BC_n-symmetric polynomials

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February 6, 2004

Abstract

We consider two important families of BC_n-symmetric polynomials, namely Okounkov's interpolation polynomials and Koornwinder's orthogonal polynomials. We give a family of difference equations satisfied by the former, as well as generalizations of the branching rule and Pieri identity, leading to a number of multivariate q-analogues of classical hypergeometric transformations. For the latter, we give new proofs of Macdonald's conjectures, as well as new identities, including an inverse binomial formula and several branching rule and connection coefficient identities. We also derive families of ordinary symmetric functions that reduce to the interpolation and Koornwinder polynomials upon appropriate specialization. As an application, we consider a number of new integral conjectures associated to classical symmetric spaces.

Contents

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1 Introduction

In the theory of multivariate orthogonal polynomials, the family of Koornwinder polynomials [11] has special significance, reducing to the Askey-Wilson polynomials [1] in the univariate case, as well as to the Macdonald polynomials for the classical root systems in appropriate limits [26]. In the present work, we prove a number of new results on these polynomials, as well as giving new proofs of Macdonald's "conjectures" for these polynomials (originally proved in [27, 22]).

The proofs in the literature of the Macdonald conjectures involve heavy use of "double affine Hecke algebra" machinery, and are thus rather far removed from the standard treatment of the Askey-Wilson case. In contrast, we will be taking a more classical approach; just as Askey-Wilson polynomials are naturally represented and studied as hypergeometric polynomials, we take as our starting point the multivariate analogue of this representation, namely Okounkov's "binomial formula" [16]. To be precise, we define Koornwinder polynomials via the binomial formula (or, equivalently, via the "evaluation" and "duality" portions of the Macdonald conjectures, together called "evaluation symmetry" below), and show that the resulting polynomials are orthogonal with respect to both the Koornwinder [11] and multivariate q-Racah [30] inner products. In addition to the new results on Koornwinder polynomials that this approach allows us to prove, we will see in a follow-up paper [17] that the hypergeometric approach generalizes to the case of elliptic hypergeometric functions, including an elliptic analogue (biorthogonal abelian functions) of Koornwinder polynomials. (The elliptic case is also treated in [18], using a combination of difference and (contour) integral operators.)

The binomial formula expands Koornwinder polynomials in terms of Okounkov's BC_n-symmetric interpolation polynomials [16], which we must therefore study first. Our treatment of these polynomials is based on a certain family of commuting q-difference operators; the joint eigenfunctions $\bar{P}^{*(n)}_\lambda(;q,t,s)$ satisfy the "extra vanishing property"

$$\bar{P}^{*(n)}_\lambda(\mu;q,t,s) := \bar{P}^{*(n)}_\lambda(q^{\mu_i}t^{n-i}s;q,t,s) = 0 \quad \lambda \not\subset \mu, \tag{1.1}$$

and thus agree with Okounkov's polynomials, up to rescaling. The resulting difference equations lead to a new proof of the branching rule for interpolation polynomials; more generally, we obtain a "bulk" version of the branching rule (in which some of the variables are set to v, tv, t^2v,...). Analogously, we also obtain a bulk Pieri identity, containing both the known e-type Pieri identity and a new g-type Pieri identity as special cases. One important consequence of these bulk identities is a formula for connection coefficients between interpolation polynomials for different values of s.

One of the interpolation polynomials in the binomial formula shows up as a "binomial coefficient":

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} := \frac{\bar{P}^{*(n)}_\mu(\lambda;q,t,st^{1-n})}{\bar{P}^{*(n)}_\mu(\mu;q,t,st^{1-n})}. \tag{1.2}$$

If we evaluate the bulk Pieri identity at a partition, the resulting sum can be expressed in terms of binomial coefficients, and turns out to be a multivariate q-analogue of the Saalschütz summation formula for a ${}_3F_2$ hypergeometric series. The symmetries of this generalized q-Saalschütz formula lead to a formula for the inverse of the matrix of binomial coefficients, as well as the duality symmetry

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} = \begin{bmatrix}\lambda'\\ \mu'\end{bmatrix}_{t,q,1/\sqrt{qt}s}. \tag{1.3}$$

[1]

Furthermore, the formula is general enough that a number of arguments [7, Chapter 2] lift from the univariate case; in particular, we obtain a multivariate analogue of Watson's transformation between a very-well-poised, terminating ${}_8\phi_7$ and a balanced, terminating ${}_4\phi_3$.

As we mentioned, we take evaluation symmetry as the defining property of Koornwinder polynomials; this then immediately implies a version of Okounkov's binomial formula. Applying the difference equations to the interpolation polynomials in this formula gives a number of new identities, notably special cases of branching rules and connection coefficients. Moreover, we find that the corresponding difference operators act nicely on Koornwinder polynomials, and have nice adjoints with respect to the Koornwinder inner product; these facts combine to show that our Koornwinder polynomials are orthogonal with respect to the correct inner product, and thus the Koornwinder polynomials as usually defined satisfy evaluation symmetry. A different approach works in the q-Racah case; here, our generalized hypergeometric transformations can be used to directly lift the univariate proof [7, Section 7.2]. Other consequences of our theory include an inverse to the binomial formula (expanding an interpolation polynomial in Koornwinder polynomials), a connection coefficient formula for Koornwinder polynomials, and a generalization of the Nasrallah-Rahman integral representation of a very-well-poised ${}_8\phi_7$ [7, Section 6.3].

Most of the formulas alluded to above are expressed in terms of ordinary Macdonald polynomials; more precisely, in terms of principal specializations of skew Macdonald polynomials. This suggests that there may be further connections between interpolation polynomials and ordinary symmetric functions. Indeed, it turns out that there are two four-parameter families of symmetric functions that reduce to the three-parameter family of interpolation polynomials upon appropriate specialization. Similarly, via the binomial formula, one obtains two seven-parameter families that reduce to Koornwinder polynomials. In addition to some curious symmetries of these lifted Koornwinder polynomials, having no counterparts for integer n, we also obtain a refinement of the fact that the leading terms of interpolation and Koornwinder polynomials are Macdonald polynomials, namely that the BC_n-symmetric polynomials can be expressed in the triangular form

$$\sum_{\mu\subset\lambda} c_{\lambda\mu} P_\mu(x_1^{\pm 1}, x_2^{\pm 1}, \dots x_n^{\pm 1}; q, t). \tag{1.4}$$

We also consider the limit $n \to \infty$ of the n-variable Koornwinder polynomials.

As an application of our results, we consider a pair of "vanishing conjectures". These are analogues for Macdonald polynomials of the fact that the integral of a Schur function over the orthogonal (or symplectic) group vanishes unless the corresponding irreducible representation of $U(n)$ has a vector fixed by the orthogonal group. More precisely, we conjecture that the integral of $P_\lambda(;q,t)$ over a suitable Koornwinder weight vanishes unless all parts of λ are even; dually, the integral over a different Koornwinder weight vanishes unless all parts of λ' are even. In addition to the Schur case $q = t$, we can prove a several other special cases, most notably that

[1] A number of symmetries of Koornwinder polynomials have been called "duality" in the literature; in the present work, we reserve the term for the symmetry under conjugation of partitions, e.g., the action of Macdonald's involution on symmetric functions.

the first conjecture holds if either $\lambda_1 \le 4$ or $\ell(\lambda) \le 1$. The latter case involves a quadratic transformation of basic hypergeometric series, and thus our conjectures can be interpreted as multivariate analogues of quadratic transformations. We also give a number of analogous conjectures related to other classical symmetric spaces, obtained by various *ad hoc* methods. (In fact, most of the conjectures of section 8 have since been proved; see [19]. More precisely, that paper proves Conjectures 2, 4, and 7 using (single) affine Hecke algebra techniques; Conjectures 1 and 6 are not amenable to that approach, but are equivalent to Conjectures 2 and 7 as shown below.)

The paper is organized as follows. Section 2 introduces the notations for partitions and (ordinary) Macdonald polynomials we will need below (for basic hypergeometric series, see [7]), as well as proving a few transformation results for principal specializations of skew Macdonald polynomials. We then introduce the interpolation polynomials in Section 3, proving their main properties; then Section 4 treats the corresponding hypergeometric transformations. Section 5 introduces Koornwinder polynomials and proves the main results discussed above (in particular, the Macdonald conjectures). Sections 6 and 7 then consider the corresponding families of symmetric functions, with special attention to the case $n \to \infty$ of lifted Koornwinder polynomials. Finally, Section 8 states the various vanishing conjectures, and proves the known special cases.

Acknowledgements

Throughout this work, Peter Forrester has been a very helpful sounding board and guide to the literature; many thanks are thus due. Appreciation is also due to the organizers of the conference on Applications of Macdonald Polynomials at the Newton Institute in April 2001, which is where the idea of looking for vanishing integrals like those of Section 8 first arose.

2 Notation

For partitions and Macdonald polynomials, we use the notations of [12], Chapters I and VI, supplemented as follows. First, we will denote the number of parts of λ by $\ell(\lambda)$ (being somewhat clearer than λ'_1). Next, we extend the dominance partial order $\le$ to partitions not of the same size in the obvious way; note that conjugation is no longer order-reversing ($311 \ge 22$, even though both partitions are self-conjugate). We also define relations $\prec$ and $\succ$ such that $\kappa \prec \lambda$ (equivalently $\lambda \succ \kappa$) for two partitions iff λ/κ is a vertical strip; that is, $\kappa_i \le \lambda_i \le \kappa_i + 1$ for all i. We also introduce some transformations of partitions. For a partition λ, we define partitions 2λ and λ^2 by $(2\lambda)_i = 2\lambda_i$, $(\lambda^2)_i = \lambda_{\lceil i/2 \rceil}$. Finally, if $\ell(\lambda) \le n$, $m^n + \lambda$ denotes the partition such that $(m^n + \lambda)_i = m + \lambda_i$; if also $\lambda_1 \le m$, $m^n - \lambda$ denotes the partition such that $(m^n - \lambda)_i = m - \lambda_{n+1-i}$. Note that $(m^n - \lambda)' = n^m - \lambda'$.

The ring Λ of symmetric functions will, unless otherwise noted, have coefficients in $\mathbb{F} = \mathbb{Q}(q,t)$; we also consider the completion $\hat{\Lambda}$ with respect to the natural grading (i.e., a sequence $f_k \to 0$ iff the degree of the lowest degree term of f_k tends to ∞). There are thus three natural types of basis: homogeneous bases, inhomogeneous bases of Λ, and inhomogeneous bases of Λ'. In the latter two cases, we require that the leading terms (the nonzero homogeneous components of largest/smallest degree) form a homogeneous basis. If the polynomials f_λ form a basis, we write $[f_\lambda]g$ for the coefficient of f_λ in the expansion of g in terms of that basis; this notation is mildly abusive, but the specific basis meant should always be clear. A similar notation applies to BC_n-symmetric polynomials. We will also need a notation for plethystic substitution in symmetric functions; if c_k is a sequence of elements of some $\mathbb{F}$-algebra and $f \in \Lambda$, then $f([c_k])$ denotes the image of f under the homomorphism $p_k \mapsto c_k$.

The formulas in the sequel will frequently involve products of the form

$$\prod_{(i,j)\in\lambda} f(i,j), \tag{2.1}$$

where $(i,j) \in \lambda$ means that $1 \le i$ and $1 \le j \le \lambda'_i$. We will also have frequent recourse to q-symbols,

$$(a;q)_n := \prod_{0\le i<n} (1-aq^i). \tag{2.2}$$

If n is omitted, the product is over all $i \ge 0$; also, $(x_1, x_2, \dots x_m; q)_n := \prod_{1\le i\le m}(x_i;q)_n$. (A number of useful q-symbol identities are given in [7, Appendix I].) The following identity will be useful:

Lemma 2.1. *Let a, b, c, d, e be arbitrary quantities. Then*

$$\prod_{(i,j)\in\lambda} (1-a^{\lambda_i} b^j c^{\lambda'_j} d^i e) = \prod_{1\le i\le j\le l} (a^{\lambda_i} b^{\lambda_{j+1}+1} c^j d^i e; b)_{\lambda_j-\lambda_{j+1}} \tag{2.3}$$

$$= \prod_{1\le i\le j\le l} (a^{\lambda_i} b^{\lambda_j} c^j d^i e; b^{-1})_{\lambda_j-\lambda_{j+1}}, \tag{2.4}$$

for any $l \ge \ell(\lambda)$. If $|b|<1$, then

$$\prod_{(i,j)\in\lambda} (1-a^{\lambda_i} b^j c^{\lambda'_j} d^i e) = \prod_{1\le i\le l} \frac{(a^{\lambda_i} b c^l d^i e; b)}{((ab)^{\lambda_i} b (cd)^i e; b)} \prod_{1\le i<j\le l} \frac{(a^{\lambda_i} b^{\lambda_j+1} c^{j-1} d^i e; b)}{(a^{\lambda_i} b^{\lambda_j+1} c^j d^i e; b)}, \tag{2.5}$$

while if $|b|>1$, then

$$\prod_{(i,j)\in\lambda} (1-a^{\lambda_i} b^j c^{\lambda'_j} d^i e) = \prod_{1\le i\le l} \frac{((ab)^{\lambda_i} (cd)^i e; b^{-1})}{(a^{\lambda_i} c^l d^i e; b^{-1})} \prod_{1\le i<j\le l} \frac{(a^{\lambda_i} b^{\lambda_j} c^j d^i e; b^{-1})}{(a^{\lambda_i} b^{\lambda_j} c^{j-1} d^i e; b^{-1})} \tag{2.6}$$

In particular,

$$\prod_{(i,j)\in\lambda} (1-b^j d^i e) = \prod_{1\le i\le l} (bd^i e; b)_{\lambda_i} \tag{2.7}$$

$$= \prod_{1\le i\le l} (b^{\lambda_i} d^i e; b^{-1})_{\lambda_i}. \tag{2.8}$$

Proof. We find

$$\prod_{(i,j)\in\lambda} (1-a^{\lambda_i} b^j c^{\lambda'_j} d^i e) = \prod_{1\le i\le l} \prod_{1\le j\le\lambda_i} (1-a^{\lambda_i} b^j c^{\lambda'_j} d^i e) \tag{2.9}$$

$$= \prod_{1\le i\le k\le l} \prod_{\substack{1\le j\le\lambda_i\\ \lambda'_j=k}} (1-a^{\lambda_i} b^j c^{\lambda'_j} d^i e) \tag{2.10}$$

$$= \prod_{1\le i\le k\le l} \prod_{\lambda_{k+1}<j\le \lambda_k} (1 - a^{\lambda_i} b^j c^{\lambda'_j} d^i e) \tag{2.11}$$

$$= \prod_{1\le i\le k\le l} (a^{\lambda_i} b^{\lambda_k} c^k d^i e; b^{-1})_{\lambda_k-\lambda_{k+1}} \tag{2.12}$$

$$= \prod_{1\le i\le k\le l} (a^{\lambda_i} b^{\lambda_{k+1}+1} c^k d^i e; b)_{\lambda_k-\lambda_{k+1}}. \tag{2.13}$$

If $|b| < 1$, then

$$\prod_{1\le i\le j\le l} (a^{\lambda_i} b^{\lambda_{j+1}+1} c^j d^i e; b)_{\lambda_j-\lambda_{j+1}} = \prod_{1\le i\le j\le l} \frac{(a^{\lambda_i} b^{\lambda_{j+1}+1} c^j d^i e; b)}{(a^{\lambda_i} b^{\lambda_j+1} c^j d^i e; b)} \tag{2.14}$$

$$= \prod_{1\le i\le l} \frac{(a^{\lambda_i} bc^l d^i e; b)}{(a^{\lambda_i} b^{\lambda_i+1} (cd)^i e; b)} \frac{\prod_{1\le i\le j<l} (a^{\lambda_i} b^{\lambda_{j+1}+1} c^j d^i e; b)}{\prod_{1\le i<j\le l} (a^{\lambda_i} b^{\lambda_j+1} c^j d^i e; b)} \tag{2.15}$$

$$= \prod_{1\le i\le l} \frac{(a^{\lambda_i} bc^l d^i e; b)}{(a^{\lambda_i} b^{\lambda_i+1} (cd)^i e; b)} \prod_{1\le i<j\le l} \frac{\prod_{1\le i<j\le l} (a^{\lambda_i} b^{\lambda_j+1} c^{j-1} d^i e; b)}{\prod_{1\le i<j\le l} (a^{\lambda_i} b^{\lambda_j+1} c^j d^i e; b)}, \tag{2.16}$$

and similarly if $|b| > 1$. □

Remark. This is essentially the argument used in section VI.6 of Macdonald for the special case $a = q$, $b = 1/q$, $c = t$, $d = 1/t$.

Three special cases are of particular importance; we define

$$C^+_\lambda(x; q, t) := \prod_{(i,j)\in\lambda} (1 - q^{\lambda_i+j-1} t^{2-\lambda'_j-i} x) \tag{2.17}$$

$$= \prod_{1\le i\le l} \frac{(q^{\lambda_i} t^{2-l-i} x; q)}{(q^{2\lambda_i} t^{2-2i} x; q)} \prod_{1\le i<j\le l} \frac{(q^{\lambda_i+\lambda_j} t^{3-i-j} x; q)}{(q^{\lambda_i+\lambda_j} t^{2-i-j} x; q)}, \tag{2.18}$$

$$C^-_\lambda(x; q, t) := \prod_{(i,j)\in\lambda} (1 - q^{\lambda_i-j} t^{\lambda'_j-i} x) \tag{2.19}$$

$$= \prod_{1\le i\le l} \frac{(x; q)}{(q^{\lambda_i} t^{l-i} x; q)} \prod_{1\le i<j\le l} \frac{(q^{\lambda_i-\lambda_j} t^{j-i} x; q)}{(q^{\lambda_i-\lambda_j} t^{j-i-1} x; q)}, \tag{2.20}$$

$$C^0_\lambda(x; q, t) := \prod_{(i,j)\in\lambda} (1 - q^{j-1} t^{1-i} x) \tag{2.21}$$

$$= \prod_{1\le i\le l} (t^{1-i} x; q)_{\lambda_i}. \tag{2.22}$$

Thus, for instance, we have the following expressions for standard Macdonald polynomial quantities in this notation.

$$b_\lambda(q, t) = \frac{C^-_\lambda(t; q, t)}{C^-_\lambda(q; q, t)} \tag{2.23}$$

$$P_\lambda(\left[\frac{1-u^k}{1-t^k}\right]; q, t) = \frac{t^{n(\lambda)} C^0_\lambda(u; q, t)}{C^-_\lambda(t; q, t)} \tag{2.24}$$

$$\langle P_\lambda, P_\lambda \rangle''_n = \frac{C^0_\lambda(t^n; q, t) C^-_\lambda(q; q, t)}{C^0_\lambda(q t^{n-1}; q, t) C^-_\lambda(t; q, t)}. \tag{2.25}$$

Again, by convention, multiple arguments before the semicolon indicate a product; e.g.,

$$C^+_\lambda(x, y; q, t) = C^+_\lambda(x; q, t)C^+_\lambda(y; q, t). \tag{2.26}$$

We list some useful transformations of these quantities.

Lemma 2.2. *For any partition λ,*

$$C^+_\lambda(1/x; 1/q, 1/t) = (-qx)^{-|\lambda|}t^{3n(\lambda)}q^{-3n(\lambda')}C^+_\lambda(x; q, t) \tag{2.27}$$

$$C^-_\lambda(1/x; 1/q, 1/t) = (-1/x)^{|\lambda|}t^{-n(\lambda)}q^{-n(\lambda')}C^-_\lambda(x; q, t) \tag{2.28}$$

$$C^0_\lambda(1/x; 1/q, 1/t) = (-1/x)^{|\lambda|}t^{n(\lambda)}q^{-n(\lambda')}C^0_\lambda(x; q, t) \tag{2.29}$$

Lemma 2.3. *For any partition λ,*

$$C^+_{\lambda'}(x; q, t) = C^+_\lambda(qtx; 1/t, 1/q) \tag{2.30}$$

$$C^-_{\lambda'}(x; q, t) = C^-_\lambda(x; t, q) \tag{2.31}$$

$$C^0_{\lambda'}(x; q, t) = C^0_\lambda(x; 1/t, 1/q) \tag{2.32}$$

Lemma 2.4. *For any integers $m, n \geq 0$ and partition λ with $\ell(\lambda) \leq n$,*

$$\frac{C^+_{m^n+\lambda}(x; q, t)}{C^+_{m^n}(x; q, t)} = \frac{C^0_\lambda(q^{2m}t^{1-n}x; q, t)C^+_\lambda(q^{2m}x; q, t)}{C^0_\lambda(q^m t^{1-n}x; q, t)} \tag{2.33}$$

$$\frac{C^-_{m^n+\lambda}(x; q, t)}{C^-_{m^n}(x; q, t)} = \frac{C^0_\lambda(q^m t^{n-1}x; q, t)C^-_\lambda(x; q, t)}{C^0_\lambda(t^{n-1}x; q, t)} \tag{2.34}$$

$$\frac{C^0_{m^n+\lambda}(x; q, t)}{C^0_{m^n}(x; q, t)} = C^0_\lambda(q^m x; q, t). \tag{2.35}$$

Lemma 2.5. *For any integers $m, n \geq 0$ and partition $\lambda \subset m^n$,*

$$\frac{C^+_{m^n-\lambda}(x; q, t)}{C^+_{m^n}(x; q, t)} = \frac{C^+_\lambda(q^{2m-1}t^{-2n+3}x; 1/q, 1/t)C^0_\lambda(q^{m-1}t^{2-2n}x, q^{2m-1}t^{2-n}x; 1/q, 1/t)}{C^0_{2\lambda^2}(q^{2m-1}t^{2-2n}x; 1/q, 1/t)} \tag{2.36}$$

$$\frac{C^-_{m^n-\lambda}(x; q, t)}{C^-_{m^n}(x; q, t)} = \frac{C^-_\lambda(x; q, t)}{C^0_\lambda(t^{n-1}x; q, t)C^0_\lambda(q^{m-1}x; 1/q, 1/t)} \tag{2.37}$$

$$\frac{C^0_{m^n-\lambda}(x; q, t)}{C^0_{m^n}(x; q, t)} = \frac{1}{C^0_\lambda(q^{m-1}t^{1-n}x; 1/q, 1/t)} \tag{2.38}$$

Lemma 2.6. *For any partition λ,*

$$C^+_{2\lambda}(x; q, t) = C^+_\lambda(x, qx; q^2, t) \tag{2.39}$$

$$C^-_{2\lambda}(x; q, t) = C^-_\lambda(x, xq; q^2, t) \tag{2.40}$$

$$C^0_{2\lambda}(x; q, t) = C^0_\lambda(x, xq; q^2, t) \tag{2.41}$$

$$C^+_{\lambda^2}(x; q, t) = C^+_\lambda(x/t, x/t^2; q, t^2) \tag{2.42}$$

$$C^-_{\lambda^2}(x; q, t) = C^-_\lambda(x, xt; q, t^2) \tag{2.43}$$

$$C^0_{\lambda^2}(x; q, t) = C^0_\lambda(x, x/t; q, t^2) \tag{2.44}$$

We will also need analogous results for principal specializations of skew Macdonald polynomials.

Lemma 2.7. *For any partitions $\kappa \subset \lambda$,*

$$P_{\lambda/\kappa}(\left[\frac{1-u^{-k}}{1-t^{-k}}\right];1/q,1/t) = (t/u)^{|\lambda/\kappa|} P_{\lambda/\kappa}(\left[\frac{1-u^k}{1-t^k}\right];q,t) \tag{2.45}$$

$$P_{\lambda'/\kappa'}(\left[\frac{1-u^k}{1-q^k}\right];t,q) = (-u)^{|\lambda/\kappa|}\frac{b_\lambda(q,t)}{b_\kappa(q,t)} P_{\lambda/\kappa}(\left[\frac{1-u^{-k}}{1-t^k}\right];q,t). \tag{2.46}$$

Let $m,n \geq 0$ be integers. If $\ell(\lambda) \leq n$, then

$$P_{(m^n+\lambda)/(m^n+\kappa)}(\left[\frac{1-u^k}{1-t^k}\right];q,t) = \frac{C^0_\lambda(q^{m+1}t^{n-1},t^n;q,t)C^0_\kappa(q^m t^n, qt^{n-1};q,t)}{C^0_\kappa(q^{m+1}t^{n-1},t^n;q,t)C^0_\lambda(q^m t^n, qt^{n-1};q,t)} P_{\lambda/\kappa}(\left[\frac{1-u^k}{1-t^k}\right];q,t). \tag{2.47}$$

If also $\lambda_1 \leq m$, then

$$P_{(m^n-\kappa)/(m^n-\lambda)}(\left[\frac{1-u^k}{1-t^k}\right];q,t) = (q/t)^{|\lambda/\kappa|}\frac{C^0_\lambda(q^{-m};q,t)C^0_\kappa(q^{1-m}/t;q,t)b_\lambda(q,t)}{C^0_\kappa(q^{-m};q,t)C^0_\lambda(q^{1-m}/t;q,t)b_\kappa(q,t)} P_{\lambda/\kappa}(\left[\frac{1-u^k}{1-t^k}\right];q,t). \tag{2.48}$$

Proof. The first two transformations are straightforward. For the third transformation, we claim that in fact

$$P_{(m^n+\lambda)/(m^n+\kappa)}(;q,t) = \frac{C^0_\lambda(q^{m+1}t^{n-1},t^n;q,t)C^0_\kappa(q^m t^n, qt^{n-1};q,t)}{C^0_\kappa(q^{m+1}t^{n-1},t^n;q,t)C^0_\lambda(q^m t^n, qt^{n-1};q,t)} P_{\lambda/\kappa}(;q,t) \tag{2.49}$$

It suffices to compare coefficients of $P_\mu(;q,t)$ for $\ell(\mu) \leq n$. We have

$$[P_\mu(;q,t)]P_{(m^n+\lambda)/(m^n+\kappa)}(;q,t) = [Q_{m^n+\lambda}(;q,t)](Q_\mu(;q,t)Q_{m^n+\kappa}(;q,t)) \tag{2.50}$$

$$= \frac{\langle P_{m^n+\lambda}(;q,t), Q_\mu(;q,t)Q_{m^n+\kappa}(;q,t)\rangle''_n}{\langle P_{m^n+\lambda}(;q,t), Q_{m^n+\lambda}(;q,t)\rangle''_n} \tag{2.51}$$

$$= \left(\frac{C^0_\lambda(q^{m+1}t^{n-1},t^n;q,t)C^0_\kappa(q^m t^n, qt^{n-1};q,t)}{C^0_\kappa(q^{m+1}t^{n-1},t^n;q,t)C^0_\lambda(q^m t^n, qt^{n-1};q,t)}\right)\frac{\langle P_\lambda(;q,t), Q_\mu(;q,t)Q_\kappa(;q,t)\rangle''_n}{\langle P_\lambda(;q,t), Q_\lambda(;q,t)\rangle''_n} \tag{2.52}$$

$$= \frac{C^0_\lambda(q^{m+1}t^{n-1},t^n;q,t)C^0_\kappa(q^m t^n, qt^{n-1};q,t)}{C^0_\kappa(q^{m+1}t^{n-1},t^n;q,t)C^0_\lambda(q^m t^n, qt^{n-1};q,t)}[P_\mu(;q,t)]P_{\lambda/\kappa}(;q,t). \tag{2.53}$$

Similarly,

$$P_{(m^n-\kappa)/(m^n-\lambda)}(;q,t) = (q/t)^{|\lambda/\kappa|}\frac{C^0_\lambda(q^{-m};q,t)C^0_\kappa(q^{1-m}/t;q,t)b_\lambda(q,t)}{C^0_\kappa(q^{-m};q,t)C^0_\lambda(q^{1-m}/t;q,t)b_\kappa(q,t)} P_{\lambda/\kappa}(;q,t), \tag{2.54}$$

and thus the fourth claim follows. □

Corollary 2.8. *For any integers $m,n \geq 0$ and partition λ with $\ell(\lambda) \leq n$, $\lambda_n \geq m$,*

$$P_{\lambda/m^n}([(1-u^k)/(1-t^k)];q,t) = \lim_{Q\to q^m}\frac{C^0_{m^n}(t^n, qt^{n-1}/Q;q,t)C^0_\lambda(qt^{n-1},u/Q;q,t)}{C^0_{m^n}(qt^{n-1},u/Q;q,t)C^0_\lambda(qt^{n-1}/Q;q,t)C^-_\lambda(t;q,t)} \tag{2.55}$$

Similarly, if $\lambda \subset m^n$,

$$\frac{P_{m^n/\lambda}([(1-u^k)/(1-t^k)];q,t)}{P_{m^n}([(1-u^k)/(1-t^k)];q,t)} = \frac{t^{n(\lambda)}(q/u)^{|\lambda|}C^0_\lambda(t^n,q^{-m};q,t)}{C^-_\lambda(q;q,t)C^0_\lambda(t^{n-1}q^{1-m}/u;q,t)} \tag{2.56}$$

3 Interpolation polynomials

We define a (q-)difference operator $D^{(n)}(u_1,u_2;q,t)$ acting on $\mathbb{F}[x_i^{\pm 1}]^{BC_n}$, as follows:

Definition 1. The operator $D^{(n)}(u_1,u_2;q,t)$ acts by:

$$(D^{(n)}(u_1,u_2;q,t)f)(x_1,x_2,\dots x_n) = \sum_{\sigma\in\{\pm 1\}^n}\prod_{1\le i\le n}\frac{(1-u_1x_i^{\sigma_i})(1-u_2x_i^{\sigma_i})}{(1-x_i^{2\sigma_i})}\prod_{1\le i<j\le n}\frac{(1-tx_i^{\sigma_i}x_j^{\sigma_j})}{(1-x_i^{\sigma_i}x_j^{\sigma_j})}f(x_1q^{\sigma_1/2},x_2q^{\sigma_2/2}\dots x_nq^{\sigma_n/2}) \tag{3.1}$$

Remark. This is one of the difference operators associated to the BC/C Macdonald polynomials; in particular, for each u_1, u_2, the eigenfunctions of $D^{(n)}(u_1,u_2;q,t)$ are the Koornwinder polynomials

$$K_\mu^{(n)}(;q,t;u_1,u_1\sqrt{q},u_2,u_2\sqrt{q}). \tag{3.2}$$

See also Lemma 5.6 below.

Now, consider the ring $\mathbb{F}[s,1/s][x_i^{\pm 1}]^{BC_n}$, with basis of the form $s^k m_\lambda(x)$ for $k\in\mathbb{Z}$ and λ a partition (where $m_\lambda(x)$ is the orbit sum of $\prod_i x_i^{\lambda_i}$); we extend the dominance ordering to such "monomials" by taking

$$(k,\lambda)\ge(l,\mu) \tag{3.3}$$

when $\lambda\ge\mu$, $|l-k|\le|\lambda|-|\mu|$. Now, define a difference operator $D_s^{(n)}(u;q,t)$ by:

$$(D_s^{(n)}(u;q,t)f)(x_1,x_2,\dots x_n;s) = D^{(n)}(s,u/s;q,t)f(x_1,x_2,\dots x_n;s\sqrt{q}), \tag{3.4}$$

thus acting on s in addition to the x variables.

