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RESEARCH AND DEVELOPMENT WITH A GENERALIZED HAZARD FUNCTION

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SOCIAL SCIENCE WORKING PAPER 455

October 1982
Revised September 1983

ABSTRACT

Previous work in this area has analyzed research and development as a stochastic racing game where the strategy is the rate of investment on the innovation, conditional on no success to date. This paper generalizes this work in several ways; first, we use a more general hazard function, although we retain the assumption that it depends only upon current investment. We find that when patent protection is perfect, equilibrium investment rates are monotonically increasing over time. Second, we allow for the possibility that some firms are currently receiving profits from the sale of a product which will be replaced by the innovation. This allows us to determine whether current industry leaders will tend to be more or less innovative than firms with smaller current market shares. We find that, in a stationary equilibrium, current industry leaders will tend to invest at a lower rate than those firms which currently have smaller market shares. We also remark that a stationary equilibrium implies that the random success date follows an exponential distribution, an assumption which is ubiquitous in the earlier theoretical work on this subject.

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I. Introduction

The purpose of this paper is to serve primarily as a technical appendix for recent work by Loury (1979), Lee and Wilde (1980), Dasgupta and Stiglitz (1980), Feichtinger (1981) and Reinganum (1981,1982). These papers analyze research and development as a stochastic racing game where the strategy is the rate of investment on the innovation. In the first three papers, this rate is assumed to be constant, resulting in an exponentially-distributed waiting time for the innovation. The last two authors have allowed time and/or state dependence of the strategies, but have assumed specific functional forms for the hazard function, or conditional density of success. The most important feature of these hazard functions is that they depend only upon current investment, and not upon accumulated previous investment. Reinganum provides an extended example, while Feichtinger assumes convex costs (equivalently, a concave hazard function), and computes and analyzes via a phase diagram an example with a constant elasticity cost function. Both find that, when the planning horizon is finite, in equilibrium firms will invest in R and D at an increasing rate over time, and will consequently experience an increasing hazard rate over time. That is, given no success to date, each firm will be increasingly likely to succeed in the next time increment. When the planning horizon is infinite and the (potential) market for the innovation is stationary, then a constant rate of investment is consistent with equilibrium behavior. Constancy of the investment rate implies an exponentially-distributed waiting time. Thus under the aforementioned circumstances the commonly-made assumption of an exponentially-distributed waiting time will actually be a consequence of equilibrium behavior in a somewhat more general setting.

This paper generalizes previous work in several ways; first, we allow the hazard function to have an initial region of increasing returns, although we retain the restriction that the hazard rate depends only upon current investment. We find that when patent protection is perfect (as assumed in Reinganum (1981) and Feichtinger (1981)), equilibrium investment rates are monotonically increasing over time. Thus the results from these extended examples hold true more generally, so long as the hazard function depends only upon current investment (and for finite time horizons). When firms suffer immediate imitation, at most one firm may invest at a decreasing rate and then only for small t; near the terminal date, both must invest at an increasing rate. If firms are identical and the equilibrium is symmetric, then both must invest at an increasing rate over time. Thus the results of these extended examples are robust to more general formulations <u>if</u> the hazard function depends upon current investment only. This suggests that it is particularly important to attempt to include accumulated expenditure in the hazard function, to see if this result may be reversed under more general circumstances. (This generalization, while desirable, makes it impossible to use two-dimensional phase diagrammatic analysis and is beyond the scope of this paper). Second, we allow for the possibility that some firms are currently receiving profits from the sale of a product which will be replaced by the innovation. This allows us to determine whether current industry leaders will tend to be more or less innovative than firms with smaller current market shares. While we cannot make general conclusions on this question, for firms which are identical except for their current revenue flows, and in a stationary equilibrium, we find that the higher the current rate of revenue, the lower is the equilibrium rate of investment on the new product. That is, current industry leaders will tend to be less innovative than firms which currently have relatively small market shares. Finally, we conduct some analysis for an arbitrary finite number of firms; however, eventually we restrict ourselves to two firms in order to follow Feichtinger's use of phase diagrams to perform detailed analysis of the equilibrium paths.

II. The Model

A key technological assumption in models of research and development is that invention is uncertain; investment in R and D is only stochastically related to the date of success. That is, a firm can stochastically hasten its date of invention by increasing its rate of investment, but it can never guarantee itself a particular success

date.

Let τ_i denote firm i's date of invention, with probability distribution $F_i(t) = \Pr\{\tau_i \leq t\}$. Define the <u>state variable for i</u> $x_i(t) = 1 - F_i(t)$, and the <u>control variable for i</u> $u_i(t)$; $u_i(t)$ represents firm i's rate of investment at date t. The state and control variables are related by the <u>hazard function</u> $h_i(u_i)$:

$$\dot{F}_i(t)/(1 - F_i(t)) = h_i(u_i(t)),$$

or

$$\dot{x}_i(t) = - h_i(u_i(t))x_i(t).$$

<u>Assumption 1</u>. The function $h_i(\cdot)$ is assumed to satisfy

(a) $h_i(0) = 0, \qquad \lim_{u_i \to \infty} h_i(u_i) = M_i$

(b) $h_i'(u_i) > 0 \qquad$ for all $u_i \in [0,\infty)$, $\lim_{u_i \to \infty} h_i'(u_i) = 0$

and

(c) there exists $\underline{u}_i < \infty$ such that

$$h_i''(u_i) \geq (\leq)\ 0 \quad \text{as} \quad u_i \leq (\geq)\ \underline{u}_i.$$

Thus we assume that no progress is made without a commitment of resources, and a greater investment results in a higher conditional density over the success date, but a very high investment rate yields only a finite hazard rate. We allow for an initial region of increasing returns, but eventually the firm experiences a declining marginal return to increased investment. Previous models have assumed convex cost functions (or, equivalently, concave production functions $h_i(\cdot)$) of the constant elasticity class (see, e.g., Feichtinger (1981), Reinganum (1981), (1982)).

Let P_{ij} denote the value of j's success to firm i. That is, firm i receives (in present value terms) $e^{-r_i t} P_{ij}$ if firm j succeeds at t, j = 1,2,...,n. Thus patent protection need not be perfect, but suppose that $P_{ii} \geq P_{ij}$. Let R_i denote <u>flow</u> profits to firm i which are received so long as no firm has completed the innovation. Thus the firms may be operating currently in a market which will be affected by the successful development of the innovation. (In previous models, R_i has always been assumed to be zero). Suppose $P_{ii} \geq R_i/r_i$. The inclusion of this pre-innovation revenue term allows us to assess the impact of current market power upon the incentive to invest in research and development.

Then the value of firm i's profits can be written

$$V^i(u) = \int_0^T e^{-r_i t} \prod_j x_j \, [\sum_j h_j(u_j)P_{ij} + R_i - u_i]dt \qquad (1)$$

where $\dot{x}_j = -h_j(u_j)x_j$, $x_j(0) = 1$, $x_j(T) \geq 0$, j = 1,2,...,n.

Notice that $F_i(t) = 1 - \exp\{-\int_0^t h_i(u_i(s))ds\} > 0$ for all $t < \infty$ since h_i

is bounded. Thus $x_j(T) > 0$ automatically. This is a consequence of the assumption that h_i depends upon current investment u_i only. A more general specification would have $h_i = h_i(u_i, x_i)$. While it would be desirable to obtain results at this higher level of generality, it does not appear to be possible aside from computational examples, since the differential equations characterizing the Nash equilibrium investment rates will be nonseparable in (x,u) and will therefore be impossible to diagram either in state or control space.

