

GROUPS AND LIE ALGEBRAS CORRESPONDING TO THE YANG-BAXTER EQUATIONS

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ABSTRACT. For a positive integer n we introduce quadratic Lie algebras \mathfrak{tr}_n , \mathfrak{qtr}_n and finitely generated groups \mathbf{Tr}_n , \mathbf{QTr}_n naturally associated with the classical and quantum Yang-Baxter equation, respectively.

We prove that the universal enveloping algebras of the Lie algebras \mathfrak{tr}_n , \mathfrak{qtr}_n are Koszul, and compute their Hilbert series. We also compute the cohomology rings of these Lie algebras (which by Koszulity are the quadratic duals of the enveloping algebras).

We construct cell complexes which are classifying spaces for the groups \mathbf{Tr}_n and \mathbf{QTr}_n , and show that the boundary maps in them are zero, which allows us to compute the integral cohomology of these groups.

We show that the Lie algebras \mathfrak{tr}_n , \mathfrak{qtr}_n map onto the associated graded algebras of the Malcev Lie algebras of the groups \mathbf{Tr}_n , \mathbf{QTr}_n , respectively, and conjecture that this map is actually an isomorphism. (This conjecture was recently proved by P. Lee, [L]). At the same time, we show that the groups \mathbf{Tr}_n and \mathbf{QTr}_n are not formal for $n \geq 4$.

1. INTRODUCTION

In this paper we consider certain discrete groups and Lie algebras associated to the Yang-Baxter equations.

Namely, we define the n -th quasitriangular Lie algebra \mathfrak{qtr}_n to be generated by r_{ij} , $1 \leq i \neq j \leq n$, with defining relations given by the classical Yang-Baxter equation

$$(1.1) \quad [r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] = 0$$

for distinct i, j, k , and $[r_{ij}, r_{kl}] = 0$ for distinct i, j, k, l . We define the n -th triangular Lie algebra \mathfrak{tr}_n by the same generators and relations, with the additional relations

$$r_{ij} = -r_{ji}.$$

We define the n -th quasitriangular group \mathbf{QTr}_n to be generated by R_{ij} , $1 \leq i \neq j \leq n$, with defining relations given by the quantum Yang-Baxter equation

$$(1.2) \quad R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij}.$$

and $R_{ij}R_{kl} = R_{kl}R_{ij}$ if i, j, k, l are distinct. We define the n -th triangular group \mathbf{Tr}_n by the same generators and relations, with the additional relations

$$R_{ij} = R_{ji}^{-1}.$$

These groups and Lie algebras are trivial for $n = 0, 1$.

Remark. The groups \mathbf{QTr}_n and \mathbf{Tr}_n are also called pure virtual braid groups and pure flat braid groups, respectively, see [L] and references therein.

These definitions are motivated by the theory of quantum groups, as explained in Section 2.

The main results of the paper are as follows.

1. We prove that the universal enveloping algebras of the Lie algebras $\mathfrak{tr}_n, \mathfrak{qtr}_n$ are Koszul, and compute their Hilbert series. We also compute the cohomology rings of these Lie algebras (which by Koszulity are the quadratic duals of the enveloping algebras).

2. We construct classifying spaces of the groups \mathbf{Tr}_n and \mathbf{QTr}_n . More specifically, a classifying space for the group \mathbf{Tr}_n can be obtained by gluing faces of the $(n - 1)$ -th permutohedron corresponding to the same set partition, and a similar construction works for \mathbf{QTr}_n . Moreover, the boundary maps in the resulting cell complexes are both zero, which allows one to compute the cohomology of the groups $\mathbf{Tr}_n, \mathbf{QTr}_n$ with integer coefficients.

3. We show that the Lie algebras $\mathfrak{tr}_n, \mathfrak{qtr}_n$ map onto the associated graded algebras of the Malcev Lie algebras of the groups $\mathbf{Tr}_n, \mathbf{QTr}_n$, respectively.

4. The quantum group intuition suggests a conjecture that these maps are isomorphisms; in other words, that the ranks of the lower central series quotients for the groups $\mathbf{Tr}_n, \mathbf{QTr}_n$ are equal to the dimensions of the homogeneous components of the Lie algebras $\mathfrak{tr}_n, \mathfrak{qtr}_n$.¹

We note, however, that the groups \mathbf{Tr}_n for $n \geq 4$ are not formal in the sense of Sullivan [Su] (i.e. their Malcev Lie algebras are not isomorphic to their rational holonomy Lie algebras). The same holds for the groups $\mathbf{QTr}_n, n \geq 4$. Summarizing, we may say that the properties of the groups $\mathbf{Tr}_n, \mathbf{QTr}_n$ are similar to those of the pure cactus group Γ_n studied in [EHKR].

Remark. 1. We warn the reader that the published version of this paper, [BEER], contains a serious error. Namely, in [BEER], the above conjecture for \mathbf{Tr}_n is stated as a theorem (Theorem 2.3), and a proof is given, which is incorrect. Namely, the proof rests on Proposition 5.1 of [BEER], which is false (we are grateful to P. Lee for discovering this error). As a result, the proof of Theorem 8.5 given in [BEER] contains a gap, as it rests on the incorrectly proved Theorem 2.3. Similarly, the proof of Proposition 6.1 of [BEER] contains a gap, as it rests on the wrong Proposition 5.1. In the present version, these errors are corrected: Propositions 5.1 and 6.1 are deleted, and Theorems 2.3 and 8.5 are stated as conjectures (Conjectures 2.3 and 6.5 below, respectively).

¹We used the computer system ‘‘GAP’’ [GAP] to show that the conjecture was true for \mathbf{Tr}_4 up to degree 7.

2. Luckily, Conjectures 2.3 and 6.5 were recently proved by P. Lee, [L], which effectively corrects the errors in [BEER]. In fact, he also proved Conjectures 2.4 and 6.6.

2. LIE ALGEBRAS AND GROUPS CORRESPONDING TO THE YANG-BAXTER EQUATIONS

2.1. ***I*-objects.** Let I be the category of finite sets where morphisms are inclusions. Similarly, let J be the category of ordered finite sets, where morphisms are increasing inclusions.

An I -object (respectively, a J -object) of a category \mathcal{C} is a covariant functor $I \rightarrow \mathcal{C}$ (respectively, $J \rightarrow \mathcal{C}$). Thus, an I -object (respectively, a J -object) is the same thing as a sequence of objects X_1, X_2, \dots in \mathcal{C} and a collection of maps $X(f) : X_m \rightarrow X_n$ for every injective (respectively, strictly increasing) map $f : [m] \rightarrow [n]$, such that $X(f)X(g) = X(fg)$.² A morphism between I - and between J -objects is, by definition, a morphism of functors.

Obviously, J is a subcategory of I (with the same isomorphism classes of objects but fewer morphisms). Thus every I -object is also a J -object.

Example. Let A be a unital associative algebra. Then we can define the I -algebra $T(A)$, such that $T(A)_n = A^{\otimes n}$, and for any $a \in A$ and $1 \leq k \leq m$, $T(A)(f)(a_k) = a_{f(k)}$, where a_k denotes the element $1 \otimes \dots \otimes a \otimes \dots \otimes 1 \in A^{\otimes m}$, with a being in the k -th component.