Lemma 3.1. *The operator $D_s^{(n)}(u;q,t)$ gives a well-defined operator on $\mathbb{F}[s,1/s][x_i^{\pm 1}]^{BC_n}$, and acts on monomials triangularly with respect to the dominance ordering. In particular,*

$$D_s^{(n)}(u;q,t)s^k m_\lambda = q^{k/2}E_\lambda^{(n)}(u;q,t)s^k m_\lambda + \textit{dominated terms}, \tag{3.5}$$

with

$$E_\lambda^{(n)}(u;q,t) = q^{-|\lambda|/2}\prod_{1\le i\le n}(1-q^{\lambda_i}t^{n-i}u). \tag{3.6}$$

Proof. Since

$$D_s^{(n)}(u;q,t)s^k f = q^{k/2}s^k D_s^{(n)}(u;q,t)f, \tag{3.7}$$

it suffices to consider the case $k=0$. If we multiply $D_s^{(n)}(u;q,t)m_\lambda$ by the BC_n-antisymmetric product

$$\prod_{1\le i\le n}(x_i-1/x_i)\prod_{1\le i<j\le n}(x_i+1/x_i-x_j-1/x_j), \tag{3.8}$$

the result is manifestly a polynomial; moreover, we can use the symmetry of m_λ to write it as

$$\sum_{\sigma\in\{\pm 1\}^n}\Big(\prod_{1\le i\le n}\sigma_i R_{x_i}(\sigma_i)\Big)F(u;x_1,x_2\dots x_n;s)m_\lambda(\sqrt{q}x_1,\sqrt{q}x_2,\dots\sqrt{q}x_n;\sqrt{q}s), \tag{3.9}$$

where $R_{x_i}(\pm 1)$ are the homomorphisms such that $R_{x_i}(\pm 1)x_j = x_j$, $R_{x_i}(\pm 1)x_i = x_i^{\pm 1}$, and where F is a Laurent polynomial in the ring

$$\mathbb{F}[u, s, 1/s][x_i^{\pm 1}]^{S_n}. \tag{3.10}$$

Moreover, the monomials of F are all dominated by the monomial

$$x_1^n x_2^{n-1} \dots x_n + \cdots, \tag{3.11}$$

and thus the monomials of Fm_λ are dominated by

$$\prod_i x_i^{n+1-i+\lambda_i} + \cdots. \tag{3.12}$$

Since the operators $\prod_i R_{x_i}(\sigma_i)$ form a normal subgroup of BC_n, the resulting sum is BC_n-antisymmetric, and we may thus divide back out the factor (3.8), to obtain a BC_n-symmetric polynomial dominated by m_λ.

For the leading coefficient, we compute:

$$[\prod_i x_i^{\sigma_i(n+1-i)}]F(u; x_1, x_2, \dots x_n; s) = (-1)^n \prod_{1\le i\le n:\sigma_i=1} ut^{n-i}, \tag{3.13}$$

and thus

$$[\prod_i x_i^{\lambda_i+n+1-i}] \sum_{\sigma\in\{\pm 1\}^n} \Bigl(\prod_{1\le i\le n} \sigma_i R_{x_i}(\sigma_i)\Bigr) Fm_\lambda = \sum_{\sigma\in\{\pm 1\}^n} (-1)^n \Bigl(\prod_{1\le i\le n} \sigma_i q^{\sigma_i\lambda_i/2}\Bigr) \prod_{1\le i\le n:\sigma_i=1} ut^{n-i} \tag{3.14}$$

$$= q^{-|\lambda|/2} \prod_{1\le i\le n} (1 - q^{\lambda_i} ut^{n-i}), \tag{3.15}$$

as required. ∎

Note that for generic u, the diagonal coefficients are all distinct, and thus for each (k, λ), there is a unique eigenfunction of $D_s^{(n)}(u; q, t)$ of the form $s^k m_\lambda +$ dominated terms; clearly multiplying such an eigenfunction by s^j yields another eigenfunction. It turns out that these eigenfunctions are independent of u, and are essentially just the BC-type interpolation polynomials of Okounkov [16]. Given a polynomial $p \in \mathbb{F}(s)[x_i^{\pm 1}]^{BC_n}$, we define $p(\mu; s) := p(q^{\mu_i} t^{n-i} s; s)$.

Theorem 3.2. *The following three facts hold for all partitions λ.*

a. The operators $D_s^{(n)}(u; q, t)$ for different u commute on the monomials dominated by m_λ, and thus have a common eigenfunction $s^k \bar{P}_\mu^{*(n)}(; s)$ for each leading monomial $s^k m_\mu$ dominated by m_λ.

b. For any partition μ, $\bar{P}_\lambda^{*(n)}(\mu; s) = 0$ unless $\lambda \subset \mu$.

c. $\bar{P}_\lambda^{*(n)}(\lambda; s) \neq 0$.

Proof. First, fix a partition λ, and suppose that (a) holds for λ. Then we claim that (b) and (c) hold as well. Indeed, let μ be any partition different from λ; in particular, fix l such that $\mu_l \neq \lambda_l$. By inspection of the eigenvalues, the operator

$$D_s^{(n)}(t^{l-n} q^{-\lambda_l}; q, t) \tag{3.16}$$

annihilates $\bar{P}^{*(n)}_\lambda$; on the other hand, we compute

$$(D^{(n)}_s(t^{l-n}q^{-\lambda_l};q,t)\bar{P}^{*(n)}_\lambda)(\mu;s/\sqrt{q}) = \sum_{\nu\prec\mu} C_{\mu/\nu}\bar{P}^{*(n)}_\lambda(\nu;s), \tag{3.17}$$

for some coefficients $C_{\mu/\nu}$; the terms not corresponding to partitions vanish, since then either $\sigma_n=-1$, $\nu_n=0$, and thus $(1-s/x_n)=0$, or $\sigma_i=-1$, $\sigma_{i+1}=1$, $\nu_i=\nu_{i+1}$, and thus $(1-tx_{i+1}/x_i)=0$. Moreover, we compute

$$C_{\mu/\mu} = \prod_{1\le i\le n}\frac{(1-q^{\mu_i-1}t^{n-i}s^2)(1-q^{\mu_i-\lambda_l}t^{l-i})}{1-q^{2\mu_i-1}t^{2n-2i}s^2}\prod_{1\le i<j\le n}\frac{1-q^{\mu_i+\mu_j-1}t^{2n-i-j+1}s^2}{1-q^{\mu_i+\mu_j-1}t^{2n-i-j}s^2}\ne 0 \tag{3.18}$$

In other words, we can expand $\bar{P}^{*(n)}_\lambda(\mu;s)$ in terms of the values $\bar{P}^{*(n)}_\lambda(\nu;s)$ for $\nu\subsetneq\mu$. By induction, we find that $\bar{P}^{*(n)}_\lambda(\mu;s)=0$ whenever μ does not contain λ, thus proving (b). Furthermore, if $\bar{P}^{*(n)}_\lambda(\lambda;s)=0$, the induction would then prove $\bar{P}^{*(n)}_\lambda(\mu;s)=0$ for all μ, impossible since $\bar{P}^{*(n)}_\lambda\ne 0$; we thus have (c) as well.

Now, suppose that (a) (and thus (b) and (c)) holds for all $\mu<\lambda$. In particular, it follows that the operators $D^{(n)}_s(u;q,t)$ commute on the space of polynomials *strictly* dominated by m_λ, and it will thus suffice to show that they commute on m_λ. Let $f(x;s)$ be the unique BC_n-symmetric polynomial with coefficients in $\mathbb{F}(s)$ of the form

$$f(x;s) = m_\lambda(x) + \sum_{\mu<\lambda} c_{\lambda\mu}(s)\bar{P}^{*(n)}_\mu(x;s) \tag{3.19}$$

such that $f(\mu;s)=0$ for $\mu<\lambda$; the resulting equations for $c_{\lambda\mu}(s)$ are triangular with nonzero diagonal, by claims (b) and (c).

Extend the action of $D^{(n)}_s(u;q,t)$ to polynomials with coefficients in $k(s)$ in the obvious way. The same inner induction as before proves

$$(D^{(n)}_s(u;q,t)f)(\mu;s) = 0 \tag{3.20}$$

whenever $\mu<\lambda$; it follows that $D^{(n)}_s(u;q,t)f$ is proportional to f, and by comparing leading monomials, that:

$$D^{(n)}_s(u;q,t)f = E^{(n)}_\lambda(u;q,t)f. \tag{3.21}$$

We conclude that

$$D^{(n)}_s(u;q,t)D^{(n)}_s(v;q,t)m_\lambda = D^{(n)}_s(u;q,t)D^{(n)}_s(v;q,t)(f-\sum_{\mu<\lambda}c_{\lambda\mu}(s)\bar{P}^{*(n)}_\mu) \tag{3.22}$$

$$= E^{(n)}_\lambda(u;q,t)E^{(n)}_\lambda(v;q,t)f - \sum_{\mu<\lambda}c_{\lambda\mu}(qs)E^{(n)}_\mu(u;q,t)E^{(n)}_\mu(v;q,t)\bar{P}^{*(n)}_\mu \tag{3.23}$$

$$= D^{(n)}_s(v;q,t)D^{(n)}_s(u;q,t)m_\lambda, \tag{3.24}$$

as required. □

In the sequel, we will write the common eigenfunctions of $D^{(n)}_s(u;q,t)$ as $\bar{P}^{*(n)}_\lambda(;q,t,s)$, and refer to them as interpolation polynomials.

Remark. In particular, it follows that these polynomials agree up to a factor in $\mathbb{F}(s)$ and a shifting of the arguments with the interpolation polynomials of [16]. It was shown there that the interpolation polynomials are not common eigenfunctions of any rational difference operators in $x_1\dots x_n$; the loophole, of course, is that our operators also act on s.

Remark. The interpolation polynomials are, naturally, independent of the choice of square root of q used to define $D_s^{(n)}(u;q,t)$; the proof shows them to be characterized by the triangularity and vanishing properties, neither of which depends on that choice.

Corollary 3.3. *The operators $D^{(n)}(u_1,u_2;q,t)$ satisfy the quasi-commutation relation*

$$D^{(n)}(u_1,\sqrt{q}u_2;q,t)D^{(n)}(\sqrt{q}u_1,u_3;q,t)=D^{(n)}(u_1,\sqrt{q}u_3;q,t)D^{(n)}(\sqrt{q}u_1,u_2;q,t) \tag{3.25}$$

Proof. For any function $f\in\mathbb{F}(s)[x_i^{\pm 1}]^{BC_n}$, we have:

$$(D_s^{(n)}(u;q,t)D_s^{(n)}(v;q,t)f)(x_1,x_2,\dots x_n;s)=(D^{(n)}(s,u/s;q,t)D^{(n)}(s\sqrt{q},\frac{v}{s\sqrt{q}};q,t)f)(x_1,x_2,\dots x_n;sq); \tag{3.26}$$

as this must be symmetric in u and v, the corrolary follows. □

Corollary 3.4. *The interpolation polynomials satisfy the difference equation*

$$q^{-|\lambda|/2}\Big(\prod_{1\le i\le n}(1-q^{\lambda_i}t^{n-i}u)\Big)\bar{P}^{*(n)}_\lambda(x_1,x_2,\dots x_n;q,t,s)= \tag{3.27}$$
$$\sum_{\sigma\in\{\pm1\}^n}\prod_{1\le i\le n}\frac{(1-sx_i^{\sigma_i})(1-ux_i^{\sigma_i}/s)}{(1-x_i^{2\sigma_i})}\prod_{1\le i<j\le n}\frac{(1-tx_i^{\sigma_i}x_j^{\sigma_j})}{(1-x_i^{\sigma_i}x_j^{\sigma_j})}\bar{P}^{*(n)}_\lambda(x_1q^{\sigma_1/2},x_2q^{\sigma_2/2}\dots x_nq^{\sigma_n/2};q,t,s\sqrt{q})$$

Two limiting cases of the theorem are of special interest. First, we find that the limit

$$\bar{P}^{*(n)}_\lambda(x_i;q,t)=\lim_{s\to\infty}s^{-|\lambda|}\bar{P}^{*(n)}_\lambda(x_is;q,t,s) \tag{3.28}$$

is well-defined, and produces an ordinary symmetric polynomial vanishing when $x_i=q^{\mu_i}t^{n-i}$ for $\mu\not\subset\lambda$. We thus recover the symmetric version of the shifted Macdonald polynomials. In this limit, our difference operators converge to those of Knop and Sahi [10, 21]. In particular, we see immediately that the shifted Macdonald polynomials are limits of BC/C-type Macdonald polynomials in multiple ways [4], as they are eigenfunctions of limiting versions of $D^{(n)}(u_1,u_2;q,t)$; the action on s becomes trivial in the limit. Similarly, the “leading term” limit

$$\lim_{a\to\infty}a^{-|\lambda|}\bar{P}^{*(n)}_\lambda(ax_i;q,t,s) \tag{3.29}$$

satisfies the difference equation of the ordinary Macdonald polynomials, and thus

$$\lim_{a\to\infty}a^{-|\lambda|}\bar{P}^{*(n)}_\lambda(ax_i;q,t,s)=P_\lambda(x_i;q,t). \tag{3.30}$$

We will prove a refinement of this fact in Theorem 6.16 below.

We will now recall some basic properties of the interpolation polynomials. First, some symmetries:

Lemma 3.5. *[16] The interpolation polynomials satisfy the identities*

$$\bar{P}^{*(n)}_\lambda(x_1,\dots x_n;q,t,s)=\bar{P}^{*(n)}_\lambda(x_1,\dots x_n;1/q,1/t,1/s) \tag{3.31}$$
$$=(-1)^{|\lambda|}\bar{P}^{*(n)}_\lambda(-x_1,\dots -x_n;q,t,-s) \tag{3.32}$$

for all partitions λ.

Next, identities for shrinking the dimension or the indexing partition:

Lemma 3.6. *[16] For any partition λ,*

$$\bar{P}^{*(n+m)}_{\lambda}(x_1,\dots x_n,s,st,\dots st^{m-1};q,t,s)=\begin{cases}\bar{P}^{*(n)}_{\lambda}(x_1,\dots x_n;q,t,st^m) & \lambda_{n+1}=0\\ 0 & \lambda_{n+1}>0\end{cases}\tag{3.33}$$

and

$$\bar{P}^{*(n)}_{m^n+\lambda}(x_1,x_2\dots x_n;q,t,s)=\prod_{(i,j)\in m^n}(x_i+1/x_i-q^{j-1}s-q^{1-j}/s)\bar{P}^{*(n)}_{\lambda}(x_1,x_2,\dots x_n;q,t,sq^m).\tag{3.34}$$

Proof. In each case, both sides are monic, triangular, and vanish on the appropriate partitions, so must be the same. □

The next corollary then follows by induction.

Corollary 3.7. *We have the normalization*

$$\bar{P}^{*(n)}_{\lambda}(\lambda;q,t,s)=(qt^{n-1}s)^{-|\lambda|}t^{n(\lambda)}q^{-2n(\lambda')}C^-_{\lambda}(q;q,t)C^+_{\lambda}(t^{2n-2}s^2;q,t)\tag{3.35}$$

Remark 1. In particular, we have:

$$P^{*(n)}_{\lambda}(x;q,t,s)=(t^{n-1}s)^{-|\lambda|}\bar{P}^{*(n)}_{\lambda}(x_it^{n-i}s;q,t,s),\tag{3.36}$$

where $P^{*(n)}_{\lambda}$ is Okounkov's interpolation polynomial in n variables; this also follows by comparing leading terms.

Remark 2. If we compute $\bar{P}^{*(n)}_{\lambda}$ by solving the appropriate triangular system of equations, we find that the denominators of the coefficients of $\bar{P}^{*(n)}_{\lambda}$ must divide the determinant of the system, i.e.

$$\prod_{\mu<\lambda}\bar{P}^{*(n)}_{\mu}(\mu;q,t,s).\tag{3.37}$$

Since the coefficients of $\bar{P}^{*(n)}_{\lambda}$ are in $\mathbb{F}[s,1/s]$, we conclude that the only possible denominator factors are q, t, s, and $(1-q^it^j)$ for $i,j\geq 0$.

We also note the following special case of the difference equation; this is of interest because the difference operator involved is independent of s.

Corollary 3.8. *The interpolation polynomials satisfy the difference equation*

$$\sum_{\sigma\in\{\pm1\}^n}\prod_{1\leq i\leq n}\frac{x_i^{\sigma_i}}{(1-x_i^{2\sigma_i})}\prod_{1\leq i<j\leq n}\frac{(1-tx_i^{\sigma_i}x_j^{\sigma_j})}{(1-x_i^{\sigma_i}x_j^{\sigma_j})}\bar{P}^{*(n)}_{\lambda}(x_1q^{\sigma_1/2},x_2q^{\sigma_2/2}\dots x_nq^{\sigma_n/2};q,t,s)$$
$$=\begin{cases}q^{-|\lambda|/2}\Big(\prod_{1\leq i\leq n}(1-q^{\lambda_i}t^{n-i})\Big)\bar{P}^{*(n)}_{\mu}(x_1,x_2,\dots x_n;q,t,s\sqrt{q}) & \lambda=1^n+\mu,\\ 0 & \textit{otherwise.}\end{cases}\tag{3.38}$$

Proof. Take $u=1$ in the full difference equation, then divide both sides by $\prod_{1\leq i\leq n}(x_i+1/x_i-s-1/s)$. □

The branching rule for interpolation polynomials extends to the following "bulk" branching rule.

Theorem 3.9. *We have*

$$\bar{P}^{*(n+m)}_{\lambda}(x_1,x_2,\dots x_n,t^{m-1}v,t^{m-2}v,\dots v;q,t,s) = \sum_{\substack{\mu\subset\lambda\\ \ell(\mu)\le n}} \psi^{(B)}_{\lambda/\mu}(v,vt^m;q,t,st^n)\bar{P}^{*(n)}_{\mu}(x_1,x_2,\dots x_n;q,t,s), \tag{3.39}$$

where

$$\psi^{(B)}_{\lambda/\mu}(v,v';q,t,s) = \frac{C^0_\lambda(s/v;q,t)C^0_\lambda(t/sv';1/q,1/t)}{C^0_\mu(s/v;q,t)C^0_\mu(t/sv';1/q,1/t)}P_{\lambda/\mu}(\left[\frac{v^k-v'^k}{1-t^k}\right];q,t) \tag{3.40}$$

Proof. Apply the difference equation with $u=vs$. The only terms that contribute are those with $\sigma_{n+i}=1$ for $1\le i\le m$, in which case the difference operator simplifies to an n-dimensional operator. We thus find

$$q^{-|\lambda|/2}\prod_{1\le i\le n+m}(1-q^{\lambda_i}t^{n+m-i}vs)\bar{P}^{*(n+m)}_{\lambda}((x_i),(t^{m-j}v);q,t,s) = \tag{3.41}$$
$$\prod_{n<i\le n+m}(1-t^{n+m-i}vs)D^{(n)}(t^m v,s;q,t)\bar{P}^{*(n+m)}_{\lambda}((x_i),(t^{m-j}v\sqrt{q});q,t,s\sqrt{q}).$$

Now, we have an expansion of the form

$$\bar{P}^{*(n+m)}_{\lambda}(x_i,v;q,t,s) = \sum_{\substack{\mu\le\lambda\\ \ell(\mu)\le n}} c_{\lambda\mu}(v;q,t,s)\bar{P}^{*(n)}_{\mu}(x_i;q,t,s), \tag{3.42}$$

with unknown coefficients $c_{\lambda\mu}$, since the interpolation polynomials are monic and triangular. As

$$D^{(n)}(t^m v,s)\bar{P}^{*(n)}_{\mu}(x_i;q,t,s\sqrt{q}) = q^{-|\mu|/2}\prod_{1\le i\le n}(1-q^{\mu_i}t^{n+m-i}vs)\bar{P}^{*(n)}_{\mu}(x_i;q,t,s), \tag{3.43}$$

we can substitute this expansion into (3.41), and compare coefficients of $\bar{P}^{*(n)}_{\mu}(x_i;q,t,s)$. We thus find

$$q^{-|\lambda|/2}\prod_{1\le i\le n+m}(1-q^{\lambda_i}t^{n+m-i}vs)c_{\lambda\mu}(v;q,t,s) = q^{-|\mu|/2}\prod_{1\le i\le n+m}(1-q^{\mu_i}t^{n+m-i}vs)c_{\lambda\mu}(v\sqrt{q};q,t,s\sqrt{q}), \tag{3.44}$$

and by symmetry ($v\mapsto 1/(t^{m-1}v)$),

$$q^{-|\lambda|/2}\prod_{1\le i\le n+m}(1-q^{\lambda_i}t^{n+1-i}s/v)c_{\lambda\mu}(v;q,t,s) = q^{-|\mu|/2}\prod_{1\le i\le n+m}(1-q^{\mu_i}t^{n+1-i}s/v)c_{\lambda\mu}(v/\sqrt{q};q,t,s\sqrt{q}). \tag{3.45}$$

Solving these difference equations gives

$$c_{\lambda\mu}(v;q,t,s) = c_{\lambda\mu}(q,t)\frac{\prod_{(i,j)\in\lambda}(-s)^{-1}q^{1-j}t^{i-n-m}(1-q^{j-1}t^{n+1-i}s/v)(1-q^{j-1}t^{n+m-i}sv)}{\prod_{(i,j)\in\mu}(-s)^{-1}q^{1-j}t^{i-n-m}(1-q^{j-1}t^{n+1-i}s/v)(1-q^{j-1}t^{n+m-i}sv)}, \tag{3.46}$$

where $c_{\lambda\mu}(q,t)$ is independent of s and v. This remaining factor is then determined by the Macdonald polynomial limit; we find that

$$c_{\lambda\mu}(q,t) = \lim_{v\to\infty} v^{-|\lambda|+|\mu|}c_{\lambda\mu}(v;q,t,s) = P_{\lambda/\mu}(1,t,\dots t^{m-1};q,t), \tag{3.47}$$

as required. ∎

Corollary 3.10. *[16] We have*

$$\bar{P}^{*(n+1)}_{\lambda}(x_1, x_2, \dots x_n, v; q, t, s) = \sum_{\substack{\mu' \prec \lambda' \\ \mu_{n+1}=0}} \psi^{(b)}_{\lambda/\mu}(v; q, t, st^n) \bar{P}^{*(n)}_{\mu}(x_1, x_2, \dots x_n; q, t, s), \tag{3.48}$$

where

$$\psi^{(b)}_{\lambda/\mu}(v; q, t, s) = \psi_{\lambda/\mu}(q, t) \prod_{(i,j)\in\lambda/\mu} (v + 1/v - q^{j-1}t^{1-i}s - q^{1-j}t^{i-1}/s) \tag{3.49}$$

Corollary 3.11. *For any partition λ and integer $n \geq \ell(\lambda)$,*

$$\bar{P}^{*(n)}_{\lambda}(xt^{n-1}s, xt^{n-2}s, \dots xs; q, t, s) = (-st^{n-1})^{-|\lambda|} t^{2n(\lambda)} q^{-n(\lambda')} \frac{C^0_{\mu}(t^n, 1/x, xs^2t^{n-1}; q, t)}{C^-_{\mu}(t; q, t)} \tag{3.50}$$

Remark. Equivalently, this gives a formula for evaluating Okounkov's original version of the interpolation polynomials at a constant.

The bulk branching rule also implies the following connection coefficient result.

Theorem 3.12. *For any partitions λ, μ,*

$$[\bar{P}^{*(n)}_{\mu}(;q,t,s)]\bar{P}^{*(n)}_{\lambda}(;q,t,s') = \frac{C^0_{\lambda}(t^n; q, t)C^0_{\lambda}(t^{1-n}/ss'; 1/q, 1/t)}{C^0_{\mu}(t^n; q, t)C^0_{\mu}(t^{1-n}/ss'; 1/q, 1/t)} P_{\lambda/\mu}([\frac{s^k - s'^k}{1-t^k}]; q, t). \tag{3.51}$$

Proof. Both sides are rational in q, t, s, s', so it suffices to prove this under the assumption $s' = st^m$ for some integer $m \geq 0$. Then by Lemma 3.6 and the bulk branching rule,

$$\bar{P}^{*(n)}_{\lambda}(x_1, \dots x_n; q, t, st^m) = \bar{P}^{*(n+m)}_{\lambda}(x_1, \dots x_n, s, st, st^2, \dots st^{m-1}; q, t, s) \tag{3.52}$$

$$= \sum_{\substack{\mu\subset\lambda \\ \ell(\mu)\leq n}} \psi^{(B)}_{\lambda/\mu}(s, s'; q, t, st^n) \bar{P}^{*(n)}_{\mu}(x_1, x_2, \dots x_n; q, t, s). \tag{3.53}$$

The theorem follows immediately. ☐

Remark. If we expand an interpolation polynomial using the connection coefficient identity, we cannot in general insist that the polynomials on both sides are evaluated at a partition. A notable exception is when $ss' = t^{n-1}q^{-m}$, since then $\bar{P}^{*(n)}(\mu; q, t, s)$ and $\bar{P}^{*(n)}(m^n - \mu; q, t, s')$ are evaluated at the same point.

There is also a bulk version of the Pieri identity.

Theorem 3.13. *For any integer $n \geq 0$ and partition μ of length $\leq n$, the following identity holds in the power series ring $\mathbb{F}(s)[x_1^{\pm 1}, \dots x_n^{\pm 1}][[u, v]]$.*

$$\prod_{1\leq i\leq n} \frac{(vx_i, v/x_i; q)}{(ux_i, u/x_i; q)} \bar{P}^{*(n)}_{\mu}(x_1, \dots x_n; q, t, s) = \prod_{1\leq i\leq n} \frac{(vq^{\mu_i}t^{n-i}s, vq^{-\mu_i}t^{i-n}/s; q)}{(uq^{\mu_i}t^{n-i}s, uq^{-\mu_i}t^{i-n}/s; q)} \sum_{\lambda\supset\mu} \psi^{(P)}_{\lambda/\mu}(u, v; q, t; st^n) \bar{P}^{*(n)}_{\lambda}(x_1, \dots x_n; q, t, s), \tag{3.54}$$

where

$$\psi^{(P)}_{\lambda/\mu}(u, v; q, t; s) = \frac{C^0_{\mu}(sv/t; q, t)C^0_{\mu}(tu/qs; 1/q, 1/t)}{C^0_{\lambda}(sv/t; q, t)C^0_{\lambda}(tu/qs; 1/q, 1/t)} Q_{\lambda/\mu}([(u^k - v^k)/(1-t^k)]; q, t). \tag{3.55}$$

Proof. It suffices to consider the case $u = q^m v$ for an integer $m \geq 0$. We certainly have an expansion of the form

$$\prod_{1\leq i\leq n} (vx_i, v/x_i; q)_m \bar{P}^{*(n)}_\mu(x_1, \dots x_n; q, t, s) = \sum_\lambda c^{(n,m)}_{\lambda\mu}(v; q, t, s)\bar{P}^{*(n)}_\lambda(x_1, \dots x_n; q, t, s), \tag{3.56}$$

for $c^{(n,m)}_{\lambda\mu}(v; q, t, s) \in \mathbb{F}[s, 1/s, v]$; the content of the theorem is that

$$c^{(n,m)}_{\lambda\mu}(v; q, t, s) = Q_{\lambda/\mu}([\frac{q^{mk}-1}{1-t^k}]; q, t)v^{|\lambda|-|\mu|} \prod_{(i,j)\in(m^n+\mu)/\lambda} (1 - vq^{j-1}t^{n-i}s)(1 - vq^{m-j}t^{i-n}/s). \tag{3.57}$$

By the Macdonald limit,

$$\lim_{v\to 0} v^{|\mu|-|\lambda|} c^{(n,m)}_{\lambda\mu}(v; q, t, s) = Q_{\lambda/\mu}([\frac{q^{mk}-1}{1-t^k}]; q, t) \tag{3.58}$$

$$\lim_{v\to\infty} v^{|\lambda|-|\mu|-2mn} c^{(n,m)}_{\lambda\mu}(v; q, t, s) = q^{(m-1)(mn+|\mu|-|\lambda|)} Q_{\lambda/\mu}([\frac{q^{mk}-1}{1-t^k}]; q, t), \tag{3.59}$$

and thus $v^{|\mu|-|\lambda|}c^{(n,m)}_{\lambda\mu}(v; q, t, s)$ is a polynomial of degree at most $2mn + 2|\mu| - 2|\lambda|$, with constant term $Q_{\lambda\mu}([(q^{mk} - 1)/(1 - t^k)]; q, t)$.

Now, if we evaluate both sides of (3.56) at κ, the left-hand side vanishes if either $\mu \not\subset \kappa$ or

$$v \in \{t^{n-i}q^{\kappa_i+j-m}s, t^{i-n}q^{1-j-\kappa_i}/s : (i, j) \in (m^n + \kappa)/\kappa\}. \tag{3.60}$$

Thus by induction in λ, we find $v^{|\mu|-|\lambda|}c^{(n,m)}_{\lambda\mu}(v; q, t, s)$ vanishes if $\mu \not\subset \lambda$, and is otherwise a multiple of

$$\prod_{\substack{(i,j)\in m^n+\mu\\(i,j)\notin\lambda}} (1 - vq^{j-1}t^{n-i}s)(1 - vq^{m-j}t^{i-n}/s). \tag{3.61}$$

This has degree $\geq 2mn + 2|\mu| - 2|\lambda|$, and thus it follows immediately that

$$v^{|\mu|-|\lambda|}c^{(n,m)}_{\lambda\mu}(v; q, t, s) = Q_{\lambda/\mu}([\frac{q^{mk}-1}{1-t^k}]; q, t) \prod_{\substack{(i,j)\in m^n+\mu\\(i,j)\notin\lambda}} (1 - vq^{j-1}t^{n-i}s)(1 - vq^{m-j}t^{i-n}/s) \tag{3.62}$$

as required; note that the skew Macdonald polynomial vanishes unless $\lambda \subset m^n + \mu$. ☐

Remark. When $m = 1$, this is essentially the proof of [16] for the ordinary Pieri identity.

The case $u = qv$ gives the ordinary e-type Pieri identity.

Corollary 3.14. *[16] For any integer $n \geq 0$ and partition μ of length $\leq n$,*

$$\prod_{1\leq i\leq n} (v + 1/v + x_i + 1/x_i)\bar{P}^{*(n)}_\mu(x_1, \dots x_n; q, t, s) \tag{3.63}$$

$$= \prod_{1\leq i\leq n} (v + 1/v + q^{\mu_i}t^{n-i}s + q^{-\mu_i}t^{i-n}/s) \sum_{\lambda\succ\mu} \psi^{(e)}_{\lambda/\mu}(v; q, t, st^n)\bar{P}^{*(n)}_\lambda(x_1, \dots x_n; q, t, s),$$

where

$$\psi^{(e)}_{\lambda/\mu}(v; q, t, s) = \psi'_{\lambda/\mu}(q, t) \prod_{(i,j)\in\lambda/\mu} (v + 1/v + q^{j-1}t^{-i}s + q^{1-j}t^i/s)^{-1}. \tag{3.64}$$

We also obtain a g-type Pieri identity, by taking $v = tu$.