<u>Assumption 2</u>. Suppose that $h_i'(0) > 1/P_{ii}$.

<u>Lemma 1</u>. There exists a unique value $\bar{u}_i < \infty$ such that $h_i'(\bar{u}_i) = 1/P_{ii}$; moreover, $\underline{u}_i < \bar{u}_i$. If P_{ii} is the capitalized value of a constant revenue stream π_{ii}, $P_{ii} = \pi_{ii}/r_i$, then $\partial\bar{u}_i/\partial\pi_{ii} > 0$ and $\partial\bar{u}_i/\partial r_i < 0$.

<u>Proof</u>. Since $h_i'(0) > 1/P_{ii}$ and $h_i''(u_i) > 0$ for all $u_i \leq \underline{u}_i$, $h_i'(u_i) > 1/P_{ii}$ for all $u_i \in [0,\underline{u}_i]$. After $\underline{u}_i$, $h_i'(u_i)$ is monotonically decreasing and continuous. Since $h_i'(\underline{u}_i) > 1/P_{ii}$ and $\lim_{u_i \to \infty} h_i'(u_i) = 0$, there exists a unique value $\bar{u}_i < \infty$ (and $> \underline{u}_i$) such that $h_i'(\bar{u}_i) = 0$. Differentiating the equation $h_i'(\bar{u}_i)P_{ii} - 1 = h_i'(\bar{u}_i)\pi_{ii}/r_i - 1 = 0$ and solving yields

$$\partial\bar{u}_i/\partial\pi_{ii} = -r_i/(\pi_{ii})^2 h_i''(\bar{u}_i) > 0$$

and

$$\partial\bar{u}_i/\partial r_i = 1/\pi_{ii}h_i'(\bar{u}_i) < 0.$$

Q. E. D.

III. Strategy Space and Equilibrium Concept

In differential games there are (at least) two alternative strategy spaces of interest, corresponding to alternative assumptions regarding the information structure and/or players' ability to commit themselves. These are path (or open-loop) strategies and decision rule (or closed-loop or feedback) strategies. The distinction is important mathematically and economically. The two strategy spaces can lead to quite different conclusions and one must be careful to use the most appropriate one in applications. However, previous work (Reinganum 1981,1982) has shown that when the hazard function depends upon current investment only, the equilibrium strategies will depend only upon time, and not upon the state variables. Thus in this particular case, we are justified in considering open-loop or path strategies to be the objects of choice.

Definition 1. Define the set of admissible strategies for i to be

$$U_i = \{u_i:[0,T] \rightarrow [0,\infty) \mid u_i \text{ is differentiable}\}.$$

Definition 2. Firm i's payoff function is $V^i(u)$ as defined above in equation (1).

Definition 3. A strategy $\hat{u}_i(t;u_{-i}) \in U_i$ is a best response to the vector $u_{-i} = (u_j)_{j\neq i}$ of rival strategies if

$$V^i(u_1,..,u_{i-1},\hat{u}_i,u_{i+1},..,u_n) \geq V^i(u_1,..,u_{i-1},u_i,u_{i+1},..,u_n)$$

for all $u_i \in U_i$.

Definition 4. A strategy vector $(u_i^*)_{i=1}^n \in U_1 \times \cdots \times U_n$ is a Nash equilibrium if, for i = 1,2,...,n,

$$V^i(u_1^*,..,u_i^*,...,u_n^*) \geq V^i(u_1^*,...,u_{i-1}^*,u_i,u_{i+1}^*,...,u_n^*)$$

for all $u_i \in U_i$. Clearly $u_i^*(t) = \hat{u}_i(t;u_{-i}^*)$.

IV. Necessary and Sufficient Conditions for Equilibrium Play

The behavioral assumption which facilitates characterization of the Nash equilibrium is that each firm maximizes its own payoff function, taking the strategies of its rivals as given. In this case, firm i chooses u_i so as to maximize $V^i(u)$, taking $(u_j)_{j\neq i}$ (and $(x_j)_{j\neq i}$) to be arbitrary functions of time subject to the constraint that $u_j \in U_j$ (and thus $x_j = \exp\{-\int_0^t h_j(u_j(s))ds\}$). Since $x_i(T) > 0$ automatically for $T < \infty$, we may disregard the constraint that $x_i(T) \geq 0$. Define the Hamiltonian for firm i

$$H_i(t,\lambda_i,x,u) = e^{-r_i t}[\prod_j x_j \sum_j h_j(u_j)P_{ij} + R_i - u_i] - \lambda_i h_i(u_i)x_i.$$

Since $(u_j,x_j)_{j\neq i}$ are functions of time only and are insensitive to choices of u_i, we can apply standard optimal control theory to obtain the following necessary conditions (2)-(5). Let ϕ_i

denote the Hamiltonian-maximizing value of u_i.

$$\frac{\partial H_i}{\partial u_i} = e^{-r_i t} \prod_{j \neq i} x_j (h_i'(\sigma_i)P_{ii} - 1) - \lambda_i h_i'(\sigma_i) = 0 \tag{2}$$

$$\frac{\partial^2 H_i}{\partial u_i^2} = e^{-r_i t} \prod_{j \neq i} x_j h_i''(\sigma_i)P_{ii} - \lambda_i h_i''(\sigma_i) \leq 0 \tag{3}$$

$$\frac{-\partial H_i}{\partial x_i} = \dot{\lambda}_i = -\left[e^{-r_i t} \prod_{j \neq i} x_j (h_i(\sigma_i)P_{ii} + \sum_{j \neq i} h_j(u_j)P_{ij} + R_i - \sigma_i) - \lambda_i h_i(\sigma_i)\right], \quad \lambda_i(T) = 0 \tag{4}$$

and

$$\dot{x}_i = - h_i(\sigma_i)x_i, \; x_i(0) = 1, \; x_i(T) \text{ free.} \tag{5}$$

Notice that σ_i depends upon (t, λ_i) only; x_i does not appear. Define the <u>maximized Hamiltonian</u>

$$H_i^o = e^{-r_i t} \prod x_j \left[h_i(\sigma_i)P_{ii} + \sum_{j \neq i} h_j(u_j)P_{ij} + R_i - \sigma_i\right] - \lambda_i(h_i(\sigma_i)x_i.$$

Since $\partial^2 H_i^o / \partial x_i^2 = 0$, the maximized Hamiltonian is concave in x_i for given (t, λ_i). Therefore the necessary conditions are also sufficient to determine a best response $\hat{u}_i(t;u_{-i})$ to the rival

strategy vector u_{-i} (Arrow and Kurz (1970), p. 45, Proposition 6). To find $\hat{u}_i$, solve equations (4) and (5) jointly for $\hat{x}(t;u_{-i})$ and $\hat{\lambda}_i(t;u_{-i})$. Then $\hat{u}_i(t;u_{-i})$ can be found by using the relationship $\hat{u}_i(t;u_{-i}) = \sigma_i(t, \hat{\lambda}_i(t;u_{-i});u_{-i})$.

<u>Lemma 2.</u> Firm i's best response to $(u_j)_{j \neq i}$, $\hat{u}_i(t;u_{-i}) \geq \underline{u}_i$ for all t.

