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PATENT RACES WITH A SEQUENCE OF INNOVATIONS

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ABSTRACT

The theoretical literature on patent races has been an interesting and fast-evolving one, moving from largeley heuristic discussion to quite rigorous analysis within the space of the past two decades. This literature has been characterized by a pattern of interesting results which are subsequently reversed under alternative behavioral and/or structural assumptions. This sensitivity of key results to mutually exclusive but perhaps equally plausible modeling assumptions has kept conclusions and policy recommendations in a constant state of revision. All of these papers have been concerned with a single innovation produced by a number of identical agents. This paper generalizes this literature in two important ways.

First, we consider a market in which one firm is the current patent- holder -- the incumbent, while the remaining firms are nonincumbents; firms are entirely symmetric in every other sense. Second, we consider a sequence of innovations, so that success does not imply that the successful firm rcaps monopoly profits forever after, but only until the next, better innovation is developed.

PATENT RACES WITH A SEQUENCE OF INNOVATIONS

Jennifer F. Reinganum[1]

I. Introduction

The theoretical literature on patent races has been an interesting and fast-evolving one, moving from largely heuristic discussion to quite rigorous analysis within the space of the past two decades. Early analytical models took a decision theoretic approach (with the exception of a game-theoretic analysis due to Scherer [1967]). Important contributions include Horowitz [1963], Barzel [1968] and Kamien and Schwartz [e.g., 1972, 1974, 1976]. The more recent game-theoretic literature on patent races includes work by Loury [1979], Lee and Wilde [1980], Dasgupta and Stiglitz [1980] and Reinganum [1981,1982]. This literature has been characterized by a pattern of interesting results which are subsequently reversed under alternative behavioral and/or structural assumptions. This sensitivity of key results to mutually exclusive but perhaps equally plausible modeling assumptions has kept conclusions and policy recommendations in a constant state of revision. For example, one important issue concerns the impact of increasing competition upon individual firm (and industry) incentives to invest in R and D. The sequence of papers by Kamien and Schwartz [1976], Loury [1979], Lee and Wilde [1980] and Reinganum [1982] provides an illuminating exchange on this question. Interesting spinoffs of these models include recent work by Mortensen [1981] and Wilde [1982]. All of these papers have been concerned with a single innovation produced by a number of identical agents, all of whom can be interpreted as ''outsiders'' to the industry. This paper generalizes this literature in two important ways.

First, we consider a market in which one firm is the current patent-holder -- the incumbent or insider -- while the remaining firms are nonincumbents or outsiders; the firms are entirely symmetric in every other sense.

Second, we consider a sequence of innovations, so that a single success does not imply that the successful firm reaps monopoly profits forever after, but rather it does so only until the next, ''better'' innovation is developed. Using dynamic programming, we compute the (subgame perfect) Nash equilibrium in a game with such a sequence of patent races.

The model is developed in Section II below. Each innovative success initiates a new stage; within each stage firms engage in a continuous-time patent race. The game with t stages remaining is constructed recursively from shorter horizon games under the assumption of subgame perfect Nash equilibrium play. Nash equilibria are shown to be symmetric among the nonincumbents, while the incumbent

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always invests less than the nonincumbents. Section III discusses comparative statics and equilibrium with free entry and exit. In Section IV we show that, under reasonable assumptions, a Nash equilibrium exists for the multi-stage game. Section V concludes and discusses related work by Futia [1980] and Rogerson [1979].

II. The Model

The multi-stage model developed below is based upon the single-innovation model of Lee and Wilde [1980], which is a reformulation of a model originally due to Loury [1979]. The Lee and Wilde model appears as a specialization of the model below for the case of a single stage and no incumbent firm.