Corollary 3.15. *For any integer $n \ge 0$ and partition μ of length $\le n$, the following identity holds in the power series ring $\mathbb{F}(s)[x_1^{\pm 1}, \dots x_n^{\pm 1}][[u]]$.*

$$\prod_{1\le i\le n} \frac{(tux_i, tu/x_i; q)}{(ux_i, u/x_i; q)} \bar{P}^{*(n)}_\mu(x_1, \dots x_n; q, t, s) = \prod_{1\le i\le n} \frac{(tuq^{\mu_i}t^{n-i}s, tuq^{-\mu_i}t^{i-n}/s; q)}{(uq^{\mu_i}t^{n-i}s, uq^{-\mu_i}t^{i-n}/s; q)} \sum_{\lambda\supset\mu} \varphi^{(g)}_{\lambda/\mu}(u; q, t; st^n) \bar{P}^{*(n)}_\lambda(x_1, \dots x_n; q, t, s), \tag{3.65}$$

where

$$\varphi^{(g)}_{\lambda/\mu}(u; q, t; s) = \varphi_{\lambda/\mu}(q, t) \prod_{(i,j)\in\lambda/\mu} \frac{u}{(1 - uq^{j-1}t^{1-i}s)(1 - uq^{-j}t^i/s)} \tag{3.66}$$

As observed in [16], the branching rule and Pieri identity are connected via the Cauchy identity.

Theorem 3.16. *[16] For any integers m, n,*

$$\sum_{\lambda\subset m^n} (-1)^{mn-|\lambda|} \bar{P}^{*(n)}_\lambda(x; q, t, s) \bar{P}^{*(m)}_{n^m-\lambda'}(y; t, q, s) = \prod_{1\le i\le n} \prod_{1\le j\le m} (x_i + 1/x_i - y_j - 1/y_j) \tag{3.67}$$

Proof. By induction in n; if we multiply both sides by

$$\prod_{1\le j\le m} (x_{n+1} + 1/x_{n+1} - y_j - 1/y_j), \tag{3.68}$$

then expanding via the (e-type) Pieri identity and simplifying via the branching rule turns the resulting left-hand side into the left-hand side of the Cauchy identity for $n + 1$. □

4 Hypergeometric transformations

Define the "binomial coefficients"

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} := \frac{\bar{P}^{*(n)}_\mu(\lambda; q, t, t^{1-n}s)}{\bar{P}^{*(n)}_\mu(\mu; q, t, t^{1-n}s)} \tag{4.1}$$

$$\begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,s} := \frac{\langle P_\mu, P_\mu\rangle''_n \bar{P}^{*(n)}_\lambda(\lambda; q, t, t^{1-n}s) \bar{P}^{*(n)}_{m^n-\lambda}(m^n - \mu; q, t, q^{-m}/s)}{\langle P_\lambda, P_\lambda\rangle''_n \bar{P}^{*(n)}_\mu(\mu; q, t, t^{1-n}s) \bar{P}^{*(n)}_{m^n-\mu}(m^n - \mu; q, t, q^{-m}/s)}, \tag{4.2}$$

where m and n are chosen so that $\lambda, \mu \subset m^n$. Note that each binomial coefficient vanishes unless $\mu \subset \lambda$, and is equal to 1 when $\mu = \lambda$; furthermore, each is preserved by the substitutions $s \mapsto -s$ and $(q, t, s) \mapsto (1/q, 1/t, 1/s)$. That the first binomial coefficient is independent of n and the second is independent of m follows from Lemma 3.6; that the second is also independent of n follows from Theorem 4.5 below. First, though, some transformations and special values.

Proposition 4.1. *For any partitions $\mu \subset \lambda$,*

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} = \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{1/q,1/t,1/s} \qquad \begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,s} = \begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{1/q,1/t,1/s} \tag{4.3}$$

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} = \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,-s} \qquad \begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,s} = \begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,-s} \tag{4.4}$$

For integers $m, n \geq 0$ with $\ell(\lambda) \leq n$,

$$\begin{bmatrix} m^n+\lambda \\ m^n+\mu \end{bmatrix}_{q,t,s} = q^{-m|\lambda/\mu|} \frac{C^0_\lambda(t^{1-n}q^{2m}s^2, t^{n-1}q^{m+1};q,t) C^0_\mu(t^{1-n}q^m s^2, t^{n-1}q;q,t)}{C^0_\mu(t^{1-n}q^{2m}s^2, t^{n-1}q^{m+1};q,t) C^0_\lambda(t^{1-n}q^m s^2, t^{n-1}q;q,t)} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t,q^m s} \tag{4.5}$$

$$\begin{Bmatrix} m^n+\lambda \\ m^n+\mu \end{Bmatrix}_{q,t,s} = q^{-m|\lambda/\mu|} \frac{C^0_\lambda(t^{1-n}q^{2m}s^2, t^{n-1}q^{m+1};q,t) C^0_\mu(t^{1-n}q^m s^2, t^{n-1}q;q,t)}{C^0_\mu(t^{1-n}q^{2m}s^2, t^{n-1}q^{m+1};q,t) C^0_\lambda(t^{1-n}q^m s^2, t^{n-1}q;q,t)} \begin{Bmatrix} \lambda \\ \mu \end{Bmatrix}_{q,t,q^m s}. \tag{4.6}$$

If in fact $\lambda \subset m^n$,

$$\begin{bmatrix} m^n-\mu \\ m^n-\lambda \end{bmatrix}_{q,t,s} = \frac{\langle P_\lambda, P_\lambda\rangle''_n \bar{P}^{*(n)}_\mu(\mu;q,t,q^{-m}/s) \bar{P}^{*(n)}_{m^n-\mu}(m^n-\mu;q,t,t^{1-n}s)}{\langle P_\mu, P_\mu\rangle''_n \bar{P}^{*(n)}_\lambda(\lambda;q,t,q^{-m}/s) \bar{P}^{*(n)}_{m^n-\lambda}(m^n-\lambda;q,t,t^{1-n}s)} \begin{Bmatrix} \lambda \\ \mu \end{Bmatrix}_{q,t,t^{n-1}/q^m s} \tag{4.7}$$

$$\begin{Bmatrix} m^n-\mu \\ m^n-\lambda \end{Bmatrix}_{q,t,s} = \frac{\langle P_\lambda, P_\lambda\rangle''_n \bar{P}^{*(n)}_\mu(\mu;q,t,q^{-m}/s) \bar{P}^{*(n)}_{m^n-\mu}(m^n-\mu;q,t,t^{1-n}s)}{\langle P_\mu, P_\mu\rangle''_n \bar{P}^{*(n)}_\lambda(\lambda;q,t,q^{-m}/s) \bar{P}^{*(n)}_{m^n-\lambda}(m^n-\lambda;q,t,t^{1-n}s)} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t,t^n/q^m s} \tag{4.8}$$

For any partition $\lambda \subset m^n$,

$$\begin{bmatrix} \lambda \\ 0 \end{bmatrix}_{q,t,s} = 1 \tag{4.9}$$

$$\begin{Bmatrix} \lambda \\ 0 \end{Bmatrix}_{q,t,s} = (-1)^{|\lambda|} t^{n(\lambda)} q^{-n(\lambda')} \frac{C^+_\lambda(s^2;q,t)}{C^0_\lambda(qs^2;q,t)} \tag{4.10}$$

$$\begin{bmatrix} m^n \\ \lambda \end{bmatrix}_{q,t,s} = (-q)^{|\lambda|} t^{n(\lambda)} q^{n(\lambda')} \frac{C^0_\lambda(t^n, q^{-m}, q^m s^2/t^{n-1};q,t)}{C^-_\lambda(q,t;q,t) C^+_\lambda(s^2;q,t)} \tag{4.11}$$

$$\begin{Bmatrix} m^n \\ \lambda \end{Bmatrix}_{q,t,s} = \frac{(-1)^{mn} t^{n(m^n)} C^0_{m^n}(q^m s^2/t^{n-1};q,t)(q^m/t^{n-1})^{|\lambda|} t^{2n(\lambda)} C^0_\lambda(t^n, q^{-m};q,t) C^0_{2\lambda^2}(s^2 q;q,t)}{q^{n(n^m)} C^0_{m^n}(qs^2;q,t) C^-_\lambda(q,t;q,t) C^+_\lambda(s^2, s^2q/t;q,t) C^0_\lambda(q^{m+1}s^2, s^2q/t^n;q,t)} \tag{4.12}$$

When $n=1$, $\lambda = l$,

$$\begin{bmatrix} m \\ l \end{bmatrix}_{q,t,s} = (-1)^l q^{l(l+1)/2} \frac{(q^{-m}, q^m s^2;q)_l}{(q^l s^2, q;q)_l} \tag{4.13}$$

$$\begin{Bmatrix} m \\ l \end{Bmatrix}_{q,t,s} = (-1)^m q^{lm-m(m-1)/2} \frac{(q^{-m}, s^2;q)_l (1-q^{2l}s^2)(q^{m+1}s^2;q)_m}{(q^{m+1}s^2, q;q)_l (1-q^{2m}s^2)(s^2;q)_m} \tag{4.14}$$

If we state the bulk Pieri identity in terms of binomial coefficients, we obtain the following generalized q-Saalschütz formula.

Theorem 4.2. *For any partitions $\kappa \subset \lambda$,*

$$\sum_{\kappa\subset\mu\subset\lambda} q^{n(\kappa')-n(\mu')} \frac{(-1)^{|\mu/\kappa|} C^0_\kappa(b,c;q,t) C^-_\mu(t;q,t) C^+_\mu(a;q,t)}{C^0_\mu(qa/b, qa/c;q,t) C^-_\kappa(t;q,t) C^+_\kappa(a;q,t)} P_{\mu/\kappa}([\frac{1-(qa/bc)^k}{1-t^k}];q,t) \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t,\sqrt{a}} \tag{4.15}$$

$$= (qa/bc)^{|\lambda/\kappa|} \frac{C^0_\lambda(b,c;q,t)}{C^0_\lambda(qa/b, qa/c;q,t)} \begin{bmatrix} \lambda \\ \kappa \end{bmatrix}_{q,t,\sqrt{a}}$$

Proof. For fixed λ, κ, both sides are rational functions of b and c; moreover, if we multiply both sides by $c^{-|\kappa|}$, the results are well defined in the limit $(b,c) \to (0,\infty)$. We may thus work in the power series ring $\mathbb{F}(\sqrt{a})[[b,1/c]]$. If we evaluate both sides of the bulk Pieri identity (Theorem 3.13) at a partition, substitute

$$(s,u,v) \mapsto (t^{1-n}\sqrt{a}, b/\sqrt{a}, q\sqrt{a}/c), \tag{4.16}$$

and simplify, the result follows. □

One consequence is the following symmetry ("duality").

Corollary 4.3. *For any partitions μ and λ,*

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} = \begin{bmatrix}\lambda'\\ \mu'\end{bmatrix}_{t,q,1/\sqrt{qt}s} \tag{4.17}$$

$$\begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,s} = \begin{Bmatrix}\lambda'\\ \mu'\end{Bmatrix}_{t,q,1/\sqrt{qt}s}. \tag{4.18}$$

Proof. For the first equation, we observe that if we conjugate κ, μ, λ in the generalized q-Saalschütz formula and substitute

$$(q,t,a,b,c) \mapsto (t,q,1/qta,1/b,1/c), \tag{4.19}$$

we obtain the generalized q-Saalschütz formula again, except with

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,\sqrt{a}} \quad \text{replaced by} \quad \begin{bmatrix}\lambda'\\ \mu'\end{bmatrix}_{t,q,1/\sqrt{qta}}. \tag{4.20}$$

Thus both binomial coefficients satisfy the same set of recurrences, with the same initial conditions, and must therefore be the same. This proves the first equation; the second equation follows immediately. ☐

Remark. An alternate proof is given in Corollary 6.6 below.

The q-Saalschütz formula can also be written in the following form, obtained by "reversing the order of summation"; that is, replacing λ, μ, κ by their complements $m^n - \lambda$, $m^n - \mu$, $m^n - \kappa$ for sufficiently large m and n.

Corollary 4.4. *For any partitions $\kappa \subset \lambda$,*

$$\sum_{\kappa\subset\mu\subset\lambda} q^{n(\mu')-n(\lambda')} \frac{(-1)^{|\lambda/\mu|} C^0_\mu(b,c;q,t) C^-_\lambda(t;q,t) C^+_\lambda(a;q,t)}{C^0_\lambda(qa/b,qa/c;q,t) C^-_\mu(t;q,t) C^+_\mu(a;q,t)} P_{\lambda/\mu}([\frac{(bc/qa)^k-1}{1-t^k}];q,t) \begin{Bmatrix}\mu\\ \kappa\end{Bmatrix}_{q,t,\sqrt{a}} \tag{4.21}$$

$$= (bc/aq)^{|\lambda/\kappa|} \frac{C^0_\kappa(b,c;q,t)}{C^0_\kappa(qa/b,qa/c;q,t)} \begin{Bmatrix}\lambda\\ \kappa\end{Bmatrix}_{q,t,\sqrt{a}}$$

Our definition of the second kind of binomial coefficients is justified by the following result.

Theorem 4.5. *The binomial coefficients satisfy the inversion identities*

$$\sum_{\kappa\subset\mu\subset\lambda} \begin{bmatrix}\mu\\ \kappa\end{bmatrix}_{q,t,s} \begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,s} = \delta_{\lambda\kappa} \tag{4.22}$$

and

$$\sum_{\kappa\subset\mu\subset\lambda} \begin{Bmatrix}\mu\\ \kappa\end{Bmatrix}_{q,t,s} \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} = \delta_{\lambda\kappa}. \tag{4.23}$$

Proof. Fix integers $m,n \geq 0$, and define the matrix $\left\{{\lambda \atop \mu}\right\}'_{q,t,s}$ indexed by partitions $\lambda,\mu \subset m^n$ to be the inverse of the matrix $\left[{\mu \atop \kappa}\right]_{q,t,s}$. The theorem is then equivalent to the equation

$$\begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}'_{q,t,s} = \begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,s}. \tag{4.24}$$

If we multiply both sides of the q-Saalschütz formua by

$$(bc/qa)^{|\rho/\kappa|}\left\{\begin{matrix}\rho\\ \lambda\end{matrix}\right\}'_{q,t,\sqrt{a}}\left\{\begin{matrix}\kappa\\ \nu\end{matrix}\right\}'_{q,t,\sqrt{a}} \tag{4.25}$$

and sum over λ and κ, we find that Corollary 4.4 is satisfied by the alternate binomial coefficients as well; as in Corollary 4.3, this implies that the two sets of binomial coefficients are the same. □

Remark. When $\lambda \neq \kappa$ both have length 1, the first sum becomes a ${}_4\phi_3$, summed to 0 by [7, Eq. 2.3.4]. The second sum is a ${}_2\phi_1$ in the univariate case.

Together with inversion, the generalized q-Saalschütz formula implies the following identity, which generalizes the sum of a terminating very-well-poised ${}_6\phi_5$.

Corollary 4.6. *For any partitions $\kappa \subset \lambda$,*

$$\sum_{\kappa\subset\mu\subset\lambda}(qa/bc)^{|\mu/\kappa|}\frac{C^0_\mu(b,c;q,t)}{C^0_\mu(qa/b,qa/c;q,t)}\left\{\begin{matrix}\lambda\\ \mu\end{matrix}\right\}_{q,t,\sqrt{a}}\begin{bmatrix}\mu\\ \kappa\end{bmatrix}_{q,t,\sqrt{a}}$$
$$=(-1)^{|\lambda/\kappa|}q^{n(\kappa')-n(\lambda')}\frac{C^-_\lambda(t;q,t)C^+_\lambda(a;q,t)C^0_\kappa(b,c;q,t)}{C^-_\kappa(t;q,t)C^+_\kappa(a;q,t)C^0_\lambda(aq/b,aq/c;q,t)}P_{\lambda/\kappa}([\frac{1-(qa/bc)^k}{1-t^k}];q,t). \tag{4.26}$$

Proof. In the left-hand side, expand

$$(qa/bc)^{|\mu/\kappa|}\frac{C^0_\mu(b,c;q,t)}{C^0_\mu(qa/b,qa/c;q,t)}\begin{bmatrix}\mu\\ \kappa\end{bmatrix}_{q,t,\sqrt{a}} \tag{4.27}$$

by applying the generalized q-Saalschütz formula in reverse. We can then sum over μ using inversion, obtaining the desired sum. □

Remark. This generalizes equation (2.4.2) of [7]; the above proof is a direct adaptation of the proof in the univariate case. The special case $\kappa = 0$, $\lambda = m^n$ was shown in [28].

Iterating the above argument gives a generalization of Watson's tranformation between a terminating very-well-poised ${}_8\phi_7$ and a balanced terminating ${}_4\phi_3$.

Theorem 4.7. *For any partitions $\kappa \subset \lambda$*

$$\sum_{\kappa\subset\mu\subset\lambda}\left(\frac{a^2q^2}{bcde}\right)^{|\mu|-|\kappa|}\frac{C^0_\mu(b,c,d,e;q,t)}{C^0_\mu(aq/b,aq/c,aq/d,aq/e;q,t)}\left\{\begin{matrix}\lambda\\ \mu\end{matrix}\right\}_{q,t,\sqrt{a}}\begin{bmatrix}\mu\\ \kappa\end{bmatrix}_{q,t,\sqrt{a}}$$
$$=(-1)^{|\lambda|-|\kappa|}q^{n(\kappa')-n(\lambda')}\frac{C^0_\kappa(b,c;q,t)C^-_\lambda(t;q,t)C^+_\lambda(a;q,t)}{C^0_\lambda(aq/d,aq/e;q,t)C^-_\kappa(t;q,t)C^+_\kappa(a;q,t)}$$
$$\sum_{\kappa\subset\mu\subset\lambda}\frac{C^0_\mu(d,e;q,t)}{C^0_\mu(aq/b,aq/c;q,t)}P_{\lambda/\mu}([\frac{1-(aq/de)^k}{1-t^k}];q,t)P_{\mu/\kappa}([\frac{(aq/de)^k-(a^2q^2/bcde)^k}{1-t^k}];q,t) \tag{4.28}$$

Remark. Taking $b = aq/c$ or $d = aq/e$ recovers Corollary 4.6.

If we exchange c and d, the left-hand side is unchanged, thus leading to a transformation of the right-hand side, a multivariate analogue of Sears' transformation of a balanced terminating ${}_4\phi_3$.

Corollary 4.8. *For any partitions $\kappa \subset \lambda$,*

$$\sum_{\kappa\subset\mu\subset\lambda} \frac{C^0_\lambda(aq/b,aq/c;q,t)C^0_\mu(d,e;q,t)}{C^0_\mu(aq/b,aq/c;q,t)C^0_\kappa(d,e;q,t)} P_{\lambda/\mu}([\frac{1-(aq/de)^k}{1-t^k}];q,t)P_{\mu/\kappa}([\frac{(aq/de)^k-(a^2q^2/bcde)^k}{1-t^k}];q,t) \tag{4.29}$$
$$= \sum_{\kappa\subset\mu\subset\lambda} \frac{C^0_\lambda(aq/b,aq/d;q,t)C^0_\mu(c,e;q,t)}{C^0_\mu(aq/b,aq/d;q,t)C^0_\kappa(c,e;q,t)} P_{\lambda/\mu}([\frac{1-(aq/ce)^k}{1-t^k}];q,t)P_{\mu/\kappa}([\frac{(aq/ce)^k-(a^2q^2/bcde)^k}{1-t^k}];q,t)$$

This implies two more transformations, the first of which is another generalized q-Saalschütz formula (not used below).

Corollary 4.9. *For any partitions $\kappa \subset \lambda$,*

$$\sum_{\kappa\subset\mu\subset\lambda} \frac{C^0_\mu(a;q,t)}{C^0_\mu(c;q,t)} P_{\lambda/\mu}([\frac{a^k-b^k}{1-t^k}];q,t)P_{\mu/\kappa}([\frac{b^k-c^k}{1-t^k}];q,t) = \frac{C^0_\kappa(a;q,t)C^0_\lambda(b;q,t)}{C^0_\kappa(b;q,t)C^0_\lambda(c;q,t)} P_{\lambda/\kappa}([\frac{a^k-c^k}{1-t^k}];q,t) \tag{4.30}$$

Similarly,

$$\sum_{\kappa\subset\mu\subset\lambda} \frac{C^0_\lambda(aq/b,aq/c;q,t)C^0_\mu(d,e;q,t)}{C^0_\mu(aq/b,aq/c;q,t)C^0_\kappa(d,e;q,t)} P_{\lambda/\mu}([\frac{1-(aq/de)^k}{1-t^k}];q,t)P_{\mu/\kappa}([\frac{(aq/de)^k-(a^2q^2/bcde)^k}{1-t^k}];q,t) \tag{4.31}$$
$$= \sum_{\kappa\subset\mu\subset\lambda} \frac{C^0_\lambda(aq/d,aq/e;q,t)C^0_\mu(b,c;q,t)}{C^0_\mu(aq/d,aq/e;q,t)C^0_\kappa(b,c;q,t)} P_{\lambda/\mu}([\frac{1-(aq/bc)^k}{1-t^k}];q,t)P_{\mu/\kappa}([\frac{(aq/bc)^k-(a^2q^2/bcde)^k}{1-t^k}];q,t)$$

Proof. The first identity follows from the special case $d = aq/e$ of Corollary 4.8; the second identity follows from two applications of that corollary. □

Remark 1. Setting $\kappa = 0$ in (4.30), multiplying both sides by $Q_\lambda(x;q,t)$ and and summing over λ gives a multivariate analogue of Euler's transformation (see [3] for an alternate proof).

Remark 2. These two identities are precisely the conditions required for the bulk branching rule (or the bulk Pieri identity) to be self-consistent: the first allows one to combine two adjoining applications into one, while the second allows one to exchange consecutive applications.

The case $\lambda = m^n$, $\kappa = 0$ of Theorem 4.7 is of special interest:

Corollary 4.10. *For any integers $m, n \geq 0$,*

$$\sum_{\mu\subset m^n} \frac{C^0_{2\mu^2}(aq;q,t)C^0_\mu(t^n,q^{-m},b,c,d,e;q,t)t^{2n(\mu)}(a^2q^{m+2}/t^{n-1}bcde)^{|\mu|}}{C^+_\mu(a,qa/t;q,t)C^0_\mu(aq/t^n,q^{m+1}a,aq/b,aq/c,aq/d,aq/e;q,t)C^-_\mu(q,t;q,t)}$$
$$= \frac{C^0_{m^n}(aq,aq/de;q,t)}{C^0_{m^n}(aq/d,aq/e;q,t)} \sum_{\mu\subset m^n} \frac{C^0_\mu(t^n,q^{-m},d,e,aq/bc;q,t)t^{2n(\mu)}q^{|\mu|}}{C^0_\mu(aq/b,aq/c,t^{n-1}q^{-m}de/a;q,t)C^-_\mu(q,t;q,t)}. \tag{4.32}$$

For future use, we define (horizontally) nonterminating versions of these sums:

$${}_8W^{(n)}_7(a;b,c,d,e,f;q,t;z) := \sum_{\ell(\mu)\leq n} \frac{C^0_{2\mu^2}(aq;q,t)C^0_\mu(t^n,b,c,d,e,f;q,t)t^{2n(\mu)}z^{|\mu|}}{C^+_\mu(a,qa/t;q,t)C^0_\mu(aq/t^n,aq/b,aq/c,aq/d,aq/e,aq/f;q,t)C^-_\mu(q,t;q,t)} \tag{4.33}$$

$${}_4\Phi^{(n)}_3\left(\begin{matrix}a,b,c,d\\e,f,g\end{matrix};q,t;z\right) := \sum_{\ell(\mu)\leq n} \frac{C^0_\mu(t^n,a,b,c,d;q,t)t^{2n(\mu)}z^{|\mu|}}{C^0_\mu(e,f,g;q,t)C^-_\mu(q,t;q,t)}. \tag{4.34}$$

Both sums converge if $|q|, |z| < 1$; in contrast, convergence of a similar vertically nonterminating sum would require $|t| > 1$. When $n = 1$, we have

$$ {}_8W_7^{(1)}(a; b, c, d, e, f; q, t; z) = {}_8W_7(a; b, c, d, e, f; q; z) \tag{4.35}$$

$$ {}_4\Phi_3^{(1)}\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; q, t; z\right) = {}_4\phi_3\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; q; z\right). \tag{4.36}$$

Corollary 4.11. *[29] If $a^2q^{m+1} = t^{n-1}bcde$, then*

$$ {}_8W_7^{(n)}(a; b, c, d, e, q^{-m}; q, t; q) = \frac{C^0_{m^n}(aq, aq/cd, aq/ce, aq/de; q, t)}{C^0_{m^n}(aq/c, aq/d, aq/e, aq/cde; q, t)} \tag{4.37}$$

Remark. This generalizes Jackson's sum of a balanced, very-well-poised, terminating ${}_8\phi_7$. In [17], we will derive a version of this indexed by skew diagrams, as well as an associated analogue of Bailey's transformation; in particular, we obtain Warnaar's conjectured multivariate elliptic Bailey transform, [31, Conjecture 6.1]. Warnaar also conjectured the elliptic analogue of the above sum, since proved in [20].

In the sequel, it will be useful to know how the difference equation is expressed in terms of the binomial coefficients.

Theorem 4.12. *For any partitions $\mu \subset \lambda$,*

$$ \psi^{(d)}_{\mu/\mu}(u; q, t, s)\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} = \sum_{\kappa \prec \lambda} \psi^{(d)}_{\lambda/\kappa}(u; q, t, s)\begin{bmatrix}\kappa\\ \mu\end{bmatrix}_{q,t,s\sqrt{q}} \tag{4.38}$$

$$ \psi^{(d)}_{\lambda/\lambda}(u; q, t, s)\begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,s\sqrt{q}} = \sum_{\kappa \succ \mu} \psi^{(d)}_{\kappa/\mu}(u; q, t, s)\begin{Bmatrix}\lambda\\ \kappa\end{Bmatrix}_{q,t,s}, \tag{4.39}$$

where

$$\begin{aligned} \psi^{(d)}_{\lambda/\kappa}(u; q, t, s) = (-u/t)^{|\lambda/\kappa|} t^{n(\kappa)-n(\lambda)} &\frac{C^0_\lambda(qts^2/u; q, t)C^0_\kappa(qu/t; q, t)}{C^0_\lambda(u/t; q, t)C^0_\kappa(qts^2/u; q, t)} \\ &\prod_{\substack{(i,j)\in\lambda\\ \lambda_i=\kappa_i}} \frac{1 - q^{\lambda_i+j-1}t^{2-\lambda'_j-i}s^2}{1 - q^{\kappa_i-j}t^{\kappa'_j-i+1}} \prod_{\substack{(i,j)\in\lambda\\ \lambda_i\neq\kappa_i}} \frac{1 - q^{\lambda_i-j+1}t^{\lambda'_j-i}}{1 - q^{\kappa_i+j+1}t^{1-\kappa'_j-i}s^2} \\ &\prod_{\substack{(i,j)\in\kappa\\ \lambda_i=\kappa_i}} \frac{1 - q^{\lambda_i-j}t^{\lambda'_j-i+1}}{1 - q^{\kappa_i+j}t^{2-\kappa'_j-i}s^2} \prod_{\substack{(i,j)\in\kappa\\ \lambda_i\neq\kappa_i}} \frac{1 - q^{\lambda_i+j}t^{1-\lambda'_j-i}s^2}{1 - q^{\kappa_i-j+1}t^{\kappa'_j-i}} \end{aligned} \tag{4.40}$$

$$ \psi^{(d)}_{\lambda/\lambda}(u; q, t, s) = \frac{C^+_\lambda(s^2; q, t)}{C^+_\lambda(s^2q; q, t)} \prod_{1\le i\le \ell(\lambda)} \frac{1 - q^{\lambda_i}t^{-i}u}{1 - t^{-i}u}. \tag{4.41}$$

Proof. For the first claim, multiply s and u by t^{-n} in the difference equation, divide both sides by

$$ \bar{P}^{*(n)}_\mu(\mu; q, t, t^{1-n}s\sqrt{q}) \prod_{1\le i\le n}(1 - t^{-i}u), \tag{4.42}$$

and evaluate at λ. We thus obtain an equation of the desired form; it remains to simplify the coefficients on the right hand side, namely

$$\prod_{i\in R'}\frac{(1-q^{\lambda_i}t^{2-n-i}s^2)(1-uq^{\lambda_i}t^{-i})}{(1-q^{2\lambda_i}t^{2-2i}s^2)(1-ut^{-i})}\prod_{i\in R}\frac{(1-q^{-\lambda_i}t^{i-n})(1-uq^{-\lambda_i}t^{i-2}/s^2)}{(1-q^{-2\lambda_i}t^{2i-2}s^{-2})(1-ut^{-i})}$$
$$\times\prod_{i<j\in R'}\frac{(1-q^{\lambda_i+\lambda_j}t^{3-i-j}s^2)}{(1-q^{\lambda_i+\lambda_j}t^{2-i-j}s^2)}\prod_{i\in R',j\in R}\frac{(1-q^{\lambda_i-\lambda_j}t^{j+1-i})}{(1-q^{\lambda_i-\lambda_j}t^{j-i})}\prod_{i<j\in R}\frac{(1-q^{-\lambda_i-\lambda_j}t^{i+j-1}s^{-2})}{(1-q^{-\lambda_i-\lambda_j}t^{i+j-2}s^{-2})}, \tag{4.43}$$

where

$$R=\{i: i\in\{1,2,\dots n\}|\lambda_i=\kappa_i+1\} \tag{4.44}$$
$$R'=\{i: i\in\{1,2,\dots n\}|\lambda_i=\kappa_i\}. \tag{4.45}$$

We have, for instance,

$$\frac{\prod_{i<j\in R}(1-q^{-\lambda_i-\lambda_j}t^{i+j-1}s^{-2})}{\prod_{i\le j\in R}(1-q^{-\lambda_i-\lambda_j}t^{i+j-2}s^{-2})}\propto\frac{\prod_{i<j\in R}(1-q^{\lambda_i+\lambda_j}t^{1-i-j}s^2)}{\prod_{i\le j\in R}(1-q^{\lambda_i+\lambda_j}t^{2-i-j}s^2)} \tag{4.46}$$

$$=\prod_{i\in R}\prod_{1\le k\le\kappa_i}\prod_{\substack{j\in R\\ \lambda_j=k}}\frac{1-q^{\lambda_i+k}t^{1-i-j}s^2}{1-q^{\lambda_i+k}t^{2-i-j}s^2}$$
$$\times\prod_{1\le k}\frac{\prod_{\kappa'_k<i<j\le\lambda'_k}(1-q^{2k}t^{1-i-j}s^2)}{\prod_{\kappa'_k<i\le j\le\lambda'_k}(1-q^{2k}t^{2-i-j}s^2)} \tag{4.47}$$

$$=\prod_{i\in R}\prod_{1\le k\le\kappa_i}\frac{1-q^{\lambda_i+k}t^{1-\lambda'_k-i}s^2}{1-q^{\lambda_i+k}t^{1-\kappa'_k-i}s^2}\prod_{1\le k}\prod_{\substack{j\in R\\ \lambda_j=k}}\frac{1}{1-q^{\lambda_j+k}t^{1-\kappa'_k-j}s^2} \tag{4.48}$$

$$=\prod_{i\in R}\frac{\prod_{1\le j\le\kappa_i}(1-q^{\lambda_i+j}t^{1-\lambda'_j-i}s^2)}{\prod_{1\le j\le\lambda_i}(1-q^{\kappa_i+1+j}t^{1-\kappa'_j-i}s^2)}, \tag{4.49}$$

where the constant of proportionality is

$$\frac{\prod_{i<j\in R}(-q^{-\lambda_i-\lambda_j}t^{i+j-1}s^{-2})}{\prod_{i\le j\in R}(-q^{-\lambda_i-\lambda_j}t^{i+j-2}s^{-2})}=t^{|R|(|R|-1)/2}\prod_{i\in R}(-q^{2\lambda_i}t^{2-2i}s^2). \tag{4.50}$$

We thus obtain two of the factors of (4.40). Here we used the fact that $i\in R$ and $\lambda_i=k$ if and only if $\kappa'_k<i\le\lambda'_k$; similarly, $i\in R'$ and $\lambda_i=k$ if and only if $\lambda_{k+1}<i\le\kappa'_k$. The remaining simplifications are analogous.