<u>Proof</u>. Equations (2) and (3) implicitly define $\sigma_i = \sigma_i(t, \lambda_i; u_{-i})$ (where the complicated dependence of σ_i upon the rival strategy vector is just noted by including it in the notation). Then $\hat{u}_i(.;u_{-i}) = \sigma_i(., \lambda_i(.);u_{-i})$ is i's best response to u_{-i}. Equation (2) implies

$$e^{-r_i t} \prod_{j \neq i} x_j P_{ii} - \lambda_i = e^{-r_i t} \prod_{j \neq i} x_j / h_i'(\sigma_i).$$

Since this expression is strictly greater than zero, equation (3) is

$$e^{-rt} \prod_{j \neq i} x_j h_i''(\sigma_i)/h_i'(\sigma_i) \leq 0$$

or $h_i''(\sigma_i) \leq 0$. Thus $\sigma_i(t, \lambda_i(t);u_{-i}) = \hat{u}_i(t;u_{-i}) \geq \underline{u}_i$ for all t.

Q. E. D.

<u>Assumption 3</u>. $R_i \geq \underline{u}_i$, $i = 1,2,\ldots,n$.

<u>Lemma 3</u>. $\lambda_i(t) \geq 0$ for all $t \leq T$. If $\underline{u}_i > 0$, then $\lambda_i(t) > 0$ for all $t < T$.

<u>Proof</u>. From equation (2), $\lambda_i = e^{-r_i t} \prod_{j\neq i} x_j \frac{(h_i'(\phi_i)P_{ii} - 1)}{h_i'(\phi_i)}$, so

$$\dot{\lambda}_i = \frac{-e^{r_i t}\prod_{j\neq i} x_j}{h_i'(\phi_i)}\left[h_i(\phi_i) + h_i'(\phi_i)\right]\left[\sum_{j\neq i} h_j(u_j)P_{ij} + R_i - \phi_i\right].$$

Since $h_i''(\phi_i) \leq 0$, $h_i(\phi_i) \geq h_i(\underline{u}_i) + h_i'(\phi_i)(\phi_i - \underline{u}_i)$.

Thus

$$h_i(\phi_i) + h_i'(\phi_i)\left[\sum_{j\neq i} h_j(u_j)P_{ij} + R_i - \phi_i\right]$$

$$\geq h_i(\underline{u}_i) + h_i'(\phi_i)(\phi_i - \underline{u}_i) + h_i'(\phi_i)\left[\sum_{j\neq i} h_j(u_j)p_{ij} + R_i - \phi_i\right]$$

$$= h_i(\underline{u}_i) + h_i'(\phi_i)\left[\sum_{j\neq i} h_j(u_j)P_{ij} + R_i - \underline{u}_i\right] \geq 0$$

(with strict inequalities if $\underline{u}_i > 0$) since $R_i \geq \underline{u}_i$. Then $\dot{\lambda}_i \leq 0$ for all t. Since $\lambda_i(T) = 0$, it follows that $\lambda_i(t) \geq 0$ for all $t < T$. If $\underline{u}_i > 0$, then $\dot{\lambda}_i < 0$; consequently $\lambda_i(T) = 0$ implies that $\lambda_i(t) > 0$ for all $t < T$.

Q. E. D.

<u>Lemma 4</u>. $\hat{u}_i(t;u_{-i}) \leq \bar{u}_i$, for all $t \leq T$; and $\hat{u}_i(T;u_{-i}) = \bar{u}_i$. If $\underline{u}_i > 0$, then $\hat{u}_i(t;u_{-i}) < \bar{u}_i$ for all $t < T$.

<u>Proof</u>. $\lambda_i(T) = 0$, so $h_i'(\phi_i(T, \lambda_i(T);u_{-i}))P_{ii} - 1 = 0$ by equation (2). Thus $\hat{u}_i(T;\ u_{-i}) = \phi_i(T,\ \lambda_i(T);u_{-i}) = \bar{u}_i$. From equation (1), since

$\lambda_i(t) \geq 0$ ($\lambda_i(t) > 0$ if $\underline{u}_i > 0$) by Lemma 3, we see that

$$h_i'(\phi_i(t,\lambda_i(t);u_{-i}))P_{ii} \geq 1, \text{ so}$$

$$h_i'(\phi_i(t,\lambda_i(t);u_{-i}) \geq 1/P_{ii} = h_i'(\bar{u}_i).$$

Since $h_i''(u_i) \leq 0$ for $u_i \geq \underline{u}_i$ (and $\phi_i \geq \underline{u}_i$),

$$\hat{u}_i(t;u_{-i}) = \phi_i(t,\lambda_i(t);u_{-i}) \leq \bar{u}_i.$$

If $\underline{u}_i > 0$, then all of the above inequalities are strict for $t < T$.

Q.E.D.

Note that equations (2)-(5) must hold simultaneously for all firms at a Nash equilibrium u^*. Substitute u_i^* into equations (2)-(5), differentiate (2) and equate the resulting expression for $\dot{\lambda}_i$ to that in equation (4). This yields a system of n first-order nonlinear ordinary differential equations in $(u_j^*)_{j=1}^n$.

$$-h_i''(u_i^*)\dot{u}_i^*/h_i'(u_i^*) = h_i'(u_i^*)\left[\sum_j h_j(u_j^*)P_{ij} + R_i - u_i^*\right]$$

$$- (h_i'(u_i^*)P_{ii} - 1)(r_i + \sum h_j(u_j^*)) \qquad (6)$$

<u>Definition 5</u>. A <u>stationary policy</u> u^o is a point at which $\dot{u}_i = 0$ for all i. That is, where

$$h_i'(u_i^o)\left[\sum_j h_j(u_j^o)P_{ij} + R_i - u_i^o\right]/(h_i'(u_i^o)P_{ii} - 1)$$

$$= r_i + \sum_j h_j(u_j^o) \qquad (7)$$

for $i = 1,2,\ldots,n$.

Theorem 1. A stationary policy u^o is a Nash equilibrium when $T = \infty$ if and only if $u_i^o \in [\underline{u}_i, \bar{u}_i]$ for all i.

Proof. The sufficiency theorem of Arrow and Kurz ((1970), p. 49, Proposition 8) states that the necessary conditions (2)-(5) replacing $\lambda_i(T) = 0$ with the transversality conditions

$$\lim_{t\to\infty} \lambda_i(t) \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \lambda_i(t)x_i(t) = 0$$

are also sufficient.

Equations (2) and (4) are summarized in equation (6); equation (3) will be satisfied if and only if $h_i''(u_i^o) \leq 0$; For a stationary policy, equation (5) implies $x_i(t) = \exp\{-u_i^o t\}$, so $x_i(0) = 1$ and $x_i(t) \geq 0$ for all t. We need only verify the new transversality conditions to conclude sufficiency of u_i^o, given $(u_j^o)_{j\neq i}$. The new transversality conditions are

$$\lim_{t\to\infty} e^{-r_i t}\left(\prod_{j\neq i} \exp\{-u_j^o t\}\right)(h_i'(u_i^o)P_{ii} - 1)/h_i'(u_i^o) \geq 0$$

and

$$\lim_{t\to\infty} e^{-r_i t}\left(\prod_j \exp\{-u_j^o t\}\right)(h_i'(u_i^o)P_{ii} - 1)/h_i'(u_i^o) = 0.$$

These are true for any stationary policy. But by Lemma 3, $\dot{\lambda}_i(t) \leq 0$, so that $\lim_{t\to\infty} \lambda_i(t) \geq 0$ implies that $\lambda_i(t) \geq 0$ for all t. From equation (2), this means that $h_i'(u_i^o)P_{ii} \geq 1$, or alternatively, $u_i^o \leq \bar{u}_i$. Thus if $u_i^o \in [\underline{u}_i, \bar{u}_i]$, then u_i^o is a best response to $(u_j^o)_{i\neq j}$. If this is true for all i, then u^o is a Nash equilibrium.