Consider a sequence of innovations with associated profit flows of $R_0, R_1, \ldots, R_T$ where R_t denotes the flow profit available from the current innovation when there are t innovations remaining. These profit flows are assumed to be known in advance; only the timing of the innovations is uncertain. We assume that $R_0 > R_1 > \ldots > R_T$. That is, successive innovations are becoming more profitable. At each stage t, a given number of firms n_t are competing to be the first to introduce the next innovation. Firm i invests at the constant rate x_{it} during the t^{th} stage.[2]

2. A more general specification would allow the rate of investment to depend upon time and accumulated investment, but under the stationarity assumptions we make below, equilibrium play implies a constant rate of investment (see Reinganum [1981]).

Each innovation is assumed to be patentable so that if a firm wins the $t+1^{th}$ race (that is, the race after which t innovations remain), it becomes the incumbent and receives flow profits of R_t until the next innovation occurs. If a firm loses the $t+1^{th}$ race, it loses its R and D investment and collects no reward during that stage. Let I_t denote the identity of the incumbent when t stages remain.

There are both <u>technological</u> and <u>market</u> uncertainties associated with each race. Technological uncertainty takes the form of a stochastic relationship between the amount of money invested in R and D and the eventual date of success. That is, firm i is unable to choose its exact date of success with certainty. Rather the date of success associated with an investment rate of x_{it} is $\tau_{it}(x_{it})$, a random variable with distribution function

$$\Pr\{\tau_{it}(x_{it}) \leq \tau\} = 1 - e^{-h_t(x_{it})\tau}, \quad \tau \in [0,\infty).$$

<u>Assumption 1.</u> Assume that the constant hazard rate $h_t(x_{it})$ has the properties that

(i) $h_t(0) = 0 = \lim_{x \to \infty} h_t'(x)$,

(ii) $h_t''(x) \geq (\leq)\ 0$ as $x \leq (\geq)\ \bar{x}_t < \infty$ and

(iii) $h_t(x)/x \geq (\leq)\ h_t'(x)$ as $x \geq (\leq)\ \tilde{x}_t < \infty$.

Thus there may be initial increasing returns, but eventually the technology exhibits decreasing returns to scale. A typical hazard rate function is illustrated in Figure 1.

Figure 1

Market uncertainty exists because the remaining $n_t - 1$ firms are simultaneously choosing investments of their own. Thus firm i innovates successfully by time τ with probability

$$\Pr\{\tau_{it}(x_{it}) \leq \tau\} = 1 - e^{-h_t(x_{it})\tau}$$

but firm i <u>wins</u> by time τ with probability

$$\Pr\{\tau_{it}(x_{it}) \leq \tau,\ \tau_{kt}(x_{kt}) > \tau, \forall k \neq i\} = e^{-\sum_{k\neq i} h_t(x_{kt})\tau}(1 - e^{-h_t(x_{it})\tau}).$$

Immediately upon the first successful completion of the t^{th} innovation, a new patent race commences. Thus the stages are of random length, with the expected length of the t^{th} stage being $1/\sum_k h_t(x_{kt})$.

The method of solution for the multi-stage game is to first solve the game for the last stage, and then to solve recursively for the multi-stage games.

<u>Stage t = 0</u>

Since the last innovation provides flow profits of R_0 per unit time, the value of being the incumbent in the last stage is $V_0^I = R_0/r_0$, where r_0 is a common discount rate. The value of being an outsider in the last stage is $V_0^N = 0$.

<u>Stage t = 1</u>

Suppose that at stage 1 (with one innovation remaining), firm i is the incumbent, while all other firms are nonincumbents. Firm i then receives flow profits at the rate R_1 and pays R and D costs at the rate x_{i1} so long as no one has succeeded. The probability that no firm has succeeded by time τ is

$$\Pr\{\tau_{k1}(x_{k1}) > \tau \text{ for all } k\} = e^{-\sum h_1(x_{kt})\tau},$$

where the summation is taken over k.