The second equation follows in a similar way; here we substitute $(s,u)\mapsto(q^{-m}/ts, tq^{-m}/u)$. Alternatively, it follows immediately from the first via inversion. □

Dualizing the difference equations (via Corollary 4.3) gives "integral equations".

Corollary 4.13. *For any partition $\mu\subset\lambda$,*

$$\psi^{(i)}_{\mu/\mu}(u;q,t,s)\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s\sqrt{t}}=\sum_{\kappa'\prec\lambda'}\psi^{(i)}_{\lambda/\kappa}(u;q,t,s)\begin{bmatrix}\kappa\\ \mu\end{bmatrix}_{q,t,s} \tag{4.51}$$
$$\psi^{(i)}_{\lambda/\lambda}(u;q,t,s)\begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,s}=\sum_{\nu'\succ\mu'}\psi^{(i)}_{\nu/\mu}(u;q,t,s)\begin{Bmatrix}\lambda\\ \nu\end{Bmatrix}_{q,t,s\sqrt{t}}, \tag{4.52}$$

where

$$\psi^{(i)}_{\lambda/\kappa}(u;q,t,s) = (u/t)^{|\lambda|-|\kappa|} t^{n(\kappa)-n(\lambda)} \frac{C^0_\lambda(s^2qt/u;q,t)C^0_\kappa(u/t;q,t)}{C^0_\lambda(u;q,t)C^0_\kappa(s^2qt/u;q,t)}$$
$$\prod_{\substack{(i,j)\in\lambda\\ \lambda'_j=\kappa'_j}} \frac{1-q^{\lambda_i+j-1}t^{-\lambda'_j-i+3}s^2}{1-q^{\kappa_i-j+1}t^{\kappa'_j-i}} \prod_{\substack{(i,j)\in\lambda\\ \lambda'_j\ne\kappa'_j}} \frac{1-q^{\lambda_i-j}t^{\lambda'_j-i+1}}{1-q^{\kappa_i+j}t^{-\kappa'_j-i+1}s^2}$$
$$\prod_{\substack{(i,j)\in\kappa\\ \lambda'_j=\kappa'_j}} \frac{1-q^{\lambda_i-j+1}t^{\lambda'_j-i}}{1-q^{\kappa_i+j-1}t^{2-\kappa'_j-i}s^2} \prod_{\substack{(i,j)\in\kappa\\ \lambda'_j\ne\kappa'_j}} \frac{1-q^{\lambda_i+j}t^{2-\lambda'_j-i}s^2}{1-q^{\kappa_i-j}t^{\kappa'_j-i+1}} \tag{4.53}$$

$$\psi^{(i)}_{\lambda/\lambda}(u;q,t,s) = \frac{C^0_\lambda(u/t;q,t)}{C^0_\lambda(u;q,t)} \frac{C^+_\lambda(s^2t)}{C^+_\lambda(s^2)} \tag{4.54}$$

Remark. These identities can be analytically continued to give a one-parameter family of integral equations for interpolation polynomials, having the integral representation of [16] as the case $u = t^n$, $\ell(\mu) < n$. In fact, we discovered these integral equations first (in order to prove Theorem 5.20), then deduced the likely existence of difference equations via duality. As the integral operators are rather complicated, and unnecessary for our purposes, we omit the details, and note simply that they correspond to the operators defined in [18] via contour integrals.

5 Koornwinder polynomials

Definition 2. The *Koornwinder polynomials* are the unique family of BC_n-symmetric polynomials

$$K^{(n)}_\lambda(;q,t;t_0,t_1,t_2,t_3) \tag{5.1}$$

such that

- (Triangularity) $K^{(n)}_\lambda(;q,t;t_0,t_1,t_2,t_3) = m_\lambda +$ dominated terms.
- (Evaluation symmetry) For any pair of partitions $\mu < \lambda$,

$$\frac{K^{(n)}_\lambda(q^{\mu_i}t^{n-i}t_0;q,t;t_0,t_1,t_2,t_3)}{k^0_\lambda(q,t,t^n;t_0{:}t_1,t_2,t_3)} = \frac{K^{(n)}_\mu(q^{\lambda_i}t^{n-i}\hat{t}_0;q,t;\hat{t}_0,\hat{t}_1,\hat{t}_2,\hat{t}_3)}{k^0_\mu(q,t,t^n;\hat{t}_0{:}\hat{t}_1,\hat{t}_2,\hat{t}_3)}, \tag{5.2}$$

  where

$$\hat{t}_0 = \sqrt{t_0t_1t_2t_3/q}; \qquad \hat{t}_i = t_0t_i/\hat{t}_0,\ i\in\{1,2,3\} \tag{5.3}$$

$$k^0_\lambda(q,t,T;t_0{:}t_1,t_2,t_3) = (t_0T/t)^{-|\lambda|}t^{n(\lambda)} \frac{C^0_\lambda(T,Tt_0t_1/t,Tt_0t_2/t,Tt_0t_3/t;q,t)}{C^-_\lambda(t;q,t)C^+_\lambda(T^2\hat{t}_0^2/t^2;q,t)}. \tag{5.4}$$

This differs from the definition in the literature (in which evaluation symmetry is replaced by orthogonality with respect to the Koornwinder inner product); that our definition is equivalent to the usual definition will be shown below.

Theorem 5.1. *The Koornwinder polynomials are well-defined, and are given by the expansion*

$$K_\lambda^{(n)}(;q,t;t_0,t_1,t_2,t_3) = \sum_{\mu\subset\lambda} \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,t^{n-1}\hat{t}_0} \frac{k_\lambda^0(q,t,t^n;t_0{:}t_1,t_2,t_3)}{k_\mu^0(q,t,t^n;t_0{:}t_1,t_2,t_3)} \bar{P}_\mu^{*(n)}(;q,t;t_0). \tag{5.5}$$

Proof. A monic triangular BC_n-symmetric polynomial with leading term m_λ is uniquely determined by its values at $q^{\mu_i}t^{n-i}t_0$ for $\mu < \lambda$. Indeed, we can write it as

$$m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}\bar{P}_\mu^{*(n)}, \tag{5.6}$$

where the coefficients $c_{\lambda\mu}$ are determined by a triangular system of linear equations with nonzero diagonal.

We find that, if the expansion holds for all $\mu < \lambda$, then

$$\frac{K_\lambda^{(n)}(q^{\mu_i}t^{n-i}t_0;q,t;t_0,t_1,t_2,t_3)}{k_\lambda^0(q,t,t^n;t_0{:}t_1,t_2,t_3)} = \frac{K_\mu^{(n)}(q^{\lambda_i}t^{n-i}\hat{t}_0;q,t;\hat{t}_0,\hat{t}_1,\hat{t}_2,\hat{t}_3)}{k_\mu^0(q,t,t^n;\hat{t}_0{:}\hat{t}_1,\hat{t}_2,\hat{t}_3)} \tag{5.7}$$

$$= \sum_{\nu\subset\mu} \begin{bmatrix}\mu\\ \nu\end{bmatrix}_{q,t,t^{n-1}t_0} \begin{bmatrix}\lambda\\ \nu\end{bmatrix}_{q,t,t^{n-1}\hat{t}_0} \frac{\bar{P}_\nu^{*(n)}(\nu;q,t,\hat{t}_0)}{k_\nu^0(q,t,t^n;\hat{t}_0{:}\hat{t}_1,\hat{t}_2,\hat{t}_3)} \tag{5.8}$$

$$= \sum_{\nu\subset\lambda} \begin{bmatrix}\mu\\ \nu\end{bmatrix}_{q,t,t^{n-1}t_0} \begin{bmatrix}\lambda\\ \nu\end{bmatrix}_{q,t,t^{n-1}\hat{t}_0} \frac{\bar{P}_\nu^{*(n)}(\nu;q,t,t_0)}{k_\nu^0(q,t,t^n;t_0{:}t_1,t_2,t_3)} \tag{5.9}$$

$$= \sum_{\nu\subset\lambda} \begin{bmatrix}\lambda\\ \nu\end{bmatrix}_{q,t,t^{n-1}t_0} \frac{\bar{P}_\nu^{*(n)}(\mu;q,t,t_0)}{k_\nu^0(q,t,t^n;t_0{:}t_1,t_2,t_3)}, \tag{5.10}$$

where the second-to-last step follows from the fact that

$$k_\nu^0(q,t,t^n;t_0{:}t_1,t_2,t_3)\bar{P}_\nu^{*(n)}(\nu;q,t,\hat{t}_0) \tag{5.11}$$

depends on $t_0,\dots t_3$ only through their pairwise products, and is thus preserved by the "hat" involution. In particular, both sides of (5.5) are monic triangular with leading term λ, and agree at $q^{\mu_i}t^{n-i}t_0$ whenever $\mu < \lambda$. The identity follows. ☐

Remark 1. This, of course, is essentially Okounkov's binomial formula [16]; the difference is that the principal specializations in Okounkov's formula have been replaced with the appropriate product. Note that in the univariate case, this is precisely the expansion of an Askey-Wilson polynomial as a ${}_4\phi_3$ [1].

Remark 2. We will tend to avoid the $\hat{t}_i$ notation in the sequel, as it is really only suited to contexts in which the parameters are fixed; in other contexts, it can lead to serious ambiguities.

Corollary 5.2. *The evaluation symmetry property holds without restriction on μ and λ.*

Corollary 5.3. *For any partition λ, the only possible factors of the denominators of the coefficients of the polynomial*

$$C_\lambda^+(t^{2n-2}t_0t_1t_2t_3/q;q,t)\bar{K}_\lambda(;q,t;t_0,t_1,t_2,t_3) \tag{5.12}$$

are binomials of the form $1-q^it^j$ for $i,j\geq 0$.

Proof. Taking into account the denominator factors introduced by the binomial coefficient and the interpolation polynomial in the binomial formula, we conclude that the only possible other denominator factors are q, t, t_0. Now, the Koornwinder inner product is well-defined whenever any of $q = 0$, $t = 0$, or $t_0 = 0$; thus Theorem 5.8 below implies that q and t and t_0 are not denominator factors of $K^{(n)}$. ☐

The symmetries of interpolation polynomials induce corresponding symmetries of Koornwinder polynomials.

Corollary 5.4. *For any partition λ,*

$$K^{(n)}_\lambda(;q,t;t_0,t_1,t_2,t_3) = K^{(n)}_\lambda(;1/q,1/t;1/t_0,1/t_1,1/t_2,1/t_3) \tag{5.13}$$

$$K^{(n)}_\lambda(;q,t;t_0,t_1,t_2,t_3) = (-1)^{|\lambda|}K^{(n)}_\lambda(-x;q,t;-t_0,-t_1,-t_2,-t_3) \tag{5.14}$$

(5.15)

A further symmetry follows from the q-Saalschütz formula.

Theorem 5.5. *For any partition λ,*

$$K^{(n)}_\lambda(;q,t;t_0,t_1,t_2,t_3) = K^{(n)}_\lambda(;q,t;t_1,t_0,t_2,t_3). \tag{5.16}$$

Thus $K^{(n)}_\lambda$ is invariant under permutations of t_0, t_1, t_2, t_3.

Proof. For all κ, the coefficient of $P^{(n)}_\kappa(;q,t;t_1)$ in both sides is the same. ☐

Remark. This is another multivariate analogue of Sears' ${}_4\phi_3$ transformation.

We will refer to this fact as "parameter symmetry".

Before delving further into the properties of the Koornwinder polynomials as we have defined them, we first must justify the name. We could of course simply refer to the proofs in the literature [27, 22, 23] that the usual Koornwinder polynomials satisfy evaluation symmetry; instead, we will give a direct proof (in particular, avoiding any use of double affine Hecke algebra machinery). We will, in fact, give two proofs. The first uses difference operators to show orthogonality with respect to the Koornwinder weight function, while the second uses our generalized hypergeometric transformations to show orthogonality with respect to the q-Racah weight function.

First, recall the difference operators $D^{(n)}(u_1,u_2;q,t)$ of Definition 1. These act on our polynomials $K^{(n)}$ as follows.

Lemma 5.6. *For any integer n and partition λ with $\ell(\lambda) \le n$,*

$$D^{(n)}(t_0,t_1;q,t)K^{(n)}_\lambda(;q,t;t_0\sqrt{q},t_1\sqrt{q},t_2,t_3) = E^{(n)}_\lambda(t_0t_1;q,t)K^{(n)}_\lambda(;q,t;t_0,t_1,t_2\sqrt{q},t_3\sqrt{q}), \tag{5.17}$$

where

$$E^{(n)}_\lambda(u;q,t) = q^{-|\lambda|/2}\prod_{1\le i\le n}(1-q^{\lambda_i}t^{n-i}u). \tag{5.18}$$

Proof. By the binomial formula, we have:

$$D^{(n)}(t_0,t_1;q,t)K^{(n)}_\lambda(;q,t;t_0\sqrt{q},t_1\sqrt{q},t_2,t_3)$$
$$=\sum_{\mu\subset\lambda}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,t^{n-1}\sqrt{t_0t_1t_2t_3}}\frac{k^0_\lambda(q,t,t^n;t_0\sqrt{q}{:}t_1\sqrt{q},t_2,t_3)}{k^0_\mu(q,t,t^n;t_0\sqrt{q}{:}t_1\sqrt{q},t_2,t_3)}D^{(n)}(t_0,t_1;q,t)\bar{P}^*_\mu(;q,t;t_0\sqrt{q}) \tag{5.19}$$
$$=\sum_{\mu\subset\lambda}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,t^{n-1}\sqrt{t_0t_1t_2t_3}}\frac{k^0_\lambda(q,t,t^n;t_0\sqrt{q}{:}t_1\sqrt{q},t_2,t_3)}{k^0_\mu(q,t,t^n;t_0\sqrt{q}{:}t_1\sqrt{q},t_2,t_3)}E^{(n)}_\mu(t_0t_1;q,t)\bar{P}^*_\mu(;q,t;t_0). \tag{5.20}$$

Since

$$k^0_\mu(q,t,t^n;t_0\sqrt{q}{:}t_1\sqrt{q},t_2,t_3)=\frac{E^{(n)}_\mu(t_0t_1;q,t)}{\prod_{1\le i\le n}(1-t^{n-i}t_0t_1)}k^0_\mu(q,t,t^n;t_0{:}t_1,t_2\sqrt{q},t_3\sqrt{q}), \tag{5.21}$$

the result follows. ☐

Remark. In particular, we find that the polynomials $K^{(n)}_\mu(;q,t;t_0,t_1,t_0\sqrt{q},t_1\sqrt{q})$ are eigenfunctions of the operator $D^{(n)}(t_0,t_1;q,t)$, and thus are BC_n/C_n-Macdonald polynomials. More generally, the polynomials $K^{(n)}_\mu(;q,t;t_0,t_1,t_2,t_3)$ are eigenfunctions of $D^{(n)}(t_0,t_1;q,t)D^{(n)}(q^{-1/2}t_2,q^{-1/2}t_3;q,t)$.

Let $w^{(n)}_K(;q,t;t_0,t_1,t_2,t_3)$ denote the Koornwinder weight function [11]

$$w^{(n)}_K(x_1,x_2,\dots x_n;q,t;t_0,t_1,t_2,t_3)=\prod_{1\le i\le n}\frac{(x_i^{\pm 2};q)}{(t_0x_i^{\pm1},t_1x_i^{\pm1},t_2x_i^{\pm1},t_3x_i^{\pm1};q)}\prod_{1\le i<j\le n}\frac{(x_i^{\pm1}x_j^{\pm1};q)}{(tx_i^{\pm1}x_j^{\pm1};q)}. \tag{5.22}$$

Also, for a multivariate function f analytic in a neighborhood of the unit torus (the locus in which all variables have magnitude 1), $\int f d\mathbb{T}$ denotes the integral of f with respect to the uniform density on the torus; i.e.,

$$\int f d\mathbb{T}=\int_{[-\pi,\pi]^n}f(e^{i\theta_1},e^{i\theta_2},\dots e^{i\theta_n})\prod_j\frac{d\theta_j}{2\pi} \tag{5.23}$$
$$=\int_{|x_j|=1}f(x_1,x_2,\dots x_n)\prod_j\frac{dx_j}{2\pi i x_j}, \tag{5.24}$$

or equivalently the constant coefficient of the Laurent expansion of f.

A straightforward adaptation of the standard adjointness argument for BC_n/C_n-Macdonald polynomials [13] proves the following.

Lemma 5.7. *Let q, t, t_0, t_1, t_2, t_3 be arbitrary complex numbers of magnitude <1. Then for any integer n and BC_n-symmetric polynomials f, g,*

$$\int (D^{(n)}(t_0,t_1;q,t)g)f\,w^{(n)}(;q,t;t_0,t_1,t_2\sqrt{q},t_3\sqrt{q})d\mathbb{T}=\int (D^{(n)}(t_2,t_3;q,t)f)g\,w^{(n)}(;q,t;t_2,t_3,t_0\sqrt{q},t_1\sqrt{q})d\mathbb{T}.$$

Proof. Factor the two weight functions as:

$$w^{(n)}(x_1,x_2,\dots x_n;q,t;t_0,t_1,t_2\sqrt{q},t_3\sqrt{q})=\Delta^{(n)}_1(x_1,x_2,\dots x_n)\Delta^{(n)}_1(x_1^{-1},x_2^{-1},\dots x_n^{-1}) \tag{5.25}$$
$$w^{(n)}(x_1,x_2,\dots x_n;q,t;t_2,t_3,t_0\sqrt{q},t_1\sqrt{q})=\Delta^{(n)}_2(x_1,x_2,\dots x_n)\Delta^{(n)}_2(x_1^{-1},x_2^{-1},\dots x_n^{-1}), \tag{5.26}$$

where

$$\Delta_1^{(n)}(x_1, x_2, \dots x_n) = \prod_{1\le i\le n} \frac{(x_i^2;q)}{(t_0x_i, t_1x_i, t_2\sqrt{q}x_i, t_3\sqrt{q}x_i;q)} \prod_{1\le i<j\le n} \frac{(x_ix_j^{\pm 1};q)}{(tx_ix_j^{\pm 1};q)} \tag{5.27}$$

$$\Delta_2^{(n)}(x_1, x_2, \dots x_n) = \prod_{1\le i\le n} \frac{(x_i^2;q)}{(t_0\sqrt{q}x_i, t_1\sqrt{q}x_i, t_2x_i, t_3x_i;q)} \prod_{1\le i<j\le n} \frac{(x_ix_j^{\pm 1};q)}{(tx_ix_j^{\pm 1};q)}. \tag{5.28}$$

The difference operators can then be expressed in the form

$$(D^{(n)}(t_0, t_1; q, t)g)(x_1, x_2, \dots x_n) = \sum_{\sigma\in\{\pm 1\}^n} \prod_{1\le i\le n} R_{x_i}(\sigma_i) \frac{(\Delta_2^{(n)}g)(\sqrt{q}x_1, \sqrt{q}x_2, \dots \sqrt{q}x_n)}{\Delta_1^{(n)}(x_1, x_2, \dots x_n)} \tag{5.29}$$

$$(D^{(n)}(t_2, t_3; q, t)f)(x_1, x_2, \dots x_n) = \sum_{\sigma\in\{\pm 1\}^n} \prod_{1\le i\le n} R_{x_i}(\sigma_i) \frac{(\Delta_1^{(n)}f)(\sqrt{q}x_1, \sqrt{q}x_2, \dots \sqrt{q}x_n)}{\Delta_2^{(n)}(x_1, x_2, \dots x_n)}, \tag{5.30}$$

where as in the proof of Lemma 3.1, $R_{x_i}(\pm 1)$ are homomorphisms defined by $R_{x_i}(\pm 1)x_j = x_j$, $R_{x_i}(\pm 1)x_i = x_i^{\pm 1}$. Since f, w, and $d\mathbb{T}$ are preserved by these operators, we find:

$$\int (D^{(n)}(t_0, t_1; q, t)g) f\, w^{(n)}(;q, t; t_0, t_1, t_2\sqrt{q}, t_3\sqrt{q}) d\mathbb{T}$$

$$= 2^n \int (\Delta_1^{(n)}f)(x_1^{-1}, x_2^{-1}, \dots x_n^{-1})(\Delta_2^{(n)}g)(\sqrt{q}x_1, \sqrt{q}x_2, \dots \sqrt{q}x_n) d\mathbb{T} \tag{5.31}$$

$$= 2^n \int (\Delta_1^{(n)}f)(\sqrt{q}x_1, \sqrt{q}x_2, \dots \sqrt{q}x_n)(\Delta_2^{(n)}g)(x_1^{-1}, x_2^{-1}, \dots x_n^{-1}) d\mathbb{T} \tag{5.32}$$

$$= \int (D^{(n)}(t_2, t_3; q, t)f) g\, w^{(n)}(;q, t; t_2, t_3, t_0\sqrt{q}, t_1\sqrt{q}) d\mathbb{T}. \tag{5.33}$$

□

Theorem 5.8. *Let q, t, t_0, t_1, t_2, t_3 be complex numbers of magnitude at most 1. Then the polynomials $K_\mu^{(n)}(;q, t; t_0, t_1, t_2, t_3)$ are orthogonal with respect to the density $w_K^{(n)}(;q, t; t_0, t_1, t_2, t_3)$ on the unit torus.*

Proof. On the one hand, the polynomials $K_\mu^{(n)}(;q, t; t_0, t_1, t_2, t_3)$ are eigenfunctions of the difference operator

$$D^{(n)}(t_0, t_1; q, t) D^{(n)}(q^{-1/2}t_2, q^{-1/2}t_3; q, t), \tag{5.34}$$

with generically distinct eigenvalues. On the other hand, this difference operator is self-adjoint with respect to the Koornwinder weight. Orthogonality follows immediately. □

Remark. In particular, we have shown that our Koornwinder polynomials agree with the usual Koornwinder polynomials, and thus the latter satisfy evaluation symmetry.

We now turn to the q-Racah case. Following [30], suppose $t_0t_1 = t^{1-n}q^{-m}$, and define a function Δ^{qR} on partitions $\mu \subset m^n$ by

$$\Delta^{qR}(\mu) = q^{-2n(\mu')} t^{2n(\mu)} (t^{2n-2}qt_0^2)^{-|\mu|} \frac{C_\mu^0(t^{n-1}t_0t_2, t^{n-1}t_0t_3; q, t) C_{m^n-\mu}^0(t^{n-1}t_1t_2, t^{n-1}t_1t_3; q, t)\langle P_\mu, P_\mu\rangle_n''}{\bar{P}_\mu^{*(n)}(\mu; q, t, t_0)\bar{P}_{m^n-\mu}^{*(n)}(m^n-\mu; q, t, t_1)}. \tag{5.35}$$

Aside from an overall constant, this is the weight function for the multivariate q-Racah polynomials of [30]. If we define a linear functional on BC_n-symmetric functions by

$$\langle f\rangle_{qR} = \sum_{\mu\subset m^n} f(q^{\mu_i}t^{n-i}t_0)\Delta^{qR}(\mu), \tag{5.36}$$

then

$$\langle \bar{P}^{*(n)}_\kappa(;q,t,t_0)\bar{P}^{*(n)}_{m^n-\lambda}(;q,t,t_1)\rangle_{qR} = \sum_{\mu\subset m^n} \Delta^{qR}(\mu)\bar{P}^*_\kappa(\mu;q,t,t_0)\bar{P}^*_{m^n-\lambda}(m^n-\mu;q,t,t_1) \tag{5.37}$$

$$= \frac{\bar{P}^{*(n)}_\kappa(\kappa;q,t,t_0)\langle P_\lambda,P_\lambda\rangle''_n}{\bar{P}^{*(n)}_\lambda(\lambda;q,t,t_0)} \sum_{\mu\subset m^n} \frac{(q/t_2t_3)^{|\mu|}C^0_\mu(t^{n-1}t_0t_2,t^{n-1}t_0t_3;q,t)}{C^0_\mu(t^{n-1}qt_0/t_2,t^{n-1}qt_0/t_3;q,t)} \begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,t_0t^{n-1}} \begin{bmatrix}\mu\\ \kappa\end{bmatrix}_{q,t,t_0t^{n-1}}. \tag{5.38}$$

This sums via Corollary 4.6 to give

$$\langle \bar{P}^{*(n)}_\kappa(;q,t,t_0)\bar{P}^{*(n)}_{m^n-\lambda}(;q,t,t_1)\rangle_{qR} \propto \frac{k^0_\kappa(q,t,t^n;t_0,t_1,t_2,t_3)P^{*(n)}_\kappa(\kappa;q,t;\hat{t}_0)}{\hat{t}_0^{|\kappa|}C^0_\kappa(t^n;q,t)C^0_\kappa(q^m;1/q,1/t)} P_{\lambda/\kappa}([\frac{1-(q/t_2t_3)^k}{1-t^k}];q,t), \tag{5.39}$$

where the constant of proportionality is independent of κ, and $\hat{t}_0 = \sqrt{t_0t_1t_2t_3/q}$ as usual. Now, from the binomial formula, we find

$$\langle K^{(n)}_{m^n-\mu}(;q,t;t_0,t_1,t_2,t_3)\bar{P}^{*(n)}_{m^n-\lambda}(;q,t,t_1)\rangle_{qR}$$

$$\propto \sum_{\kappa\subset\lambda} P^{*(n)}_\kappa(m^n-\mu;q,t,\hat{t}_0)\frac{C^0_\lambda(t^n;q,t)C^0_\lambda(q^m;1/q,1/t)}{C^0_\kappa(t^n;q,t)C^0_\kappa(q^m;1/q,1/t)} P_{\lambda/\kappa}([\frac{\hat{t}_0^k-\hat{t}_1^k}{1-t^k}];q,t) \tag{5.40}$$

$$\propto P^{*(n)}_\lambda(\mu;q,t,\hat{t}_1), \tag{5.41}$$

where $\hat{t}_1 = t^{1-n}q^{-m}/\hat{t}_0$, and 5.41 follows from the connection coefficient identity for interpolation polynomials. In particular,

$$\langle K^{(n)}_{m^n-\mu}(;q,t;t_0,t_1,t_2,t_3)\bar{P}^{*(n)}_{m^n-\lambda}(;q,t,t_1)\rangle_{qR} = 0 \tag{5.42}$$

unless $\lambda\subset\mu$; it follows that the polynomials $K^{(n)}_\mu(;q,t;t_0,t_1,t_2,t_3)$ are orthogonal with respect to the given inner product. If we keep track of the constants when $\lambda=\mu$, we obtain the following theorem.

Theorem 5.9. *[30] If $t_0t_1 = t^{1-n}q^{-m}$ and $\lambda\subset m^n$, then*

$$\frac{\langle K^{(n)}_\lambda(;q,t;t_0,t_1,t_2,t_3)K^{(n)}_\lambda(;q,t;t_1,t_0,t_2,t_3)\rangle_{qR}}{\langle 1\rangle_{qR}} = \delta_{\lambda\mu}N_\lambda(;q,t,t^n;t_0,t_1,t_2,t_3), \tag{5.43}$$

where

$$N_\lambda(;q,t,T;t_0,t_1,t_2,t_3) = \frac{C^-_\lambda(q;q,t)C^+_\lambda(T^2t_0t_1t_2t_3/t^3;q,t)C^0_\lambda(T,Tt_0t_1t_2t_3/t^2;q,t)\prod_{0\le i<j\le 3}C^0_\lambda(Tt_it_j/t;q,t)}{C^-_\lambda(t;q,t)C^+_\lambda(T^2t_0t_1t_2t_3/qt^2;q,t)C^0_{2\lambda^2}(T^2t_0t_1t_2t_3/t^2;q,t)} \tag{5.44}$$

Remark 1. The proof of [30] was based on the usual definition of Koornwinder polynomials, and involved showing that Koornwinder's difference operator is self-adjoint with respect to the q-Racah inner product. One can presumably construct a similar proof based on our difference operator.

Remark 2. Except for the formula for the norm, we could have derived this from orthogonality with respect to the Koornwinder weight, by reference to the results of [24].

Remark 3. The normalization $\langle 1\rangle_{qR}$ can be computed using the rectangle case of the generalized ${}_6\phi_5$ sum: take $\kappa = 0$, $\lambda = m^n$ above. (This normalization was first computed in [28] by proving this case of the ${}_6\phi_5$ sum.)

Remark 4. Similarly, we can express the inner product

$$\langle \bar{P}^{*(n)}_{\kappa}(;q,t,t_0)\bar{P}^{*(n)}_{m^n-\lambda}(;q,t,t_1)\prod_{1\le i\le n}\frac{(ux_i,u/x_i;q)}{(vx_i,v/x_i;q)}\rangle_{qR} \tag{5.45}$$

as a generalized very-well-poised ${}_8\phi_7$; that is, in terms of the special case

$$(a;b,c,d,e)\mapsto (t^{2n-2}t_0^2;t^{n-1}t_0t_2,t^{n-1}t_0t_3,t^{n-1}t_0u,qt_0t^{n-1}/v) \tag{5.46}$$

of Theorem 4.7. Is there a similar interpretation of this sum without the q-Racah constraint $t_0t_1 = t^{1-n}q^{-m}$?

If we define the *virtual Koornwinder integral*

$$I^{(n)}_K(f;q,t;t_0,t_1,t_2,t_3) := [K^{(n)}_0(;q,t;t_0,t_1,t_2,t_3)]f, \tag{5.47}$$

for any BC_n-symmetric function f, the two orthogonality results imply the following expressions.

Corollary 5.10. *For q, t, $t_0\dots t_3$ of magnitude <1, and all BC_n-symmetric polynomials f,*

$$I^{(n)}_K(f;q,t;t_0,t_1,t_2,t_3) = Z^{-1}\int f(x_1,x_2,\dots x_n)w^{(n)}_K(x;q,t;t_0,t_1,t_2,t_3)d\mathbb{T}, \tag{5.48}$$

where

$$Z=\int w^{(n)}_K(x;q,t;t_0,t_1,t_2,t_3)d\mathbb{T}. \tag{5.49}$$

Similarly, if $t_0t_1 = t^{1-n}q^{-m}$, then

$$I^{(n)}_K(f;q,t;t_0,t_1,t_2,t_3) = \frac{\langle f\rangle_{qR}}{\langle 1\rangle_{qR}}. \tag{5.50}$$

As the virtual integral is a rational function of the parameters, we also conclude:

Corollary 5.11. *For all partitions λ, μ,*

$$I^{(n)}_K(K^{(n)}_\lambda(;q,t;t_0,t_1,t_2,t_3)K^{(n)}_\mu(;q,t;t_0,t_1,t_2,t_3);q,t;t_0,t_1,t_2,t_3) = \delta_{\lambda\mu}N_\lambda(;q,t,t^n;t_0,t_1,t_2,t_3), \tag{5.51}$$

with N_λ as above.