Q.E.D.

To see directly why no stationary policy with $u_i^o > \bar{u}_i$ can be an equilibrium, note that in this event equation (6) implies that

$$\sum_j h_j(u_j^o)P_{ij} + R_i - u_i^o < 0.$$

But

$$V^i(u^o) = \int_0^\infty e^{-r_i t}\left[\sum_j h_j(u_j^o)P_{ij} + R_i - u_i^o\right]dt < 0.$$

Thus if $h_i'(u_i^o)P_{ii} < 1$, then u_i^o is dominated by the strategy $u_i \equiv 0$.

Corollary 1. In a stationary equilibrium, $1 - F_i(t) = \exp\{-h(u_i^o)t\}$. That is, in equilibrium, the cumulative distribution function of the i^{th} firm's success date is exponential. Thus the convenient assumption of an exponentially distributed success date (with the parameter the decision variable), which is used virtually throughout the literature on research and development, is in fact correct when the firms face an infinite horizon, a stationary environment and a hazard function which depends only on current investment.

Theorem 2. Suppose u^o is a stationary Nash equilibrium policy. Suppose $h_i = h$, $r_i = r$, $P_{ii} = P$, $P_{ij} = 0$, $(j \neq i)$. Then $R_i > R_j$ implies $u_i^o < u_j^o$.

This is the case of identical firms and perfect patent protection, the case most commonly examined. We have generalized it somewhat to examine the impact of current monopoly power upon incentives to invest in R and D. Our conclusion is that, when ranked in increasing order by current profits, firms' investment levels in innovative activity follow exactly the reverse order.

Proof. First recall that at a stationary equilibrium $u_i^o \in [\underline{u}_i, \overline{u}_i]$ for all i. If $h_i = h$, $P_{ii} = P$, then $\underline{u}_i = \underline{u}$ and $\overline{u}_i = \overline{u}$. Thus $u_i^o \in [\underline{u}, \overline{u}]$ for all i. Define

$$g_i(u_i) = h'(u_i)(h(u_i)P + R_i - u_i)/(h'(u_i)P - 1) - h'(u_j^o)(h(u_j^o)P + R_j - u_j^o)/(h'(u_j^o)P - 1)$$

Note that $g_i(u_i^o) = 0$ by equation (7).

$$g_i'(u_i) = h'(u_i) - h''(u_i)(h(u_i)P + R_i - u_i)/(h'(u_i)P - 1)^2.$$

Recall that $h''(u_i) \leq 0$ for all $u_i \in [\underline{u}, \overline{u}]$. In addition, $h(u_i)P + R_i - u_i \geq 0$ for all $u_i \in [\underline{u}, \overline{u}]$. To see this, note that $R_i \geq \underline{u}$ by Assumption 3. Moreover, $h'(u_i)P - 1 \geq 0$ for all $u_i \in [\underline{u}, \overline{u}]$, so $h(u_i)P + R_i - u_i$ is nondecreasing on $[\underline{u}, \overline{u}]$. Thus $g_i'(u_i) > 0$ on $[\underline{u}, \overline{u}]$. Since $g_i(u_j^o) = h'(u_j^o)(R_i - R_j)/(h'(u_j^o)P - 1) > 0$, it follows that $u_i^o < u_j^o$.

Q.E.D.

V. Phase Diagrammatic Analysis

By specializing our analysis to the case of n = 2, we can examine the nonstationary equilibrium (for finite T) in detail. We will be particularly interested in the behavior of the Nash equilibrium investment paths over time. Since equation (6) is independent of the state variables, we can graph the loci $\{(u_1, u_2) | \dot{u}_i = 0\}$, i = 1,2, in the control space. For simplicity, we focus upon the two special cases:

A) perfect patent protection: $P_{ii} = P_i$, $P_{ij} = 0$, $j \neq i$

B) immediate imitation: $P_{ij} = P$ for all i,j.

Recall that

$$\text{sgn } \dot{u}_i = \text{sgn } \{h_i'(u_i)(h_i(u_i)P_{ii} + h_j(u_j)P_{ij} + R_i - u_i) - (h_i'(u_i)P_{ii} - 1)(r_i + h_1(u_1) + h_2(u_2))\}$$

The equation $\dot{u}_1 = 0$ implicitly defines u_2 as a function of u_1 : $u_2 = \alpha(u_1)$.

Case A. Solving $\dot{u}_1 = 0$ for $h_2(\alpha(u_1))$ yields

$$h_2(\alpha(u_1)) = \left[r_1 + h_1(u_1) + h_1'(u_1)(R_1 - u_1 - r_1P_1)\right]/(h'_1(u_1)P_1 - 1) \quad (8)$$

This may sometimes be a negative number; extend the function $h_2(u_2)$ linearly for negative numbers:

$$h_2(u_2) = h_2(0) + h_2'(0)(u_2 - 0) = h_2'(0)u_2.$$

While the equilibrium paths cannot specify negative investment rates, we can consider them for the purposes of the phase diagram.

Differentiating the equation $\dot{u}_1 = 0$ totally, and solving for du_2/du_1 yields

$$\alpha'(u_1) = h_1''(u_1)\left[R_1 - u_1 - r_1P_1 - h_2(\alpha)P_1\right]/h_2'(\alpha)(h_1'(u_1)P_1 - 1). \quad (9)$$

Substituting $h_2(\alpha)$ from equation (8) into equation (9) yields

$$\alpha'(u_1) = h_1''(u_1)\left[u_1 - R_1 - h_1(u_1)P_1\right]/h_2'(\alpha)(h_1'(u_1)P_1 - 1)^2.$$

Lemma 5. There exists a unique value $\tilde{u}_1 \in (\bar{u}_1,\infty)$ such that

$$u_1 - R_1 - h_1(u_1)P_1 \le (\ge)\ 0 \quad \text{as} \quad u_1 \le (\ge)\ \tilde{u}_1.$$

Proof. $u_1 - R_1 - h_1(u_1)P_1 \le 0$ if and only if $h_1(u_1)P_1 + R_1 - u_1 \ge 0$. If $u_1 \in [0,\underline{u}_1]$, then $h_1(u_1)P_1 + R_1 - u_1 > 0$ since $R_1 \ge \underline{u}_1$ by Assumption 3. If $u_1 \in [\underline{u}_1,\bar{u}_1]$, since h_1 is concave on this region,

$$h_1(u_1) \ge h_1(\underline{u}_1) + h_1'(u_1)(u_1 - \underline{u}_1)$$

Thus

$$h_1(u_1)P_1 + R_1 - u_1$$

$$\ge h_1(\underline{u}_1)P_1 + h_1'(u_1)u_1P_1 - h_1'(u_1)\underline{u}_1P_1 + \underline{u}_1 - u_1$$

$$= h_1(\underline{u}_1)P_1 + (h_1'(u_1)P_1 - 1)(u_1 - \underline{u}_1) > 0,$$

since $h_1'(u_1)P_1 - 1 \ge 0$ and $u_1 - \underline{u}_1 \ge 0$ for $u_1 \in [\underline{u}_1,\bar{u}_1]$. Finally, consider $u_1 \in (\bar{u}_1,\infty)$. We know $h_1(\bar{u}_1)P_1 + R_1 - \bar{u}_1 > 0$ by the argument above; $\lim_{u_1\to\infty} h_1(u_1)P_1 + R_1 - u_1 = -\infty$, while $h_1(u_1)P_1 + R_1 - u_1$ is continuous and monotonically declining in u_1 for $u_1 \in (\bar{u}_1,\infty)$.