2.2. ***I*-Lie algebras associated to the classical Yang-Baxter equation.** Let us define two I -Lie algebras over \mathbb{Q} , \mathfrak{t} and \mathfrak{qt} (the triangular and quasitriangular Lie algebra). Namely, the Lie algebras \mathfrak{t}_n and \mathfrak{qt}_n have been defined above. Now for each injective map $f : [m] \rightarrow [n]$ we have the corresponding map $(\mathfrak{q})\mathfrak{t}(f) : (\mathfrak{q})\mathfrak{t}_m \rightarrow (\mathfrak{q})\mathfrak{t}_n$ given by $f(r_{ij}) = r_{f(i)f(j)}$, which gives \mathfrak{qt} and \mathfrak{t} the structure of I -Lie algebras.

We can also define the corresponding universal enveloping I -algebras $U(\mathfrak{qt})$, $U(\mathfrak{t})$ in the obvious way.

This definition is motivated by the following proposition, whose proof is straightforward.

Proposition 2.1. *Let A be a unital associative algebra. Then $\text{Mor}_I(\mathfrak{qt}, T(A))$ (in the category of I -Lie algebras) is the set of elements $r \in A^{\otimes 2}$ satisfying the classical Yang-Baxter equation*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

where r_{ij} is the image in $A^{\otimes 3}$ of r through the map $1 \mapsto i, 2 \mapsto j$. Similarly, $\text{Mor}_I(\mathfrak{t}, T(A))$ is the set of skew-symmetric elements $r \in A^{\otimes 2}$ satisfying the classical Yang-Baxter equation.

We have natural homomorphisms $\mathfrak{t}_n \rightarrow \mathfrak{qt}_n \rightarrow \mathfrak{t}_n$, whose composition is the identity. Namely, the second (surjective) map is defined by sending r_{ij} to r_{ij} for all i, j , while the first (injective) map is defined by the same condition,

²Here $[n]$ denotes the set $\{1, \dots, n\}$.

but only for $i < j$. Thus \mathfrak{tr}_n is a split quotient of \mathfrak{qtr}_n . We also note that the injection $\mathfrak{tr}_n \rightarrow \mathfrak{qtr}_n$ induces a map $\phi : \text{Mor}_I(\mathfrak{qtr}, T(A)) \rightarrow \text{Mor}_J(\mathfrak{tr}, T(A))$, which is actually an isomorphism; in particular, $\text{Mor}_J(\mathfrak{tr}, T(A))$ is the set of elements $r \in A^{\otimes 2}$ (not necessarily skew-symmetric) satisfying the classical Yang-Baxter equation.

2.3. I -groups associated to the quantum Yang-Baxter equation.

We can also define I -groups \mathbf{QTr} , \mathbf{Tr} , which are quantum analogs of the Lie algebras \mathfrak{qtr} , \mathfrak{tr} . Namely, for each injective map $f : [m] \rightarrow [n]$ we have the corresponding map $(\mathbf{QTr})(f) : (\mathbf{QTr})_m \rightarrow (\mathbf{QTr})_n$ given by $f(R_{ij}) = R_{f(i)f(j)}$, which gives \mathbf{QTr} and \mathbf{Tr} the structure of I -groups.

This definition is motivated by the following proposition, whose proof is straightforward.

Proposition 2.2. *Let A be a unital associative algebra. Then $\text{Mor}_I(\mathbf{QTr}, T(A))$ (in the category of I -monoids) is the set of invertible elements $R \in A^{\otimes 2}$ satisfying the quantum Yang-Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

and $\text{Mor}_I(\mathbf{Tr}, T(A))$ is the set of elements $R \in A^{\otimes 2}$ satisfying the quantum Yang-Baxter equation and the unitarity condition $R^{21} = R^{-1}$.

We have natural homomorphisms $\mathbf{Tr}_n \rightarrow \mathbf{QTr}_n \rightarrow \mathbf{Tr}_n$, whose composition is the identity. Namely, the second (surjective) map is defined by sending R_{ij} to R_{ij} for all i, j , while the first (injective) map is defined by the same condition, but only for $i < j$. Thus \mathbf{Tr}_n is a split quotient of \mathbf{QTr}_n . We also note that the injection $\mathbf{Tr}_n \rightarrow \mathbf{QTr}_n$ induces a map $\phi : \text{Mor}_I(\mathbf{QTr}, T(A)) \rightarrow \text{Mor}_J(\mathbf{Tr}, T(A))$, which is actually an isomorphism; in particular, $\text{Mor}_J(\mathbf{Tr}, T(A))$ is the set of elements $R \in A^{\otimes 2}$ satisfying the quantum Yang-Baxter equation (but not necessarily the unitarity condition).

Remark. Recall that if E, F are sets, then a (partially defined) function $f : E \rightarrow F$ is a pair (D_f, \bar{f}) where $D_f \subset E$, and $\bar{f} : D_f \rightarrow F$ is a map. Define I' (resp., J') as the opposite of the category whose objects are finite (resp., ordered finite) sets and morphisms are partially defined functions (resp., non-decreasing functions). Then \mathfrak{qtr} , \mathfrak{tr} are I' -Lie algebras, and \mathbf{QTr} , \mathbf{Tr} are J' -groups. A partially defined function $f : [n] \rightarrow [m]$ gives rise to a morphism $(\mathfrak{q})\mathfrak{tr}_m \rightarrow (\mathfrak{q})\mathfrak{tr}_n$ by $r_{ij} \mapsto \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} r_{i'j'}$. Similarly, a non-decreasing partially defined function $f : [n] \rightarrow [m]$ gives rise to the morphism $(\mathbf{Q})\mathbf{Tr}_m \rightarrow (\mathbf{Q})\mathbf{Tr}_n$, by $R_{ij} \mapsto \prod_{i' \in f^{-1}(i), j' \in f^{-1}(j)} R_{i'j'}$ (where the product is taken in increasing order of i', j').

2.4. Prounipotent completions. For a discrete group G , let $\text{Lie}(G)$ denote the Lie algebra of the \mathbb{Q} -prounipotent completion of G (i.e., the Malcev Lie algebra of G), and let $\text{grLie}(G)$ be the associated graded of this Lie algebra with respect to the lower central series filtration. If G is an I -group, these are I -Lie algebras.

We have natural surjective homomorphisms of I -Lie algebras $\phi_{(\mathfrak{q})\mathfrak{tr}} : (\mathfrak{q})\mathfrak{tr} \rightarrow \text{grLie}((\mathbf{Q})\mathbf{Tr})$, given by the formula $r_{ij} \rightarrow \log R_{ij}$.

Conjecture 2.3. (stated as Theorem 2.3 in the published version; now a theorem of P. Lee, [L]) *The homomorphism $\phi_{\mathfrak{tr}}$ is an isomorphism.*

Conjecture 2.4. (now a theorem of P. Lee, [L]) *The homomorphism $\phi_{\mathfrak{qtr}}$ is an isomorphism.*

Note that Conjecture 2.3 follows from Conjecture 2.4, since the group \mathbf{Tr}_n is a split quotient of the group \mathbf{QTr}_n (via $R_{ij} \mapsto R_{ij} \mapsto R_{ij}$), and similarly for the corresponding Lie algebras.

