DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

EQUILIBRIA IN MARKETS WITH A RIESZ SPACE OF COMMODITIES

Charalambos D. Aliprantis
California Institute of Technology
Indiana University and Purdue University at Indianapolis

and

Donald J. Brown
California Institute of Technology
Yale University

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ABSTRACT

Using the theory of Riesz spaces, we present a new proof of the existence of competitive equilibria for an economy having a Riesz space of commodities.

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1. INTRODUCTION

In the Arrow-Debreu model of a Walrasian economy, [3] and [7], the commodity space is $\mathbb{R}^n$ and the price space is $\mathbb{R}^n_+$, where n is the number of commodities. Agent's characteristics such as consumption sets, production sets, utility functions, the price simplex, excess demand functions, etc. are introduced in terms of subsets of $\mathbb{R}^n$ or $\mathbb{R}^n_+$ or functions on $\mathbb{R}^n$ or $\mathbb{R}^n_+$. As is well known, the Arrow-Debreu model allows consumption and production to be contingent on time and the state of the world, when there is a finite number of periods and a finite number of states.

In a world of uncertainty where there are an infinite number of states or an intertemporal economy having an infinite number of time periods (e.g., an infinite horizon), the appropriate model for the space of commodities is an infinite dimensional vector space. In particular, ordered locally convex topological vector spaces are the most frequently used models of infinite dimensional commodity spaces. In such models, the price space is the cone of positive continuous linear functionals in the dual space. See [5], [12], and [14].

The paradigmatic example being the dual pair $\langle L_\infty(\mu), L_1(\mu)\rangle$ where μ is a σ-finite measure on the underlying state space; $L_\infty(\mu)$ is the space of commodities; $L_1^+(\mu)$ is the space of prices. This model was introduced by Bewley in [4], where he proved the existence of a Walrasian equilibrium in both exchange economies and economies with production. In proving existence, Bewley considers $L_\infty(\mu)$ with the sup norm topology, and in this case the dual space is the vector space of all bounded additive functionals on $L_\infty(\mu)$.

The spaces $\mathbb{R}^n$ and $L_\infty(\mu)$ in addition to being ordered linear vector spaces are also Riesz spaces or vector lattices. In fact, considered as Banach spaces under the sup norm, they belong to the special class of Banach lattices, i.e. to the class of normed Riesz spaces which are complete under their norms. In this paper, we exploit the Riesz space structure of the commodity spaces to give a new proof of the existence of Walrasian equilibrium. The relevant facts about Riesz spaces may be found in the appendix.

In this work the space of commodities will be a Riesz space and the price simplex will be a suitable subset of the order dual. We define an excess demand function as a mapping from the price simplex into the commodity space which is weakly continuous and satisfies a boundary condition. If Δ is the price simplex and E is the excess demand function with domain $D \subseteq \Delta$, then we define the revealed preference relation $\succ$ on D by saying that $p \succ q$, whenever $p, q \in D$ and $p(E_q) > 0$. The definition of $\succ$ was suggested by Nikaido's discussion of economies with gross substitutability in [15, Section 18, p. 197].

Our main result is that whenever E satisfies Walras' law, then the set of maximal elements of $\succ$ is non-empty and every maximal element q of $\succ$ is an equilibrium price, i.e. $E_q = 0$. The existence proof uses Riesz space arguments along with an infinite dimensional version of Sonnenschein's theorem that an irreflexive, convex-valued, upper semicontinuous binary relation on a compact convex subset of $\mathbb{R}^n$ has a maximal element.

It is demonstrated in the paper that our results encompass all of the finite dimensional cases, say in [8], Bewley's result for ℓ_∞ in the pure exchange case [4], and several new infinite dimensional markets.

2. EQUILIBRIUM THEOREMS

Let $\langle L,L'\rangle$ be a Riesz dual system. A <u>price simplex</u> associated with $\langle L,L'\rangle$ is a non-empty convex w^*-compact subset Δ of $\tilde{L}_+$ (the members of which are called prices) satisfying the following two properties:

a) The convex set S of all <u>strictly positive prices</u>, i.e. the convex set

$$S = \{p \in L'_+ \cap \Delta : p \gg 0\},$$

is w^*-dense in Δ; and

b) The cone generated by S (i.e., the set $\bigcup_{\lambda \geq 0} \lambda S$) is w^*-dense in L'_+.

Now let Δ be a price simplex for a Riesz dual system $\langle L,L'\rangle$. An <u>excess demand function</u> E for Δ is a mapping $p \mapsto E_p$, from a convex subset D (=dom E) of $\tilde{L}_+$ into L, satisfying the following four properties:

1. <u>Density Condition</u>: D is a w^*-dense convex subset of Δ;

2. <u>Continuity Condition</u>: There exists a locally convex topology t on L compatible with the dual system $\langle L,L'\rangle$ (i.e., $\sigma(L,L') \subseteq t \subseteq \tau(L,L')$) such that $E:(D,w^*) \to (L,t)$ is continuous.

3. <u>Boundary Condition</u>: If a net $\{p_\alpha\} \subseteq D$ satisfies $p_\alpha \xrightarrow{w^*} q \in \Delta \sim D$, then $\overline{\lim}\, p(E_{p_\alpha}) > 0$ holds for some $p \in D$; and

4. <u>Walras' Law</u>: $p(E_p) = 0$ holds for all $p \in D$.

We define an <u>economy</u> $\mathcal{E}$ as a triplet $(\langle L,L'\rangle, \Delta, E)$, where Δ is a price simplex associated with the Riesz dual system $\langle L,L'\rangle$, and E is an excess demand function for Δ.

Consider a binary relation $\succ$ on a convex set D of a topological vector space (E,τ). As usual, for each $p \in D$ we write:

$$(-\infty, p) = \{q \in D : p \succ q\} \text{ and } (p, \infty) = \{q \in D : q \succ p\}.$$

The binary relation $\succ$ is said to be:

1. <u>irreflexive</u>, whenever $p \not\succ p$ holds for all $p \in D$;

2. <u>convex valued</u>, whenever (p, ∞) is convex for each $p \in D$; and

3. τ-<u>upper semicontinuous</u>, whenever $(-\infty, p)$ is τ-open in D for each $p \in D$.

An element $p \in D$ is said to be a <u>maximal element</u> for $\succ$, whenever $q \not\succ p$ holds for all $q \in D$.

The following theorem will be of fundamental importance.

THEOREM 2.1† *Let* (E, τ) *be a Hausdorff topological vector space, and let* D *be a non-empty, convex, and* τ*-compact subset of* E. *If* $\succ$ *is an irreflexive, convex valued, and upper semicontinuous binary relation on* D, *then the set of all maximal elements of* $\succ$ *is non-empty and* τ*-compact*.