By the intermediate value theorem, there exists a value $\tilde{u}_1 \in (\bar{u}_1,\infty)$ such that $h_1(\tilde{u}_1)P_1 + R_1 - \tilde{u}_1 = 0$. Monotonicity and the arguments above imply that $\tilde{u}_1$ is unique and $h_1(u_1)P_1 + R_1 - u \ge (\le) 0$ as $u_1 \le (\ge)\ \tilde{u}_1$.

Q. E. D.

We can now characterize the curve $u_2 = \alpha(u_1)$. The expression $\alpha'(u_1)$ undergoes several sign changes on $[0,\infty]$.

For $u_1 \in [0,\underline{u}_1)$, $\alpha'(u_1) < 0$ with $\alpha'(\underline{u}_1) = 0$. For $u_1 \in (\underline{u}_1,\bar{u}_1)$, $\alpha'(u_1) > 0$ with a vertical asymptote at $\bar{u}_1$. For

$u_1 \in (\bar{u}_1, \tilde{u}_1)$, $\alpha'(u_1) > 0$ with $\alpha'(\tilde{u}_1) = 0$. Finally, for $u_1 \in (\tilde{u}_1, \infty)$, $\alpha'(u_1) < 0$. In addition, one can compute $h_2(\alpha(0))$ and $\lim_{u_1 \to \infty} h_2(\alpha(u_1))$, using equation (8).

$$\operatorname{sgn} h_2(\alpha(0)) = \operatorname{sgn}\{r_1 + h_1'(0)(R_1 - r_1P_1)\}$$

and

$$\lim_{u_1 \to \infty} h_2(\alpha(u_1)) = \lim_{u_1 \to \infty} \frac{r_1 + h_1(u_1) - h_1'(u_1)u_1 + h_1'(u_1)(R_1 - r_1P_1)}{h_1'(u_1)P_1 - 1} < 0.$$

Above the curve $\dot{u}_1 = 0$, u_1 is decreasing if $u_1 \in [0, \bar{u}_1)$, and increasing if $u_1 \in (\bar{u}_1, \infty)$. This is illustrated in Figure 1.

Similarly, the equation $\dot{u}_2 = 0$ implicitly defines u_1 as a function of u_2 : $u_1 = \beta(u_2)$, with

$$h_1(\beta(u_2)) = \left[r_2 + h_2(u_2) + h_2'(u_2)(R_2 - u_2 - r_2P_2)\right]/(h_2'(u_2)P_2 - 1), \quad (10)$$

and

$$\beta'(u_2) = h_2''(u_2)\left[R_2 - r_2P_2 - u_1 - h_1(\beta)P_2\right]/h_1'(\beta)(h_2'(u_2)P_2 - 1). \quad (11)$$

Substituting $h_1(\beta(u_2))$ from above (again extending h_1 by $h_1(u_1) = h_1'(0)u_1$ for negative u_1) yields

$$\beta'(u_2) = h_2''(u_2)\left[u_2 - R_2 - h_2(u_2)P_2\right]/h_1'(\beta)(h_2'(u_2)P_2 - 1)^2. \quad (12)$$

The analysis of this curve $u_1 = \beta(u_2)$ parallels that of the curve $u_2 = \alpha(u_1)$. Combining these two analyses yields the completed phase diagram in Figure 2.

