is balanced, and by the definition of $\beta$, $w(y^*, x^*) \in V^B(x^*)$ for all $B \in \beta$. Thus for each $B \in \beta$ there exists $z^B \in F^B$ satisfying $w_i(y^*, x^*) \leq v_i(z_i^B, x_i^*)$ for all $i \in B$. Let $\{\lambda_B\}$ be the associated balancing weights. Since the game is balanced, $z^* \in F$ where $z^* = \sum_{i \in B} \lambda_B z_i^B$. Since $z_i^*$ is a convex combination of the $z_i^B$ for $i \in B$ and $v_i(z_i^B, x_i^*) \geq w_i(y^*, x^*)$, it follows that $v_i(z^*_i, x^*_i) \geq w_i(y^*, x^*)$.

By (1) and (3), $x^* \varepsilon$ co $E(x^*,y^*)$. Since $z^* \varepsilon F$ and $v(z^*,x^*) \geq w(y^*,x^*)$, if $z \varepsilon E(x^*,y^*)$, then $v(z,x^*) \geq w(y^*,x^*)$. Suppose that $w_i(y^*,x^*) > 0$. Then for all $z \varepsilon E(x^*,y^*)$, $v(z_i,x^*_i) > 0$ as well. Thus $z_i \varepsilon P_i(x^*_i)$. Thus $x^* \varepsilon$ co $E(x^*,y^*)$ implies that $x^*_i \varepsilon P_i(x^*_i)$, a contradiction. Thus $w(y^*,x^*) \leq 0$. Also since F is convex and $E(x^*,y^*) \subset F$, it follows that $x^* \varepsilon F$. Thus $x^*$ is a strong equilibrum. |||

## 5. POSSIBLE EXTENSIONS

The theorem on the nonemptiness of the core of an economy is not strictly stronger than that of Bohm [1974]. His assumption that preferences are representable by utility functions is traded off against the assumption that Y is convex and each $F^B$ is nonempty.

The lemma on strong equilibria is not as strong as one would like. It would be nice if the lemma could be proved under the following hypotheses, which are more along the lines of Shafer and Sonnenschein [1975]. It is desirable to make $F^B$ depend in a continuous way on x. This generalization is straightforward. The other desirable generalization would be to allow $P_i : X \to\to X_i$, i.e., to allow i's preferences over strategies to depend on the other's strategies. This is a more serious obstacle.

To see the nature of the difficulty, for $z^B \varepsilon S^B$ let $x|z^B$ be defined by

$$(x|z^B)_i = \begin{cases} x_i & i \notin B \\ z_i^B & i \varepsilon B. \end{cases}$$



To carry out the preceding argument, it would be necessary to show that if $v_i(z_i^B, x|z^B) \geq w$ for $B \varepsilon \beta$ and if z is given by $z_i = \sum_{i \varepsilon B} \lambda_B z_i^B$, that $v_i(z_i,z) \geq w$. This is true if the graph of $P_i$ is convex, but it difficult to attach any interpretation to this condition. Also the requirement of convex graph runs counter to another desired weakening, namely only assuming that $x_i \notin$ co $P_i(x)$. This latter assumption is sufficient for the existence of a noncooperative equilibrium of an abstract economy and it would be nice to carry it over to the cooperative case.

Another possible generalization would be to specify coalition structures as in Ichiishi [1981], but this seems pointless unless the hypotheses on preferences can be relaxed.

## APPENDIX

<u>Quasi-concavity of $v_i$ in its first argument.</u>

Let $v_i(z_i^k, x_i) \geq w$, $k = 1,\ldots,p$ and let $z_i = \sum_{k=1}^{p} \lambda_k z_i^k$ be a convex combination of $z_i^1,\ldots,z_i^p$. Then $v_i(z_i, x_i) \geq w$.