Remark. An alternative derivation of this (assuming evaluation symmetry) was given in [27].

The inversion formula for generalized binomial coefficients allows us to invert the binomial formula.

Theorem 5.12. *For any partition λ, we have the expansion*

$$\bar{P}^{*(n)}_\lambda(;q,t,t_0) = \sum_{\mu\subset\lambda}\begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,t^{n-1}\sqrt{t_0t_1t_2t_3/q}}\frac{k^0_\lambda(q,t,t^n;t_0{:}t_1,t_2,t_3)}{k^0_\mu(q,t,t^n;t_0{:}t_1,t_2,t_3)}K^{(n)}_\mu(;q,t;t_0,t_1,t_2,t_3) \tag{5.52}$$

This has two interesting consequences. First, we obtain a connection coefficient formula for Koornwinder polynomials, by expanding the interpolation polynomial in the binomial formula using the inverse binomial formula.

Theorem 5.13. *For any partitions $\kappa \subset \lambda$,*

$$\left[\frac{K^{(n)}_\kappa(;q,t;t_0,t_1,t_2,t_3)}{k^0_\kappa(q,t,t^n;t_0{:}t_1,t_2,t_3)}\right]\frac{K^{(n)}_\lambda(;q,t;t_0,t'_1,t'_2,t'_3)}{k^0_\lambda(q,t,t^n;t_0{:}t'_1,t'_2,t'_3)} = \sum_{\kappa\subset\mu\subset\lambda}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,t^{n-1}\sqrt{t_0t'_1t'_2t'_3/q}}\begin{Bmatrix}\mu\\ \kappa\end{Bmatrix}_{q,t,t^{n-1}\sqrt{t_0t_1t_2t_3/q}}\frac{k^0_\mu(q,t,t^n;t_0{:}t_1,t_2,t_3)}{k^0_\mu(q,t,t^n;t_0{:}t'_1,t'_2,t'_3)}. \tag{5.53}$$

Remark. If $t'_2 = t_2$, $t'_3 = t_3$, so only t_1 is changed, then the corresponding sum for Askey-Wilson polynomials has a closed form evaluation; it turns out that something similar is true for Koornwinder polynomials, in that the sum can be evaluated in terms of the generalized binomial coefficients of [17]. See also Theorem 5.20 below.

Second, we obtain an integral formula generalizing Kadell's formula (and the q-analogue) for the (normalized) integral of a Macdonald polynomial over a Jacobi ensemble.

Corollary 5.14. *For any partition λ, one has the following virtual integral.*

$$I^{(n)}_K(\bar{P}^{*(n)}_\lambda(;q,t,t_0);q,t;t_0,t_1,t_2,t_3) = (-t_0t^{n-1})^{-|\lambda|}t^{2n(\lambda)}q^{-n(\lambda')}\frac{C^0_\lambda(t^n,t^{n-1}t_0t_1,t^{n-1}t_0t_2,t^{n-1}t_0t_3;q,t)}{C^-_\lambda(t;q,t)C^0_\lambda(t^{2n-2}t_0t_1t_2t_3;q,t)} \tag{5.54}$$

This allows us to prove the following integral representation for ${}_8W^{(n)}_7$ series.

Theorem 5.15. *Choose $t_0,t_1,t_2,t_3,t_4,q,t,u\in C$ such that* $\max(|t_i|,|q|,|t|)<1$. *Then*

$$I^{(n)}_K\Big(\prod_{1\le i\le n}\frac{(ux_i,u/x_i;q)}{(t_4x_i,t_4/x_i;q)};q,t;t_0,t_1,t_2,t_3\Big) = \prod_{0\le i<n}\frac{(t^{-i}u't'_0,t^{-i}u't'_1,t^{-i}u't'_2,t^{-i}t'_0t'_1t'_2t'_4;q)}{(t^{-i}t'_0t'_4,t^{-i}t'_1t'_4,t^{-i}t'_2t'_4,t^{-i}u't'_0t'_1t'_2;q)}\, {}_8W^{(n)}_7(u't'_0t'_1t'_2/q;t'_0t'_1,t'_0t'_2,t'_1t'_2,u'/t'_3,u'/t'_4;q,t;t^{1-n}t'_3t'_4), \tag{5.55}$$

where $t'_i = t^{(n-1)/2}t_i$, $u' = t^{(n-1)/2}u$.

Proof. First, suppose that $t_4 = q^m u$. Then

$$\prod_{1\le i\le n}\frac{(ux_i,u/x_i;q)}{(t_4x_i,t_4/x_i;q)} = (-u)^{mn}q^{n(n^m)}\prod_{\substack{1\le i\le m\\1\le j\le n}}(x_j+1/x_j-q^{i-1}u-q^{1-i}/u) \tag{5.56}$$

$$= (-u)^{mn}q^{n(n^m)}\bar{P}^{*(n)}_{m^n}(x;q,t,u) \tag{5.57}$$

$$= C^0_{m^n}(t^{n-1}t_0u,u/t_0;q,t)\sum_{\lambda\subset m^n}\frac{q^{n(\lambda')}(-t^{n-1}t_0q)^{|\lambda|}C^0_\lambda(q^{-m};q,t)\bar{P}^{*(n)}_\lambda(x;q,t,t_0)}{C^0_\lambda(t^{n-1}t_0u,t^{n-1}q^{1-m}t_0/u;q,t)C^-_\lambda(q;q,t)}. \tag{5.58}$$

Integrating both sides, we find that

$$
\begin{aligned}
I_K^{(n)}(&\prod_{1\le i\le n}\frac{(ux_i,u/x_i;q)}{(q^m ux_i,q^m u/x_i;q)};q,t;t_0,t_1,t_2,t_3)\\
&= C^0_{m^n}(t^{n-1}t_0u,u/t_0;q,t)_4\Phi_3^{(n)}\left(\begin{matrix}q^{-m},t^{n-1}t_0t_1,t^{n-1}t_0t_2,t^{n-1}t_0t_3\\ t^{2n-2}t_0t_1t_2t_3,t^{n-1}t_0u,t^{n-1}q^{1-m}t_0/u\end{matrix};q,t;q\right) \qquad (5.59)\\
&= \frac{C^0_{m^n}(t^{n-1}ut_0,t^{n-1}ut_1,t^{n-1}ut_2;q,t)}{C^0_{m^n}(t^{2n-2}ut_0t_1t_2;q,t)}\\
&\quad {}_8W_7^{(n)}(t^{2n-2}ut_0t_1t_2/q;t^{n-1}t_0t_1,t^{n-1}t_0t_2,t^{n-1}t_1t_2,u/t_3,q^{-m};q,t;q^m ut_3). \qquad (5.60)
\end{aligned}
$$

Thus the desired formula holds when $t_4=q^m u$. The result follows from the fact that both sides are manifestly analytic in the stated domain. □

If we exchange t_2 and t_3, the integral is clearly unchanged; we thus obtain a transformation of ${}_8W_7^{(n)}$ series. Similarly, permuting the parameters of the ${}_8W_7^{(n)}$ leads to a transformation of the integral. The resulting symmetry groups are enlarged from S_5 to the Weyl group D_5, and there are thus a total of three different expressions for the integral, corresponding to the double cosets $S_5\backslash D_5/S_5$. The remaining two are:

$$
\begin{aligned}
&\prod_{0\le i<n}\frac{(t^{-i}u'/t'_4,t^{-i}u't'_4,t^{-i}t'_0t'_1t'_2t'_4,t^{-i}t'_0t'_1t'_3t'_4,t^{-i}t'_0t'_2t'_3t'_4,t^{-i}t'_1t'_2t'_3t'_4;q)}{(t^{-i}t'_0t'_1t'_2t'_3,t^{-i}t'_0t'_4,t^{-i}t'_1t'_4,t^{-i}t'_2t'_4,t^{-i}t'_3t'_4,t^{-i}t'_0t'_1t'_2t'_3t'^2_4;q)}\\
&\qquad {}_8W_7^{(n)}(t'_0t'_1t'_2t'_3t'^2_4/q;t'_0t'_4,t'_1t'_4,t'_2t'_4,t'_3t'_4,t'_0t'_1t'_2t'_3t'_4/u';q,t;t^{1-n}u'/t'_4),
\end{aligned} \qquad (5.61)
$$

subject to the convergence condition $|t^{1-n}u'/t'_4|<1$, and

$$
\begin{aligned}
&\prod_{0\le i<n}\frac{(t^{-i}t'_0u',t^{-i}t'_1u',t^{-i}t'_2u',t^{-i}t'_3u',t^{-i}t'_4u',t^{-i}t'_0t'_1t'_2t'_3t'_4/u';q)}{(t^{-i}t'_0t'_4,t^{-i}t'_1t'_4,t^{-i}t'_2t'_4,t^{-i}t'_3t'_4,t^{-i}u'^2,t^{-i}t'_0t'_1t'_2t'_3;q)}\\
&\qquad {}_8W_7^{(n)}(u'^2/q;u'/t'_0,u'/t'_1,u'/t'_2,u'/t'_3,u'/t'_4;q,t;t^{1-n}t'_0t'_1t'_2t'_3t'_4/u'),
\end{aligned} \qquad (5.62)
$$

subject to the convergence condition $|t^{1-n}t'_0t'_1t'_2t'_3t'_4/u'|<1$. (Compare equations (6.3.7-9) of [7].)

Corollary 5.16. *Let m, n be nonnegative integers. Then*

$$
\begin{aligned}
I_K^{(n)}(&\prod_{1\le i\le n}\frac{(t^{m/2}vx_i,t^{m/2}v/x_i;q)}{(t^{-m/2}vx_i,t^{-m/2}v/x_i;q)};q,t;t^{m/2}t_0,t^{m/2}t_1,t^{m/2}t_2,t^{m/2}t_3)\\
&= I_K^{(m)}(\prod_{1\le i\le m}\frac{(t^{n/2}vx_i,t^{n/2}v/x_i;q)}{(t^{-n/2}vx_i,t^{-n/2}v/x_i;q)};q,t;t^{n/2}t_0,t^{n/2}t_1,t^{n/2}t_2,t^{n/2}t_3).
\end{aligned} \qquad (5.63)
$$

Proof. Expand the integrals via (5.62), and use the fact that

$$
{}_8W_7^{(n)}(a;b,c,d,e,t^m;q,t;z)={}_8W_7^{(m)}(a;b,c,d,e,t^n;q,t;z). \qquad (5.64)
$$

□

Remark. The Jacobi limit is implicitly used in [6, Section 15.7].

Corollary 5.17. *Let m, n be nonnegative integers, and set $u = t^{2n+2m-2}t_0t_1t_2t_3t_4$. Then*

$$\prod_{0\le i<n} \frac{(t^it_0t_4, t^it_1t_4, t^it_2t_4, t^it_3t_4;q)}{(t^{-i}u/t_0, t^{-i}u/t_1, t^{-i}u/t_2, t^{-i}u/t_3;q)} I_K^{(n)}\Big(\prod_{1\le i\le n} \frac{(t^{-m/2}ux_i, t^{-m/2}u/x_i;q)}{(t^{m/2}t_4x_i, t^{m/2}t_4/x_i;q)}; q, t; t^{m/2}t_0, t^{m/2}t_1, t^{m/2}t_2, t^{m/2}t_3\Big) \tag{5.65}$$

is symmetric in m, n.

Proof. The same, but using (5.61). □

Remark 1. When $m = 0$ and thus $u = t^{2n-2}t_0t_1t_2t_3t_4$, we conclude

$$I_K^{(n)}\Big(\prod_{1\le i\le n} \frac{(ux_i, u/x_i;q)}{(t_4x_i, t_4/x_i;q)}; q, t; t_0, t_1, t_2, t_3\Big) = \prod_{0\le i<n} \frac{(t^{-i}u/t_0, t^{-i}u/t_1, t^{-i}u/t_2, t^{-i}u/t_3;q)}{(t^it_0t_4, t^it_1t_4, t^it_2t_4, t^it_3t_4;q)}; \tag{5.66}$$

aside from the normalization of the integral, this is Theorem 2.1 of [8].

Remark 2. Formally, there is an analogous result using (5.55); this is a special case of equation (7.24) below.

Remark 3. If we were to ignore the constraints that certain ratios be integral powers of t in the above transformations, we would find that our integral has symmetry group D_6. It is unclear how to make this rigorous, however, given the significant difficulties with convergence.

We conclude with some miscellaneous results.

Combining the inversion identity for binomial coefficients, the duality symmetry of binomial coefficients, and the Cauchy identity for interpolation polynomials gives a Cauchy identity for Koornwinder polynomials.

Theorem 5.18. *[14] For all integers $m, n \ge 0$,*

$$\sum_{\lambda\subset m^n} (-1)^{mn-|\lambda|} K_\lambda^{(n)}(x_1, \dots x_n; q, t; t_0, t_1, t_2, t_3) K_{n^m-\lambda'}^{(m)}(y_1, \dots y_m; t, q; t_0, t_1, t_2, t_3) = \prod_{1\le i\le n}\prod_{1\le j\le m} (x_i + 1/x_i - y_j - 1/y_j). \tag{5.67}$$

The action of the difference operators on the Koornwinder polynomials (Lemma 5.6) is related via evaluation symmetry to the following connection coefficient result.

Theorem 5.19. *For any partition λ,*

$$\frac{K_\lambda^{(n)}(;q, t; t_0, t_1, t_2, t_3)}{k_\lambda^0(q, t, t^n; t_0{:}t_1, t_2, t_3)} = \sum_{\kappa\prec\lambda} \psi_{\lambda/\kappa}^{(d)}(t^nt_0t_1; q, t, t^{n-1}\sqrt{t_0t_1t_2t_3/q}) \frac{K_\kappa^{(n)}(;q, t; t_0, qt_1, t_2, t_3)}{k_\kappa^0(q, t, t^n; t_0{:}qt_1, t_2, t_3)} \tag{5.68}$$

Proof. Apply the difference equation for binomial coefficients, with parameter $u = t^nt_0t_1$, and simplify the result using the fact that

$$k_\mu^0(q, t, T; t_0{:}t_1, t_2, t_3)\psi_{\mu/\mu}^{(d)}(Tt_0t_1; q, t, (T/t)\sqrt{t_0t_1t_2t_3/q}) = k_\mu^0(q, t, T; t_0{:}qt_1, t_2, t_3). \tag{5.69}$$

□

Dualizing (i.e., using (7.13)), or equivalently using the integral equation, gives another special connection coefficient.

Theorem 5.20. *For any partition* λ*,*

$$\frac{K^{(n)}_\lambda(;q,t;t_0,t_1t,t_2,t_3)}{k^0_\lambda(q,t,t^n;t_0{:}t_1t,t_2,t_3)} = \sum_{\kappa'\prec\lambda'} \psi^{(i)}_{\lambda/\kappa}(t^nt_0t_1;q,t,t^{n-1}\sqrt{t_0t_1t_2t_3/q})\frac{K^{(n)}_\kappa(;q,t;t_0,t_1,t_2,t_3)}{k^0_\kappa(q,t,t^n;t_0{:}t_1,t_2,t_3)} \tag{5.70}$$

Proof. Take $u = t^nt_0t_1$ in the integral equation, and note

$$k^0_\mu(q,t,T;t_0{:}tt_1,t_2,t_3)\psi^{(i)}_{\mu/\mu}(Tt_0t_1;q,t,(T/t)\sqrt{t_0t_1t_2t_3/q}) = k^0_\mu(q,t,T;t_0{:}t_1,t_2,t_3). \tag{5.71}$$

□

Remark. Note that this formula still depends nontrivially on t in the univariate case. Since the Askey-Wilson polynomials are naturally independent of t, we obtain the connection coefficient formula [1, Eqs. 6.4-5] for Askey-Wilson polynomials with only one parameter changed.

We similarly obtain the following quasi-branching rule.

Theorem 5.21. *For any partition* λ*,*

$$\frac{K^{(n+1)}_\lambda(x_1,\dots x_n,t_0;q,t;t_0,t_1,t_2,t_3)}{k^0_\lambda(q,t,t^{n+1};t_0{:}t_1,t_2,t_3)} = \sum_{\substack{\kappa'\prec\lambda'\\ \ell(\kappa)\le n}} \psi^{(i)}_{\lambda/\kappa}(t^{n+1};q,t,t^n\sqrt{t_0t_1t_2t_3/qt})\frac{K^{(n)}_\kappa(x_1,\dots x_n;q,t;t_0t,t_1,t_2,t_3)}{k^0_\kappa(q,t,t^n;t_0t{:}t_1,t_2,t_3)} \tag{5.72}$$

Proof. Applying Lemma 3.6, we find that it suffices to show

$$\frac{\left[\lambda \atop \mu\right]_{q,t,t^n\sqrt{t_0t_1t_2t_3/q}}}{k^0_\mu(q,t,t^{n+1};t_0{:}t_1,t_2,t_3)} = \sum_{\substack{\kappa\subset\lambda\\ \ell(\kappa)\le n}} \psi^{(i)}_{\lambda/\kappa}(t^{n+1};q,t,t^{n-1}\sqrt{t_0t_1t_2t_3t/q})\frac{\left[\kappa \atop \mu\right]_{q,t,t^{n-1}\sqrt{t_0t_1t_2t_3t/q}}}{k^0_\mu(q,t,t^n;t_0t{:}t_1,t_2,t_3)}, \tag{5.73}$$

for $\ell(\mu)\le n$. Since

$$\psi^{(i)}_{\lambda/\kappa}(t^{n+1};q,t,t^{n-1}\sqrt{t_0t_1t_2t_3t/q}) = 0 \tag{5.74}$$

for $\ell(\kappa) = n+1$, the restriction on the length of κ can be removed, at which point the integral equation may be applied. □

Remark 1. The special case of the integral equation used in the above proof corresponds to the integral representation of [16].

Remark 2. This implies via the Cauchy identity a quasi-Pieri identity of the form

$$\prod_{1\le i\le n}(x_i+1/x_i-t_0-1/t_0)K^{(n)}_\mu(x_1,\dots x_n;q,t;qt_0,t_1,t_2,t_3) = \sum_{\substack{\lambda\succ\mu\\ \ell(\lambda)\le n}} c_{\lambda\mu}K^{(n)}_\lambda(x_1,\dots x_n;q,t;t_0,t_1,t_2,t_3), \tag{5.75}$$

for suitable coefficients $c_{\lambda\mu}$.

If we combine the last two theorems, we obtain a special branching rule of the form

$$K^{(n+1)}_\lambda(x_1,\dots x_n,t_0;q,t;t_0,t_1,t_2,t_3) = \sum_{\mu'\prec\kappa'\prec\lambda'} c_{\lambda/\kappa} d_{\kappa/\mu} K^{(n)}_\mu(x_1,\dots x_n;q,t;t_0,t_1,t_2,t_3), \tag{5.76}$$

for certain coefficients $c_{\lambda/\kappa}$, $d_{\kappa/\mu}$, thus confirming the speculation in [16] that the integral representation is related to a branching rule for Koornwinder polynomials. In general, it follows by combining the Pieri identities [27] and the Cauchy identity that there exists a branching rule of the form

$$K^{(n+1)}_\lambda(x_1,\dots x_n,u;q,t;t_0,t_1,t_2,t_3) = \sum_{\substack{\mu'\subset\lambda' \\ \exists\kappa:\mu'\prec\kappa'\prec\lambda'}} c_{\lambda/\mu} K^{(n)}_\mu(x_1,\dots x_n;q,t;t_0,t_1,t_2,t_3), \tag{5.77}$$

for suitable coefficients $c_{\lambda/\mu} \in \mathbb{F}(t_0,t_1,t_2,t_3)[u,1/u]$. The expression one obtains for these coefficients is far from a closed form, however.

6 Symmetric functions from interpolation polynomials

It turns out that the families of interpolation and Koornwinder polynomials have natural lifts to families of symmetric functions; in each case, one obtains two families with an additional algebraic parameter that reduce to the given BC_n-symmetric polynomials when appropriately specialized. We will first consider the interpolation polynomial case.

The first lifting involves inverting the natural projection from Λ to $\mathbb{F}[x_i^{\pm 1}]^{BC_n}$ given by

$$f \mapsto f(x_1,1/x_1,x_2,1/x_2,\dots x_n,1/x_n). \tag{6.1}$$

For any n, this map is surjective, but is quite far from injective, and thus we have no way to define a unique lifting. It turns out, however, that if we introduce an algebraic parameter $T = t^n$, then there is a unique lifting with coefficients in $\mathbb{F}(s,T)$.

The above homomorphism acts on power-sums by

$$p_k \mapsto \sum_{1\le i\le n} x_i^k + x_i^{-k}. \tag{6.2}$$

If we evaluate this at the partition λ, we have:

$$\sum_{1\le i\le n} x_i^k + x_i^{-k} = \sum_{1\le i\le n} \Big(q^{k\mu_i} t^{(n-i)k} s^k + q^{-k\mu_i} t^{-(n-i)k} s^{-k}\Big) \tag{6.3}$$

$$= \sum_{1\le i\le \ell(\mu)} \Big((q^{k\mu_i}-1)t^{(n-i)k}s^k + (q^{-k\mu_i}-1)t^{-(n-i)k}s^{-k}\Big) + \sum_{1\le i\le n}\Big(t^{(n-i)k}s^k + t^{-(n-i)k}s^{-k}\Big) \tag{6.4}$$

$$= \sum_{1\le i\le \ell(\mu)} \big((q^{k\mu_i}-1)t^{-ki}(sT)^k + (q^{-k\mu_i}-1)t^{ki}(sT)^{-k}\big) + s^k\frac{1-T^k}{1-t^k} + s^{-k}\frac{1-1/T^k}{1-1/t^k}. \tag{6.5}$$

This motivates the following definition.

Definition 3. The *lifted interpolation polynomials* are the unique family of (inhomogeneous) symmetric functions $\tilde{P}^*_\lambda(;q,t,T;s)$ such that

- $\tilde{P}^*_\lambda(;q,t,T;s) = m_\lambda +$ dominated terms
- $\tilde{P}^*_\lambda(\langle\mu\rangle_{q,t,T;s};q,t,T;s) = 0$ for $\mu < \lambda$.

Here, for a symmetric function f, $f(\langle\mu\rangle_{q,t,T;s})$ is its image under the homomorphism such that

$$p_k(\langle\mu\rangle_{q,t,T;s}) = \sum_{1\le i\le \ell(\mu)} \big((q^{k\mu_i}-1)t^{-ki}(sT)^k + (q^{-k\mu_i}-1)t^{ki}(sT)^{-k}\big) + s^k\frac{1-T^k}{1-t^k} + s^{-k}\frac{1-1/T^k}{1-1/t^k}. \tag{6.6}$$

Theorem 6.1. *The lifted interpolation polynomials are well-defined, with coefficients in $\mathbb{F}(s,T)$, and have the property that*

$$\tilde{P}^*_\lambda(x_1,x_1^{-1},x_2,x_2^{-1},\dots x_n,x_n^{-1};q,t,t^n;s) = \begin{cases} \bar{P}^*_\lambda(x_1,\dots x_n;q,t,s) & \ell(\lambda)\le n \\ 0 & \ell(\lambda) > n. \end{cases} \tag{6.7}$$

Moreover,

$$\tilde{P}^*_\lambda(\langle\mu\rangle_{q,t,T;s};q,t,T;s) = 0 \tag{6.8}$$

unless $\lambda\subset\mu$, and

$$\tilde{P}^*_\lambda(\langle\lambda\rangle_{q,t,T;s};q,t,T;s) = (qsT/t)^{-|\lambda|}t^{n(\lambda)}q^{-2n(\lambda')}C^-_\lambda(q;q,t)C^+_\lambda((sT/t)^2;q,t) \tag{6.9}$$

Proof. Suppose that the claims are known for $\lambda < \kappa$. Then we can write

$$\tilde{P}^*_\kappa(;q,t,T;s) = m_\kappa + \sum_{\mu\le\kappa} c_\mu \tilde{P}^*_\mu(;q,t,T;s) \tag{6.10}$$

for appropriate coefficients $c_\mu \in \mathbb{F}(s,T)$; the resulting equations are triangular with nonzero diagonal by the inductive hypothesis, and thus $\tilde{P}^*_\kappa$ is well-defined. We cannot yet rule out a pole at $T = t^n$, but can certainly conclude that only finitely many such poles exist.

Now, for $n\ge\ell(\kappa)$ not hitting such a pole,

$$\tilde{P}^*_\kappa(x_1,x_1^{-1},x_2,x_2^{-1},\dots x_n,x_n^{-1};q,t,t^n;s) \tag{6.11}$$

is a BC_n-symmetric polynomial with leading monomial m_κ (since the natural projection is triangular on the monomial functions) satisfying the necessary vanishing identities, so must therefore equal the interpolation polynomial.

In particular, for any partition $\mu\not\supset\kappa$,

$$\tilde{P}^*_\kappa(\langle\mu\rangle_{q,t,T;s};q,t,T;s) = 0 \tag{6.12}$$

whenever $T = t^n$ for n sufficiently large. But then this identity must in fact hold in $\mathbb{F}(s,T)$. Similarly, the evaluation at $\mu=\kappa$ holds for sufficiently large n, and thus for all T. In particular, the diagonal coefficients in the equations for c_μ are nonzero in $\mathbb{F}(s)$ for any T of the form t^n, and thus the coefficients have no poles at such T.

Finally, we observe that for $n < \ell(\kappa)$,

$$\tilde{P}^*_\kappa(x_1, x_1^{-1}, x_2, x_2^{-1}, \dots x_n, x_n^{-1}; q, t, t^n; s) \tag{6.13}$$

is a BC_n-symmetric polynomial vanishing at all partitions μ of length at most n, and must therefore vanish. □

The next lemma indicates that the two parameters s and T give only one degree of freedom, up to homomorphism.

Lemma 6.2. *We have the plethystic identity*

$$\tilde{P}^*_\lambda(; q, t, T; sT') = \tilde{P}^*_\lambda([p_k + s^k \frac{T^k - T'^k}{1 - t^k} + s^{-k} \frac{1/T^k - 1/T'^k}{1 - 1/t^k}]; q, t, T'; sT). \tag{6.14}$$

Proof. The right hand side has leading monomial m_λ and vanishes at all the appropriate partitions. □

Corollary 6.3. *The coefficients of* $(sT)^{|\lambda|}\tilde{P}^*_\lambda(; q, t, T; s)$ *lie in* $\mathbb{F}[s, T]$.

Proof. The coefficients of $s^{|\lambda|}\bar{P}^*_\lambda(; q, t, t^n; s)$ must lie in $\mathbb{F}[s]$ for n sufficiently large, since the same is true of $s^{|\lambda|}\bar{P}^{*(n)}_\lambda(; q, t, s)$. In particular, the coefficients of $(sT)^{|\lambda|}\bar{P}^*_\lambda(; q, t, t^n; sT/t^n)$ lie in $\mathbb{F}[sT]$; applying the homomorphism

$$p_k \mapsto p_k - s^k \frac{t^{nk} - T^k}{1 - t^k} - s^{-k} \frac{t^{-nk} - T^{-k}}{1 - t^{-k}} \tag{6.15}$$

at most enlarges the coefficient ring to $\mathbb{F}[s, T]$, and produces $(sT)^{|\lambda|}\tilde{P}^*_\lambda(; q, t, T; s)$. □

Remark. In fact, an examination of $\tilde{P}^*_\lambda(\langle\lambda\rangle)$ and the homomorphism $\langle\lambda\rangle_{q,t,T;s}$ further shows that the only possible denominator factors are s, T, q, t, and $(1 - q^i t^j)$ for i, j nonnegative integers, not both 0.

We now define a slight modification of the Macdonald involution. Recall that the Macdonald involution is the homomorphism

$$\omega_{q,t} : p_k \mapsto (-1)^{k-1} \frac{1 - q^k}{1 - t^k} p_k. \tag{6.16}$$

We rescale this slightly, to give:

$$\tilde{\omega}_{q,t} : p_k \mapsto (-1)^{k-1} \frac{q^{k/2} - q^{-k/2}}{t^{k/2} - t^{-k/2}} p_k; \tag{6.17}$$

thus this acts on ordinary Macdonald polynomials as

$$\tilde{\omega}_{q,t} P_\mu(; q, t) = b_\mu(q, t)^{-1} (t/q)^{|\mu|/2} P_{\mu'}(; t, q). \tag{6.18}$$

Lemma 6.4. *For any symmetric function* f *and partition* μ,

$$(\tilde{\omega}_{q,t} f)(\langle\mu\rangle_{t,q,1/T; -\sqrt{qt}/s}) = f(\langle\mu'\rangle_{q,t,T;s}). \tag{6.19}$$

Proof. It suffices to prove this for the power-sum functions p_k. Now, we find:

$$(-1)^{k-1}\frac{q^{k/2}-q^{-k/2}}{t^{k/2}-t^{-k/2}}\sum_{1\le i\le \ell(\mu)}(t^{k\mu_i}-1)q^{-ki}(-\sqrt{qt}/sT)^k = \frac{1-q^k}{1-t^{-k}}\sum_{1\le i\le\ell(\mu)}(t^{k\mu_i}-1)(sT)^{-k} \tag{6.20}$$

$$= (1-q^k)\sum_{(i,j)\in\mu} t^{jk}(sT)^{-k} \tag{6.21}$$

$$= (1-q^k)\sum_{(j,i)\in\mu'} t^{jk}q^{-ki}(sT)^{-k} \tag{6.22}$$

$$= \sum_{1\le j\le\ell(\mu')}(q^{-\mu'_j}-1)t^{jk}(sT)^{-k}. \tag{6.23}$$

and

$$(-1)^{k-1}\frac{q^{k/2}-q^{-k/2}}{t^{k/2}-t^{-k/2}}(-\sqrt{qt}/s)^k\frac{1-T^{-k}}{1-q^k} = s^{-k}\frac{1-T^{-k}}{1-t^{-k}}. \tag{6.24}$$

The other two terms simplify analogously, and thus

$$(\tilde\omega_{q,t}p_k)(\langle\mu\rangle_{t,q,1/T;-\sqrt{qt}/s}) = p_k(\langle\mu'\rangle_{q,t,T;s}) \tag{6.25}$$

as required. □

This gives us the following result:

Theorem 6.5. *The interpolation polynomials satisfy*

$$\tilde\omega_{q,t}\tilde P_\mu(;q,t,T;s) = b_\mu(q,t)^{-1}(t/q)^{|\mu|/2}\tilde P_{\mu'}(;t,q,1/T;-\sqrt{qt}/s). \tag{6.26}$$

Proof. If we evaluate the left-hand side at $\langle\nu\rangle_{t,q,1/T;-\sqrt{qt}/s}$, the result is

$$\tilde P_\mu(\langle\nu\rangle_{q,t,T;s};q,t,T;s), \tag{6.27}$$

and thus vanishes unless $\nu\supset\mu$. It follows that

$$\tilde\omega_{q,t}\tilde P_\mu(;q,t,T;s)\propto \tilde P_{\mu'}(;t,q,1/T;-\sqrt{qt}/s), \tag{6.28}$$

with some nonzero constant. Now, by triangularity, we can write

$$\tilde P_{\mu'}(;t,q,1/T;-\sqrt{qt}/s) = P_{\mu'}(;t,q)+\sum_{\nu<\mu'}c_\nu P_\nu(;t,q); \tag{6.29}$$

similarly, applying $\tilde\omega_{qt}$ to $\tilde P_\mu$, we have

$$\tilde P_{\mu'}(;t,q,1/T;-\sqrt{qt}/s) = \sum_{\nu\le\mu}c'_\nu P_{\nu'}(;t,q). \tag{6.30}$$

Since conjugation reverses dominance for partitions of the same size, we find that the degree $|\mu|$ portion of $\tilde P_{\mu'}$ is $P_{\mu'}$, and similarly for $\tilde P_\mu$. The constant then follows immediately. □

Remark. In particular, we obtain a new proof of the leading term limit for interpolation polynomials.