PROOF. Let $F(p) = D \sim (-\infty, p)$. Since $p \notin (-\infty, p)$, it follows that $F(p)$ is non-empty for all $p \in \Delta$. Also, by the upper semicontinuity of $\succ$, we see that each $F(p)$ is τ-closed, and hence, $\bigcap_{p \in D} F(p)$ is τ-compact. Now it is a routine matter to verify that $\bigcap_{p \in D} F(p)$ is precisely the set of all maximal elements of $\succ$. Thus, it remains to be shown that $\bigcap_{p \in D} F(p)$ is non-empty.

To this end, let $p_i, \ldots, p_n \in D$. Then we claim that $co\{p_1, \ldots, p_n\} \subseteq \bigcup_{i=1}^{n} F(p_i)$ holds. Indeed, if $q = \sum_{i=1}^{n} \alpha_i p_i$ is a convex combination and $q \notin \bigcup_{i=1}^{n} F(p_i)$, then $p_i \succ q$ holds for all $i = 1, \ldots, n$, and so, (since $\succ$ is convex valued) we must have $q = \sum_{i=1}^{n} \alpha_i p_i \succ q$,

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† The argument in this proposition is due to H. F. Sonnenschein [18], who used it to prove Theorem 2.1. for a finite dimensional topological vector space. We thank Kim Border for recalling Sonnenschein's theorem to our attention.

contrary to the irreflexivity of $\succ$. Now according to [11, Lemma 1, p. 21] we have $\bigcap_{p \in D} F(p) \neq \emptyset$, as desired. ■

For the rest of the discussion in this section $\mathcal{E} = (\langle L, L' \rangle, \Delta, E)$ will be a fixed economy. The economy $\mathcal{E}$ defines a binary relation $\succ$ on D (which will be referred to as the *revealed preference relation* [15, Section 18.3]) as follows:

If $p, q \in D$, *then we say that* p *dominates* q *(in symbols,* $p \succ q$), *whenever* $p(E_q) > 0$ *holds*.

THEOREM 2.2. *The revealed preference relation for an arbitrary economy is irreflexive, convex valued, and* w^**-upper semicontinuous*.

PROOF. Since $p(E_p) = 0$ holds for all $p \in D$, we see that $p \not\succ p$ for each $p \in D$, and so, $\succ$ is irreflexive.

To see that $\succ$ is convex valued, let $q_1, q_2 \in (p, \infty)$ and $0 < \alpha < 1$. Thus, $q_1(E_p) > 0$ and $q_2(E_p) > 0$ both hold, and so, if $q = \alpha q_1 + (1-\alpha) q_2$, then $q(E_p) = \alpha q_1(E_p) + (1-\alpha) q_2(E_p) > 0$. That is, $q \succ p$ holds, proving that (p, ∞) is a convex set.

Finally, let us establish that $\succ$ is upper semicontinuous. To this end, let $q \in (-\infty, p)$. This means that $p(E_q) > 0$. Since p is a t-continuous linear functional on L, it follows that the function $r \mapsto p(E_r)$, from (D, w^*) into L, is continuous. Consequently, there exists a w^*-neighborhood V of q such that $r \in D \cap V$ implies $p(E_r) > 0$ (i.e., $p \succ r$). Therefore, q is an interior point of $(-\infty, p)$, and hence, $(-\infty, p)$ is an open subset of (D, w^*). ■

An element $p \in D$ is said to be a free disposal equilibrium price, whenever $E_p \leq 0$ holds.

The free disposal equilibrium prices are characterized as follows.

THEOREM 2.3. For an element $p \in D$ the following statements are equivalent:

1. p is a free disposal equilibrium price.
2. p is a maximal element for the revealed preference relation $\succ$.
3. $q(E_p) \leq 0$ holds for all $q \in D$.

PROOF. (1) => (2) Let $E_p \leq 0$. Then $q(E_p) \leq 0$ holds for all $q \in D$, and so, $q \not\succ p$ for all $q \in D$. That is, p is a maximal element for $\succ$.

(2) => (3) Obvious.

(3) => (1) Since D is w*-dense in Δ, statement (3) implies that $q(E_p) \leq 0$ holds for all $q \in \Delta$. From this it follows easily that $q(E_p) \leq 0$ holds for all $q \in L'_+$. But then Theorem 4.4 (in the appendix) shows that $E_p \leq 0$. Thus, p is a free disposal equilibrium price, and the proof of the theorem is finished. ■

We now come to one of the main results of this paper.

THEOREM 2.4. Every economy $\mathcal{E}$ has a non-empty w*-compact set of free disposal equilibrium prices.

PROOF. We first show that the set of free disposal equilibrium prices is non-empty. Let A denote the collection of all finite subsets of D. For each $\alpha \in A$, let D_α denote the convex hull of α. Clearly, each D_α is w*-compact, and $\bigcup_{\alpha \in A} D_\alpha = D$ holds. Also, the collection $\{D_\alpha\}$ is directed upwards by inclusion. Now let $\alpha \in A$ be fixed. By Theorem 2.2 the revealed preference relation $\succ$ restricted to D_α is irreflective, convex valued, and w*-upper semicontinuous. Hence, by Theorem 2.1, there exists some $p_\alpha \in D_\alpha$ that is maximal for $\succ$ on D_α (i.e., $p(E_{p_\alpha}) \leq 0$ holds for each $p \in D_\alpha$). Next consider the net $\{p_\alpha : \alpha \in A\}$, where A is indexed by the inclusion $\supseteq$. Since Δ is w*-compact, we can assume (by passing to a subnet if necessary) that $p_\alpha \xrightarrow{w^*} q$ holds in Δ.

We claim that $q \in D$. Indeed, if $q \in \Delta \sim D$ holds, then by our boundary condition there exists some $p \in D$ with $\overline{\lim}\, p(E_{p_\alpha}) > 0$. On the other hand, p must lie in some D_α; and since $\{D_\alpha\}$ is directed upwards, there exists some $\beta \in A$ so that $p \in D_\alpha$ holds for all $\alpha \supseteq \beta$. But then for each $\alpha \supseteq \beta$ we have $p(E_{p_\alpha}) \leq 0$, and so, $\overline{\lim}\, p(E_{p_\alpha}) \leq 0$ holds, which is a contradiction. Thus, $q \in D$.

We now establish that q is a free disposal equilibrium price. To this end, let $p \in D$. Since the function, $r \mapsto p(E_r)$ from (D, w^*) into $\mathbb{R}$ is continuous, it follows that $p(E_q) = w^*\text{-}\lim p(E_{p_\alpha})$. As above, we see that there exists some $\beta \in A$ satisfying $p(E_{p_\alpha}) \leq 0$ for all $\alpha \supseteq \beta$, and so, $p(E_q) \leq 0$ holds for all $p \in D$. By Theorem 2.3, q is a free disposal equilibrium price.