Conjecture 2.4 is motivated by the following theorem.

Theorem 2.5. *Let K be the kernel of $\phi_{\mathfrak{qtr}}$. Then K is annihilated by every morphism in $\text{Mor}(\mathfrak{qtr}, T(A))$ for any algebra A .*

Proof. By a result of [EK1, EK2], any solution $r \in A \otimes A$ of the classical Yang-Baxter equation can be quantized to a solution $R = 1 + \hbar r + O(\hbar^2)$ of the quantum Yang-Baxter equation. This implies that any morphism $(\mathfrak{q})\mathfrak{tr} \rightarrow T(A)$ can be deformed to a morphism $(\mathbf{Q})\mathbf{Tr} \rightarrow T(A)$, which implies the required statement. \square

Remark. It is obvious that \mathfrak{tr}_3 is a free product of abelian Lie algebras $\mathbb{Q}^2 * \mathbb{Q}$, and \mathbf{Tr}_3 is a free product $\mathbb{Z}^2 * \mathbb{Z}$; therefore $\text{Lie}(\mathbf{Tr}_3) = \mathfrak{tr}_3$. However, we have checked using “Magma” that $\text{Lie}(\mathbf{Tr}_n)$ is not isomorphic to \mathfrak{tr}_n for $n = 4$, already modulo elements of degree 5. Since we have split injections $\mathfrak{tr}_n \rightarrow \mathfrak{tr}_{n+1}$ and $\mathbf{Tr}_n \rightarrow \mathbf{Tr}_{n+1}$, the same statement holds for $n > 4$. This implies that the group \mathbf{Tr}_n is not formal for $n \geq 4$ (i.e. its classifying space is not a formal topological space, see [Su]). Since we have a split injection $\mathbf{Tr}_n \rightarrow \mathbf{Qtr}_n$, the same holds³ for the groups \mathbf{QTr}_n , $n \geq 4$.

3. THE KOSZULITY AND HILBERT SERIES OF $U(\mathfrak{tr}_n)$, $U(\mathfrak{qtr}_n)$.

One of the main results of this paper is the following theorem.

Theorem 3.1. (i) *The algebras $U(\mathfrak{tr}_n)$, $U(\mathfrak{qtr}_n)$ are Koszul.*

(ii) *The Hilbert series of these algebras are equal to $1/P_{(\mathfrak{q})\mathfrak{tr}_n}(-t)$, where $P_{(\mathfrak{q})\mathfrak{tr}_n}(t)$ are the polynomials with the following exponential generating functions:*

$$(3.1) \quad 1 + \sum_{n=1}^{\infty} P_{\mathfrak{tr}_n}(t) \frac{u^n}{n!} = e^{\frac{e^{tu}-1}{t}},$$

$$(3.2) \quad 1 + \sum_{n=1}^{\infty} P_{\mathfrak{qtr}_n}(t) \frac{u^n}{n!} = e^{\frac{u}{1-tu}}.$$

³For basics about formality of groups, see e.g. [PS].

(iii) The polynomials $P_{(\mathfrak{q})\mathfrak{tr}_n}(t)$ are given by the following explicit formulas:

$$P_{\mathfrak{tr}_n}(t) = \sum_{k=1}^n \frac{1}{k!} \left(\sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n \right) t^{n-k}.$$

(the palindrome of Bell's exponential polynomial, [Be]);

$$P_{\mathfrak{qtr}_n}(t) = \sum_{p=0}^{n-1} \binom{n-1}{p} \frac{n!}{(n-p)!} t^p.$$

The proof of this theorem is given in the next section.

4. PROOF OF THEOREM 3.1

Part (iii) of the theorem follows by direct computation from part (ii), so we prove only parts (i) and (ii).

4.1. The triangular case. Recall that if B is a quadratic algebra, then the quadratic dual $B^!$ is the quadratic algebra with generators $B[1]^*$ and relations $B[2]^\perp \subset B[1]^* \otimes B[1]^*$.

Consider the quadratic dual algebra A_n to $U(\mathfrak{tr}_n)$. Denote by a_{ij} the set of generators of A_n dual to the generators r_{ij} of $U(\mathfrak{tr}_n)$ (so a_{ij} are defined for distinct $i, j \in [n]$, and $a_{ij} = -a_{ji}$).

Lemma 4.1. *The algebra A_n is the supercommutative algebra generated by odd generators a_{ij} , $1 \leq i \neq j \leq n$, with defining relations $a_{ij} + a_{ji} = 0$, and*

$$a_{ij}a_{jk} = a_{jk}a_{ki}$$

for any three distinct indices i, j, k .

Proof. Let $\omega = \sum a_{ij}r_{ij} \in A_n \otimes \mathfrak{tr}_n$. The relations of A_n can be written as the Maurer-Cartan equation $[\omega, \omega] = 0$ (where the commutator is taken in the supersense). Taking components of this equation, we get the relations for a_{ij} . \square

Lemma 4.1 allows us to easily find a basis of A_n . Namely, define a monomial in A_n to be reduced if it is of the form $a_{i_1 i_2} a_{i_1 i_3} \dots a_{i_1 i_m}$, with $i_1 < i_2 < \dots < i_m$. The support of this monomial is the set $\{i_1, \dots, i_m\}$, and the root label is i_1 .

Proposition 4.2. *Products of reduced monomials with disjoint supports (in the order of increasing the root labels) form a basis in A_n .*

Proof. Take any monomial in A_n . If it is not a product of reduced ones with disjoint supports, then it has a quadratic factor of the form $a_{ij}a_{jk}$, where $j > i$ or $j > k$. Using the relations, we can then replace it with another quadratic monomial, so that the total sum of labels is reduced. This implies that products of reduced monomials span A_n . The fact that these products are linearly independent is easy, since all relations are binomial. \square

Corollary 4.3. (i) The elements $a_{ij}a_{jk} - a_{jk}a_{ki}$ with $k < i, j$ form a quadratic Gröbner basis for supercommutative algebras⁴ for A_n (for any ordering of monomials in which the sum of labels is monotonically nondecreasing).

(ii) A_n is Koszul.

(iii) The Hilbert polynomial of A_n is $P_{\text{tr}_n}(t)$.

Proof. (i) follows from the Proposition 4.2. (ii) follows from (i) since any supercommutative algebra with a quadratic Gröbner basis is Koszul (see e.g. [Yu], Theorem 6.16). To prove (iii), note that Proposition 4.2 implies that $\dim A_n[k]$ is the number of partitions of the set $[n]$ into $n - k$ nonempty parts (see [Wi]), so the result follows from standard combinatorics. \square

Corollary 4.3 and the standard theory of Koszul algebras imply Theorem 3.1 in the triangular case. Indeed, the dual of a Koszul algebra is Koszul, and the Hilbert series of a Koszul algebra and its dual are related by the equation $p(t)q(-t) = 1$.