Corollary 6.6. *For any partitions μ and λ,*

$$\tilde{P}_\mu(\langle\lambda\rangle_{q,t,T;s}; q, t, T; s) = b_\mu(q,t)^{-1}(t/q)^{|\mu|/2}\tilde{P}_{\mu'}(\langle\lambda'\rangle_{t,q,1/T;-\sqrt{qt}/s}; t, q, 1/T; -\sqrt{qt}/s). \tag{6.31}$$

In particular,

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,s} = \begin{bmatrix}\lambda'\\ \mu'\end{bmatrix}_{t,q,1/\sqrt{qt}s} \tag{6.32}$$

$$\begin{Bmatrix}\lambda\\ \mu\end{Bmatrix}_{q,t,s} = \begin{Bmatrix}\lambda'\\ \mu'\end{Bmatrix}_{t,q,1/\sqrt{qt}s}. \tag{6.33}$$

Our second lifting of interpolation polynomials to symmetric functions involves the observation that for $\lambda \subset m^n$,

$$(\prod_{1\le i\le n} x_i^m)\bar{P}^{*(n)}_{m^n-\lambda}(x_1,\dots x_n; q, t, s) \tag{6.34}$$

is an S_n-symmetric polynomial (without negative exponents). Moreover, we have the following:

Lemma 6.7. *For arbitrary positive integers m, n, and an arbitrary partition $\lambda \subset m^n$,*

$$\lim_{x_n\to 0}(\prod_{1\le i\le n} x_i^m)\bar{P}^{*(n)}_{m^n-\lambda}(x_1,\dots x_n; q, t, s) = \begin{cases}(\prod_{1\le i\le n-1} x_i^m)\bar{P}^{*(n-1)}_{m^{n-1}-\lambda}(x_1,\dots x_{n-1}; q, t, s) & \lambda_n = 0\\ 0 & \lambda_n > 0.\end{cases} \tag{6.35}$$

Proof. We apply the branching rule; the only term that does not vanish in the limit is the unique term in which the degree has been reduced by m, namely

$$\bar{P}^{*(n-1)}_{m^{n-1}-\lambda}(x_1,\dots x_{n-1}; q, t, s). \tag{6.36}$$

□

With this in mind, we define:

Definition 4. The *virtual interpolation polynomials* are the unique symmetric functions $\hat{P}^*_\lambda(; q, t, Q; s) \in \hat{\Lambda}$ with coefficients in $\mathbb{F}(Q, s)$ such that

$$\hat{P}^*_\lambda(x_1,\dots x_n; q, t, Q; s) = \prod_{1\le i\le n}\frac{x_i^m(sx_i, q^{m+1}x_i/sQ; q)}{(qx_i/s, sQx_i/q^m; q)}\bar{P}^{*(n)}_{m^n-\lambda}(x_1,\dots x_n; q, t, sQ/q^m) \tag{6.37}$$

for all positive integers m, n such that $\lambda \subset m^n$.

We immediately see that these are well-defined and that the leading term (now the term of smallest degree) is again the Macdonald polynomial. The two kinds of interpolation polynomials are related via the Cauchy identity:

Theorem 6.8. *We have the following identity of symmetric functions in $\Lambda_x \otimes \hat{\Lambda}_y$:*

$$\sum_\lambda(-1)^{|\lambda|}\tilde{P}^*_\lambda(x; q, t, T; s)\hat{P}^*_{\lambda'}(y; t, q, T; s) = \sum_\lambda(-1)^{|\lambda|}P_\lambda(x; q, t)P_{\lambda'}(y; t, q) \tag{6.38}$$

Proof. Fix integers m, $n > 0$. Then we have:

$$\sum_{\lambda}(-1)^{|\lambda|}\tilde{P}^*_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots x_n^{\pm 1}; q, t, t^n; s)\hat{P}^*_{\lambda'}(y_1, y_2, \dots y_m; t, q, t^n; s)$$

$$= \sum_{\lambda \subset m^n}(-1)^{|\lambda|}\bar{P}^*_\lambda(x_1, x_2, \dots x_n; q, t, s) \prod_{1\le j\le m} y_j^n \bar{P}^*_{n^m-\lambda'}(y_1, y_2, \dots y_m; t, q, s) \tag{6.39}$$

$$= \prod_{1\le j\le m} y_j^n \prod_{\substack{1\le i\le n\\1\le j\le m}} (y_j + 1/y_j - x_i - 1/x_i) \tag{6.40}$$

$$= \prod_{\substack{1\le i\le n\\1\le j\le m}} (1 - y_j x_i)(1 - y_j/x_i) \tag{6.41}$$

$$= \sum_{\lambda}(-1)^{|\lambda|}P_\lambda(x_1^{\pm 1}, x_2^{\pm 1}, \dots x_n^{\pm 1}; q, t)P_{\lambda'}(y_1, \dots y_m; t, q) \tag{6.42}$$

The desired identity follows by rationality of coefficients. □

Corollary 6.9.

$$\tilde{\omega}_{q,t}\hat{P}_\lambda(; q, t, Q; s) = b_\lambda(q, t)^{-1}(t/q)^{|\lambda|/2}\hat{P}_{\lambda'}(; t, q, 1/Q; -\sqrt{qt}/s) \tag{6.43}$$

Proof. In the sum

$$\sum_{\lambda}(-1)^{|\lambda|}\tilde{P}^*_{\lambda'}(x; t, q, Q; s)\hat{P}^*_\lambda(y; q, t, Q; s) = \sum_{\lambda}(-1)^{|\lambda|}P_{\lambda'}(x; t, q)P_\lambda(y; q, t), \tag{6.44}$$

apply $\tilde{\omega}_{q,t}$ to the y variables and $\tilde{\omega}_{t,q}$ to the x variables. We find:

$$\sum_{\lambda}(-1)^{|\lambda|}P_{\lambda'}(x; q, t)P_\lambda(y; t, q) = \sum_{\lambda}(-1)^{|\lambda|}b_{\lambda'}(t, q)^{-1}(q/t)^{|\lambda|/2}\tilde{P}^*_\lambda(x; q, t, 1/Q; -\sqrt{qt}/s)(\tilde{\omega}_{q,t}\hat{P}^*_\lambda(y; q, t, Q; s)), \tag{6.45}$$

and thus, taking coefficients of $\tilde{P}^*_\lambda(x; q, t, 1/Q; -\sqrt{qt}/s)$ on both sides:

$$\hat{P}^*_{\lambda'}(y; t, q, 1/Q; -\sqrt{qt}/s) = b_{\lambda'}(t, q)^{-1}(q/t)^{|\lambda|/2}(\tilde{\omega}_{q,t}\hat{P}^*_\lambda(y; q, t, Q; s)). \tag{6.46}$$

The desired result is immediate. □

Applying the involution to only one of the sets of variables gives another Cauchy identity:

Corollary 6.10.

$$\sum_{\lambda} b_\lambda(q, t)(q/t)^{|\lambda|/2}\tilde{P}^*_\lambda(x; q, t, T; s)\hat{P}^*_\lambda(y; q, t, 1/T; \sqrt{qt}/s) = \sum_{\lambda}(q/t)^{|\lambda|/2}P_\lambda(x; q, t)Q_\lambda(y; q, t) \tag{6.47}$$

Corollary 6.11. *The virtual interpolation polynomials can be expanded in terms of Macdonald polynomials as follows:*

$$\hat{P}^*_\mu(; q, t, Q; s) = \sum_{\lambda}(-1)^{|\lambda|-|\mu|}\left([\tilde{P}^*_{\mu'}(; t, q, Q; s)]P_{\lambda'}(; t, q)\right)P_\lambda(; q, t) \tag{6.48}$$

The bulk branching rule lifts to the following result.

Theorem 6.12. *For any partition λ,*

$$\tilde{P}^*_\lambda([p_k + \frac{u^k - v^k}{1-t^k} + \frac{u^{-k}-v^{-k}}{1-t^{-k}}]; q, t, Tv/u; s) = \sum_{\mu\subset\lambda} \psi^{(B)}_{\lambda/\mu}(u, v; q, t; sT)\tilde{P}^*_\mu(; q, t, T; s), \tag{6.49}$$

where $\psi^{(B)}_{\lambda/\mu}(u, v; q, t; s)$ is as in Theorem 3.9. Similarly, for any partition λ, we have the following identity in $\hat{\Lambda}$ with coefficients in $\mathbb{F}(s)[[u, v]]$.

$$\hat{P}^*_\lambda([p_k + \frac{u^k - v^k}{1-t^k}]; q, t, Q; s) = \hat{P}^*_0([\frac{u^k - v^k}{1-t^k}]; q, t, Q; s) \sum_{\mu\subset\lambda} \hat{\psi}^{(B)}_{\lambda/\mu}(u, v; q, t; sQ)\hat{P}^*_\mu(; q, t, Q; s), \tag{6.50}$$

where

$$\hat{P}^*_0([\frac{u^k - v^k}{1-t^k}]; q, t, Q; s) = \prod_{j\ge 0} \frac{(t^j us, t^j qu/sQ, t^j vsQ, t^j qv/s; q)}{(t^j vs, t^j qv/sQ, t^j usQ, t^j qu/s; q)} \tag{6.51}$$

$$\hat{\psi}^{(B)}_{\lambda/\mu}(u, v; q, t; s) = \frac{C^0_\mu(qu/ts; q, t)C^0_\mu(sv/q; 1/q, 1/t)}{C^0_\lambda(qv/ts; q, t)C^0_\lambda(su/q; 1/q, 1/t)} P_{\lambda/\mu}([(u^k - v^k)/(1 - t^k)]; q, t). \tag{6.52}$$

Corollary 3.11 (evaluation at a "constant") has the following especially pleasing lift.

Corollary 6.13. *For any partition λ,*

$$(-\sqrt{xyzt})^{|\lambda|}\tilde{P}^*_\lambda([\frac{(x^{k/2} - x^{-k/2})(y^{k/2} - y^{-k/2})(z^{k/2} - z^{-k/2})}{t^{k/2} - t^{-k/2}}]; q, t, 1; \sqrt{t/xyz}) = t^{-2n(\lambda)}q^{n'(\lambda)} \frac{C^0_\lambda(x, y, z; 1/q, 1/t)}{C^-_\lambda(1/t; 1/q, 1/t)} \tag{6.53}$$

Remark. Corollary 3.11 is the special case $y = t^n$, $z = 1/(t^{n-1}xs^2)$.

We also note the lifted versions of the connection coefficient identity.

Theorem 6.14. *For any partitions λ, μ,*

$$[\tilde{P}^*_\mu(; q, t, T; s)]\tilde{P}^*_\lambda(; q, t, T; s') = \frac{C^0_\lambda(T; q, t)C^0_\lambda(t/Tss'; 1/q, 1/t)}{C^0_\mu(T; q, t)C^0_\mu(t/Tss'; 1/q, 1/t)} P_{\lambda/\mu}([\frac{s^k - s'^k}{1-t^k}]; q, t). \tag{6.54}$$

$$[\hat{P}^*_\lambda(; q, t, Q; s)]\hat{P}^*_\mu(; q, t, Q; s') = \frac{C^0_\lambda(Q; 1/q, 1/t)C^0_\lambda(t/Qss'; q, t)}{C^0_\mu(Q; 1/q, 1/t)C^0_\mu(t/Qss'; q, t)} Q_{\lambda/\mu}([\frac{s^k - s'^k}{1-t^k}]; q, t) \tag{6.55}$$

We also have a lift of the bulk Pieri identity, albeit only to the lifted polynomials (for the virtual polynomials, there are convergence problems, even formally).

Theorem 6.15. *For any partition μ, the following identity holds in $(\Lambda \otimes \mathbb{F}(s))[[u, v]]$.*

$$\Big(\sum_\kappa Q_\kappa(\left[\frac{u^k - v^k}{1-t^k}\right]; q, t)P_\kappa(; q, t)\Big)\tilde{P}^*_\mu(; q, t, T; s) \tag{6.56}$$

$$= \Big(\sum_\kappa Q_\kappa(\left[\frac{u^k - v^k}{1-t^k}\right]; q, t)P_\kappa(\langle\mu\rangle_{q,t,T;s}; q, t)\Big) \sum_{\lambda\supset\mu} \psi^{(P)}_{\lambda/\mu}(u, v; q, t; sT)\tilde{P}^*_\lambda(; q, t, T; s),$$

where $\psi^{(P)}_{\lambda/\mu}(u, v; q, t; s)$ is as in Theorem 3.13.

We close with the following refinement of the fact that the leading terms of the interpolation polynomials are Macdonald polynomials.

Theorem 6.16. *The lifted and virtual interpolation polynomials are triangular in the Macdonald polynomial basis, with respect to the inclusion partial order. That is, we have the expansions*

$$\tilde{P}^*_\lambda(;q,t,T;s) = P_\lambda(;q,t) + \sum_{\mu\subset\lambda} c_{\lambda/\mu} P_\mu(;q,t) \tag{6.57}$$

$$\hat{P}^*_\mu(;q,t,T;s) = P_\mu(;q,t) + \sum_{\lambda\supset\mu} \hat{c}_{\lambda/\mu} P_\lambda(;q,t) \tag{6.58}$$

for suitable constants c.

Proof. We first note that by Corollary 6.11, the first claim implies the second. Now, suppose the first claim is false, and let λ be an inclusion-minimal partition such that $\tilde{P}^*_\lambda(;q,t,T;s)$ is not triangular. Furthermore, let μ be an inclusion-maximal partition not contained in λ such that

$$[P_\mu(;q,t)]\tilde{P}^*_\lambda(;q,t,T;s) \neq 0. \tag{6.59}$$

We will show that this coefficient is independent of T and s; that it is 0 (giving a contradiction) will be shown in the proof of Theorem 7.25 below.

By Theorem 6.14, for any fixed $T \neq 0$, the lifted interpolation polynomials are mutually triangular with respect to the inclusion ordering, and thus the minimality of λ implies that the coefficient is independent of s. On the other hand,

$$[P_\mu(;q,t)]\tilde{P}^*_\lambda(;q,t,T;s) = [P_\mu(;q,t)]\tilde{P}^*_\lambda([p_k + s^k\frac{T^k-1}{1-t^k} + s^{-k}\frac{1/T^k-1}{1-1/t^k}];q,t,1;sT); \tag{6.60}$$

since this homomorphism is triangular in the Macdonald basis, the maximality of μ implies that

$$[P_\mu(;q,t)]\tilde{P}^*_\lambda(;q,t,T;s) = [P_\mu(;q,t)]\tilde{P}^*_\lambda(;q,t,1;sT), \tag{6.61}$$

and is thus independent of s and T as required. ☐

Corollary 6.17. *For any integer $n \geq 0$,*

$$\tilde{P}^*_{1^n}(;q,t,T;s) = (e_n - e_{n-2})([p_k - s^k\frac{1-(T/t^{n-1})^k}{1-t^k} - s^{-k}\frac{1-(t^{n-1}/T)^k}{1-1/t^k}]), \tag{6.62}$$

where $e_{-1} = e_{-2} = 0$.

Proof. It suffices to prove this in the case $T = t^{n-1}$. Triangularity then tells us that

$$\tilde{P}^*_{1^n}(;q,t,t^{n-1};s) = e_n + \sum_{0\leq m<n} c_m e_m \tag{6.63}$$

for suitable coefficients c_m. On the other hand, we know that

$$\tilde{P}^*_{1^n}(x_1^{\pm 1}, x_2^{\pm 1}, \dots x_{n-1}^{\pm 1};q,t,t^{n-1};s) = 0. \tag{6.64}$$

The only algebraic relation satisfied by the quantities

$$e_m(x_1^{\pm 1}, x_2^{\pm 1}, \dots x_{n-1}^{\pm 1}) \tag{6.65}$$

for $0 \le m \le n$ is that $e_n = e_{n-2}$, and thus $e_n - e_{n-2}$ is the only symmetric function satisfying both requirements. ☐

7 Symmetric functions from Koornwinder polynomials

Via the binomial formula, the lifted interpolation polynomials lead immediately to a lifting for Koornwinder polynomials.

Definition 5. The *lifted Koornwinder polynomials* are defined by the expansion

$$\tilde{K}_\lambda(;q,t,T;t_0,t_1,t_2,t_3) = \sum_{\mu\subset\lambda} \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t,(T/t)\sqrt{t_0t_1t_2t_3/q}} \frac{k^0_\lambda(q,t,T;t_0{:}t_1,t_2,t_3)}{k^0_\mu(q,t,T;t_0{:}t_1,t_2,t_3)} \tilde{P}^*_\mu(;q,t,T;t_0). \tag{7.1}$$

Theorem 7.1. *For any integer $n > 0$ and partition λ, and for generic values of the parameters,*

$$\tilde{K}_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1};q,t,t^n;t_0,t_1,t_2,t_3) = \begin{cases} K^{(n)}_\lambda(x_1,\dots x_n;q,t;t_0,t_1,t_2,t_3) & \ell(\lambda)\le n\\ 0 & \textit{otherwise.}\end{cases} \tag{7.2}$$

Proof. The claim when $\ell(\lambda) \le n$ is immediate from the binomial formula for ordinary Koornwinder polynomials. Thus assume $\ell(\lambda) > n$, and consider the term of the lifted binomial formula corresponding to a partition μ. The only factors that can lead to a zero or pole at $T = t^n$ are the factors $C^0_\lambda(T;q,t)$ of k^0_λ, $C^0_\mu(T;q,t)$ of $k^0_\mu(T;q,t)$, and the lifted interpolation polynomial itself. Now, when $\ell(\mu) \le n$, $C^0_\mu(t^n;q,t) \ne 0$ while $C^0_\lambda(t^n;q,t) = 0$, and thus the μ term vanishes. On the other hand, when $\ell(\mu) > n$, we find that

$$\lim_{T\to t^n} C^0_\lambda(T;q,t)/C^0_\mu(T;q,t) \tag{7.3}$$

is well-defined and nonzero. In this case, however, the interpolation polynomial itself vanishes. Thus all terms in the expansion vanish, as required. ☐

Remark. The genericity hypothesis is necessary when $\ell(\lambda) > n$:

$$\lim_{T\to t} \tilde{K}_{1^2}(x_1, 1/x_1;q,t,T;1,-1,\sqrt{t},-\sqrt{t}) = 1, \tag{7.4}$$

not 0. However, as long as T is specialized before any of the other parameters, this will not be a problem. Indeed, by the following corollary, the only possible problems (for generic q and t) arise when

$$C^+_\lambda(t^{2n-2}t_0t_1t_2t_3/q;q,t) = 0. \tag{7.5}$$

Corollary 7.2. *For any partition λ, the only possible factors of the denominators of the coefficients of the symmetric function*

$$t^{2|\lambda|+3n(\lambda)} C^+_\lambda((T/t)^2t_0t_1t_2t_3/q;q,t)\tilde{K}_\lambda(;q,t,T;t_0,t_1,t_2,t_3) \tag{7.6}$$

are t and binomials of the form $1 - q^it^j$ for $i,j \ge 0$.

Proof. If we specialize $T = t^n$ for any sufficiently large n, then we obtain an ordinary Koornwinder polynomial, in which the only denominator factors that can appear are of the form $1 - q^i t^j$, $i, j \geq 0$. Since this is true for all sufficiently large n, the result follows. $\square$

Remark. We conjecture that, in fact,

$$t^{2|\lambda|+3n(\lambda)} C^-_\lambda(t; q, t) C^+_\lambda((T/t)^2 t_0 t_1 t_2 t_3/q; q, t) \tilde{K}_\lambda([p_k/(1-t)]; q, t, T; t_0, t_1, t_2, t_3) \tag{7.7}$$

has coefficients in $\mathbb{Z}[q, t, t_0, t_1, t_2, t_3, T]$. (This is true for the leading terms, as in that case it reduces to the corresponding integrality result for Macdonald polynomials.)

Proposition 7.3. *For any pair of partitions μ, λ,*

$$\frac{\tilde{K}_\lambda(\langle\mu\rangle_{q,t,T;t_0}; q, t, T; t_0, t_1, t_2, t_3)}{k^0_\lambda(q, t, T; t_0{:}t_1, t_2, t_3)} = \frac{\tilde{K}_\mu(\langle\lambda\rangle_{q,t,T;\hat{t}_0}; q, t, T; \hat{t}_0, \hat{t}_1, \hat{t}_2, \hat{t}_3)}{k^0_\mu(q, t, T; \hat{t}_0{:}\hat{t}_1, \hat{t}_2, \hat{t}_3)}, \tag{7.8}$$

where

$$\hat{t}_0 = \sqrt{t_0 t_1 t_2 t_3/q}; \qquad \hat{t}_i = t_0 t_i/\hat{t}_0,\ i \in \{1, 2, 3\}. \tag{7.9}$$

The symmetries of ordinary Koornwinder polynomials lift; in addition, we obtain new symmetries involving T.

Proposition 7.4. *For any partition λ,*

$$\tilde{K}_\lambda(; q, t, T; t_0, t_1, t_2, t_3) = \tilde{K}_\lambda(; 1/q, 1/t, 1/T; 1/t_0, 1/t_1, 1/t_2, 1/t_3) \tag{7.10}$$

$$\tilde{K}_\lambda(; q, t, T; t_0, t_1, t_2, t_3) = (-1)^{|\lambda|} \tilde{K}_\lambda([(-1)^k p_k]; q, t, T; -t_0, -t_1, -t_2, -t_3) \tag{7.11}$$

$$\tilde{K}_\lambda(; q, t, T; t_0, t_1, t_2, t_3) = \tilde{K}_\lambda([p_k + \frac{(t/t_0)^k + (t/t_1)^k - t_0^k - t_1^k}{(1-t^k)}]; q, t, T t_0 t_1/t; t/t_1, t/t_0, t_2, t_3) \tag{7.12}$$

$$\tilde{\omega}_{q,t}(\tilde{K}_\lambda(; q, t, T; t_0, t_1, t_2, t_3)) = b_\lambda(q, t)^{-1} (t/q)^{|\lambda|/2} \tilde{K}_{\lambda'}(; t, q, 1/T; \frac{-\sqrt{qt}}{t_0}, \frac{-\sqrt{qt}}{t_1}, \frac{-\sqrt{qt}}{t_2}, \frac{-\sqrt{qt}}{t_3}). \tag{7.13}$$

Furthermore, $\tilde{K}_\lambda$ is invariant under permutations of t_0, t_1, t_2, t_3.

Remark. We note three particularly nice special cases of (7.12):

$$\tilde{K}_\lambda(; q, t, T; \sqrt{t}, -\sqrt{t}, t_2, t_3) = \tilde{K}_\lambda(; q, t, -T; -\sqrt{t}, \sqrt{t}, t_2, t_3) \tag{7.14}$$

$$\tilde{K}_\lambda(; q, t, T; t, \sqrt{t}, t_2, t_3) = \tilde{K}_\lambda([p_k + 1]; q, t, T\sqrt{t}; \sqrt{t}, 1, t_2, t_3) \tag{7.15}$$

$$\tilde{K}_\lambda(; q, t, T; -t, -\sqrt{t}, t_2, t_3) = \tilde{K}_\lambda([p_k + (-1)^k]; q, t, T\sqrt{t}; -\sqrt{t}, -1, t_2, t_3). \tag{7.16}$$

Definition 6. The virtual Koornwinder integral $I_K(; q, t, T; t_0, t_1, t_2, t_3)$ is the linear functional on symmetric functions defined by

$$I_K(f; q, t, T; t_0, t_1, t_2, t_3) = [\tilde{K}_0(; q, t, T; t_0, t_1, t_2, t_3)] f. \tag{7.17}$$

In particular, we note that when $T = t^n$, this reduces to the virtual Koornwinder integral defined above:

$$I^{(n)}_K(; q, t; t_0, t_1, t_2, t_3) = I_K(; q, t, t^n; t_0, t_1, t_2, t_3) \tag{7.18}$$

Again, the specialization of T must occur before any other specialization. For instance, the identity

$$I_K^{(1)}(e_2; q, t; t_0, t_1, t_2, t_3) = 1 \tag{7.19}$$

holds for all values of the parameters. On the other hand,

$$\lim_{T\to t} I_K(e_2; q, t, T; \pm 1, \pm\sqrt{t}) = 0. \tag{7.20}$$

Again, as long as q and t are generic and $C_\lambda^+(t^{2n-2}t_0t_1t_2t_3/q; q, t) \neq 0$, there is no problem.

Since the set of parameters with $T = t^n$ is Zariski dense, the orthogonality of ordinary Koornwinder polynomials lifts.

Proposition 7.5.

$$I_K(\tilde{K}_\lambda(; q, t, T; t_0, t_1, t_2, t_3)\tilde{K}_\mu(; q, t, T; t_0, t_1, t_2, t_3); q, t, T; t_0, t_1, t_2, t_3) = \delta_{\lambda\mu} N_\lambda(; q, t, T; t_0, t_1, t_2, t_3), \tag{7.21}$$

with N_λ as above.

The symmetries of Proposition 7.4 carry over to the virtual integral.

Corollary 7.6. *For any partition λ and any symmetric function f,*

$$I_K(f; 1/q, 1/t, 1/T; 1/t_0, 1/t_1, 1/t_2, 1/t_3) = I_K(f; q, t, T; t_0, t_1, t_2, t_3) \tag{7.22}$$

$$I_K(f; q, t, T; -t_0, -t_1, -t_2, -t_3) = I_K(f([(-1)^k p_k]); q, t, T; t_0, t_1, t_2, t_3) \tag{7.23}$$

$$I_K(f; q, t, Tt_0t_1/t; t/t_1, t/t_0, t_2, t_3) = I_K(f([p_k - \frac{t_0^k + t_1^k - (t/t_0)^k - (t/t_1)^k}{(1-t)}]); q, t, T; t_0, t_1, t_2, t_3) \tag{7.24}$$

$$I_K(f; t, q, 1/T; \frac{-\sqrt{qt}}{t_0}, \frac{-\sqrt{qt}}{t_1}, \frac{-\sqrt{qt}}{t_2}, \frac{-\sqrt{qt}}{t_3}) = I_K(\tilde{\omega}_{q,t} f; q, t, T; t_0, t_1, t_2, t_3). \tag{7.25}$$

Furthermore, I_K is symmetric in t_0, t_1, t_2, t_3.

The next several results are immediate lifts of the analogous results above for ordinary Koornwinder polynomials.

Proposition 7.7. *For any partition λ, we have the expansion*

$$\tilde{P}_\lambda^*(; q, t, T; t_0) = \sum_{\mu \subset \lambda} \begin{Bmatrix} \lambda \\ \mu \end{Bmatrix}_{q,t,(T/t)\sqrt{t_0t_1t_2t_3/q}} \frac{k_\lambda^0(q, t, T; t_0{:}t_1, t_2, t_3)}{k_\mu^0(q, t, T; t_0{:}t_1, t_2, t_3)} \tilde{K}_\mu(; q, t, T; t_0, t_1, t_2, t_3) \tag{7.26}$$

Proposition 7.8. *For any partitions $\kappa \subset \lambda$,*

$$\left[\frac{\tilde{K}_\kappa(; q, t, T; t_0, t_1, t_2, t_3)}{k_\kappa^0(q, t, T; t_0{:}t_1, t_2, t_3)}\right] \frac{\tilde{K}_\lambda(; q, t, T; t_0, t_1', t_2', t_3')}{k_\lambda^0(q, t, T; t_0{:}t_1', t_2', t_3')} = \sum_{\kappa \subset \mu \subset \lambda} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t,(T/t)\sqrt{t_0t_1't_2't_3'/q}} \begin{Bmatrix} \mu \\ \kappa \end{Bmatrix}_{q,t,(T/t)\sqrt{t_0t_1t_2t_3/q}} \frac{k_\mu^0(q, t, T; t_0{:}t_1, t_2, t_3)}{k_\mu^0(q, t, T; t_0{:}t_1', t_2', t_3')}. \tag{7.27}$$

Proposition 7.9. *For any partition λ, one has the following virtual integral.*

$$I_K(\tilde{P}^*_\lambda(;q,t,T;t_0);q,t,T;t_0,t_1,t_2,t_3) = (-t_0T/t)^{-|\lambda|}t^{2n(\lambda)}q^{-n(\lambda')}\frac{C^0_\lambda(T,Tt_0t_1/t,Tt_0t_2/t,Tt_0t_3/t;q,t)}{C^-_\lambda(t;q,t)C^0_\lambda(T^2t_0t_1t_2t_3/t^2;q,t)} \tag{7.28}$$

Proposition 7.10. *For any partition λ,*

$$\frac{\tilde{K}_\lambda(;q,t,T;t_0,t_1,t_2,t_3)}{k^0_\lambda(q,t,T;t_0{:}t_1,t_2,t_3)} = \sum_{\kappa\prec\lambda}\psi^{(d)}_{\lambda/\kappa}(Tt_0t_1;q,t,(T/t)\sqrt{t_0t_1t_2t_3/q})\frac{\tilde{K}_\kappa(;q,t,T;t_0,qt_1,t_2,t_3)}{k^0_\kappa(q,t,T;t_0{:}qt_1,t_2,t_3)} \tag{7.29}$$

Proposition 7.11. *For any partition λ,*

$$\frac{\tilde{K}_\lambda(;q,t,T;t_0,t_1t,t_2,t_3)}{k^0_\lambda(q,t,T;t_0{:}t_1t,t_2,t_3)} = \sum_{\kappa'\prec\lambda'}\psi^{(i)}_{\lambda/\kappa}(Tt_0t_1;q,t,(T/t)\sqrt{t_0t_1t_2t_3/q})\frac{\tilde{K}_\kappa(;q,t,T;t_0,t_1,t_2,t_3)}{k^0_\kappa(q,t,T;t_0{:}t_1,t_2,t_3)} \tag{7.30}$$

Proposition 7.12. *For any partition λ,*

$$\frac{\tilde{K}_\lambda([p_k+t_0^k+t_0^{-k}];q,t,tT;t_0,t_1,t_2,t_3)}{k^0_\lambda(q,t,tT;t_0{:}t_1,t_2,t_3)} = \sum_{\kappa'\prec\lambda'}\psi^{(i)}_{\lambda/\kappa}(tT;q,t,(T/t)\sqrt{t_0t_1t_2t_3t/q})\frac{\tilde{K}_\kappa(;q,t,T;t_0t,t_1,t_2,t_3)}{k^0_\kappa(q,t,T;t_0t{:}t_1,t_2,t_3)} \tag{7.31}$$

We next turn to the Cauchy identities.