In addition, if no firm has succeeded by time τ, and firm i succeeds at time τ, then firm i receives the patent with value V_0^I. This event has instantaneous probability

$$\Pr\{\ \tau_{k1}(x_{k1}) > \tau \text{ for all } k,\ \tau_{i1}(x_{i1}) < \tau + d\tau\} = e^{-\sum h_1(x_{k1})\tau} h_1(x_{i1})d\tau.$$

Combining these terms and discounting at the rate r_1 yields the expected profit to firm i (the incumbent in stage 1) when i invests x_{it} and the rival firms invest $x_{-it} = (x_{11},\ldots,x_{(i-1)1},x_{(i+1)1},\ldots,x_{n_1 1})$. Let X_1 denote the vector (x_{i1},x_{-i1}). Then

$$V_1^i(i,X_1) = \int_0^\infty e^{-r_1\tau} e^{-\sum h_1(x_{k1})\tau}[h_1(x_{i1})V_0^I + R_1 - x_{i1}]d\tau$$
$$= [h_1(x_{i1})V_0^I + R_1 - x_{i1}]/[r_1 + \sum h_1(x_{k1})]. \qquad (1)$$

Firm j's payoff is symmetric except that it receives no flow profit R_1 in stage 1.

$$V_1^j(i,X_1) = \int_0^\infty e^{-r_1\tau} e^{-\sum h_1(x_{k1})\tau} [h_1(x_{j1})V_0^I - x_{j1}]d\tau$$
$$= [h_1(x_{j1})V_0^I - x_{j1}]/[r_1 + \sum h_1(x_{k1})]. \qquad (2)$$

Definition 1. A strategy for firm k in the game with one innovation remaining is an investment rule $x_{k1}:\{1,2,\ldots,n_1\} \rightarrow [0,\infty)^{n_1}$. The expression $x_{k1}(i)$ represents firm k's investment rate if i is the current incumbent. Let $X_1(i) = (x_{k1}(i),x_{-k1}(i))$.

Definition 2. The payoff to firm k when $I_1 = i$ and the strategy vector X_1 is played is $V_1^k(i,X_1)$ as described in equations (1) and (2), $k = 1,2,\ldots,n_1$.

Definition 3. A Nash equilibrium for the game with one stage remaining is a strategy $x^*_{k1}(\cdot)$ for each firm k such that, for all i, $V_1^k(i,x^*_{k1}(i),x^*_{-k1}(i)) \geq V_1^k(i,x_{k1},x^*_{-k1}(i))$ for all $x_{k1} \in [0,\infty)$, $k = 1,2,\ldots, n_1$.

The imposition of Nash equilibrium play in each state i implies that the Nash equilibrium of Definition 3 is actually subgame perfect (Selten [1975]). Thus each firm must choose an investment level which is a Nash equilibrium with respect to all information available at stage t. This information consists of the stage, t, and the identity of the current patent-holder, i.

Initially we will assume that a (subgame perfect) Nash equilibrium exists and we will characterize the Nash equilibrium investment levels. Existence questions are taken up separately in Section IV. Because of the symmetry of the firms we need only characterize $X_1^*(i)$ for one state i. If $I_1 = j$ instead of i, then $X_1^*(j)$ is such that $x^*_{i1}(j) = x^*_{j1}(i)$, $x^*_{i1}(i) = x^*_{j1}(j)$ and $x^*_{k1}(i) = x^*_{k1}(j)$ for $k \neq i,j$. That is, at any stage, all that matters to firm k is whether or not k is the incumbent; if k is not the incumbent it is irrelevant (to firm k) who is the incumbent.

Fix $I_1 = i$. Since $X_1^*(i)$ is a Nash equilibrium strategy vector, it must be that (at an interior $X_1^*(i)$),

$$\partial V_1^i(i,X_1^*)/\partial x_{i1} = [r_1 + \sum h_1(x^*_{k1})][h_1'(x^*_{it})V_0^I - 1]$$
$$- [h_1(x^*_{i1})V_0^I - x^*_{i1} + R_1]h_1'(x^*_{i1}) = 0 \qquad (3)$$

and for all $j \neq i$,

$$\partial V_1^j(i,X_1^*)/\partial x_{j1} = [r_1 + \sum h_1(x^*_{k1})][h_1'(x^*_{j1})V_0^I - 1]$$
$$- [h_1(x^*_{j1})V_0^I - x^*_{j1}]h_1'(x^*_{j1}) = 0. \qquad (4)$$