4.2. The quasitriangular case. The proof in the quasitriangular case is analogous although a bit more complicated. Let us split $r_{ij} \in \mathfrak{qtr}_n$ into a symmetric and skew-symmetric part: $r_{ij} = t_{ij} + \rho_{ij}$, where t_{ij} is symmetric and ρ_{ij} is skewsymmetric in i, j . Then the defining relations for \mathfrak{qtr}_n take the form:

$$[t_{ij}, t_{ik} + t_{jk}] = 0, [t_{ij}, \rho_{ik} + \rho_{jk}] = 0,$$

$$[\rho_{ij}, \rho_{jk}] + [\rho_{jk}, \rho_{ki}] + [\rho_{ki}, \rho_{ij}] = [t_{ij}, t_{jk}],$$

for distinct i, j, k , and

$$[\rho_{ij}, \rho_{kl}] = [\rho_{ij}, t_{kl}] = [t_{ij}, t_{kl}] = 0$$

if i, j, k, l are distinct.

As before, consider the quadratic dual algebra QA_n to $U(\mathfrak{qtr}_n)$. Denote by a_{ij}, b_{ij} the set of generators of QA_n dual to the generators ρ_{ij} and t_{ij} of $U(\mathfrak{qtr}_n)$ (so a_{ij}, b_{ij} are defined for distinct $i, j \in [n]$, and $a_{ij} = -a_{ji}$, $b_{ij} = b_{ji}$).

Lemma 4.4. The algebra QA_n is the supercommutative algebra generated by odd generators a_{ij}, b_{ij} with defining relations

$$a_{jk}a_{ij} = a_{ki}a_{jk} = b_{ij}b_{jk} + b_{jk}b_{ki} + b_{ki}b_{ij},$$

$$a_{ij}b_{jk} = a_{ik}b_{jk}$$

for any three distinct indices i, j, k , and

$$a_{ij}b_{ij} = 0,$$

for $i \neq j$.

Proof. The proof is analogous to the proof of Lemma 4.1. \square

⁴About Gröbner bases for supercommutative algebras, see e.g. [MV].

The algebra QA_n has a filtration in which $\deg(b_{ij}) = 1$, and $\deg(a_{ij}) = 0$. In the algebra $\text{gr}QA_n$, the graded versions of the above relations are satisfied. These graded versions are the same as the original relations, except for the first set of relations in Lemma 4.4, which is replaced by

$$a_{ij}a_{jk} = a_{jk}a_{ki}, \quad b_{ij}b_{jk} + b_{jk}b_{ki} + b_{ki}b_{ij} = 0.$$

Let QA_n^0 be the algebra with generators a_{ij}, b_{ij} , whose defining relations are the graded version of the relations of QA_n . We have a surjective homomorphism $QA_n^0 \rightarrow \text{gr}QA_n$ (we will show below that it is an isomorphism). Note that we have a split injection $OS_n \rightarrow QA_n^0$ from the Orlik-Solomon algebra of the braid arrangement (generated by b_{ij}) to QA_n^0 .

The exterior algebra in a_{ij} and b_{ij} is graded, as a space, by 2-step set partitions: partitions of $[n]$ into nonempty sets S_1, \dots, S_l , and then of each S_p into nonempty subsets S_{pq} , $q = 1, \dots, m_p$. Namely, if we are given a monomial M in a_{ij} and b_{ij} , we connect i, j by a black edge if b_{ij} is present in M , and by a red edge if b_{ij} is not present, but a_{ij} is present. Then we define the S_p to be the connected components of the obtained graph, and the S_{pq} to be the connected components of the graph of black edges only.

It is easy to see that the relations of QA_n^0 are compatible with this grading, and thus that QA_n^0 also has a grading by 2-step set partitions. This fact allows us to find a basis of QA_n^0 .

Namely, let $S = (S_{pq})$ be a 2-step set partition of $[n]$, and let i_{pq} be the minimal element of S_{pq} . Let i_p be the minimum of i_{pq} over all q , and q_p be such that $i_{pq_p} = i_p$.

For $T \subset [n]$, let $OS(T)$ be the Orlik-Solomon algebra generated by b_{ij} , $i, j \in T$. Let $\{b(T, s)\}$, $1 \leq s \leq (|T| - 1)!$, be the broken circuit basis of the top component of this algebra (see [Yu]).

Let $QA_n^0(S)$ be the degree S part of QA_n^0 .

Proposition 4.5. *The elements $\prod_{p=1}^r (\prod_{q=1}^{s_p} b(S_{pq}, s_{pq}) \prod_{q \neq q_p} a_{i_p i_{pq}})$ for all s_{pq} form a basis of $QA_n^0(S)$.*

Proof. The proof is analogous to the proof of Proposition 4.2. Namely, it is easy to show that any monomial in $QA_n^0(S)$ can be reduced, using the relations, to a monomial from Proposition 4.5. On the other hand, it is clear that the monomials in Proposition 4.2 are linearly independent (this follows from compatibility of the relations with the grading by 2-step set partitions). \square

Corollary 4.6. (i) *The algebra QA_n^0 has a quadratic Gröbner basis.*

(ii) *QA_n^0 is Koszul.*

(iii) *The Hilbert polynomial of QA_n^0 is $P_{\text{qtr}_n}(t)$.*

Proof. (i) Pick any ordering of monomials with sum of labels monotonically nondecreasing. It is well known that the Orlik-Solomon algebra OS_n has a quadratic Gröbner basis with respect to this ordering, compiled of all the relations (see [Yu]): the initial monomials for this basis are products $b_{ip}b_{jp}$

with $p > i, j$. Putting this basis together with the elements $a_{ij}a_{jk} - a_{jk}a_{ki}$ for $k < i, j$, $a_{ij}b_{ij}$, $a_{ij}b_{jk} - a_{ik}b_{jk}$ for $k < j$, we get a quadratic Gröbner basis of QA_n^0 . This implies (i).

(ii) follows from (i).

(iii) This reduces to counting 2-step partitions S with weights. We will adopt the following construction of such partitions: first we partition $[n]$ into r nonempty subsets $S(i)$, $i = 1, \dots, r$, and then pick a set partition of $[r]$ into l parts T_1, \dots, T_l to decide when we will have $S(i) = S_{pq} \subset S_p$; namely, $S(i) \subset S_p$ if and only if $i \in T_p$.

Let $s_p = |T_p|$, $p = 1, \dots, l$, and d_i be the sizes of the parts $S(i)$.

Let $P(t)$ be the Hilbert polynomial of QA_n^0 . Let

$$F(t, u) := 1 + \sum_{n \geq 1} P_n(t) \frac{u^n}{n!}.$$

We have

$$(4.1) \quad F(t, u) = 1 + \sum_{r, \ell} \sum_{\substack{d_1, \dots, d_r > 0 \\ d_1 + \dots + d_r = n}} \frac{n!}{d_1! \cdots d_r!} \frac{(d_1 - 1)! \cdots (d_r - 1)!}{r!} \\ \times \sum_{\substack{s_1, \dots, s_\ell > 0 \\ s_1 + \dots + s_\ell = r}} \frac{r!}{s_1! \cdots s_\ell!} \frac{t^{n-\ell} u^n}{\ell! n!}.$$

Here $\frac{n!}{d_1! \cdots d_r!}$ is the number of ways to choose the parts $S(i)$ once the sizes of $S(i)$ have been fixed, the factor $1/r!$ accounts for the fact that the order of the parts $S(i)$ does not matter, $(d_i - 1)!$ are the sizes of the top components of the algebras $OS(S(i))$, $\frac{r!}{s_1! \cdots s_\ell!}$ is the number of ways to choose the parts T_p once their sizes have been fixed, and $1/\ell!$ accounts for the fact that the order of the parts T_p does not matter.