For convenience, the common subscript i will be omitted. If $w = 0$, the result is trivial. If $w > 0$, let $N_w(x, z^k)$ be an open ball of radius w about $(x, z^k)$. From the definition of v, $N_w(x, z^k) \subset \mathrm{Gr}\, P$, $k = 1,\ldots,p$. Let $(x', z') \in N_w(x, z)$. Then $|(x' - x, z' - z)| < w$ so $(x + (x' - x), z^k + (z' - z)) \in N_w(x, z^k) \subset \mathrm{Gr}\, P$. Thus $z^k + z' - z \in P(x')$. Since $P(x')$ is convex, $z' = \sum_{k=1}^{p} \lambda_k (z^k + z' - z) \in P(x')$. Thus $N_w(x, z) \subset \mathrm{Gr}\, P$, so $v(z, x) \geq w$.

<u>The correspondences $\gamma$ and $\mu$ satisfy the Fan-Browder hypotheses.</u>

It is straightforward to verify that $\gamma$ and $\mu$ are upper hemi-continuous with nonempty compact convex values. It is harder to see that for every $(x,y) \in S \times \Delta$, there exist $(x',y') \in \mu(x,y)$, $(x'',y'') \subset \gamma(x,y)$ and $\bar{\lambda} > 0$ satisfying $(x,y) + \bar{\lambda}[x',y') - (x'',y'')] \in S \times \Delta$. The argument is virtually identical to one used by Ichiishi [1981] with only slightly different correspondences. Put $x'' = x$, $y' = m_N$ and choose any $x' \in \mathrm{co}\, E(x,y)$. Then $x + \lambda(x' - x'') = (1 - \lambda)x + \lambda x' \in S$ for any $\lambda \in [0,1]$. Let $B \subset N = \{i : y_i > 0\}$. It is shown below that



$\mathrm{co}\, \{a^i : i \in B\} \subset \bigcup_{C \subset B} \Gamma^C(x)$. Given this, choose $C \subset B$ so that $y \in \Gamma^C(x)$. Put $y'' = m_C$. Then $(x'',y'') \in \gamma(x,y)$. For $\lambda \in [0,1]$, define $y^\lambda = y + \lambda(y' - y'') = y + \lambda(m_N - m_C)$. Then $\sum_{i=1}^{n} y_i^\lambda = \sum_{i=1}^{n} y_i + \lambda(\sum_{i=1}^{n} m_{Ni} - \sum_{i=1}^{n} m_{Ci}) = -Mn + \lambda(-Mn + Mn) = -Mn$ and if $\bar{\lambda}$ is small enough, $y^{\bar{\lambda}} \leq 0$ so $y^{\bar{\lambda}} \in \Delta$.

The argument that $\mathrm{co}\, \{a^i \mid i \in B\} \subset \bigcup_{C \subset B} \Gamma^C(x)$ for all B is due to Shapley [1973]. If $B = N$ this just says that $w(y,x) \in V^B(x)$ for some B, so suppose that $B \neq N$, so that $|B| < n$. Let $y \in \Gamma^C(x) \cap \mathrm{co}\, \{a^i : i \in B\}$. We need to show that $C \subset B$. Since $y \in \Gamma^C(x)$, $w(y,x) \in V^C(x)$, so $w_j(y,x) = y_j + \tau(y,x) \leq M$ for all $j \in C$. But $\sum_{j \in B} y_j = -Mn$, and for some $k \in B$, $y_k$ is less than equal the average of the $y_j$'s for $j \in B$, so $y_k \leq \frac{-Mn}{|B|} < -M$ as $|B| < n$. But $w(y,x) \geq 0$ so $y_k + \tau(y,x) \geq 0$, so $\tau(y,x) > M$. This and $y_j + \tau(y,x) \leq M$ for $j \in C$ imply that $y_j < 0$ for all $j \in C$. Since $y \in \mathrm{co}\, \{a^i : i \in B\}$ and $y_j < 0$ for $j \in C$ it follows that $C \subset B$. |||

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