Finally, let us show that the set of all free disposal equilibrium prices is w*-compact. It is enough to show that this set is w*-closed in D. So, let $\{p_\alpha\}$ be a net of free disposal equilibrium prices satisfying $p_\alpha \xrightarrow{w^*} q \in \Delta$. If $q \in \Delta \sim D$, then by our boundary condition there exists some $p \in D$ with $\overline{\lim}\, p(E_{p_\alpha}) > 0$. On the other hand, since each p_α is a maximal element, $p(E_{p_\alpha}) \leq 0$ holds for each α, which contradicts $\overline{\lim}\, p(E_{p_\alpha}) > 0$. Thus, $q \in D$. From the continuity of the function $r \mapsto p(E_r)$ we get $p(E_q) \leq 0$ for all $p \in D$, and so, by Theorem 2.3, q is a free disposal equilibrium price. The proof of the theorem is now complete. ■

A strictly positive price $p \in D$ is said to be an <u>equilibrium price</u> for the economy $\mathcal{E}$, whenever $E_p = 0$ holds.

The following fundamental theorem guarantees equilibrium prices.

THEOREM 2.5. <u>Every economy</u> $\mathcal{E}$ <u>with</u> D = S <u>has a non-empty w*-compact set of equilibrium prices</u>.

PROOF. If D = S holds, then we claim that $p \in S$ satisfies $E_p \leq 0$ if and only if $E_p = 0$. Indeed, if $E_p \leq 0$ holds, then either (a) $E_p < 0$ or (b) $E_p = 0$. If (a) is true, then (since $p \in S$) we have $p(E_p) < 0$, which contradicts Walras' law. Consequently, (b) is true, i.e., $E_p = 0$.

Thus, an element $p \in D$ is a free disposal equilibrium price if and only if p is an equilibrium price. The conclusion now follows from Theorem 2.4. ■

Finally, we note in passing that if the excess demand function E satisfies Samuelson's weak axiom of revealed preference, i.e., $p, q \in D$ and $p(E_q) \leq 0$ implies $q(E_p) > 0$, then the economy $\mathcal{E}$ has precisely one free disposal equilibrium price. Indeed, note first that Samuelson's axiom is equivalent to the statement: If $p, q \in D$, satisfy $p \not> q$, then $q > p$ holds. Thus, if $p, q \in D$ are two distinct free disposal equilibrium prices, then by the maximality of p and q we have $p \not> q$ and $q \not> p$, which contradicts Samuelson's axiom.

3. EXAMPLES OF ECONOMIES

First, we present examples of price simplices. Our examples will be related one way or another to AM-spaces with unit. A Banach lattice L is said to be an AM-space (abstract M-space), whenever for each $f, g \in L^+$ we have

$$\|f \vee g\| = \operatorname{Max}\{\|f\|, \|g\|\}.$$

(The letter M stands for maximum.) The element $e > 0$ in an AM-space L is said to be the unit of L, whenever the order interval $[-e,e]$ coincides with the closed unit ball of L, i.e., whenever $[-e,e] = \{f \in L : \|f\| \leq 1\}$ holds. Here are some examples of AM-spaces with units:

1. The Riesz space $C_b(\Omega)$ of all continuous bounded real valued functions on a topological space Ω, with the sup norm

$$\|f\|_\infty = \sup\{|f(\omega)|: \omega \in \Omega\}.$$

The unit e is the constant function one on Ω.

2. The Riesz space $L_\infty(\mu)$ of all essentially bounded measurable functions on a measure space (X,Σ,μ), with the essential sup norm

$$||f||_\infty = \text{ess sup}|f|.$$

The constant function one is the unit.

3. The Riesz space ℓ_∞ of all order bounded real sequences with the sup norm. Again, the constant sequence one is the unit.

The Banach lattice c_0 of all null sequences with the sup norm is an AM-space, but it does not admit a unit. It is easy to check but important to observe that if an AM-space has a unit e, then for each positive linear functional p on L we have

$$||p|| = p(e).$$

THEOREM 3.1. <u>Let</u> $\langle L,L'\rangle$ <u>be a Riesz dual system with L an AM-space with unit</u>. <u>If the convex set</u>

$$S = \{p \in L'_+ : p \gg 0 \ \underline{\text{and}}\ ||p|| = 1\}$$

<u>is non-empty, then the w*-closure Δ of S in L* is a price simplex for</u> $\langle L,L'\rangle$. <u>In addition, if</u> L'_+ <u>is w*-dense in</u> L^*_+, <u>then</u>

$$\Delta = \{p \in L^*_+ : ||p|| = 1\}.$$

PROOF. Let $e > 0$ be the unit of L. Since in this case $L^* = L^\sim$ holds (see, [2, Thm 24.10, p. 192]), it is easy to see that

$$S = \{p \in L'_+ : p \gg 0 \text{ and } p(e) = 1\}.$$

This implies that S is a convex set. Also, since S is a subset of the closed unit ball of L*, Alaoglu's theorem shows that the w*-closure Δ of S in L* is convex and w*-compact. Clearly, $\Delta \subseteq \{p \in L^*_+ : p(e) = 1\}$.

Now assume that $S \neq \emptyset$. Fix some $p \in S$. Then for each $q \in L'_+$ with $q(e) = 1$ we have $\alpha p + (1-\alpha)q \in S$ for all $0 < \alpha < 1$, and so, from $w^*-\lim_{\alpha \to 0+} [\alpha p + (1-\alpha)q] = q$ we see that $\bigcup_{\lambda \geq 0} \lambda S$ is w*-dense in L'_+ (and that $q \in \Delta$). Thus, Δ is a price simplex for $\langle L,L'\rangle$.

Finally, assume that L'_+ is w*-dense in L^*_+. Let $p \in L^*_+$ satisfy $p(e) = 1$. Pick a net $\{p_\alpha\} \subseteq L'_+$ with $p_\alpha \xrightarrow{w^*} p$; we can assume $p_\alpha \neq 0$ for all α. From $\lim p_\alpha(e) = p(e) = 1$, $\{p_\alpha/p_\alpha(e)\} \subseteq \Delta$, and $w^*-\lim [p_\alpha/p_\alpha(e)] = p$, it follows that $p \in \Delta$. Thus, $\Delta = \{p \in L^*_+ : p(e) = 1\}$ holds in this case. ■

The specific examples of price simplices we have in mind are all special cases of the preceding theorem. The finite dimensional case comes first.

EXAMPLE 3.2. <u>The convex set</u>

$$\Delta = \{p = (p_1,\ldots,p_n) \in \mathbb{R}^n : p_i \geq 0 \ \underline{\text{for}}\ i = 1,\ldots,n \ \underline{\text{and}} \sum_{i=1}^n p_i = 1\}$$

<u>is a price simplex for the Riesz dual system</u> $\langle \mathbb{R}^n, \mathbb{R}^n\rangle$.