Definition 7. The *virtual Koornwinder polynomials* are the basis of $\hat{\Lambda}$ given by

$$\hat{K}_\lambda(;q,t,Q;t_0,t_1,t_2,t_3) = \sum_{\mu\supset\lambda}(-1)^{|\mu|-|\lambda|}\begin{Bmatrix}\mu\\ \lambda\end{Bmatrix}_{q,t,1/Q\hat{t}_0}\frac{k^0_{\mu'}(t,q,Q;t_0{:}t_1,t_2,t_3)}{k^0_{\lambda'}(t,q,Q;t_0{:}t_1,t_2,t_3)}\hat{P}^*_\mu(;q,t,Q;t_0), \tag{7.32}$$

where $\hat{t}_0 = \sqrt{t_0t_1t_2t_3/q}$.

The name is justified by the following result, a straightforward verification from the binomial formula.

Theorem 7.13. *If $\lambda \subset m^n$, then*

$$\hat{K}_\lambda(x_1,\dots x_n;q,t,q^m;t_0,t_1,t_2,t_3) = \prod_{1\le i\le n}x_i^m K^{(n)}_{m^n-\lambda}(x_1,\dots x_n;q,t;t_0,t_1,t_2,t_3) \tag{7.33}$$

The Cauchy identities follow immediately from the inversion formula for binomial coefficients and the Cauchy identities for interpolation polynomials.

Theorem 7.14. *We have the following identities in $\Lambda_x\otimes\hat{\Lambda}_y$.*

$$\sum_\lambda(-1)^{|\lambda|}\tilde{K}_\lambda(x;q,t,T;t_0,t_1,t_2,t_3)\hat{K}_{\lambda'}(y;t,q,T;t_0,t_1,t_2,t_3) = \sum_\lambda(-1)^{|\lambda|}P_\lambda(x;q,t)P_{\lambda'}(y;t,q) \tag{7.34}$$

$$\sum_\lambda b_\lambda(q,t)(\frac{q}{t})^{|\lambda|/2}\tilde{K}_\lambda(x;q,t,T;t_0,t_1,t_2,t_3)\hat{K}_\lambda(y;q,t,1/T;\frac{\sqrt{qt}}{t_0},\frac{\sqrt{qt}}{t_1},\frac{\sqrt{qt}}{t_2},\frac{\sqrt{qt}}{t_3})$$
$$= \sum_\lambda(\frac{q}{t})^{|\lambda|/2}P_\lambda(x;q,t)Q_\lambda(y;q,t) \tag{7.35}$$

The special case $\hat{K}_0$ is of particular interest, as a result of the following integrals (obtained by integrating the Cauchy identities):

Corollary 7.15.

$$I_K^{(n)}(\prod_{\substack{1\le i\le n\\1\le j\le m}}(1-y_jx_i)(1-y_j/x_i);q,t;t_0,t_1,t_2,t_3)=\hat{K}_0(y_1,\dots y_m;t,q,t^n;t_0,t_1,t_2,t_3) \tag{7.36}$$

$$I_K^{(n)}(\prod_{\substack{1\le i\le n\\1\le j\le m}}\frac{(\sqrt{qt}y_jx_i^{\pm 1};q)}{(\sqrt{q/t}y_jx_i^{\pm 1};q)};q,t;t_0,t_1,t_2,t_3)=\hat{K}_0(y_1,\dots y_m;q,t,t^{-n};\frac{\sqrt{qt}}{t_0},\frac{\sqrt{qt}}{t_1},\frac{\sqrt{qt}}{t_2},\frac{\sqrt{qt}}{t_3}) \tag{7.37}$$

Corollary 7.16. *The following identity holds for all sufficiently small y.*

$$\begin{aligned}\hat{K}_0(y;q,t,Q;t_0,t_1,t_2,t_3)=&\frac{(qy/t_0Q,qy/t_1Q,qy/t_2Q,q^2y/t_0t_1t_2Q;q)}{(qy/t_0,qy/t_1,qy/t_2,q^2y/t_0t_1t_2Q^2;q)}\\&{}_8W_7(\frac{qy}{t_0t_1t_2Q^2};\frac{q}{t_0t_1Q},\frac{q}{t_0t_2Q},\frac{q}{t_1t_2Q},yt_3,1/Q;q,qy/t_3)\end{aligned} \tag{7.38}$$

Proof. Applying Corollary 5.16 to equation (7.37), we see that the identity holds whenever $Q=t^{-n}$, and thus for all Q. □

The case $T=0$ of the lifted and virtual Koornwinder polynomials turns out to be especially nice. This is somewhat surprising, considering that the binomial formula behaves very badly in that case: a rather large amount of cancellation is required to eliminate the apparent singularities at $T=0$. The key to dealing with the $T=0$ case turns out to be the inner product.

Definition 8. Let μ and σ be sequences of complex numbers, such that $\Re(\sigma_j)>0$ for all j. Then the Gaussian functional $I_G(;\mu;\sigma)$ is the linear functional on symmetric functions defined by

$$I_G(f;\mu;\sigma)=\int_{\mathbb{R}^{\deg(f)}}f\prod_{1\le j\le\deg(f)}(2\pi\sigma_j)^{-1/2}e^{-(p_j-\mu_j)^2/2\sigma_j}dp_j; \tag{7.39}$$

in particular, $I_G(1;\mu;\sigma)=1$.

This has a probabilistic interpretation: $I_G(f;\mu;\sigma)$ is the expected value of f if the power sum functions p_k are independent and normally distributed random variables with mean μ_k and variance σ_k. In particular, $I_G(f;\mu;\sigma)$ is polynomial in μ and σ; we extend it to arbitrary μ and σ accordingly.

Our reason for introducing Gaussian functionals is the following theorem.

Theorem 7.17. *For any symmetric function f,*

$$I_K(f;q,t,0;t_0,t_1,t_2,t_3)=I_G(f;\mu;\sigma), \tag{7.40}$$

where

$$\mu_{2k-1}=\frac{t_0^{2k-1}+t_1^{2k-1}+t_2^{2k-1}+t_3^{2k-1}}{1-t^{2k-1}} \tag{7.41}$$

$$\mu_{2k}=\frac{t_0^{2k}+t_1^{2k}+t_2^{2k}+t_3^{2k}-1-t^k-q^k-(qt)^k}{1-t^{2k}} \tag{7.42}$$

$$\sigma_k=\frac{1-t^k}{1-q^k}. \tag{7.43}$$

Proof. Since $I_K(f)$ is a rational function of the parameters for any f, it suffices to consider the limit

$$\lim_{n\to\infty} Z_n^{-1} \int f(x_i^{\pm 1}) w_K^{(n)}(x;q,t;t_0,t_1,t_2,t_3) d\mathbb{T} \tag{7.44}$$

after expanding the integrand as a formal power series in the parameters.

Now, if $q = t = t_0 = t_1 = t_2 = t_3 = 0$, $w_K^{(n)}$ reduces to the distribution function for the eigenvalues of a (Haar-distributed) random symplectic matrix. It follows from Theorem 6 of [5] (see also [2, Section 8]) that

$$\frac{\int f(x_i^{\pm 1}) w_K^{(n)}(x;0,0;0,0,0,0) d\mathbb{T}}{\int w_K^{(n)}(x;0,0;0,0,0,0) d\mathbb{T}} = I_G(f;\mu^{(0)},\sigma^{(0)}), \tag{7.45}$$

where $\mu^{(0)}_{2k-1} = 0$, $\mu^{(0)}_{2k} = -1$, $\sigma^{(0)}_k = k$, for all sufficiently large n. Thus for any power series f with coefficients in Λ, we have the formal limit

$$\lim_{n\to\infty} I_K^{(n)}(f;0,0;0,0,0,0) = I_G(f;\mu^{(0)},\sigma^{(0)}). \tag{7.46}$$

Now, we have

$$\frac{w_K^{(n)}(x;q,t;t_0,t_1,t_2,t_3)}{w_K^{(n)}(x;0,0;0,0,0,0)} = \tag{7.47}$$
$$(q;q)^{-n} \prod_{1\le k} \exp\left(-\frac{q^k+t^k}{2k(1-q^k)} p_{2k}(x_i^{\pm 1}) + \frac{t_0^k+t_1^k+t_2^k+t_3^k}{k(1-q^k)} p_k(x_i^{\pm 1}) - \frac{q^k-t^k}{2k(1-q^k)} p_k^2(x_i^{\pm 1}) \right).$$

This combines with the Gaussian density from $I_G(f;\mu^{(0)},\sigma^{(0)})$ to give the desired Gaussian density. □

In the sequel, we will evaluate the Gaussian integral via the following expansions.

Lemma 7.18. *We have the following integrals:*

$$I_K\Big(\prod_{j,k} \frac{(tx_jy_k;q)}{(x_jy_k;q)};q,t,0;t_0,t_1,t_2,t_3\Big)_y = \prod_{j<k} \frac{(tx_jx_k;q)}{(x_jx_k;q)} \prod_j \frac{(tx_j^2;q)}{(t_0x_j,t_1x_j,t_2x_j,t_3x_j;q)} \tag{7.48}$$

$$I_K\Big(\prod_{j,k}(1+x_jy_k);q,t,0;t_0,t_1,t_2,t_3\Big)_y = \prod_{j<k} \frac{(qx_jx_k;t)}{(x_jx_k;t)} \prod_j \frac{(-t_0x_j,-t_1x_j,-t_2x_j,-t_3x_j;t)}{(x_j^2;t)}. \tag{7.49}$$

Proof. Complete the square in the Gaussian integrals. □

The lifted Koornwinder polynomials for $T = 0$ are given by the following generating function.

Theorem 7.19. *We have the following identity in $\hat{\Lambda}_x \otimes \Lambda_y$.*

$$\sum_\lambda Q_\lambda(x;q,t) \tilde{K}_\lambda(y;q,t,0;t_0,t_1,t_2,t_3) = \prod_{j,k} \frac{(tx_jy_k;q)}{(x_jy_k;q)} \prod_{j<k} \frac{(x_jx_k;q)}{(tx_jx_k;q)} \prod_j \frac{(t_0x_j,t_1x_j,t_2x_j,t_3x_j;q)}{(tx_j^2;q)} \tag{7.50}$$

Proof. It is equivalent to show that the polynomials

$$K'_\lambda(y) = [Q_\lambda(x;q,t)] \prod_{j,k} \frac{(tx_jy_k;q)}{(x_jy_k;q)} \prod_{j<k} \frac{(x_jx_k;q)}{(tx_jx_k;q)} \prod_j \frac{(t_0x_j,t_1x_j,t_2x_j,t_3x_j;q)}{(tx_j^2;q)} \tag{7.51}$$

are orthogonal with respect to the inner product $I_K(;q,t,0;t_0,t_1,t_2,t_3)$. Evaluating the integral

$$I_K(\prod_{j,k}\frac{(tx_jy_k,tz_jy_k;q)}{(x_jy_k,z_jy_k;q)}\prod_{j<k}\frac{(x_jx_k,z_jz_k;q)}{(tx_jx_k,tz_jz_k;q)}\prod_j\frac{(t_0x_j,t_1x_j,t_2x_j,t_3x_j,t_0z_j,t_1z_j,t_2z_j,t_3z_j;q)}{(tx_j^2,tz_j^2;q)};q,t,0;t_0,t_1,t_2,t_3)_y \tag{7.52}$$

using Lemma 7.18 produces the result

$$\prod_{j,k}\frac{(tx_jz_k;q)}{(x_jz_k;q)}=\sum_\lambda Q_\lambda(x;q,t)P_\lambda(z;q,t); \tag{7.53}$$

it follows that

$$I_K(K'_\lambda K'_\mu;q,t,0;t_0,t_1,t_2,t_3)=\delta_{\lambda\mu}b_\lambda(q,t)^{-1}. \tag{7.54}$$

□

Remark 1. Note, in particular, that

$$N_\lambda(q,t,0;t_0,t_1,t_2,t_3)=b_\lambda(q,t)^{-1} \tag{7.55}$$

as required.

Remark 2. The special cases

$$(q,t;t_0,t_1,t_2,t_3)\mapsto(q,q;1,-1,\sqrt{q},-\sqrt{q}) \tag{7.56}$$

$$(q,t;t_0,t_1,t_2,t_3)\mapsto(q,q;q,-q,\sqrt{q},-\sqrt{q}) \tag{7.57}$$

correspond to the Cauchy identities for orthogonal and symplectic characters, respectively; in those cases, the lifted Koornwinder polynomials are independent of T, and equal to the "virtual characters" surveyed in section 5 of [25]. Similar comments apply to the $T=0$ results below.

By duality, we obtain:

Corollary 7.20. *We have the following identity in $\hat{\Lambda}_x\otimes\Lambda_y$.*

$$\sum_\lambda P_{\lambda'}(x;t,q)\tilde{K}_\lambda(y;q,t,0;t_0,t_1,t_2,t_3)=\prod_{j,k}(1+x_jy_k)\prod_{j<k}\frac{(x_jx_k;t)}{(qx_jx_k;t)}\prod_j\frac{(x_j^2;t)}{(t_0x_j,t_1x_j,t_2x_j,t_3x_j;t)}. \tag{7.58}$$

Comparing this with the Cauchy identity, we find:

Corollary 7.21. *The virtual Koornwinder polynomials are, when $T=0$, given by the formula*

$$\hat{K}_\lambda(x;q,t,0;t_0,t_1,t_2,t_3)=\prod_j\frac{(t_0x_j,t_1x_j,t_2x_j,t_3x_j;q)}{(x_j^2;q)}\prod_{j<k}\frac{(tx_jx_k;q)}{(x_jx_k;q)}P_\lambda(x;q,t) \tag{7.59}$$

Proof. Here we use the fact that if x is a formal variable, then $(x;q)(x/q;1/q)=1$. □

The generating function also gives us the following branching rule.

Theorem 7.22. *For any partition λ, we have the following identity in $\Lambda_x \otimes \Lambda_y$.*

$$\tilde{K}_\lambda(x, y; q, t, 0; t_0, t_1, t_2, t_3) = \sum_{\mu \subset \lambda} P_{\lambda/\mu}(x; q, t)\tilde{K}_\mu(y; q, t, 0; t_0, t_1, t_2, t_3). \tag{7.60}$$

Proof. We have

$$\sum_\lambda Q_\lambda(z; q, t)\tilde{K}_\lambda(x, y; q, t, 0; t_0, t_1, t_2, t_3) = \sum_\mu Q_\mu(z; q, t)P_\mu(x)\sum_\lambda Q_\lambda(z; q, t)\tilde{K}_\lambda(y; q, t, 0; t_0, t_1, t_2, t_3). \tag{7.61}$$

Taking coefficients of $Q_\kappa(z; q, t)$ and $\tilde{K}_\nu(y; q, t, 0; t_0, t_1, t_2, t_3)$ gives

$$[\tilde{K}_\nu(y; q, t, 0; t_0, t_1, t_2, t_3)]\tilde{K}_\kappa(x, y; q, t, 0; t_0, t_1, t_2, t_3) = \sum_\mu [Q_\kappa(z; q, t)](Q_\mu(z; q, t)Q_\nu(z; q, t))P_\mu(x) \tag{7.62}$$

$$= P_{\kappa/\nu}(x). \tag{7.63}$$

□

Setting $y = 0$ gives us the following expansion.

Corollary 7.23. *For any partition λ,*

$$\tilde{K}_\lambda(; q, t, 0; t_0, t_1, t_2, t_3) = \sum_{\mu \subset \lambda} P_{\lambda/\mu}(; q, t)\tilde{K}_\mu(0; q, t, 0; t_0, t_1, t_2, t_3). \tag{7.64}$$

The following is then immediate from the fact that $[P_\kappa]P_{\lambda/\mu} = 0$ unless $\kappa \subset \lambda$.

Corollary 7.24. *When $T = 0$, the lifted and virtual Koornwinder polynomials are triangular in the Macdonald polynomial basis, with respect to the inclusion partial order.*

This allows us to finish the proof of Theorem 6.16, as well as prove a corresponding result for Koornwinder polynomials.

Theorem 7.25. *The lifted and virtual Koornwinder polynomials are triangular in the Macdonald polynomial basis, with respect to the inclusion partial order.*

Proof. Let λ and μ be chosen as in the proof of Theorem 6.16. Recall that we had shown there that the coefficient

$$[P_\mu(; q, t)]\tilde{P}^*_\lambda(; q, t, T; s) \tag{7.65}$$

is independent of s and T, and still need to show that it is 0. From the binomial formula, it follows that

$$[P_\mu(; q, t)]\tilde{K}_\lambda(; q, t, T; t_0, t_1, t_2, t_3) = [P_\mu(; q, t)]\tilde{P}^*_\lambda(; q, t, T; t_0) \tag{7.66}$$

It thus suffices to show that the left-hand-side is 0 for some value of the parameters. Taking $T = 0$ suffices, by Corollary 7.24, and thus Theorem 6.16 holds. The present result follows from the binomial formulas. □

The branching rule also implies the following expansion.

Corollary 7.26. *For any partition λ,*

$$P_\lambda(;q,t) = \sum_{\mu\subset\lambda} I_K(P_{\lambda/\mu}(x;q,t);q,t,0;t_0,t_1,t_2,t_3)\tilde{K}_\mu(;q,t,0;t_0,t_1,t_2,t_3) \tag{7.67}$$

Proof. We observe that

$$\sum_{\mu\subset\lambda} P_{\lambda/\mu}(x;q,t)\tilde{K}_\mu(y;q,t,0;t_0,t_1,t_2,t_3) = \tilde{K}_\mu(x,y;q,t,0;t_0,t_1,t_2,t_3) \tag{7.68}$$

$$= \sum_{\mu\subset\lambda} P_{\lambda/\mu}(y;q,t)\tilde{K}_\mu(x;q,t,0;t_0,t_1,t_2,t_3). \tag{7.69}$$

Integrating the y variables gives the desired result. □

The fact that the $T=0$ branching rule is independent of t_0, t_1, t_2, t_3 is related to the following plethystic symmetry, which follows easily from the generating function.

Theorem 7.27. *For any partition λ,*

$$\tilde{K}_\lambda(;q,t,0;t'_0,t'_1,t'_2,t'_3) = \tilde{K}_\lambda(\left[p_k + \frac{t_0^k+t_1^k+t_2^k+t_3^k-t_0'^k-t_1'^k-t_2'^k-t_3'^k}{1-t^k}\right];q,t,0;t_0,t_1,t_2,t_3). \tag{7.70}$$

In particular,

$$\tilde{K}_\lambda(;q,t,0;t'_0,t_1,t_2,t_3) = \sum_{\mu\subset\lambda} P_{\lambda/\mu}(\left[\frac{t_0^k-t_0'^k}{1-t^k}\right];q,t)\tilde{K}_\mu(;q,t,0;t_0;t_1,t_2,t_3). \tag{7.71}$$

Remark. The special case $t'_0 = t_0 t$ gives

$$\tilde{K}_\lambda(;q,t,0;t_0t,t_1,t_2,t_3) = \sum_{\mu\subset\lambda} \psi_{\lambda/\mu}(q,t)t_0^{|\lambda/\mu|}\tilde{K}_\mu(;q,t,0;t_0;t_1,t_2,t_3), \tag{7.72}$$

a limiting case of Theorem 5.20. Similarly,

$$\tilde{K}_\lambda([p_k+t_0^k+t_0^{-k}];q,t,0;t_0,t_1,t_2,t_3) = \sum_{\mu\subset\lambda} \psi_{\lambda/\mu}(q,t)t_0^{-|\lambda/\mu|}\tilde{K}_\mu([p_k+t_0^k];q,t,0;t_0,t_1,t_2,t_3) \tag{7.73}$$

$$= \sum_{\mu\subset\lambda} \psi_{\lambda/\mu}(q,t)t_0^{-|\lambda/\mu|}\tilde{K}_\mu(;q,t,0;t_0t,t_1,t_2,t_3), \tag{7.74}$$

a limiting case of Theorem 5.21. (In fact, these identities are what originally suggested that something like those theorems should be true.) Compare also the plethystic symmetry of lifted Koornwinder polynomials (equation (7.12)).

The Pieri identities are also nice when $T=0$.

Theorem 7.28. *For any partition λ and integer $n \ge 0$,*

$$(\sum_n u^n g_n)\tilde{K}_\lambda(;q,t,0;t_0,t_1,t_2,t_3) \tag{7.75}$$

$$= \frac{(tu^2;q)}{(t_0u,t_1u,t_2u,t_3u;q)} \sum_{\substack{\nu\prec\lambda\\ \mu\succ\nu}} u^{|\lambda|-2|\nu|+|\mu|}\psi_{\lambda/\nu}(q,t)\varphi_{\mu/\nu}(q,t)\tilde{K}_\mu(;q,t,0;t_0,t_1,t_2,t_3)$$

Proof. Let f_λ denote the left-hand side, and consider the generating function

$$\sum_{\lambda,\mu}([\tilde{K}_\mu(;q,t,0;t_0,t_1,t_2,t_3)]f_\lambda)Q_\lambda(x;q,t)P_\mu(y;q,t). \tag{7.76}$$

Writing

$$[\tilde{K}_\mu(;q,t,0;t_0,t_1,t_2,t_3)]f_\lambda = b_\mu(q,t)^{-1}I_K(\tilde{K}_\mu(;q,t,0;t_0,t_1,t_2,t_3)f_\lambda;q,t,0;t_0,t_1,t_2,t_3), \tag{7.77}$$

we find that the generating function can be written as a Gaussian integral. Applying Lemma 7.18, we find

$$\begin{aligned}\sum_{\lambda,\mu}&([\tilde{K}_\mu(;q,t,0;t_0,t_1,t_2,t_3)]f_\lambda)Q_\lambda(x;q,t)P_\mu(y;q,t)\\ &= \frac{(tu^2;q)}{(t_0u,t_1u,t_2u,t_3u;q)}\prod_j\frac{(tux_j,tuy_j;q)}{(ux_j,uy_j;q)}\sum_\nu Q_\nu(x;q,t)P_\nu(y;q,t). &(7.78)\\ &= \frac{(tu^2;q)}{(t_0u,t_1u,t_2u,t_3u;q)}\sum_\nu\sum_{\lambda\succ\nu}\sum_{\mu\succ\nu}\psi_{\lambda/\nu}(q,t)Q_\lambda(x;q,t)\varphi_{\mu/\nu}(q,t)P_\mu(y;q,t). &(7.79)\end{aligned}$$

Comparing coefficients of $Q_\lambda(x;q,t)P_\mu(y;q,t)$ on both sides gives the desired result. □

Similarly,

Theorem 7.29. *For any partition λ and integer $n \geq 0$,*

$$\begin{aligned}(\sum_n &u^n e_n)\tilde{K}_\lambda(;q,t,0;t_0,t_1,t_2,t_3) &(7.80)\\ &= \frac{(-t_0u,-t_1u,-t_2u,-t_3u;t)}{(u^2;t)}\sum_{\substack{\nu'\prec\lambda'\\ \mu'\succ\nu'}}u^{|\lambda|-2|\nu|+|\mu|}\varphi'_{\lambda/\nu}(q,t)\psi'_{\mu/\nu}(q,t)\tilde{K}_\mu(;q,t,0;t_0,t_1,t_2,t_3).\end{aligned}$$

8 Vanishing conjectures

If we substitute $(t_0,t_1,t_2,t_3)\mapsto(\pm\sqrt{t},\pm\sqrt{qt})$ in equation (7.48), the right-hand side becomes

$$\prod_{j<k}\frac{(tx_jx_k;q)}{(x_jx_k;q)}. \tag{8.1}$$

This is the right-hand side of a generalized Littlewood identity due to Macdonald [12, Ex. VI.7.4]; we thus obtain the following proposition.

Proposition 8.1. *For any partition λ,*

$$I_K(P_\lambda(;q,t);q,t,0;\pm\sqrt{t},\pm\sqrt{qt}) = 0 \tag{8.2}$$

unless λ is of the form μ^2, in which case

$$I_K(P_{\mu^2}(;q,t);q,t,0;\pm\sqrt{t},\pm\sqrt{qt}) = \frac{C^-_\mu(qt;q,t^2)}{C^-_\mu(t^2;q,t^2)}. \tag{8.3}$$

The dual identity is also related to an integral.

Proposition 8.2. *For any partition λ,*

$$I_K(P_\lambda(;q,t);q,t,0;\pm 1,\pm\sqrt{t}) = 0 \tag{8.4}$$

unless λ is of the form 2μ, in which case

$$I_K(P_{2\mu}(;q,t);q,t,0;\pm 1,\pm\sqrt{t}) = \frac{C^-_\mu(q;q^2,t)}{C^-_\mu(t;q^2,t)}. \tag{8.5}$$

This leads us to formulate the following conjectures.

Conjecture 1. *For any partition λ,*

$$I_K(P_\lambda(;q,t);q,t,T;\pm\sqrt{t},\pm\sqrt{qt}) = 0 \tag{8.6}$$

unless λ is of the form μ^2, in which case

$$I_K(P_{\mu^2}(;q,t);q,t,T;\pm\sqrt{t},\pm\sqrt{qt}) = \frac{C^0_\mu(T^2;q,t^2)C^-_\mu(qt;q,t^2)}{C^0_\mu(qT^2/t;q,t^2)C^-_\mu(t^2;q,t^2)} \tag{8.7}$$

Conjecture 2. *For any partition λ,*

$$I_K(P_\lambda(;q,t);q,t,T;\pm 1,\pm\sqrt{t}) = 0 \tag{8.8}$$

unless λ is of the form 2μ, in which case

$$I_K(P_{2\mu}(;q,t);q,t,T;\pm 1,\pm\sqrt{t}) = \frac{C^0_\mu(T^2;q^2,t)C^-_\mu(q;q^2,t)}{C^0_\mu(qT^2/t;q^2,t)C^-_\mu(t;q^2,t)} \tag{8.9}$$

(Note that these conjectures are equivalent, by duality.) The vanishing of these integrals is, of course, a natural conjecture; the nonzero values will be justified below.

Proposition 8.3. *Conjecture 1 is equivalent to the following claim: For all integers $n \ge 0$ and partitions λ with $\ell(\lambda) \le 2n$,*

$$I^{(n)}_K(P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1};q,t);q,t;\pm\sqrt{t},\pm\sqrt{qt}) = 0 \tag{8.10}$$

unless λ is of the form μ^2.

Proof. The given claim is the specialization $T = t^n$ of the vanishing part of Conjecture 1, and thus by rationality is equivalent to the vanishing claim. It thus remains to show that vanishing implies

$$I^{(n)}_K(P_{\mu^2}(x_1^{\pm 1},\dots x_n^{\pm 1};q,t);q,t;\pm\sqrt{t},\pm\sqrt{qt}) = \frac{C^0_\mu(t^n;q^2,t)C^-_\mu(q;q^2,t)}{C^0_\mu(qt^{n-1};q^2,t)C^-_\mu(t;q^2,t)}. \tag{8.11}$$

Suppose this is true for $\mu \subsetneq \lambda$, and choose $\mu \subset \lambda$ so that $|\lambda/\mu| = 1$. Let ν be the unique partition such that $\mu^2 \subsetneq \nu \subsetneq \lambda^2$. Now, $e_1 - e_{2n-1}$ is in the kernel of the homomorphism $f \mapsto f(x_1^{\pm 1},\dots x_n^{\pm 1})$, and thus

$$I^{(n)}_K((e_1 P_\nu(;q,t))(x_1^{\pm 1},\dots x_n^{\pm 1});q,t;\pm\sqrt{t},\pm\sqrt{qt}) = I^{(n)}_K((e_{2n-1}P_\nu(;q,t))(x_1^{\pm 1},\dots x_n^{\pm 1};q,t);q,t;\pm\sqrt{t},\pm\sqrt{qt}), \tag{8.12}$$

Now, if we expand each side via the Pieri identity, only one term on each side has a nonvanishing integral; we thus find

$$\psi'_{\lambda^2/\nu} I_K^{(n)}(P_{\lambda^2}(x_1^{\pm 1},\dots x_n^{\pm 1};q,t);q,t;\pm\sqrt{t},\pm\sqrt{qt}) = \psi'_{(1^{2n}+\mu^2)/\nu} I_K^{(n)}(P_{\mu^2}(x_1^{\pm 1},\dots x_n^{\pm 1};q,t);q,t;\pm\sqrt{t},\pm\sqrt{qt}). \tag{8.13}$$

Solving this for

$$I_K^{(n)}(P_{\lambda^2}(;q,t)(x_1^{\pm 1},\dots x_n^{\pm 1});q,t;\pm\sqrt{t},\pm\sqrt{qt}) \tag{8.14}$$

gives the desired result. □

Remark. Compare the computation of the nonzero value of the Macdonald inner product in [12, Section VI.9], and the computation of the nonzero value of the Koornwinder inner product in [27].

For the other version of the conjecture, we have the following refinement.

Proposition 8.4. *Let m be a nonnegative integer. Then the following claims are equivalent.*

- *Conjecture 2 holds for all partitions λ with $\lambda_1 \le m$.*
- *For all integers $n \ge 0$ and partitions $\lambda \subset m^{2n}$,*

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1};q,t);q,t;\pm 1,\pm\sqrt{t}) + I_K^{(n-1)}(P_\lambda(x_1^{\pm 1},\dots x_{n-1}^{\pm 1},1,-1;q,t);q,t;\pm t,\pm\sqrt{t}) = 0 \tag{8.15}$$

  unless λ is of the form 2μ.
- *For all integers $n \ge 0$ and partitions $\lambda \subset m^{2n+1}$,*

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1},1;q,t);q,t;t,-1,\pm\sqrt{t}) + I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_{n-1}^{\pm 1},-1;q,t);q,t;1,-t,\pm\sqrt{t}) = 0 \tag{8.16}$$

  unless λ is of the form 2μ.