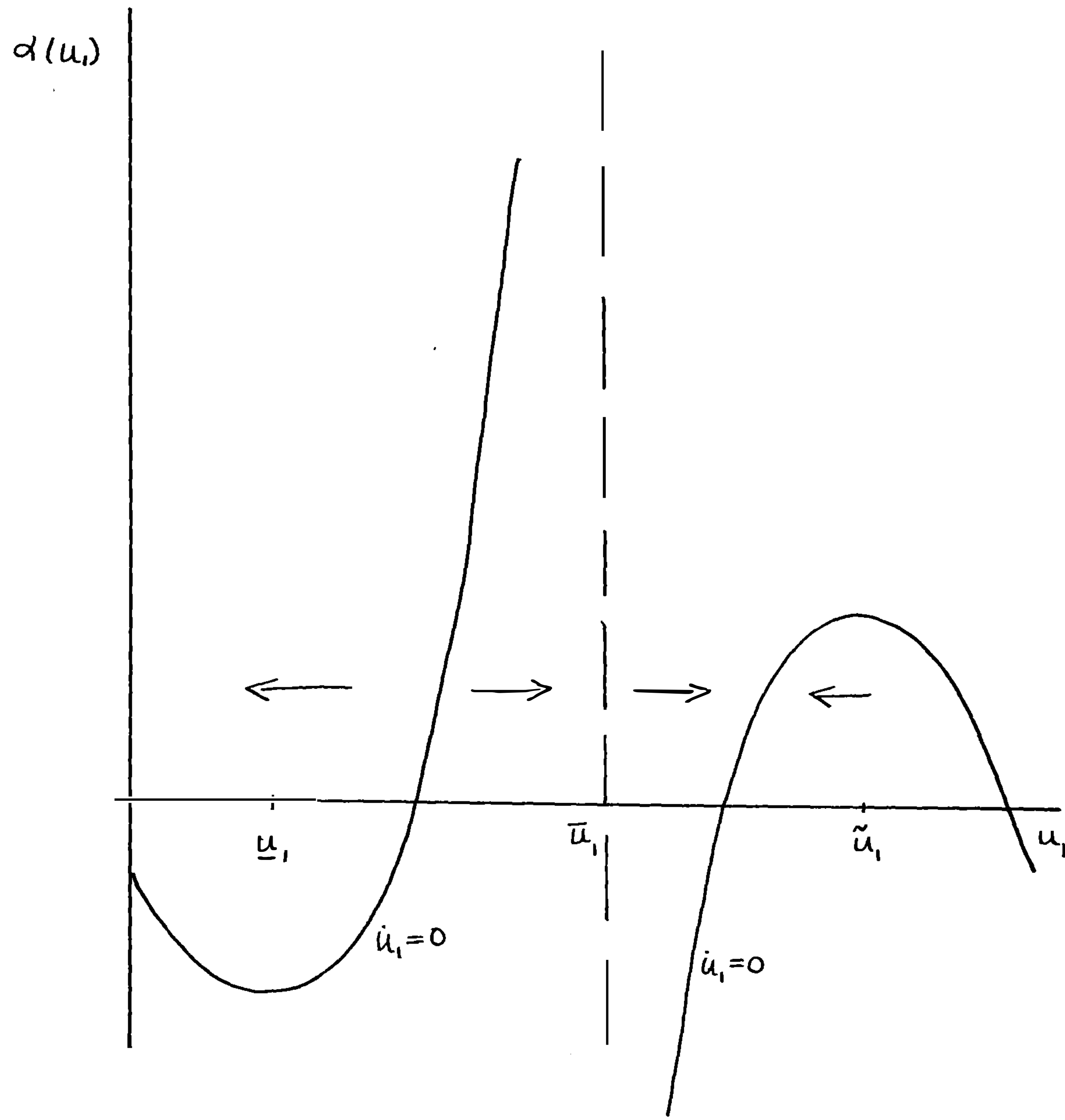


Figure 1

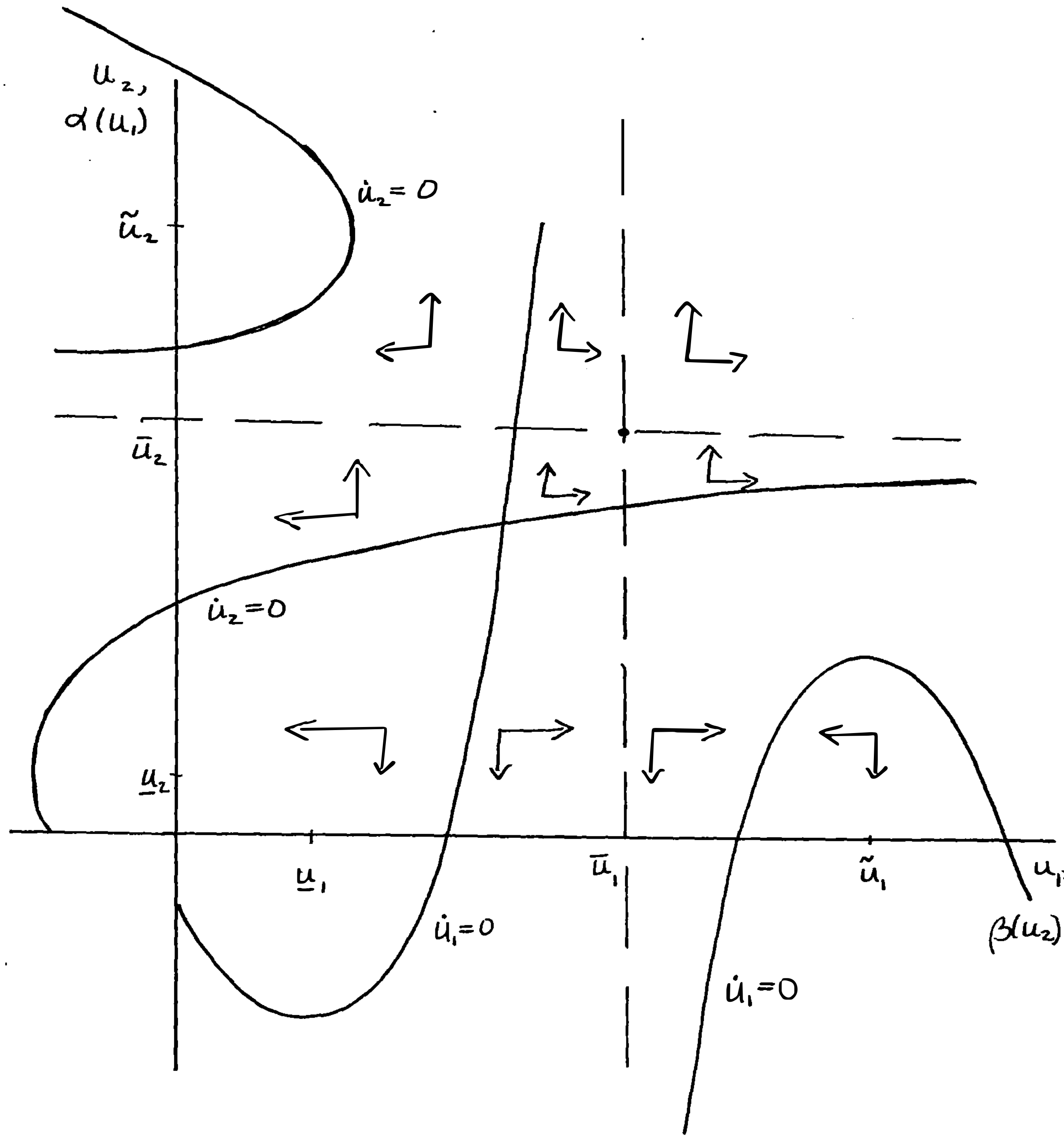


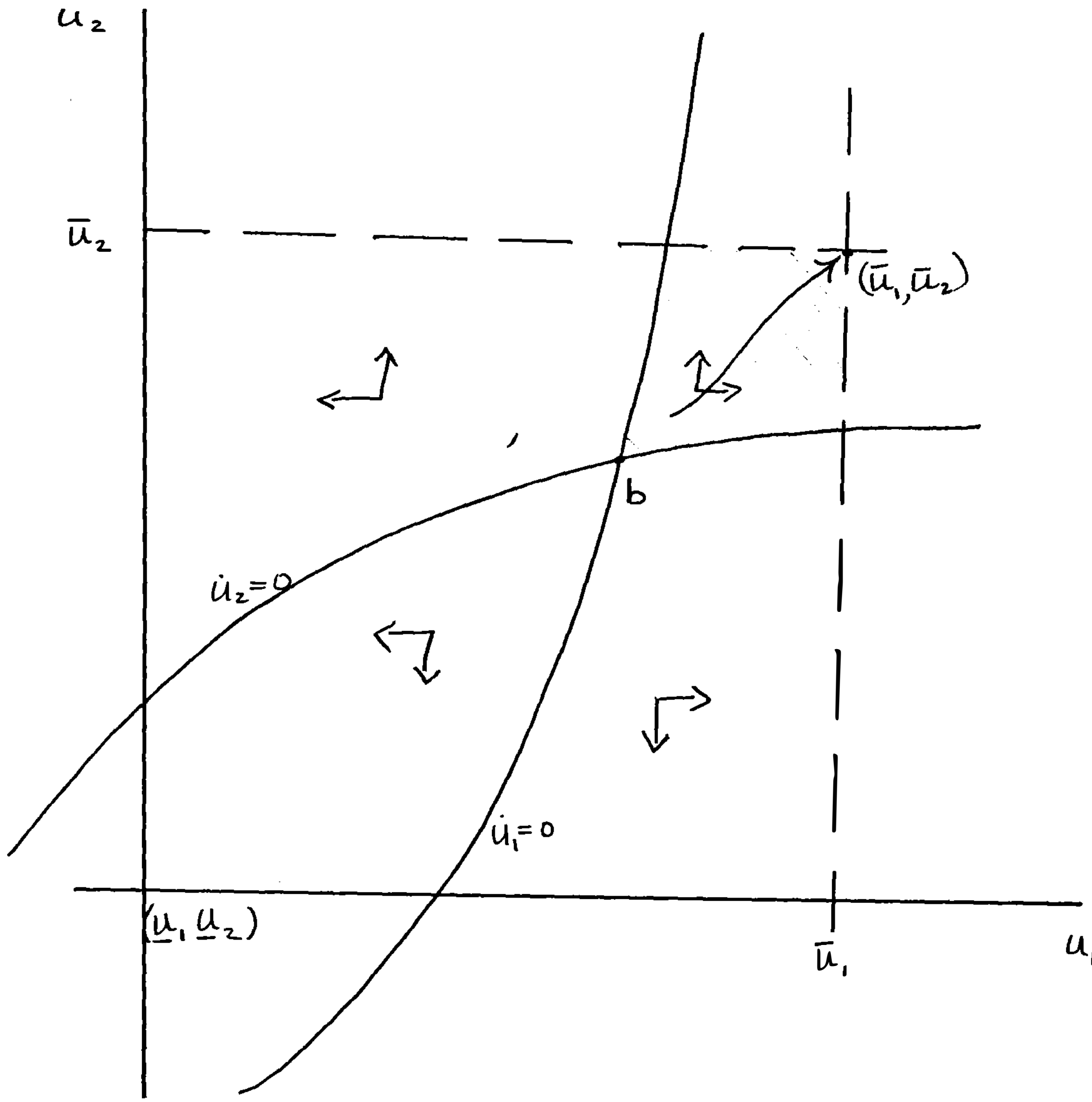
Figure 2

Recall that along any Nash equilibrium path, $u_i^* \in [\underline{u}_i,\bar{u}_i]$ for all t. Thus only the portion of Figure 2 excerpted below in Figure 3 is relevant for the characterization of the equilibrium paths. For the nonstationary (finite horizon) case, the only region which can contain the Nash equilibrium paths is the shaded region in Figure 3. This is because $u_i^*(t) < \bar{u}_i$ for $t < T$ and $u_i^*(T) = \bar{u}_i$. No paths beginning in another region can reach $(\bar{u}_1,\bar{u}_2)$ from below. The point b is a stationary Nash equilibrium policy.