To see this, consider $\mathbb{R}^n$ as an AM-space with unit. (The norm is $||x||_\infty = \max\{|x_i|: i = 1,\ldots,n\}$ and the unit $e = (1,\ldots,1)$.) The rest follows from Theorem 3.1 by observing that if $p = (p_1,\dots,p_n) \geq 0$ holds in the norm dual of $(\mathbb{R}^n, \|\cdot\|_\infty)$, then $\|p\| = p(e) = \sum_{i=1}^{n} p_i$. Also, note that

$$S = \{p = (p_1,\dots,p_n) \in \mathbb{R}^n : p_i > 0 \text{ for } i = 1,\dots,n \text{ and } \sum_{i=1}^{n} p_i = 1\}. \blacksquare$$

This next case is that of $C(\Omega)$ with Ω a compact metric space.

EXAMPLE 3.3. <u>If</u> $L = C(\Omega)$ <u>for some compact metric space, then the convex set</u>

$$\Delta = \{p \in L_+^* : \|p\| = 1\}$$

<u>is a price simplex for the Riesz dual system</u> $\langle L, L^*\rangle$, <u>where</u> L^* <u>is the norm dual of</u> $C(\Omega)$ <u>equipped with the sup norm</u>.

To see this, note that according to Theorem 3.1 it is enough to show that there exists a strictly positive linear functional on L.

To this end, start by observing that since Ω is a compact metric space, $C(\Omega)$ is a separable Banach lattice; see, for instance [16, Prop. 7.5, p. 105]. This implies that $B_+^* = \{p \in L_+^* : \|p\| \leq 1\}$ with the w*-topology is a compact metric space [10, Thm 1, p. 426], and in particular, that (B_+^*, w^*) is separable. Fix a countable w*-dense subset $\{p_1, p_2, \dots\}$ of B_+^*, and put $p = \sum_{n=1}^{\infty} 2^{-n} p_n \in L_+^*$. Then p is strictly positive on L.

For if $p(f) = 0$ holds for some $f \geq 0$, then $p_n(f) = 0$ likewise holds for all n, and so, by the w*-denseness of $\{p_1, p_2, \dots\}$ in B_+^* we get $p(f) = 0$ for all $p \in B_+^*$, from which it follows that $f = 0$. This implies that p is strictly positive on L. $\blacksquare$

Now consider the Banach lattice $L = L_1(\mu)$ for some measure space (X, Σ, μ). It is a fact that L is an ideal in its second norm dual; see [1, Thm 9.2, p. 61]. If, in addition, (X, Σ, μ) is σ-finite, then $L_1^*(\mu) = L_\infty(\mu)$ holds (see, for example [2, Thm 27.10, p. 234]), and so, in the σ-finite case $\langle L_\infty(\mu), L_1(\mu)\rangle$ is a Riesz dual system. This Riesz dual system admits always a price simplex. The details follow.

EXAMPLE 3.4. If (X, Σ, μ) <u>is a</u> σ <u>-finite measure space, then the convex set</u>

$$\Delta = \{p \in L_\infty^*(\mu) : p \geq 0 \text{ and } \|p\| = 1\}$$

<u>is a price simplex for the Riesz dual system</u> $\langle L_\infty(\mu), L_1(\mu)\rangle$.

Note that

$$S = \{f \in L_1^+(\mu) \cap \Delta : f \gg 0\}$$

$$= \{f \in L_1(\mu) : f \gg 0 \text{ on } L_\infty(\mu) \text{ and } \|f\|_1 = 1\}.$$

We show next that S is non-empty. To this end, pick a disjoint sequence $\{A_n\}$ of Σ satisfying $\bigcup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty$ for each n. Next for each n choose $0 < \lambda_n < 1$ with $\lambda_n \mu(A_n) < 2^{-n}$, and then let $g = \sum_{n=1}^{\infty} \lambda_n \chi_{A_n} \in L_1(\mu)$. Now the linear functional

$$p(f) = \int fg\,d\mu, \quad f \in L_1(\mu),$$

is strictly positive on $L_\infty(\mu)$, and so, $g/||g||_1 \in S$.

The rest now follows from Theorem 3.1 by observing that L_1^+ is w^*-dense in $(L_\infty^*)_+$ (see Theorem 4.5 in the appendix). ■

An important special case of the previous example is the Riesz dual system $\langle \ell_\infty, \ell_1 \rangle$.

Now let us consider a completely regular Hausdorff topological space Ω. In [17] F. D. Sentilles defined a notion of a topology β on $C_b(\Omega)$ which extends the notion of the strict topology introduced by R. C. Buck [6] for locally compact Ω. It was shown in [17] that the strict topology β has the following properties:

a) β is the finest locally convex topology on $C_b(\Omega)$ for which Dini's theorem holds (i.e., $\{f_\alpha\} \subseteq (\Omega)$ and $f_\alpha(\omega) \downarrow 0$ for all $\omega \in \Omega$ imply $f_\alpha \xrightarrow{\beta} 0$) [17, p. 328];

b) β is locally solid [17, Thm 6.1, p. 327]; and

c) when Ω is either σ-compact or complete separable metric, then the topological dual of $(C_b(\Omega), \beta)$ is precisely the vector space M_t of all linear functionals on $C_b(\Omega)$ that are representable as an integral with respect to a unique compact-regular Borel measure Ω [17, Thm 9.1, p. 332].

Thus, according to Theorem 4.3 of the appendix, if Ω is a complete separable metric space, then M_t is an ideal of $C_b^*(\Omega)$ (the norm dual of $C_b(\Omega)$ with the sup norm), and hence, $\langle C_b(\Omega), M_t \rangle$ is a Riesz dual system.

EXAMPLE 3.5. *If Ω is a complete separable metric space, then the Riesz dual system $\langle C_b(\Omega), M_t \rangle$ admits a price simplex.*

The proof goes as follows: By [19, Thm 14, p. 192 and Thm 18, p. 194], (M_t^+, w^*) is complete, separable, and metrizable. If $\{p_1, p_2, \ldots,\}$ is a countable w^*-dense subset of $M_t^+ \sim \{0\}$, then $p = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{p_n}{||p_n||} \in M_t^+$ is strictly positive on $C_b(\Omega)$ (see the corresponding proof in Example 3.3). The rest follows from Theorem 3.1. ■

An interesting special case of the preceding example is when $\Omega = C[0,1]$ with the topology generated by the sup norm.

Finally, we close this section with several examples of economies.

EXAMPLE 3.6. Consider the Riesz dual system $\langle \mathbb{R}^n, \mathbb{R}^n \rangle$ of Example 3.2 with the price simplex

$$\Delta = \{p = (p_1,\ldots,p_n) \in \mathbb{R}_+^n : \sum_{i=1}^{n} p_i = 1\}.$$

In Section 3 of [8], Debreu derives the excess demand correspondence for an economy having a finite number of profit maximizing firms and a finite number of utility maximizing consumers. If, in addition to his assumptions, we assume that production sets are strictly convex and that utility functions are strictly quasi-concave, then the excess demand correspondence is a function, which we denote by E. Debreu shows that the domain of E is Δ; E is continuous; and E satisfies Walras' law. Hence, $(\langle \mathbb{R}^n, \mathbb{R}^n\rangle, \Delta, E)$ is an economy in our sense, and so, by Theorem 2.4 there exists some $p \in \Delta$ with $E_p \leq 0$ and $p(E_p) = 0$. This proves Theorem 6 of [8] for the special case where the excess demand correspondence is a function. ■

EXAMPLE 3.7. Again consider the Riesz dual system $\langle \mathbb{R}^n, \mathbb{R}^n\rangle$ with Δ and S as in Example 3.2.