Cancelling $n!, r!, (d_i - 1)!$ and summing over n, d_i , we get

$$F(t, u) = 1 + \sum_{r, \ell} \sum_{\substack{s_1, \dots, s_\ell > 0 \\ s_1 + \dots + s_\ell = r}} \frac{(-\log(1 - tu))^r t^{-\ell}}{s_1! \cdots s_\ell! \ell!}.$$

Now summing over $r > 0, s_p$ we get

$$F(t, u) = 1 + \sum_{\ell} ((1 - tu)^{-1} - 1)^\ell \frac{t^{-\ell}}{\ell!} = e^{\frac{u}{1-tu}}.$$

This completes the proof. \square

Remark. Let $P(t, v) = \sum D_{pq} t^p v^q$, where D_{pq} is the dimension of the space of elements of QA_n^0 of degree p , which have degree $n - q$ with respect to the variables b_{ij} . Set $F(t, u, v) = 1 + \sum_n P(t, v) u^n / n!$. Then the expression for $F(t, u, v)$ is obtained as above, except that we need to insert a factor v^r .

This implies that

$$F(t, u, v) = \exp \frac{(1 - tu)^{-v} - 1}{t}$$

Proposition 4.7. (i) *The natural map $QA_n^0 \rightarrow \text{gr}(QA_n)$ is an isomorphism.*
(ii) *QA_n is Koszul.*

Proof. (i) By Koszulity of QA_n^0 , it is sufficient to check (see e.g. [BG]) that the map is bijective in degrees ≤ 3 , which is a direct computation. Part (ii) follows from (i), since if $\text{gr}(A)$ is Koszul, so is A . \square

Similarly to the previous section, Proposition 4.7 implies Theorem 3.1 in the quasitriangular case. Thus Theorem 3.1 is proved.

4.3. Connection with the pure braid groups. Let \mathbf{PB}_n be the pure braid group on n strands. Let \mathfrak{pb}_n be the Lie algebra of its pronipotent completion. According to the results of Kohno [Ko], this Lie algebra is isomorphic to its graded, and is generated by $t_{ij} = t_{ji}$, $i \neq j$, $i, j \in [n]$, with defining relations

$$[t_{ij}, t_{ik} + t_{jk}] = 0,$$

and $[t_{ij}, t_{kl}] = 0$ if i, j, k, l are distinct.

We have a group homomorphism $\Psi_n : \mathbf{PB}_n \rightarrow \mathbf{QTr}_n$ defined by

$$T_{ij} \mapsto R_{j-1,j} \dots R_{i+1,j} R_{ij} R_{ji} (R_{j-1,j} \dots R_{i+1,j})^{-1}$$

where $T_{ij} = (\sigma_{j-1} \dots \sigma_{i+1}) \sigma_i^2 (\sigma_{j-1} \dots \sigma_{i+1})^{-1}$ are the Artin-Burau generators of \mathbf{PB}_n , and σ_i are the Artin generators of the full braid group.

The infinitesimal analog of Ψ_n is the Lie algebra homomorphism $\psi_n : \mathfrak{pb}_n \rightarrow \mathfrak{qtr}_n$ defined by the formula $t_{ij} \mapsto r_{ij} + r_{ji}$.

It is clear that the kernel of the projection $\mathbf{QTr}_n \rightarrow \mathbf{Tr}_n$ is the normal closure of the image of Ψ_n . Similarly, the kernel of the projection $\mathfrak{qtr}_n \rightarrow \mathfrak{tr}_n$ is the normal closure of the image of ψ_n .

Proposition 4.8. *The homomorphisms ψ_n, Ψ_n are injective.*

The proof of Proposition 4.8 is given below in subsection 5.

Remark. Here is another proof of the fact that ψ_n is injective. Let \mathfrak{qtr}_n^0 be the associated graded of \mathfrak{qtr}_n under the filtration defined by $\deg(t_{ij}) = 0$, $\deg(\rho_{ij}) = 1$. Proposition 4.7 implies that the defining relations for this Lie algebra are

$$\begin{aligned} [t_{ij}, t_{ik} + t_{jk}] &= 0, [t_{ij}, \rho_{ik} + \rho_{jk}] = 0, \\ [\rho_{ij}, \rho_{jk}] + [\rho_{jk}, \rho_{ki}] + [\rho_{ki}, \rho_{ij}] &= 0, \end{aligned}$$

for distinct i, j, k , and

$$[\rho_{ij}, \rho_{kl}] = [\rho_{ij}, t_{kl}] = [t_{ij}, t_{kl}] = 0$$

if i, j, k, l are distinct, and its universal enveloping algebra is the quadratic dual to QA_n^0 . Let $\psi_n^0 = \text{gr} \psi_n : \mathfrak{pb}_n \rightarrow \mathfrak{qtr}_n^0$. Then it follows from the above relations that ψ_n^0 (unlike ψ_n) is a *split* homomorphism, hence it is injective. Thus ψ_n is injective as well.

5. PROOF OF PROPOSITION 4.8

In this section, we fix an integer $n \geq 1$. In [En1], the second author introduced

$$U_n^{\text{univ}} = (U(\mathfrak{g})^{\otimes n})_{\text{univ}} := \bigoplus_{N \geq 0} ((\mathcal{F}\mathcal{A}_N^{\otimes n})_{\sum_i \delta_i} \otimes (\mathcal{F}\mathcal{A}_N^{\otimes n})_{\sum_i \delta_i})_{\mathfrak{S}_N}.$$

Here $\mathcal{F}\mathcal{A}_N$ is the free algebra generated by x_1, \dots, x_N . It is graded by $\bigoplus_{i=1}^N \mathbb{Z}_{\geq 0} \delta_i$ ($\deg(x_i) = \delta_i$), and equipped with the action of \mathfrak{S}_N permuting the generators. The index $\sum_i \delta_i$ means the part of degree $\sum_i \delta_i$, and the index \mathfrak{S}_N means the coinvariants of the diagonal action of \mathfrak{S}_N . We also defined an algebra structure on U_n^{univ} . It has the following property: if (A, r_A) is an algebra equipped with a solution $r_A = \sum_{i \in I} a(i) \otimes b(i) \in A^{\otimes 2}$ of the classical Yang-Baxter equation, then the linear map $U_n^{\text{univ}} \rightarrow A^{\otimes n}$ taking

$$\begin{aligned} & \left(x_1 \cdots x_{k_1} \otimes x_{k_1+1} \cdots x_{k_2} \otimes \cdots \otimes x_{k_{n-1}+1} \cdots x_N \right) \\ & \otimes \left(x_{\sigma(1)} \cdots x_{\sigma(\ell_1)} \otimes x_{\sigma(\ell_1+1)} \cdots x_{\sigma(\ell_2)} \otimes \cdots \otimes x_{\sigma(\ell_{n-1}+1)} \cdots x_{\sigma(N)} \right) \end{aligned}$$

to

$$\begin{aligned} & \sum_{i_1, \dots, i_N \in I} a(i_1) \cdots a(i_{k_1}) b(i_{\sigma(1)}) \cdots b(i_{\sigma(\ell_1)}) \otimes a(i_{k_1+1}) \cdots a(i_{k_2}) b(i_{\sigma(\ell_1+1)}) \cdots b(i_{\sigma(\ell_2)}) \\ & \otimes \cdots \otimes a(i_{k_{n-1}+1}) \cdots a(i_N) b(i_{\sigma(\ell_{n-1}+1)}) \cdots b(i_{\sigma(N)}) \end{aligned}$$

is an algebra morphism.