<u>Theorem 3.</u> For the game with perfect patent protection, any Nash equilibrium strategy $u_i^*(\cdot)$ must be monotonically increasing over time (for $T < \infty$).

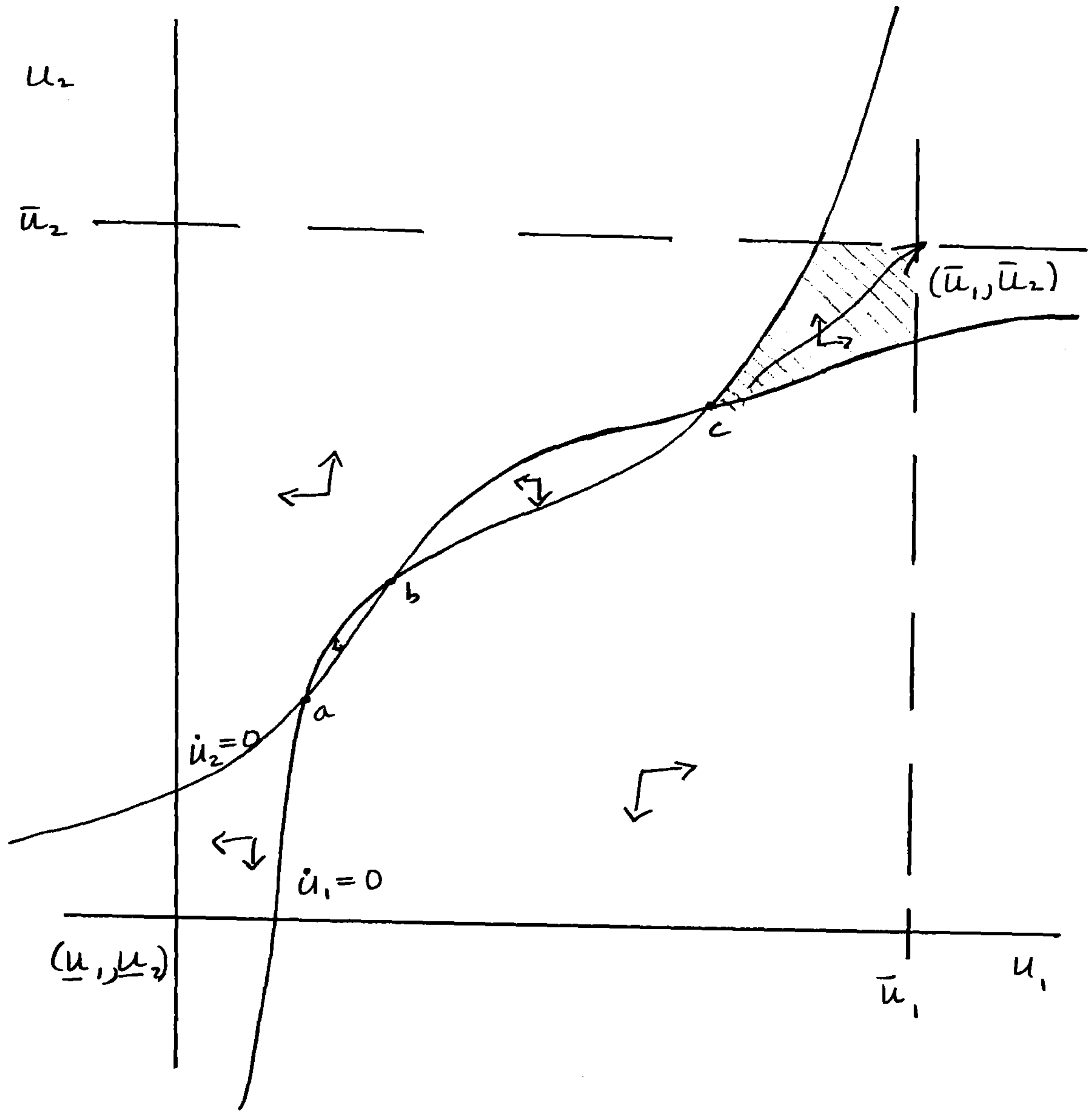
Thus firms will invest so that the conditional density of success — the hazard rate $h_i(u_i^*(\cdot))$ — is an increasing function of time. That is, given no success to date, firm i will be increasingly likely to succeed in the next time increment dt as the current date t increases.

There <u>may</u> be multiple stationary points, as in Figure 4. A sufficient condition for the uniqueness of the stationary point is that $\alpha'(u_1) > 1$ and $\beta'(u_2) > 1$ on $[\underline{u}_1,\bar{u}_1] \times [\underline{u}_2,\bar{u}_2]$. Then <u>if</u> there exists a stationary point $u^o \in [\underline{u}_1,\bar{u}_1] \times [\underline{u}_2,\bar{u}_2]$, it will be unique. This is because the function $u_1 - \beta(\alpha(u_1))$ is monotonically decreasing on $[\underline{u}_1,\bar{u}_1]$.



u_2
$\bar{u}_2$
$(\bar{u}_1, \bar{u}_2)$
b
$\dot{u}_2 = 0$
$\dot{u}_1 = 0$
$(\underline{u}_1, \underline{u}_2)$
$\bar{u}_1$
u_1

Figure 3



u_2
$\bar{u}_2$
$(\bar{u}_1, \bar{u}_2)$
c
b
a
$\dot{u}_2 = 0$
$\dot{u}_1 = 0$
$(\underline{u}_1, \underline{u}_2)$
$\bar{u}_1$
u_1

Figure 4

<u>Case B.</u> $P_{ij} = P$ for all i, j. Then solving $\dot{u}_1 = 0$ for u_2 as a function of $u_1 : u_2 = \alpha(u_1)$.

$$h_2(\alpha(u_1)) = - r_1 - h_1(u_1) + h_1'(u_1)(Pr_1 - R_1 + u_1).$$

Again sgn $\alpha(0)$ = sgn$\{h_1'(0)(Pr_1 - R_1) - r_1\}$ and $\lim_{u_1 \to \infty} h_2(\alpha(u_1)) < 0$. Differentiating the equation $\dot{u}_1 = 0$ and solving for du_2/du_1 implies

$$\alpha'(u_1) = h_1''(u_1)(r_1P - R_1 + u_1)/h_2'(\alpha(u_1)).$$

This expression changes sign only once, at $\underline{u}_1$. First $\alpha'(u_1) > 0$, then $\alpha'(u_1) < 0$ for $u_1 \in (\underline{u}_1, \infty)$. Above the locus $\dot{u}_1 = 0$, u_1 is increasing; below it, decreasing. Similarly, the equation $\dot{u}_2 = 0$ implicitly defines $u_1 = \beta(u_2)$, with

$$h_1(\beta(u_2)) = - r_2 - h_2(u_2) \text{ u } h_2'(u_2)(Pr_2 - R_2 + u_2)$$

and

$$\beta'(u_2) = h_2''(u_2)(Pr_2 - R_2 + u_2)/h_1'(\beta(u_2)).$$

<u>Lemma 6.</u> $h_2(\alpha(\bar{u}_1)) < 0$.