Proof. First, assume the vanishing portion of Conjecture 2. We cannot directly specialize $T \to t^n$, since this is a case in which the order of specialization is important. A consideration of possible poles shows that the only partitions for which

$$\tilde{K}_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1};q,t,t^n;\pm 1,\pm\sqrt{t}) \tag{8.17}$$

is ill-defined are the partitions $\lambda = 1^k$ for $n+1 \le k \le 2n$. In this case, we find (by the Cauchy identity, say) that

$$\tilde{K}_{1^k}(;q,t,T;\pm 1,\pm\sqrt{t}) = e_k \tag{8.18}$$

for all k, independent of T. We thus find that

$$I_K^{(n)}(\tilde{K}_{1^k}(x_1^{\pm 1},\dots x_n^{\pm 1};q,t,T;\pm 1,\pm\sqrt{t}) = 0 \tag{8.19}$$

unless $k = 2n$. It follows that for $\ell(\lambda) < 2n$,

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1};q,t);q,t;\pm 1,\pm\sqrt{t}) = \lim_{T\to t^n} I_K(P_\lambda(;q,t);q,t,T;\pm 1,\pm\sqrt{t}). \tag{8.20}$$

Similarly, by the plethystic symmetry of Koornwinder integrals,

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_{n-1}^{\pm 1},1,-1;q,t);q,t;\pm t,\pm\sqrt{t}) = \lim_{T\to t^n} I_K(P_\lambda([p_k+1+(-1)^k];q,t);q,t,T;\pm t,\pm\sqrt{t}) \tag{8.21}$$

$$= \lim_{T\to t^n} I_K(P_\lambda(p_k;q,t);q,t,T;\pm 1,\pm\sqrt{t}), \tag{8.22}$$

while for $\ell(\lambda) < 2n+1$,

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1},1;q,t);q,t;t,-1,\pm\sqrt{t}) = \lim_{T\to t^{n+1/2}} I_K(P_\lambda(;q,t);q,t,T;\pm 1,\pm\sqrt{t}) \tag{8.23}$$

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1},-1;q,t);q,t;1,-t,\pm\sqrt{t}) = \lim_{T\to t^{n+1/2}} I_K(P_\lambda(;q,t);q,t,T;\pm 1,\pm\sqrt{t}). \tag{8.24}$$

More generally, if $\lambda = k^{2n}+\mu$ with $\ell(\mu) < 2n$,

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1};q,t);q,t;\pm 1,\pm\sqrt{t}) = \lim_{T\to t^n} I_K(P_\mu(;q,t);q,t,T;\pm 1,\pm\sqrt{t}) \tag{8.25}$$

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_{n-1}^{\pm 1},1,-1;q,t);q,t;\pm t,\pm\sqrt{t}) = (-1)^k \lim_{T\to t^n} I_K(P_\mu(;q,t);q,t,T;\pm 1,\pm\sqrt{t}), \tag{8.26}$$

and if $\lambda = k^{2n+1}+\mu$ with $\ell(\mu) < 2n+1$,

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1},1;q,t);q,t;t,-1,\pm\sqrt{t}) = \lim_{T\to t^{n+1/2}} I_K(P_\mu(;q,t);q,t,T;\pm 1,\pm\sqrt{t}) \tag{8.27}$$

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_{n-1}^{\pm 1},-1;q,t);q,t;1,-t,\pm\sqrt{t}) = (-1)^k \lim_{T\to t^{n+1/2}} I_K(P_\mu(;q,t);q,t,T;\pm 1,\pm\sqrt{t}). \tag{8.28}$$

Thus the vanishing portion of Conjecture 2 is equivalent to the other two claims.

It remains to consider the nonzero values of the integrals. We consider the $2n$ version; the other is analogous. It suffices to consider

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots x_{n-1}^{\pm 1},1,-1;q,t);q,t;\pm t,\pm\sqrt{t}) \tag{8.29}$$

when λ is an even partition with $\lambda_1 \le m$, $\ell(\lambda) < 2n$. Let $\mu \subset \lambda$ be any partition such that $\mu_1 < m$ and λ/μ is a horizontal strip. There is then a unique even partition ν such that μ/ν is a horizontal strip. Now, the function

$$(e_{|\lambda/\mu|} + e_{2n-|\lambda/\mu|})P_\mu(;q,t). \tag{8.30}$$

is annihilated by the integral, since

$$(e_{|\lambda/\mu|} + e_{2n-|\lambda/\mu|})(x_1^{\pm}1,\dots x_{n-1}^{\pm 1},1,-1) = 0. \tag{8.31}$$

On the other hand, every term $P_\kappa(;q,t)$ in the Pieri expansion has $\kappa_1 \le m$, and thus the only nonvanishing terms are

$$\psi_{\lambda/\mu}P_\lambda(;q,t) + \psi_{(1^{2n}+\nu)/\mu}P_{1^{2n}+\nu}(;q,t) \tag{8.32}$$

(plus an additional factor of 2 if $|\mu/\nu| = n$). Solving the resulting recurrence gives the desired nonzero values of the integral. □

We can show these conjectures in a number of special cases. We begin with the Schur case.

Theorem 8.5. *Conjectures 1 and 2 hold when $q = t$.*

Proof. We apply the propositions, and observe that the resulting integrals can be expressed in terms of integrals over classical groups. To be precise, the $q = t$ cases of the conjectures are equivalent to the claims

$$\int_{U\in Sp(2n)} s_\lambda(U)dU = 0 \quad \text{unless } \lambda = \mu^2 \tag{8.33}$$

$$\int_{U\in O(n)} s_\lambda(U)dU = 0 \quad \text{unless } \lambda = 2\mu, \tag{8.34}$$

where the integrals are with respect to the corresponding Haar measures. These follow from the theory of zonal polynomials [12, Chapter VI], or equivalently from the theory of symmetric spaces [9] (specifically $U(2n)/Sp(2n)$ and $U(n)/O(n)$); this also shows that the integrals are 1 when nonzero, agreeing with the general formula. ☐

Remark. In fact, our original motivation for the above conjectures was to generalize these results to Macdonald polynomials. The connection to generalized Littlewood identities was then suggested by the results of [2, Section 5].

The theory of zonal polynomials gives us another special case.

Theorem 8.6. *Conjecture 1 holds in the case* $(q,t) \mapsto \lim_{q\to 1}(q^2, q)$*. Conjecture 2 holds in the case* $(q,t) \mapsto \lim_{q\to 1}(q, q^2)$.

Proof. The expression of zonal polynomials in terms of Jack polynomials gives, for partitions μ with $\ell(\mu) \le n$,

$$\int_{U\in Sp(2n)} s_{\mu^2}(AU) \propto \lim_{q\to 1} P_\mu(AJA^t; q, q^2) \tag{8.35}$$

$$\int_{U\in O(2n)} s_{2\mu}(AU) \propto \lim_{q\to 1} P_\mu(AA^tA^t; q^2, q), \tag{8.36}$$

where J is the symplectic inner product; the integrals vanish on Schur functions not of the stated form. Now, consider the integral

$$\int_{U\in Sp(n)} \int_{U'\in O(2n)} s_\lambda(UU'). \tag{8.37}$$

This vanishes unless λ has both the form 2κ and the form κ^2; that is, unless $\lambda = 2\mu^2$ for some μ. Thus the integrals

$$\int_{U\in Sp(2n)} \lim_{q\to 1} P_\lambda(UU^t; q^2, q) \quad \text{and} \quad \int_{U\in O(2n)} \lim_{q\to 1} P_\lambda(UJU^t; q, q^2) \tag{8.38}$$

vanish unless λ has the form μ^2 and 2μ respectively. As these can be written as integrals over the spaces $Sp(2n)/U(n)$ and $O(2n)/U(n)$, we can express them as limiting cases of Koornwinder integrals (more precisely, Jacobi integrals); the theorem follows. ☐

Remark. Since the theory of zonal polynomials extends via quantum groups to the cases $P_\lambda(;q,q^2)$, $P_\lambda(;q^2,q)$ (without limits) [15], it should be possible to extend this argument accordingly.

We next turn to special cases with generic parameters, but for which λ has been constrained.

Theorem 8.7. *Conjecture 1 holds if* $\lambda_1 \le 1$*; Conjecture 2 holds if* $\ell(\lambda) \le 1$.

Proof. We consider the second claim; the first will follow by duality. We thus need to show, for u a formal variable:

$$I_K(\sum_k u^k g_k; q, t, T; \pm 1, \pm\sqrt{t}) = \sum_k u^{2k} \frac{(T; q^2)_k (qt; q^2)_k}{(qT/t; q^2)_k (q^2; q^2)_k} = {}_2\phi_1(T^2, qt; qT^2/t; q^2, u^2). \tag{8.39}$$

In fact, we claim that, more generally,

$$I_K(\sum_k u^k g_k; q, t, T; \pm a, \pm\sqrt{t}) = {}_2\phi_1(T^2, qt/a^2; T^2 qa^2/t; q^2, u^2 a^2). \tag{8.40}$$

By rationality, we may assume $T = t^n$ for some integer $n \geq 0$. Then

$$I_K(\sum_k u^k g_k; q, t, T; \pm a, \pm\sqrt{t}) = I_K^{(n)}(\prod_j \frac{(tux_j^{\pm 1}; q)}{(ux_j^{\pm 1}; q)}; q, t; \pm a, \pm\sqrt{t}) \tag{8.41}$$

$$= I_K^{(1)}(\frac{(t^{(n+1)/2}ux^{\pm 1}; q)}{(t^{-(n-1)/2}ux^{\pm 1}; q)}; q, t; \pm t^{(n-1)/2}a, \pm t^{n/2}) \tag{8.42}$$

$$= \frac{(T^2a^2u^2, T^2u^2t; q^2)(u^2t, Ta^2/t; q)}{(a^2u^2, u^2t; q^2)(tTu^2, a^2T^2/t; q)} \tag{8.43}$$

$${}_8W_7(tTu^2/q; tu/a, -tu/a, \sqrt{t}u, -\sqrt{t}u, T; q, Ta^2b^2/t^2), \tag{8.44}$$

as long as $|a^2b^2T/t^2| < 1$. The claim follows by quadratic transformation (equation (3.5.4) of [7]). ∎

Remark. The conjectures can thus be viewed as multivariate analogues of quadratic transformations.

The same argument shows that Conjecture 2 holds whenever $\lambda_1 \leq 1$. It turns out that we can show something much stronger; to do so, we will need yet another equivalent form of the conjectures.

Theorem 8.8. *Let m be a nonnegative integer. Then the following statements are equivalent.*

- *Conjecture 2 holds for all partitions λ with $\lambda_1 \leq m$.*
- *For all integers $n \geq 0$, we have*

$$[P_\lambda(x_1, \dots x_m; q, t)](x_1 \dots x_m)^{n/2} P^{D_m}_{n\omega_m}(x_1^{-1}, \dots x_m^{-1}; q, t) = 0 \tag{8.45}$$

unless $\lambda = \mu^2$ for some μ, where $P^{D_m}_{n\omega_m}(; q, t)$ is the D_m-type Macdonald polynomial associated to the weight

$$n\omega_m = (n/2, n/2, \dots n/2). \tag{8.46}$$

Proof. We will show that the second claim is equivalent to the second and third claims of Proposition 8.4. By orthogonality of Koornwinder polynomials, these may be written in the form

$$[K_0^{(n)}(x_1, \dots x_n; q, t; \pm 1, \pm\sqrt{t})]P_\lambda(x_1^{\pm 1}, \dots x_n^{\pm 1}; q, t) + [K_0^{(n-1)}(x_1, \dots x_{n-1}; q, t; \pm t, \pm\sqrt{t})]P_\lambda(x_1^{\pm 1}, \dots x_{n-1}^{\pm 1}, 1, -1; q, t) = 0 \tag{8.47}$$

$$[K_0^{(n)}(x_1, \dots x_n; q, t; t, -1, \pm\sqrt{t})]P_\lambda(x_1^{\pm 1}, \dots x_n^{\pm 1}, 1; q, t) + [K_0^{(n)}(x_1, \dots x_n; q, t; 1, -t, \pm\sqrt{t})]P_\lambda(x_1^{\pm 1}, \dots x_n^{\pm 1}, -1; q, t) = 0 \tag{8.48}$$

(unless $\lambda = 2\mu$). By the Cauchy identity for Koornwinder polynomials, we compute

$$\begin{aligned}
&[K_0^{(n)}(x_1,\dots x_n;q,t;\pm 1,\pm\sqrt{t})]P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1};q,t) \\
&\qquad = (-1)^{|\lambda|}[P_{\lambda'}(y_1,\dots y_m;t,q)] \prod_{1\le j\le m} y_j^n K_{n^m}^{(m)}(y_1,\dots y_m;t,q;\pm 1,\pm\sqrt{t}) && (8.49)\\
&[K_0^{(n-1)}(x_1,\dots x_{n-1};q,t;\pm t,\pm\sqrt{t})]P_\lambda(x_1^{\pm 1},\dots x_{n-1}^{\pm 1},1,-1;q,t) \\
&\qquad = (-1)^{|\lambda|}[P_{\lambda'}(y_1,\dots y_m;t,q)] \prod_{1\le j\le m} y_j^n (y_j^{-1}-y_j) K_{(n-1)^m}^{(m)}(y_1,\dots y_m;t,q;\pm t,\pm\sqrt{t}) && (8.50)\\
&[K_0^{(n)}(x_1,\dots x_n;q,t;t,-1,\pm\sqrt{t})]P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1},1;q,t) && (8.51)\\
&\qquad = (-1)^{|\lambda|}[P_{\lambda'}(y_1,\dots y_m;t,q)] \prod_{1\le j\le m} y_j^{n+1/2}(y_j^{-1/2}-y_j^{1/2}) K_{n^m}^{(m)}(y_1,\dots y_m;t,q;t,-1,\pm\sqrt{t}) \\
&[K_0^{(n)}(x_1,\dots x_n;q,t;1,-t,\pm\sqrt{t})]P_\lambda(x_1^{\pm 1},\dots x_n^{\pm 1},-1;q,t) && (8.52)\\
&\qquad = (-1)^{|\lambda|}[P_{\lambda'}(y_1,\dots y_m;t,q)] \prod_{1\le j\le m} y_j^{n+1/2}(y_j^{-1/2}+y_j^{1/2}) K_{n^m}^{(m)}(y_1,\dots y_m;t,q;1,-t,\pm\sqrt{t}).
\end{aligned}$$

By the considerations of [26, Section 5.4], we find

$$K_{n^m}^{(m)}(y_1,\dots y_m;t,q;\pm 1,\pm\sqrt{t}) + \prod_{1\le j\le m}(y_j^{-1}-y_j)K_{n^m}^{(m)}(y_1,\dots y_m;t,q;\pm t,\pm\sqrt{t}) = P_{2n\omega_m}^{D_m}(y_1^{-1},\dots y_m^{-1};t,q), \tag{8.53}$$

and similarly

$$\begin{aligned}
&\prod_{1\le j\le m}(y_j^{-1/2}-y_j^{1/2})K_{n^m}^{(m)}(y_1,\dots y_m;t,q;t,-1,\pm\sqrt{t}) && (8.54)\\
&\qquad + \prod_{1\le j\le m}(y_j^{-1/2}+y_j^{1/2})K_{n^m}^{(m)}(y_1,\dots y_m;t,q;-t,1,\pm\sqrt{t}) = P_{(2n+1)\omega}^{D_m}(y_1^{-1},\dots y_m^{-1};t,q).
\end{aligned}$$

The theorem follows. ☐

Corollary 8.9. *Conjecture 1 holds whenever $\ell(\lambda) \le 4$; Conjecture 2 holds whenever $\lambda_1 \le 4$.*

Proof. By the theorem, we must show that

$$[P_\lambda(x_1,\dots x_4;q,t)](x_1\dots x_4)^{n/2}P_{n\omega_4}^{D_4}(x_1^{-1},\dots x_4^{-1};q,t) = 0 \tag{8.55}$$

unless λ is of the form μ^2. Now, the triality automorphism of D_4 (which still applies in the Macdonald setting) implies the identity

$$\begin{aligned}
(x_1x_2x_3x_4)^{n/2}P_{n\omega_4}^{D_4}(x_1^{-1},\dots x_4^{-1};q,t) &= u^n P_n^{D_4}(u,x_1x_2/u,x_1x_3/u,x_1x_4/u;q,t), && (8.56)\\
&= u^n K_n^{(4)}(u,x_1x_2/u,x_1x_3/u,x_1x_4/u;q,t;\pm 1,\pm\sqrt{q}), && (8.57)
\end{aligned}$$

where $u = \sqrt{x_1x_2x_3x_4}$. By triangularity, this is a linear combination of the polynomials

$$u^n P_k((u)^{\pm 1},(x_1x_2/u)^{\pm 1},(x_1x_3/u)^{\pm 1},(x_1x_4/u)^{\pm 1};q,t) \tag{8.58}$$

for $k \leq n$; by symmetry, only those k having the same parity as n occur. Since for $\ell(\lambda) \leq 4$,

$$u^{2l} P_\lambda(x_1, \dots x_4) = P_{l^4+\lambda}(x_1, \dots x_4), \tag{8.59}$$

preserving the constraint $\lambda = \mu^2$, we find that it suffices to show that

$$[P_\lambda(x_1, x_2, \dots x_4; q, t)] u^k P_k((u)^{\pm 1}, (x_1x_2/u)^{\pm 1}, (x_1x_3/u)^{\pm 1}, (x_1x_4/u)^{\pm 1}; q, t) \tag{8.60}$$

for $\lambda \neq \mu^2$. Now, we have the generating function

$$\begin{aligned}
&\sum_k v^k u^k P_k((u)^{\pm 1}, (x_1x_2/u)^{\pm 1}, (x_1x_3/u)^{\pm 1}, (x_1x_4/u)^{\pm 1}; q, t) \\
&\qquad = \frac{(tvx_1x_2x_3x_4, tv, tvx_1x_2, tvx_1x_3, tvx_1x_4, tvx_2x_3, tvx_2x_4, tvx_3x_4; q)}{(vx_1x_2x_3x_4, v, vx_1x_2, vx_1x_3, vx_1x_4, vx_2x_3, vx_2x_4, vx_3x_4; q)} &(8.61)\\
&\qquad = \frac{(tvx_1x_2x_3x_4, tv; q)}{(vx_1x_2x_3x_4, v; q)} \sum_{\ell(\mu)\leq 2} \frac{C^-_\mu(t; q, t^2)}{C^-_\mu(q; q, t^2)} P_{\mu^2}(x_1, x_2, x_3, x_4; q, t), &(8.62)
\end{aligned}$$

by Macdonald's generalized Littlewood conjecture. The factors out front have no effect on the vanishing requirement; the corollary follows. □

Remark. In particular, the conjectures hold if $|\lambda| \leq 5$.

We observed above that the case $q = t$ of Conjectures 1 and 2 follows from the theory of symmetric spaces, specifically the spaces $U(2n)/Sp(2n)$ and $U(n)/O(n)$. It is thus natural to wonder whether one can formulate similar conjectures for other symmetric spaces. This indeed is the case; for instance, the analogous "conjecture" for spaces of the form $G \times G/G$ results is simply the orthogonality of the Macdonald polynomials for the associated root system. For the other classical symmetric spaces, the situation turns out to be more complicated, as we shall see below.

One approach to generating such conjectures is simply to make an educated guess based on the form of the integral for $q = t$. For the Grassmannian $U(m+n)/U(m) \times U(n)$ with $m \leq n$, the Schur case is

$$\int_{U_1 \in U(m), U_2 \in U(n)} s_\lambda(U_1 \oplus U_2) dU_1 dU_2 = 0, \tag{8.63}$$

unless the dominant weight λ of $U(m+n)$ satisfies

$$\lambda_i + \lambda_{m+n+1-i} = 0, \quad 1 \leq i \leq m \tag{8.64}$$

$$\lambda_i = 0, \quad m+1 \leq i \leq n-m, \tag{8.65}$$

in which case the integral is 1. This condition can be stated more concisely as $\lambda = \mu\overline{\mu}$ for $\ell(\mu) \leq m$, where $\mu\overline{\nu}$ denotes the dominant weight of $U(m+n)$ with positive part μ and negative part $0^{m+n} - \nu$. This immediately suggests the following conjecture. Here and for the remainder of this section, we take the convention that a factor $1/Z$ in front of an integral of a Macdonald or Koornwinder polynomial over a weight function is the constant that makes the integral 1 when the polynomial is trivial.

Conjecture 3. *Let m and n be integers with $0 \le m \le n$. Then for a dominant weight $\mu\overline{\nu}$ of $U(m+n)$,*

$$\frac{1}{Z}\int P_{\mu\overline{\nu}}(x_1,\dots x_m,y_1,\dots y_n)\prod_{1\le i\ne j\le m}\frac{(x_i/x_j;q)}{(tx_i/x_j;q)}\prod_{1\le i\ne j\le n}\frac{(y_i/y_j;q)}{(ty_i/y_j;q)}\prod_{1\le i\le m}\frac{dx_i}{2\pi\sqrt{-1}x_i}\prod_{1\le i\le n}\frac{dy_i}{2\pi\sqrt{-1}y_i}=0 \tag{8.66}$$

unless $\mu=\nu$ and $\ell(\mu)\le m$, in which case the integral is

$$\frac{C^-_\mu(q;q,t)C^+_\mu(t^{m+n-2}q;q,t)C^0_\mu(t^n,t^m;q,t)}{C^-_\mu(t;q,t)C^+_\mu(t^{m+n-2}t;q,t)C^0_\mu(qt^{n-1},qt^{m-1};q,t)}. \tag{8.67}$$

Remark 1. Unlike Conjectures 1 and 2 (as well as the other conjectures below), the nonzero value here has not been computed via Pieri identities, but has merely been guessed from low-order examples.

Remark 2. There is an obvious analogue for Koornwinder polynomials (related to the Grassmannians $O(m+n)/O(m)\times O(n)$ and $Sp(2m+2n)/Sp(2m)\times Sp(2n)$) but we have not been able to test it enough to justify making a formal conjecture.

The reason why it was relatively easy to formulate conjectures for the spaces $U(n)/O(n)$, $U(2n)/Sp(2n)$ is that in those cases, the rank of the smaller group is about half the rank of the bigger group. This, for instance, is what allowed us to compute the nonzero values via Pieri identities. In the remaining cases, the rank differs, if at all, by only 1, and thus the vanishing condition is not enough to determine the weight function. In a number of cases, however, the small group is most naturally taken to be disconnected, and while the rank of the identity component is indeed large, the effective rank of the nonidentity component is often much smaller.

The simplest example of this is the case $U(2n)/U(n)\times U(n)$. As the stabilizer group of a symmetric space, $U(n)\times U(n)$ is the subgroup preserved by an involution acting on $U(2n)$; to be precise, it is the centralizer of the element

$$\begin{pmatrix} I_n & 0\\ 0 & -I_n\end{pmatrix}. \tag{8.68}$$

Now, the element

$$\begin{pmatrix} 0 & I_n\\ I_n & 0\end{pmatrix}, \tag{8.69}$$

while not preserved by the involution, is at least preserved up to sign; furthermore, it normalizes $U(n)\times U(n)$, acting by switching the two unitary groups. If we integrate a Schur function over the corresponding coset of $U(n)\times U(n)$, the integral vanishes on the same weights, and evaluates to ±1 where nonzero. To extend this to the Macdonald case, we observe that the eigenvalues of an element of this coset come in $\pm$ pairs; we thus wish an integral of the form

$$\int P_{\mu\overline{\nu}}(\pm\sqrt{x_1},\pm\sqrt{x_2},\cdots\pm\sqrt{x_n};q,t)w(x)\prod_{1\le j\le n}\frac{dx_j}{2\pi\sqrt{-1}x_j}, \tag{8.70}$$

vanishing unless $\mu=\nu$. Since

$$e_1(\pm\sqrt{x_1},\pm\sqrt{x_2},\cdots\pm\sqrt{x_n})=0, \tag{8.71}$$

the Pieri identity argument applies here, and thus the weight function (if it exists) is unique. By examining low-rank cases, we are led to the following conjecture.

Conjecture 4. *For any integer $n \geq 0$, and any dominant weight $\mu\overline{\nu}$ of $U(2n)$,*

$$\frac{1}{Z}\int P_{\mu\overline{\nu}}(\pm\sqrt{x_1}, \pm\sqrt{x_2}, \cdots \pm\sqrt{x_n}; q, t) \prod_{1\leq i\neq j\leq n} \frac{(x_i/x_j; q^2)}{(t^2 x_i/x_j; q^2)} \prod_{1\leq i\leq n} \frac{dx_i}{2\pi\sqrt{-1}x_i} = 0 \tag{8.72}$$

unless $\mu = \nu$, when the integral is

$$\frac{(-1)^{|\mu|} C^-_\mu(q; q, t) C^+_\mu(t^{2n-2}q; q, t) C^0_\mu(t^n, -t^n; q, t)}{C^-_\mu(t; q, t) C^+_\mu(t^{2n-2}t; q, t) C^0_\mu(qt^{n-1}, -qt^{n-1}; q, t)} \tag{8.73}$$

By analogy with Proposition 8.4, we would have expected the nonzero values of this integral to differ from the nonzero values in Conjecture 3 by only a sign factor. It turns out that these values are (conjecturally) attained by another nice integral.

Conjecture 5. *For any integer $n \geq 0$, and any dominant weight $\mu\overline{\nu}$ of $U(2n)$,*

$$\frac{1}{Z}\int P_{\mu\overline{\nu}}(x_1, \ldots x_n, y_1, \ldots y_n; q, t) \prod_{1\leq i,j\leq n} \frac{(qx_i/y_j, qy_i/x_j; q^2)}{(tx_i/y_j, ty_i/x_j; q^2)} \prod_{1\leq i\neq j\leq n} \frac{(x_i/x_j, y_i/y_j; q^2)}{(qtx_i/x_j, qty_i/y_j; q^2)} \prod_{1\leq i\leq n} \frac{dx_i}{2\pi\sqrt{-1}x_i} = 0 \tag{8.74}$$

unless $\mu = \nu$, when the integral is

$$\frac{C^-_\mu(q; q, t) C^+_\mu(t^{2n-2}q; q, t) C^0_\mu(t^n, -t^n; q, t)}{C^-_\mu(t; q, t) C^+_\mu(t^{2n-2}t; q, t) C^0_\mu(qt^{n-1}, -qt^{n-1}; q, t)} \tag{8.75}$$

Remark. Note that the weight function in this case is not of a standard Macdonald or Koornwinder form. The associated orthogonal polynomials may be of interest.

For the spaces $O(2n)/O(n) \times O(n)$ and $Sp(2n)/U(n)$, we have the following conjecture. Here $[2p_{k/2}]$ represents the homomorphism such that $p_{2k+1} \to 0$, $p_{2k} \to 2p_k$; this is just the infinite variable analogue of the specialization $\pm\sqrt{x_1}, \pm\sqrt{x_2}, \cdots \pm\sqrt{x_n}$.

Conjecture 6. *For any partition λ,*

$$I_K(\tilde{K}_\lambda([2p_{k/2}]; q, t, T; a, -a, b, -b); q^2, t^2, T; -1, -t, a^2, b^2) = 0 \tag{8.76}$$

unless λ is of the form 2μ, in which case the integral is

$$\frac{(-1)^{|\mu|} C^-_\mu(q; q^2, t) C^+_\mu(a^2b^2T^2/t^3; q^2, t) C^0_\mu(T, -a^2T/t, -b^2T/t, a^2b^2T/t^2; q^2, t)}{C^-_\mu(t; q^2, t) C^+_\mu(a^2b^2T^2/qt^2; q^2, t) C^0_{2\mu}(a^2b^2T^2/t^3; q^2, t^2)}. \tag{8.77}$$

Similarly, for the spaces $O(4n)/U(2n)$ and $Sp(4n)/Sp(2n) \times Sp(2n)$,

Conjecture 7. *For any partition λ,*

$$I_K(\tilde{K}_\lambda([2p_{k/2}]; q, t, T; a, -a, b, -b); q^2, t^2, T; -t, -qt, a^2, b^2) = 0 \tag{8.78}$$

unless λ is of the form μ^2, in which case the integral is

$$\frac{(-1)^{|\mu|} C^-_\mu(qt; q, t^2) C^+_\mu(a^2b^2T^2/t^4; q, t^2) C^0_\mu(T, -a^2T/t, -b^2T/t, a^2b^2T/t^2; q, t^2)}{C^+_\mu(a^2b^2T^2/qt^3; q, t^2) C^-_\mu(t^2; q, t^2) C^0_{\mu^2}(a^2b^2T^2q/t^2; q^2, t^2)}. \tag{8.79}$$

In these cases, we have no conjectured weight function corresponding to the values with the sign factors removed; the problem is that the two Schur cases associated to each integral not only break the BC_n symmetry, but do so in different ways.

It turns out that Propositions 7.11 and 7.12 produce integrals associated to orthogonal group Grassmannians. For $O(2n)/O(1) \times O(2n-1)$:

Proposition 8.10. *For any partition λ,*

$$I_K(\tilde{K}_\lambda([p_k + t_0^k + t_0^{-k}]; q, t, tT; t_0, t_1, t_2, t_3); q, t, T; t_0 t, t_1, t_2, t_3) = 0 \tag{8.80}$$

unless $\ell(\lambda) \leq 1$, in which case the integral is

$$t_0^{-\lambda_1} \frac{(Tt_0t_1, Tt_0t_2, Tt_0t_3, Tt_0t_1t_2t_3/t; q)_{\lambda_1}}{(q^{\lambda_1-1}T^2t_0t_1t_2t_3, T^2t_0t_1t_2t_3/t; q)_{\lambda_1}}. \tag{8.81}$$

Similarly, for $O(2n+1)/O(1) \times O(2n)$,

Proposition 8.11. *For any partition λ,*

$$I_K(\tilde{K}_\lambda(; q, t, T; t_0 t, t_1, t_2, t_3); q, t, T; t_0, t_1, t_2, t_3) = 0 \tag{8.82}$$

unless $\ell(\lambda) \leq 1$, in which case the integral is

$$t_0^{\lambda_1} \frac{(T, Tt_1t_2/t, Tt_1t_3/t, Tt_2t_3/t; q)_{\lambda_1}}{(q^{\lambda_1-1}T^2t_0t_1t_2t_3/t, T^2t_0t_1t_2t_3/t^2; q)_{\lambda_1}}. \tag{8.83}$$

Less trivially, for the nonidentity component of $O(2n+1)/O(2) \times O(2n-1)$,

Theorem 8.12. *For any partition λ,*

$$I_K(\tilde{K}_\lambda([p_k + a^k + (-a)^k + a^{-k} + (-a)^{-k}]; q, t, t^2T; a, -a, b, -b); q, t, T; at, -at, b, -b) = 0 \tag{8.84}$$

unless $\ell(\lambda) \leq 2$ and $|\lambda|$ is even, in which case the integral is generically nonzero and admits a factorization into q-symbols.

Proof. By two consecutive applications of the quasi-branching rule, the integral evaluates to

$$\sum_{0 \prec \kappa' \prec \lambda'} \psi^{(i)}_{\lambda/\kappa}(tT; q, t, T\sqrt{a^2b^2t/q})\psi^{(i)}_{\kappa/0}(tT; q, t, T\sqrt{a^2b^2/q}) \frac{k^0_\kappa(q, t, tT; a{:}-at, b, -b)k^0_\lambda(q, t, t^2T; -a{:}a, b, -b)}{k^0_\kappa(q, t, tT; -at{:}a, b, -b)}. \tag{8.85}$$

This is clearly nonzero unless $\ell(\lambda) \leq 2$, in which case $\ell(\kappa) \leq 1$. The sum turns out to be proportional to a terminating very-well-poised ${}_8W_7$, summable by Equation II.16 of [7]. ☐

Finally, for the nonidentity component of $O(2n+2)/O(2) \times O(2n)$, a similar calculation gives

Theorem 8.13. *For any partition λ,*

$$I_K(\tilde{K}_\lambda(; q, t, T; at, -at, b, -b); q, t, T; a, -a, b, -b) = 0 \tag{8.86}$$

unless $\ell(\lambda) \leq 2$ and $|\lambda|$ is even, in which case the integral is generically nonzero, and admits a factorization into q-symbols.

It is likely that there are a number of other "nice" integrals satisfying appropriate vanishing conditions, but a more systematic method of construction will likely be needed to find them. It is, however, unclear to what extent our existing conjectures can be systematized, especially given the multiple (untwisted) integrals associated to $U(2n)/U(n) \times U(n)$.

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