Consider $\mathbb{R}^n$ as the commodity space of an exchange economy having a finite number of consumers whose consumption sets are $\mathbb{R}^n_+$. Each agent's endowment is in the interior of $\mathbb{R}^n_+$, and we assume that agents are utility maximizers having strictly quasi-concave, continuous, monotone utility functions. Again in [8], Debreu shows that for this economy the excess demand function E has as its domain S; that E is continuous; and that it satisfies Walras' law. Moreover, E is bounded from below (i.e., there exists some $x \in \mathbb{R}^n$ satisfying $x \leq E_p$ for all $p \in S$), and $\|E_{p_n}\| \to \infty$ whenever $\{p_n\} \subseteq S$ satisfies $p_n \to p \in \Delta \sim S$. It is easy to see that the latter two properties of E imply our boundary condition: $\{p_n\} \subseteq S$ and $p_n \to p \in \Delta \sim S$ imply $\overline{\lim}\, p(E_{p_n}) > 0$ for some $p \in S$. Hence, by Theorem 2.5 there exists some $p \in S$ with $E_p = 0$, and this is the conclusion of Theorem 8.3 of [9]. ■

EXAMPLE 3.8. Let $\langle \ell_\infty, \ell_1\rangle$ be the special Riesz dual system of Example 3.4 with the price simplex $\Delta = \{p \in \ell_\infty^* : p \geq 0 \text{ and } \|p\| = 1\}$. Note that $S = \{p \in \ell_1 : p \gg 0 \text{ on } \ell_\infty \text{ and } \|p\|_1 = 1\}$.

Consider ℓ_∞ with the Mackey topology $\tau(\ell_\infty, \ell_1)$ as the commodity space of an exchange economy having a finite number of consumers. Each consumer has ℓ_∞^+ with the relative $\tau(\ell_\infty, \ell_1)$-topology as his consumption set. Endowments are in $\ell_\infty^+ \sim \{0\}$. Agents maximize strictly quasi-concave, monotone, and $\tau(\ell_\infty, \ell_1)$-continuous preferences.

The social endowment ω is a vector of ℓ_∞^+ uniformly bounded away from zero. By Alaoglu's theorem $[0, \omega]$ is $\sigma(\ell_\infty, \ell_1)$-compact. Consequently, each agent's attainable consumption set is $\sigma(\ell_\infty, \ell_1)$-compact. We choose a $\sigma(\ell_\infty, \ell_1)$-compact convex subset of ℓ_∞^+ containing all attainable consumption sets, say the interval $[0,2\omega]$. Consider a new exchange economy where each agent has $[0,2\omega]$ as his consumption set and retains his original endowment and preferences. For this economy, the "excess demand function E" is well defined, with domain $D = \{p \in \ell_1 : p \geq 0 \text{ and } \|p\|_1 = 1\}$. However, since E need not be continuous on D, we cannot invoke our existence Theorem 2.4 directly. Rather we shall use the argument in the proof of Theorem 2.4 to demonstrate the existence of an equilibrium price for this example.

As in the proof of Theorem 2.4, let $\{D_\alpha : \alpha \in A\}$ be the family of all finite dimensional simplices contained in D. It is easy to show that $E:(D_\alpha, w^*) \to (\ell_\infty, \tau(\ell_\infty, \ell_1))$ is continuous, using arguments--say in Debreu [8]. Then the revealed preference relation $\succ$ restricted to D_α is irreflexive, convex valued, and w*-upper semicontinuous (see the proof of Theorem 2.2). Thus, by Theorem 2.1, $\succ$ has a maximal element of D_α, say q_α. Now consider the nets $\{q_\alpha\} \subseteq \Delta$ and $\{E_{q_\alpha}\} \subseteq [-\omega, \omega]$. Since Δ is $\sigma(\ell_\infty^*, \ell_\infty)$-compact and $[-\omega, \omega]$ is $\sigma(\ell_\infty, \ell_1)$-compact, by passing to two subnets (if necessary) we can assume that $q_\alpha \xrightarrow{\sigma(\ell_\infty^*,\ell_\infty)} q \in \Delta$ and $E_{q_\alpha} \xrightarrow{\sigma(\ell_\infty,\ell_1)} Z_q \in [-\omega, \omega]$.

We shall show that $q \in S$ and $Z_q = E_q = 0$, i.e., that q is an equilibrium price.

Suppose for some $p \in D$ we have $p(Z_q) > 0$. Then $p(E_{q_\alpha}) > 0$ holds for all sufficiently large α. But for all sufficiently large α we also have $p \in D_\alpha$, which contradicts the maximality of q_α with respect to $\succ$ on D_α. Hence, $p(Z_q) \leq 0$ holds for all $p \in D$, and this implies $Z_q \leq 0$. We can write $Z_q = X_q - \omega$, where $X_{q_\alpha} \to X_q$(for $\sigma(\ell_\infty, \ell_1)$) and X_{q_α} is the aggregate demand at prices q_α. Hence, $X_q \leq \omega$. A standard argument now shows that $q(X_q) = q(\omega)$.

Now by the Yosida-Hewitt theorem (see Proposition 2.6 in the appendix of [4]) we can write $q = q_c + q_\infty$, where $q_c \in \ell_1^+$ and q_∞ is purely finitely additive. Next we claim that $q_\infty = 0$. To see this, assume by way of contradiction that $q_\infty > 0$. We have two cases.

CASE I: $q_\infty(X_q) > 0$.

In this case, we have two possibilities.

a) $q_c > 0$.

This means that there exists some i with $q_c(i) > 0$. Put $\lambda = (\min\{q_\infty(X_q), \|\omega\|_\infty\})/2q_c(i)$, and note that $y = \lambda e_i + X_q$ satisfies $y \succ_t X_q$ for each t. By the $\tau(\ell_\infty, \ell_1)$-continuity of the $\succ_t$, there exists some n so that $\check{y}_n = (y_1,\ldots,y_n,0,0,\ldots)$ satisfies $\check{y}_n \succ_t X_q$ for all t and $q(\check{y}_n) = q_c(\check{y}_n) < q(X_q) = q(\omega)$. Hence, for some t we have $q_\alpha(\check{y}_n) < q_\alpha(\omega_t)$ and $\check{y}_n \succ_t X_{q_\alpha}(t)$ for all sufficiently large α, a contradiction.

b) $q_c = 0$.