<u>Proof.</u>

$$h_2(\alpha(\bar{u}_1)) = - r_1 - h_1(\bar{u}_1) + \frac{1}{P}(Pr_1 - R_1 + \bar{u}_1)$$
$$= - h_1(\bar{u}_1) - (R_1 - \bar{u}_1)/P$$

$$= [- h_1(\bar{u}_1)P - R_1 + \bar{u}_1]/P$$

In the proof of Lemma 5 we established that $h_1'(\bar{u}_1)P + R_1 - \bar{u}_1 > 0$. Thus $-h_1(\bar{u}_1)P - R_1 + \bar{u}_1 < 0$.

Q. E. D.

The phase diagram summarizing these results is in Figure 5. Now equilibrium paths may begin in either region contained in the dotted-line box. Thus <u>at most one</u> equilibrium path can decline over time, and then only initially. Eventually, both are increasing functions of time. Neither point a,b nor c is a stationary Nash equilibrium policy (for $T = \infty$); point d is a stationary Nash equilibrium policy. If firms are identical and the equilibrium is symmetric (that is, $u_i^*(t) = u_j^*(t)$ for all t), then both firms must invest at an increasing rate throughout [0,T].

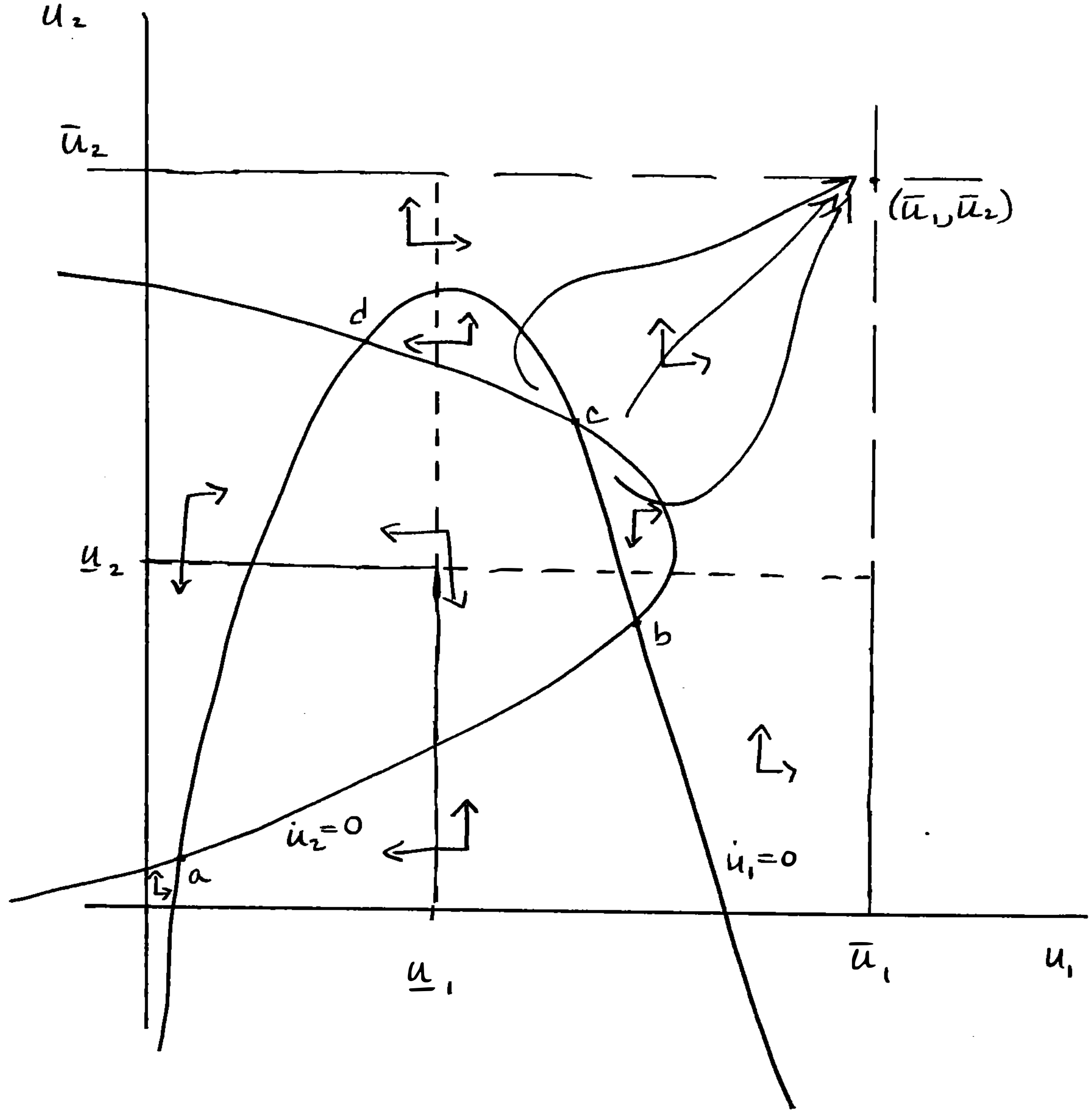

u_2
$\bar{u}_2$
$(\bar{u}_1, \bar{u}_2)$
d
c
$\underline{u}_2$
b
a
$\dot{u}_2 = 0$
$\dot{u}_1 = 0$
$\underline{u}_1$
$\bar{u}_1$
u_1

Figure 5

VI. Conclusion

We have presented a model of investment in research and development which generalizes previous work in this area. The qualitative properties of the equilibrium investment rates are similar to those discovered in this earlier research; that is, firms invest at an increasing rate over time when patent protection is perfect, or in symmetric equilibrium. This suggests that the assumption which is responsible for this monotonicity is that the hazard function depends only upon the current investment rate. Thus relaxation of this assumption seems the most important (albeit the most difficult) direction for subsequent investigation to take.

By including the possibility that firms are currently active in the market for a substitute product, we were able to determine that, in a stationary equilibrium, firms with higher current revenues will invest at lower rates than firms with relatively lower current revenues (assuming that the firms are alike in all other respects).

VII. References

Arrow, Kenneth and Mordecai Kurz, Public Investment, the Rate of Return, and Optimal Fiscal Policy, Baltimore: The John Hopkins Press, 1970.

Dasgupta, Partha and Joseph E. Stiglitz, "Uncertainty, Industrial Structure and the Speed of R and D," Bell Journal of Economics 11 (Autumn 1980), 1-28.

Feichtinger, Gustav, "Optimal Research Policies in a Noncooperative Project Undertaken by Two Firms," Institute Fur Okonometrie und Operations Research Forschungsbericht Nr. 37, Technische Universitat Wien, February 1981.

Kamien, Morton I., and Nancy L. Schwartz, "A Generalized Hazard Rate," Economics Letters 5 (1980), 245-249.

Lee, Tom and Louis L. Wilde, "Market Structure and Innovation: A Reformulation," Quarterly Journal of Economics (March 1980), 429-436.

Loury, Glenn C., "Market Structure and Innovation," Quarterly Journal of Economics (August 1979), 395-410.

Reinganum, Jennifer F., "Dynamic Games of Innovation," Journal of Economic Theory 25 (August 1981), 21-41.

Reinganum, Jennifer F., "A Dynamic Game of R and D: Patent Protection and Competitive Behavior," Econometrica 50 (May 1982), 671-688.