Let us set $r_{ij}^{\text{univ}} = (1^{\otimes i-1} \otimes x_1 \otimes 1^{\otimes n-i}) \otimes (1^{\otimes j-1} \otimes x_1 \otimes 1^{\otimes n-j}) \in U_n^{\text{univ}}$. Then we have a Lie algebra morphism $\zeta_n : \mathfrak{qtr}_n \rightarrow U_n^{\text{univ}}$, $r_{ij} \mapsto r_{ij}^{\text{univ}}$.

By [En2], Section 1.13, the composition $\zeta_n \circ \psi_n$ is injective. Hence ψ_n is injective.

Let $Z_n : \mathbf{QTr}_n \rightarrow (\widehat{U}_n^{\text{univ}})_1^\times$ be the homomorphism sending R_{ij} to R_{ij}^{univ} . We denote by $G(\mathbb{Q}) = \exp(\text{Lie}(G))$ the Malcev \mathbb{Q} -completion of a group G . As $(\widehat{U}_n^{\text{univ}})_1^\times$ is a pronipotent \mathbb{Q} -Lie group, Z_n factors through $Z_n(\mathbb{Q}) : \mathbf{QTr}_n(\mathbb{Q}) \rightarrow (\widehat{U}_n^{\text{univ}})_1^\times$. Also, let $\Psi_n(\mathbb{Q}) : \mathbf{PB}_n(\mathbb{Q}) \rightarrow \mathbf{QTr}_n(\mathbb{Q})$ be the extension of Ψ_n to Malcev \mathbb{Q} -completions. It is easy to see that the associated graded map of $Z_n(\mathbb{Q}) \circ \Psi_n(\mathbb{Q}) : \mathbf{PB}_n(\mathbb{Q}) \rightarrow (\widehat{U}_n^{\text{univ}})_1^\times$ is $\zeta_n \circ \psi_n$. Hence $Z_n(\mathbb{Q}) \circ \Psi_n(\mathbb{Q})$ is injective, and therefore $\Psi_n(\mathbb{Q})$ is injective. But the group \mathbf{PB}_n is an iterated cross product of free groups, which implies that the natural map $\mathbf{PB}_n \rightarrow \mathbf{PB}_n(\mathbb{Q})$ is injective. Thus Ψ_n is injective, as desired.

Question 5.1. *Is the map ζ_n injective?*

6. CLASSIFYING SPACES FOR THE GROUPS \mathbf{Tr}_n , \mathbf{QTr}_n .

6.1. The Permutohedron. Let P_n be the convex hull of $\mathfrak{S}_n \cdot (n, n-1, \dots, 1)$ in the affine hyperplane \mathbb{A}_n defined by the equation

$$\sum_{i=1}^n x_i = 1 + \dots + n = n(n+1)/2$$

in \mathbb{R}^n . This is a polyhedron, containing the points (x_1, \dots, x_n) such that, for every $S \subset [n]$, we have $\sum_{s \in S} x_s \in [1 + \dots + |S|, (n - |S| + 1) + \dots + n]$.

For S a finite set, we write P_S for the permutohedron $P_{|S|}$ constructed in \mathbb{R}^S .

The faces of P_n can be determined as follows. The $(n-r)$ -dimensional faces are in bijection with the ordered partitions $[n] = S_1 \sqcup \dots \sqcup S_r$; the face corresponding to such a choice is the set of the points (x_1, \dots, x_n) satisfying

$$\text{for all } i \in \{1, \dots, r\}: \sum_{s \in S_i} x_s = (|S_1| + \dots + |S_{i-1}| + 1) + \dots + (|S_1| + \dots + |S_i|).$$

This face is therefore the Cartesian product $P_{S_1} \times \dots \times P_{S_r}$, with the coordinates of P_{S_i} shifted $|S_1| + \dots + |S_{i-1}|$ away from the origin. For example, the vertex with coordinates $(\pi(1), \dots, \pi(n))$ corresponds to the partition $[n] = \{\pi^{-1}(1)\} \sqcup \dots \sqcup \{\pi^{-1}(n)\}$. Geometric inclusion of faces corresponds combinatorially to ordered refinement of partitions; namely, $S_1 \sqcup \dots \sqcup S_r$ is a face of $T_1 \sqcup \dots \sqcup T_s$ if there is an order-preserving surjection $f: [r] \rightarrow [s]$ with $T_i = \bigcup_{j \in f^{-1}(i)} S_j$. The set of faces is partially ordered by this relation; partitions into singletons are atoms, and the one-part partition $[n]$ is the maximal element.

6.2. The classifying space C_n . For every r there is a natural action of the symmetric group \mathfrak{S}_r on the disjoint union of the $(n-r)$ -dimensional faces of P_n . In the combinatorial model, it is the natural permutation action

$$\pi(S_1 \sqcup \dots \sqcup S_r) = S_{\pi^{-1}(1)} \sqcup \dots \sqcup S_{\pi^{-1}(r)},$$

which moves S_i from position i to position $\pi(i)$. In the geometric model, the action is given by piecewise translations: in the face associated with $S_1 \sqcup \dots \sqcup S_r$,

$$\pi(x_s) = x_s - (|S_1| + \dots + |S_{i-1}|) + (|S_{\pi^{-1}(1)}| + \dots + |S_{\pi^{-1}(\pi(i)-1)}|).$$

These actions fit together, in the sense that if $S_1 \sqcup \dots \sqcup S_r$ is a face of $T_1 \sqcup \dots \sqcup T_s$ via the surjection f as above, then f interlaces the \mathfrak{S}_r - and \mathfrak{S}_s -actions. We let C_n be the quotient of P_n by these actions.

Theorem 6.1. *C_n is a classifying space for the group \mathbf{Tr}_n .*

Proof. This amounts to showing that C_n has a contractible universal cover, and has fundamental group \mathbf{Tr}_n .

For the first assertion, we show that C_n is locally a non-positively curved space, whence [BH, Corollary II.1.5] yields that the universal cover of C_n

is contractible. Since C_n is a quotient of Euclidean space, it suffices to show by [BH, Theorem II.5.2] that the link $Lk(*, C_n)$ of the vertex $* \in C_n$ has curvature ≤ 1 ; applying iteratively [BH, Theorem 5.4], this amounts to showing that for every face F of C_n the link $Lk(F, C_n)$ contains no isometrically embedded circles of length $< 2\pi$.

Recall that the faces of P_n are indexed by ordered set-partitions of $[n] = \{1, 2, \dots, n\}$, and that faces of C_n are indexed by unordered set-partitions of $[n]$.