This means that $q = q_\infty$. Choose λ such that $0 < \lambda < \|\omega\|_\infty$, and note that $y = \lambda e_i + X_q$ satisfies $y \succ_t X_q$ for all agents t, and moreover $y < 2\omega$. By the $\tau(\ell_\infty, \ell_1)$-continuity of the $\succ_t$ there exists some n so that $\check{y}_n = (y_1,\ldots,y_n,0,0,\ldots)$ satisfies $\check{y}_n \succ_t X_q$ for all t. Moreover, $q(\check{y}_n) = 0$ and $q(\omega) > 0$ (since ω is uniformly bounded away from zero). Hence, for some agent t we have $q_\alpha(\check{y}_n) \leq q_\alpha(\omega_t)$ and $\check{y}_n \succ_t X_\alpha(t)$ for all sufficiently large α, a contradiction.

CASE II: $q_\infty(X_q) = 0$.

Note that (since ω is uniformly bounded away from zero) $q_\infty(\omega) > 0$ holds. But then the relation

$$\begin{aligned} q_c(X_q) &= q_c(X_q) + q_\infty(X_q) = q(X_q) \\ &= q(\omega) = q_c(\omega) + q_\infty(\omega) > q_c(\omega), \end{aligned}$$

contradicts $X_q \leq \omega$ and $q_c \geq 0$.

Thus, $q = q_c \in D$ holds. Next we claim that $q \in S$. To see this, assume that $q(i) = 0$ holds for some i. Choose some $0 < \lambda < \|\omega\|_\infty$ with $\lambda e_i + X_q < 2\omega$. If $y = \lambda e_i + X_q$, then $\check{y}_n \underset{t}{\succ} X_q$, and by arguing as in Case I(a) above, there exists some $\check{y}_n = (y_1,\dots,y_n,0,0,\dots)$ with $q(\check{y}_n) < q(\omega)$ and $\check{y}_n \underset{t}{\succ} X_q$ for all t. Hence, for some t, $q_\alpha(\check{y}_n) < q_\alpha(\omega_t)$ and $\check{y}_n \underset{t}{\succ} X_{q_\alpha}(t)$ hold for all sufficiently large α, a contradiction. Therefore, $q \in S$.

Finally, a standard argument shows that for each t, $q(X_q(t)) = q(\omega_t)$ and that $X_q(t)$ is maximal in t's budget set with respect to prices q. Hence, by Walras' law q is an equilibrium price, and so, $Z_q = E_q = 0$ holds. ■

The next example presents a method of constructing excess demand functions.

EXAMPLE 3.9. Let $\langle L, L^* \rangle$ be a Riesz dual system with L an AM-space with unit e and let $\Delta = \{p \in L^*_+ : p(e) = 1\}$ be a price simplex for $\langle L, L^* \rangle$.

Fix a continuous function $F : (\Delta, w^*) \to (L, \|\cdot\|)$. (For instance if $\{e_n\}$ and $\{u_n\}$ are two sequences of L with $\{e_n\}$ norm bounded and $\sum_{n=1}^{\infty} \|u_n\| < \infty$, then put $F_p = \sum_{n=1}^{\infty} p(e_n)u_n$ for each $p \in \Delta$.) Now it is a routine matter to verify that the function $E : (\Delta, w^*) \to (L, \|\ \|)$ defined by $E_p = F_p - p(F_p)e$, $p \in \Delta$, is an excess demand function (with domain Δ).

Finally, for a specific example. Let $L = C(\Omega)$ for some compact metric space Ω. Fix a countable norm dense subset $\{f_n\}$ of

$C(\Omega)$ consisting of nonzero functions and define $e_n = u_n = \frac{2^{-n}}{\|f_n\|_\infty} |f_n|$. Then it is easy to see that $\frac{p(e)}{p(F_p)} F_p - e$ is an excess demand function (with domain $L^*_+ \sim \{0\}$) which is homogeneous of degree 0. ■

Our final example uses the method of Example 3.9 to construct an excess demand function E on a simplex Δ where the domain D of E is a proper subset of Δ.

EXAMPLE 3.10. Consider the Riesz dual system $\langle \ell_\infty, \ell_1 \rangle$ with price simplex $\Delta = \{p \in \ell^*_\infty : p \geq 0 \text{ and } \|p\| = 1\}$. Let $D = \{p \in \ell_1^+ : \|p\|_1 = 1\}$, $u_1 = (1,1,\dots) = e$, $u_2 = (0,1,1,\dots)$, $u_3 = (0,0,1,1,\dots)$, etc. and $e_1 = (1,0,0,\dots)$, $e_2 = (0,1,0,0,\dots)$, etc.

Let $\xi : [0,1] \to [0,\infty]$ be a strictly increasing continuous function; for instance, let $\xi(x) = x^p (0 < p < \infty)$, $\xi(x) = x$, or $\xi(x) = 1 - e^{-x}$, etc. Also, let $\{\lambda_n\}$ be a sequence of real numbers with $\lambda_n > 0$ for all n and $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Now define $F : \Delta \to \ell_\infty^+$ by

$$F_p = \sum_{n=1}^{\infty} \lambda_n \xi[p(u_n)]\, e_n.$$

Then it is easy to verify that the functions $p \mapsto F_p$, from (Δ, w^*) into $(\ell_\infty, \|\cdot\|)$, and $p \mapsto p(F_p)$ from (Δ, w^*) into $\mathbb{R}$, are both continuous. Also, if $p = (p_1, p_2, \dots) \in D$, then from the inequality

$$\lambda_n \xi[p(u_n)] = \lambda_n \xi(\sum_{i=n}^{\infty} p_i) \geq \lambda_n \xi(p_n),$$

if follows that $p(F_p) \geq \sum_{n=1}^{\infty} \lambda_n \xi(p_n) p_n > 0$. Thus, the function $E:(D, w^*) \to (\ell_\infty, ||\cdot||)$ defined by

$$E_p = \frac{F_p}{p(F_p)} - e$$

is continuous and satisfies Walras' law. In addition, we claim that it satisfies our boundary condition, i.e. if a net $\{p_\alpha\} \subseteq D$ satisfies $p_\alpha \xrightarrow{w^*} q \in \Delta \sim D$, then $\overline{\lim}\ p(E_{p_\alpha}) > 0$ holds for some $p \in D$.

To this end, let $\{p_\alpha\} \subseteq D$ satisfy $p_\alpha \xrightarrow{w^*} p \in \Delta \sim D$. Write $p = p_c + p_\infty$, with $p_c \in \ell_1^+$ and p_∞ purely finitely additive.

CASE I: $p_c = 0$ (i.e., $p = p_\infty$).

Clearly, $p(F_p) = 0$ holds, and so, the element $e_1 \in D$ satisfies

$$\lim e_1(E_{p_\alpha}) = \lim\left[\frac{\lambda_1 \xi(1)}{p_\alpha(F_{p_\alpha})} - 1\right] = \infty.$$

CASE II: $p_c > 0$.