Let therefore $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$ be an unordered set partition, with $S_i \subset [n]$, and let F denote the corresponding face of C_n . Note that F is of dimension $n - r$. The link $L = Lk(F_S, C_n)$ is a spherical simplicial complex of dimension $r - 1$, combinatorially isomorphic to $Lk(*, C_r)$, which can be described as follows:

The vertex set V of L consists of ordered pairs of distinct elements of S ; the pair (S_i, S_j) represents the partition $\{S_1, \dots, S_i \cup S_j, \dots, S_r\}$. A subset Θ of V spans a simplex if and only if there exists a permutation (T_1, \dots, T_r) of \mathcal{S} such that $\Theta \subset \{(t_1, t_2), (t_2, t_3), \dots, (t_{r-1}, t_r)\}$.

An isometrically embedded circle in L is necessarily a subset of the 1-skeleton of L , so it is important to compute the lengths of the edges of L . These lengths can be described geometrically as dihedral angles between certain $(n - r + 1)$ -faces incident to F . By considering normal vectors to these $(n - r + 1)$ -faces inside the $(n - r + 2)$ -face in \mathbb{A}_n containing them, these angles can be computed explicitly.

Let (S_i, S_j) and (S_k, S_l) be two vertices of L . For $S \subset [n]$, let e_S denote the vector having 1's in coordinates $\in S$ and 0 elsewhere. Two cases can occur:

- $|\{i, j, k, l\}| = 4$: normal vectors to (S_i, S_j) and (S_k, S_l) may be chosen respectively as $|S_j|e_{S_i} - |S_i|e_{S_j}$ and $|S_k|e_{S_l} - |S_l|e_{S_k}$, so that the arclength between these two vertices is $\pi/2$.
- $j = k, |\{i, j, l\}| = 3$: normal vectors may be chosen respectively as $v_{ij} = (|S_k| + |S_l|)e_{S_i} - |S_i|e_{S_k} - |S_i|e_{S_l}$ and $v_{kl} = (|S_i| + |S_j|)e_{S_l} - |S_l|e_{S_i} - |S_l|e_{S_j}$, and the arclength between these vertices is the angle between these vectors. The exact value will not be needed, but since $\langle v_{ij}, v_{kl} \rangle < 0$ the angle is strictly $> \pi/2$.

Consider now an isometrically embedded circle in L ; we wish to show that its length is at least 2π . Since all edge lengths in the 1-skeleton of L have length at least $\pi/2$, it suffices to consider triangles in L . The only geodesic triangles are those which do not bound a 2-face. There is one such triangle for every cyclically ordered triple of elements of S ; the vertices of this triangle are of the form (S_i, S_j) , (S_j, S_k) and (S_k, S_i) . Now the arclength between these vertices add up to 2π , because their respective vectors v_{ij} , v_{jk} and v_{ki} are coplanar: $v_{ij} + v_{jk} + v_{ki} = 0$.

We next compute the fundamental group of C_n . It is generated by simple loops in the 1-skeleton of C_n , and has relations given by the 2-cells. A simple

loop in the 1-skeleton is of the form $\{a, b\}$, neglecting singletons. We identify it with the generator R_{ab} , with the ordering $a < b$. The other generators R_{ba} are redundant, because of the relation $R_{ab}R_{ba} = 1$.

A typical 2-cell is either of the form $\{a, b\} \sqcup \{c, d\}$, in which case it gives the relation $R_{ab}R_{cd} = R_{cd}R_{ab}$, or of the form $\{a, b, c\}$, in which case it gives the relation $R_{ab}R_{ac}R_{bc} = R_{bc}R_{ac}R_{ab}$, again assuming the ordering $a < b < c$.

The other relations of \mathbf{Tr}_n , namely those of the form $R_{ab}R_{ac}R_{bc} = R_{bc}R_{ac}R_{ab}$ with a, b, c not in the order $a < b < c$, are cyclic permutations of the one in the standard ordering. \square

Remark. The quotient C_n of P_n may be constructed in two steps. First, let T_n be the space obtained from P_n by identify opposing faces $S_1 \sqcup S_2$ and $S_2 \sqcup S_1$. In terms of partitions, this corresponds to identifying an ordered partition with all of its cyclic permutations. There is a lattice of translations of \mathbb{A}_n , isomorphic to \mathbb{Z}^{n-1} , and spanned by all vectors $(-1, \dots, -1, n-1, -1, \dots, -1)$. It is easy to see that P_n is a fundamental domain for this lattice. Indeed two translates with non-trivial intersection are of the form $P_n + v$ and $P_n + v + |S| \sum_{s \notin S} x_s - (n - |S|) \sum_{s \in S} x_s$, for some $S \subset [n]$, and these two translates intersect in the $(n-2)$ -dimensional face $S \sqcup ([n] \setminus S)$.

There are natural copies of $P_{i_1} \times \dots \times P_{i_r}$ with $i_1 + \dots + i_r = n$ in T_n , for example a copy of $P_{n-1} \times P_1 \cong P_{n-1}$ spanned by all $S_1 \sqcup \dots \sqcup S_r \sqcup \{n\}$ for any face $S_1 \sqcup \dots \sqcup S_r$ of P_{n-1} . One may construct C_n by quotienting in T_n each of these $P_{i_1} \times \dots \times P_{i_r}$ into $C_{i_1} \times \dots \times C_{i_r}$.

These considerations are sufficient to describe the spaces C_n for small n : C_1 is a point; C_2 is a circle, the quotient of the line segment $[(1, 2), (2, 1)]$ by identification of its endpoints; and C_3 is the quotient of a 2-torus obtained by gluing two distinct points together. It is homotopic to the connected sum of a 2-torus and a circle.

6.3. The classifying space QC_n . Consider now the space QC_n constructed as follows. On the disjoint union $C_n \times \mathfrak{S}_n$, identify all faces $(S_1 \sqcup \dots \sqcup S_r, \sigma)$ and $(S_1 \sqcup \dots \sqcup S_r, \tau)$ precisely when

$$\text{for all } i \in \{1, \dots, r\} : \text{for all } x, y \in S_r : \sigma(x) < \sigma(y) \Leftrightarrow \tau(x) < \tau(y).$$

Theorem 6.2. QC_n is a classifying space for the group \mathbf{QTr}_n .

Proof. The proof that QC_n is locally a non-positively curved space is similar to the proof for C_n : the links in QC_n are covers of links in C_n . We omit the details.

A simple loop in the 1-skeleton of QC_n is of the form $(\{a, b\}, \sigma)$, neglecting singletons, where σ specifies an orientation of $\{a, b\}$. We identify it with the generator R_{ab} , with the ordering specified by $\sigma(a) < \sigma(b)$.

A typical 2-cell is either of the form $(\{a, b\} \sqcup \{c, d\}, \sigma)$, in which case it gives the relation $R_{ab}R_{cd} = R_{cd}R_{ab}$, in the order specified by $\sigma(a) < \sigma(b)$ and $\sigma(c) < \sigma(d)$, or is of the form $(\{a, b, c\}, \sigma)$, in which case it gives the relation $R_{ab}R_{ac}R_{bc} = R_{bc}R_{ac}R_{ab}$, again in the order $\sigma(a) < \sigma(b) < \sigma(c)$. \square

Now the fact that \mathbf{Tr}_n is a split quotient of \mathbf{QTr}_n acquires a geometric interpretation. Namely, C_n is both a quotient of QC_n , by projecting on the first coordinate, and a subspace of QC_n , embedded as $C_n \times \{1\}$.

6.4. Homology of C_n and QC_n . The permutohedron P_n is homeomorphic to a ball, and therefore has non-trivial homology only in dimension 0. Consider the chain complex spanned by all faces of the polyhedron. In the combinatorial model, the boundary operator is given by

$$(6.1) \quad \begin{aligned} \partial(S_1 \sqcup \cdots \sqcup S_r) &= \sum_{i=1}^r (-)^i S_1 \sqcup \cdots \sqcup \partial(S_i) \sqcup \cdots \sqcup S_r, \\ \partial(S) &= \sum_{S=S' \sqcup S''} (-)^? S' \sqcup S'', \end{aligned}$$

for appropriate signs in $\partial(S)$. We fix an orientation for the top-dimensional face in P_n . Each face $S_1 \sqcup \cdots \sqcup S_r$ of P_n is then a Cartesian product of translates of P_{S_i} , and we give that face the product orientation. The exact signs in $\partial(S)$ are not important, but we note that $S' \sqcup S''$ and $S'' \sqcup S'$ are translates of each other, and have opposite signs in $\partial(S)$.

Consider now the quotient C_n . It admits a natural chain complex, where the $(n-r)$ -dimensional complex is spanned by \mathfrak{S}_r -orbits of $(n-r)$ -dimensional faces, i.e. by unordered partitions of $[n]$ in r parts. The boundary operator on that complex is induced from the boundary on P_n , since by the choice above all faces in the same \mathfrak{S}_r -orbit have the same orientation.

Proposition 6.3. (i) *The boundary map on P_n induces the zero map on the complex of C_n ; in other words, the complex for C_n is minimal.*

(ii) *The homology group $H_r(\mathbf{Tr}_n; \mathbb{Z})$ is free of rank the number $\{ \binom{n}{n-r} \}$ of unordered partitions of n in $n-r$ parts⁵. The Hilbert-Poincaré series of $H_*(\mathbf{Tr}_n; \mathbb{Z})$ is the polynomial $P_{\mathbf{tr}_n}(t)$ of Theorem 3.1.*

Proof. The boundary map on any face of C_n is calculated from the boundary on top-dimensional faces, by formula (6.1). Now the top-dimensional face of P_n is mapped bijectively to C_n , while the faces of dimension one less are identified by pairs in C_n . Furthermore, the boundary operator assigns sign $+1$ to one and sign -1 to the other. Therefore all boundary operators are trivial on top-dimensional faces, and by extension on all faces.

The consequence for homology of \mathbf{Tr}_n follows immediately, since it is by definition the homology of a classifying space. \square

The same arguments hold for QC_n , namely:

Proposition 6.4. (i) *The boundary map on P_n induces a trivial map on the complex of QC_n ; in other words, the complex for QC_n is minimal.*

(ii) *The homology group $H_r(\mathbf{QTr}_n; \mathbb{Z})$ is free of rank the number of unordered partitions of n in r ordered parts. The Hilbert-Poincaré series of $H_*(\mathbf{QTr}_n; \mathbb{Z})$ is the polynomial $P_{\mathbf{qtr}_n}(t)$ of Theorem 3.1.*

⁵a.k.a. the “Stirling numbers of the second kind”

Proof. Since QC_n is a quotient of a disjoint union of copies of C_n , the induced boundary map is *a fortiori* trivial.

The r -faces of QC_n are given by the numbers $A_{n,n-r}$ of partitions of $[n]$ in $n-r$ parts, with an ordering on each of the parts. These numbers obviously satisfy the recursion

$$A_{n,p} = A_{n-1,p-1} + (n+p-1)A_{n-1,p},$$

since given a partition of $[n]$ in p parts one may remove n from the partition and obtain either a partition of $[n-1]$ in $p-1$ parts, or a partition of $[n-1]$ in p parts, where the number n appeared in any of $n+p-1$ positions. This recursion is also satisfied by the coefficients $\binom{n-1}{p} \frac{n!}{(n-p)!}$ appearing in Theorem 3.1, finishing the proof. \square

6.5. The cohomology rings of the groups \mathbf{Tr}_n , \mathbf{QTr}_n . We now relate the cohomology of the Lie algebras $(\mathfrak{q})\mathfrak{t}_n$ and the groups $(\mathbf{Q})\mathbf{Tr}_n$.

We have natural homomorphisms $\xi_n : A_n \rightarrow H^*(\mathbf{Tr}_n, \mathbb{Q})$ and $\eta_n : QA_n \rightarrow H^*(\mathbf{QTr}_n, \mathbb{Q})$, defined as follows. The generators a_{ij} of $H^*(\mathbf{Tr}_n, \mathbb{Q})$ and a_{ij}, b_{ij} of $H^*(\mathbf{QTr}_n, \mathbb{Q})$ are obtained by pulling back standard generators via the projections $\mathbf{Tr}_n \rightarrow \mathbf{Tr}_2$, $\mathbf{QTr}_n \rightarrow \mathbf{QTr}_2$ (note that the groups \mathbf{Tr}_2 , \mathbf{QTr}_2 are free in one, respectively two generators). The fact that these generators satisfy the relations of A_n, QA_n respectively follows from consideration of the projections $\mathbf{Tr}_n \rightarrow \mathbf{Tr}_3$, $\mathbf{QTr}_n \rightarrow \mathbf{QTr}_3$, and the structure of the cohomology rings of \mathbf{Tr}_3 , \mathbf{QTr}_3 , which is easy to determine by looking at the 2-dimensional complexes C_3, QC_3 . This defines the homomorphisms ξ_n, η_n .

Conjecture 6.5. (stated as Theorem 8.5 in the published version; now a theorem of P. Lee, [L]) ξ_n is an isomorphism.

Let us show that Conjecture 6.5 follows from Conjecture 2.3. Indeed, by the above results, ξ_n is a map between spaces of the same dimension. Thus it suffices to show that ξ_n is injective.

Let $\mathfrak{g}_n = \text{Lie } \mathbf{Tr}_n$. We have a natural map $\theta_n : H^*(\mathbf{Tr}_n, \mathbb{Q}) \rightarrow H_{cts}^*(\mathfrak{g}_n, \mathbb{Q})$ (the subscript *cts* means continuous cohomology). By Conjecture 2.3, we have $\text{gr } \mathfrak{g}_n = \mathfrak{g}_n^0 = \mathfrak{t}_n$. Thus, since the algebra $U(\mathfrak{g}_n^0)$ is Koszul, all the differentials of the spectral sequence computing the cohomology of $H_{cts}^*(\mathfrak{g}_n, \mathbb{Q})$ starting from $H^*(\mathfrak{g}_n^0, \mathbb{Q})$ are zero. Thus the injectivity of ξ_n will follow from the injectivity of the natural map $\xi_n^0 : A_n \rightarrow H^*(\mathfrak{g}_n^0, \mathbb{Q})$ (as $\theta_n \circ \xi_n$ is a deformation of ξ_n^0). But we know (by the results of Sections 3,4) that ξ_n^0 is an isomorphism, as desired.

The same argument shows that Conjecture 2.4 implies the following

Conjecture 6.6. (now a theorem of P. Lee, [L]) η_n is an isomorphism.

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