Since $||p_c|| < ||p_c|| + ||p_\infty|| = ||p|| = 1$, there exists some $q \in D$ satisfying $p_c < q$. Also, note that $p_\infty(u_n) = p_\infty(e) = ||p_\infty|| > 0$ holds for each n. Therefore,

$$F_p = \sum_{n=1}^{\infty} \lambda_n \xi[p(u_n)]\ e_n \geq \sum_{n=1}^{\infty} \lambda_n \xi[p_\infty(e)] e_n >> 0.$$

In particular, $p_c(F_p) > 0$ holds, and so, from $p_c(F_p) = p(F_p)$ we see

that $0 < p(F_p) < q(F_p)$. This implies

$$\lim q(E_{p_\alpha}) = \frac{q(F_p)}{p(F_p)} - 1 > 0,$$

and we are finished. ■

These examples suggest that the appropriate commodity space for economies are the AM-spaces with unit, which admit a strictly positive linear functional.

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4. APPENDIX: RIESZ SPACES

As we have said in the introduction, this paper utilizes the theory of Riesz spaces (vector lattices). For this reason, this section deals exclusively with the fundamental properties of Riesz spaces. For details concerning the lattice properties of Riesz spaces we refer the reader to [13], and for the topological concepts on Riesz spaces to [1] and [16].

A relation $\geq$ on a non-empty set X is said to be an order relation, whenever

a) $x \geq x$ holds for all $x \in X$;

b) $x \geq y$ and $y \geq x$ imply $x = y$; and

c) $x \geq y$ and $y \geq z$ imply $x \geq z$.

An <u>ordered set</u> is a non-empty set X together with an order relation $\geq$. The symbol $x \leq y$ is an alternative notation for $y \geq x$.

Now let A be a non-empty set in an ordered set X. Then an element $y \in X$ is said to be a least upper bound (or a supremum) for A, whenever

1) y is an upper bound for A, i.e., $x \leq y$ holds for all $x \in A$; and

2) if $x \leq z$ holds for all $x \in A$, then $y \leq z$.

Clearly, a set A can have at most one least upper bound, and if it does have one, then it is denoted by sup A. The greatest lower bound (or infimum) of a set A is defined similarly, and is denoted by inf A.

A <u>lattice</u> is an ordered set X such that $\sup\{x,y\}$ and $\inf\{x,y\}$ exist for each pair $x,y \in X$. As usual we write

$$x \vee y = \sup\{x,y\} \text{ and } x \wedge y = \inf\{x,y\}.$$

In this paper, all vector spaces are real vector spaces. The symbol $\mathbb{R}$ will stand for the set of real numbers. An <u>ordered vector space</u> L is a vector space L together with an order relation $\geq$ which is compatible with the algebraic structure of L in the following manner:

i) $f \geq g$ in L implies $f + h \geq g + h$ for all $h \in L$; and

ii) $f \geq g$ in L implies $\alpha f \geq \alpha g$ for all $\alpha \geq 0$.

The set $L^+ = \{f \in L : f \geq 0\}$ is called the <u>positive cone</u> of L, and its members are called the positive elements of L. An ordered vector space L which is also a lattice is referred to as a <u>Riesz space</u> (or a <u>vector lattice</u>).

Typical examples of Riesz spaces are provided by the function spaces. A <u>function space</u> L is a vector space of real-valued functions defined on a non-empty set Ω such that for each $f, g \in L$ the two functions $f \vee g$ and $f \wedge g$ defined by

$$f \vee g(\omega) = \max\{f(\omega), g(\omega)\}, \text{ and}$$
$$f \wedge g(\omega) = \min\{f(\omega), g(\omega)\}$$

belong to L. Clearly, every function space L under the ordering $f \geq g$ whenever $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$, is a Riesz space. Also, $f \geq 0$ in L means $f(\omega) \geq 0$ for all $\omega \in \Omega$. Here are some examples of function spaces.

1. $\mathbb{R}^{\Omega}$, all real-valued functions on a set Ω;

2. $C(\Omega)$, all continuous functions on a topological space Ω;

3. $C_b(\Omega)$, all bounded continuous functions on a topological space Ω;

4. $B(\Omega)$, all bounded real-valued functions on a set Ω;

5. ℓ_p $(1 \leq p < \infty)$, all sequences $(x_1, x_2, \ldots)$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$;

6. ℓ_∞, all bounded sequences.

Let L be a Riesz space. Then for each $f \in L$ we put

$$f^+ = f \vee 0, \; f^- = (-f) \vee 0, \text{ and } |f| = f \vee (-f).$$

The element f^+ is called the <u>positive part</u> of f, f^- the <u>negative part</u>, and $|f|$ the <u>absolute value</u> of f. The following identities hold:

$$f^+ \wedge f^- = 0;$$

$$f = f^+ - f^-; \text{ and}$$

$$|f| = f^+ + f^-.$$

In particular, every element of L can be written as a difference of two positive elements. Every set of the form $[-u,u] = \{f \in L : -u \leq f \leq u\}$ $(u \in L^+)$ is called an order interval of L. A linear functional $\varphi : L \to \mathbb{R}$ is said to be <u>order bounded</u>, whenever φ maps order intervals of L onto bounded subsets of $\mathbb{R}$; i.e., whenever $\varphi([-u,u])$ is a bounded subset of $\mathbb{R}$ for each $u \in L^+$. Clearly, the set of all order bounded linear functionals on L is a vector space. This vector space is called the <u>order dual</u> of L, and is denoted by $L^\sim$. In $L^\sim$ an ordering $\geq$ is introduced by saying that $\varphi \geq \psi$ whenever $\varphi(f) \geq \psi(f)$ holds for all $f \in L^+$. Under this ordering $L^\sim$ becomes an ordered vector space. In actuality, $L^\sim$ is a Riesz space. For the proof of the next result of F. Riesz see [1, Thm 3.3, p. 20] or [2, Thm 24.2, p. 189].

THEOREM 4.1. (F. Riesz) <u>If L is a Riesz space, then its order dual</u> $L^\sim$ <u>is likewise a Riesz space</u>. <u>Moreover, if</u> $\varphi \in L^\sim$ <u>and</u> $f \in L^+$, <u>then</u>

$$\varphi^+(f) = \sup\{\varphi(g) : 0 \leq g \leq f\};$$

$$\varphi^-(f) = \sup\{-\varphi(g) : 0 \leq g \leq f\}; \ \underline{\text{and}}$$

$$|\varphi|(f) = \sup\{\varphi(g) : -f \leq g \leq f\}.$$

Let $L_+^\sim$ denote the positive cone of $L^\sim$. If should be clear that $L_+^\sim$ consists precisely of all linear functionals $\varphi : L \to \mathbb{R}$ for which $\varphi(f) \geq 0$ holds for all $f \in L^+$. The members of $L_+^\sim$ are called the <u>positive linear functionals</u> on L. By Theorem 4.1, every order bounded linear functional on L can be written as a difference of two positive linear functionals on L. A positive linear functional $\varphi \in L_+^\sim$ is said

to be <u>strictly positive</u>, in symbols $\varphi \gg 0$, whenever $f > 0$ in L (i.e., $f \geq 0$ and $f \neq 0$) implies $\varphi(f) > 0$.

The next result is the dual of Theorem 4.1. For a proof see [2, Thm 24.3, p. 190].

THEOREM 4.2. <u>Let L be a Riesz space</u>. <u>If</u> $\varphi \in L_+^\sim$, <u>then for each</u> $f \in L$ <u>we have</u>

$$\varphi(f^+) = \sup\{\psi(f) : 0 \leq \psi \leq \varphi\};$$

$$\varphi(f^-) = \sup\{-\psi(f) : 0 \leq \psi \leq \varphi\}; \text{ and}$$

$$\varphi(|f|) = \sup\{\psi(f) : -\varphi \leq \psi \leq \varphi\}.$$

An <u>ideal</u> A of a Riesz space L is a vector subspace of L such that $|f| \leq |g|$ and $g \in A$ imply $f \in A$.

We now turn our attention to topological concepts on Riesz spaces. A seminorm $\|\cdot\|$ on a Riesz space L is said to be a <u>lattice</u> (or a <u>Riesz</u>) <u>seminorm</u>, whenever

$$|f| \leq |g| \text{ in L implies } \|f\| \leq \|g\|.$$

A <u>locally convex-solid Riesz space</u> (L,τ) is a Riesz space L equipped with a Hausdorff linear topology τ that is generated by a family of lattice seminorms. Recall that the vector space of all τ-continuous linear functionals on L is known as the topological dual of (L,τ).

Regarding topological duals of locally convex-solid Riesz spaces the following result holds. (For a proof see [1, Thm 5.7, p. 36].)

THEOREM 4.3. The topological dual L' of a locally convex-solid Riesz space (L,τ) is an ideal of its order dual $L^{\sim}$, that is, $|\psi| \leq |\varphi|$ and $\varphi \in L'$ imply $\psi \in L'$. In particular, L' is a Riesz space in its own right.

If L is a Riesz space and L' is an ideal of $L^{\sim}$ separating the points of L, then the pair $\langle L,L'\rangle$, under the duality $\langle f,\varphi\rangle = \varphi(f)$, is called a Riesz dual system.

THEOREM 4.4. Let $\langle L,L'\rangle$ be a Riesz dual system, and let $f \in L$. Then $f \geq 0$ holds if and only if we have $\varphi(f) \geq 0$ for all $0 \leq \varphi \in L'$.

PROOF. Assume that $\varphi(f) \geq 0$ holds for all $\varphi \in L'_+$. Fix $0 \leq \psi \in L'$. Since L' is an ideal of $L^{\sim}$, it follows from Theorem 4.2 that

$$\psi(f^-) = \sup\{-\varphi(f) : \varphi \in L^{\sim} \text{ and } 0 \leq \varphi \leq \psi\}$$

$$= \sup\{-\varphi(f) : \varphi \in L' \text{ and } 0 \leq \varphi \leq \psi\} \leq 0.$$

On the other hand, $\psi(f^-) \geq 0$ holds trivially, and so, $\psi(f^-) = 0$ holds for all $0 \leq \psi \in L'$. Since L' is a Riesz space, this implies $\psi(f^-) = 0$ for all $\psi \in L'$, from which it follows that $f^- = 0$. But then $f = f^+ - f^- = f^+ \geq 0$, as desired. ■

A special class of locally convex-solid Riesz spaces are the Banach lattices. A Riesz space that under a lattice norm is a

complete metric space is referred to as a Banach lattice. Some important examples of Banach lattices are:

a) The $C(\Omega)$ spaces, Ω compact, with the sup norm

$$||f||_{\infty} = \sup\{|f(\omega)| : \omega \in \Omega\}.$$

b) The $L_p(\mu)$ spaces $(1 \leq p < \infty)$ under the norm

$$||f||_p = (\int |f|^p d\mu)^{1/p}.$$

c) The $L_{\infty}(\mu)$ spaces under the essential sup norm.

d) The $C_b(\Omega)$ spaces under the sup norm.

The Banach lattices have a number of remarkable properties. For instance, a given Riesz space admits at most one lattice norm (up to an equivalence, of course) that makes it a Banach lattice. (The sup norm, for example, is the only lattice norm that makes $C[0,1]$ a Banach lattice.) On the other hand, every positive linear functional on a Banach lattice is norm continuous; see [2, Thm 24.10, p. 192]. Thus, if L is a Banach lattice and L^* denotes its norm dual, then $L^* = L^{\sim}$ holds, and moreover L^* is a Banach lattice.

Now let L be a Banach lattice. Then there is a natural embedding $f \mapsto \hat{f}$ of L into its double norm dual L^{**} defined by

$$\hat{f}(\varphi) = \varphi(f) \text{ for } f \in L \text{ and } \varphi \in L'.$$

The embedding is linear, norm and lattice preserving. Thus, if L is sublattice of the Banach lattice L^{**}. The following property of this identification will be needed.

THEOREM 4.5. If L is a Banach lattice, then L^+ is $\sigma(L^{**},L^*)$ -dense in L_+^{**}.

PROOF. Let $\overline{L^+}$ be the $\sigma(L^{**},L^*)$-closure of L^+ in L^{**}. Clearly, $\overline{L^+} \subseteq L_+^{**}$. Assume by way of contradiction that there exists $0 < \varphi \in L_+^{**} \sim \overline{L^+}$. Then by the Hahn-Banach theorem there exist $c \in \mathbb{R}$ and a $\sigma(L^{**},L^*)$-continuous linear functional on L^{**} (i.e., some $f \in L^*$) satisfying $\varphi(f) < c$ and $f(x) \geq c$ for all $x \in L^+$. Since $\alpha x \in L^+$ for all $x \in L^+$ and all $\alpha \geq 0$, it follows that $c \leq 0$ and $f \in L_+^*$. But then we have $0 \leq \varphi(f) < c \leq 0$, which is impossible. Hence, $\overline{L^+} = L_+^{**}$ holds. ■

For more about Banach lattices the reader can consult the book of H. H. Schaefer [16]. A locally convex topology t on L is compatible with $\langle L,L'\rangle$ if (L,t) has topological dual L'. The Mackey topology $\tau(L,L')$ on L is the finest locally convex topology on L compatible with $\langle L,L'\rangle$. The weak topology $\sigma(L,L')$ is the coarsest locally convex topology on L compatible with $\langle L,L'\rangle$. It is well known that a locally convex topology t on L is compatible with $\langle L,L'\rangle$ if and only if $\sigma(L,L') \subseteq t \subseteq \tau(L,L')$ holds. The weak topology $\sigma(L^\sim,L)$ on $L^\sim$ will be denoted by w^*.

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