Equations (3) and (4) can be solved to yield the following useful relations.

$$V_1^i(i,X_1^*) = [h_1(x^*_{i1})V_0^I - x^*_{i1} + R_1]/[r_1 + \sum h_1(x^*_{k1})]$$
$$= [h_1'(x^*_{i1})V_0^I - 1]/h_1'(x^*_{i1}) \qquad (5)$$

and

$$V_1^j(i,X_1^*) = [h_1(x_{j1}^*)V_0^I - x_{j1}^*]/[r_1 + \sum h_1(x_{k1}^*)]$$
$$= [h_1'(x_{j1}^*)V_0^I - 1]/h_1'(x_{j1}^*). \quad (6)$$

From (5) and (6) it is clear that nonnegativity of $V_1^i(i,X_1^*)$ and $V_1^j(i,X_1^*)$ requires $h_1'(x_{i1}^*)V_0^I - 1 \geq 0$ and $h_1'(x_{j1}^*)V_0^I - 1 \geq 0$, respectively.

An alternative simplification of (3)-(4) yields

$$h_1'(x_{i1}^*)[r_1 V_0^I + \sum_{k\neq i} h_1(x_{k1}^*)V_0^I + x_{i1}^* - R_1] = r_1 + \sum h_1(x_{k1}^*) \quad (7)$$

and for all $j\neq i$,

$$h_1'(x_{j1}^*)[r_1 V_0^I + \sum_{k\neq j} h_1(x_{k1}^*)V_0^I + x_{j1}^*] = r_1 + \sum h_1(x_{k1}^*). \quad (8)$$

Second-order necessary conditions for a maximum at x_1^* are

$$\partial^2 V_1^i/\partial x_{i1}^2 = h_1''(x_{i1}^*)[r_1 V_0^I + \sum_{k\neq i} h_1(x_{k1}^*)V_0^I + x_{i1}^* - R_1] \leq 0 \quad (9)$$

and for $j\neq i$,

$$\partial^2 V_1^j/\partial x_{j1}^2 = h_1''(x_{j1}^*)[r_1 V_0^I + \sum_{k\neq j} h_1(x_{k1}^*)V_0^I + x_{j1}^*] \leq 0. \quad (10)$$

Since $h_1'(x) \geq 0$ for all x and $r_1 + \sum h_1(x_{k1}) > 0$, equations (7) and (8) imply that $h_1''(x_{i1}^*) \leq 0$ and $h_1''(x_{j1}^*) \leq 0$. That is, $x_{i1}^* \geq \bar{x}_1$ and $x_{1j}^* \geq \bar{x}_1$. Thus all firms operate their R and D technologies in the decreasing returns portion of the technology production function $h_1(\cdot)$.

<u>Proposition 1.</u> Under assumption 1, any Nash equilibrium in the game with one stage remaining is symmetric among the nonincumbents. That is, $x_{j1}^*(i) = x_{\ell 1}^*(i) = x_{N1}$ for all j, $\ell\neq i$.

<u>Proof.</u> Fix $I_1 = i$ and suppress the argument i from $X_1^*(i)$. Define the function

$$g_{j1}(x_{j1}) = h_1'(x_{j1})[r_1 V_0^I + \sum_{k\neq j} h_1(x_{k1}^*)V_0^I + x_{j1}]$$
$$- h_1'(x_{\ell 1}^*)[r_1 V_0^I + (\sum_{k\neq \ell, j} h_1(x_{k1}^*) + h_1(x_{j1}))V_0^I + x_{\ell 1}^*]. \quad (11)$$

Equation (8) implies that $g_{j1}(x_{j1}^*) = 0$. Moreover, $g_{j1}(x_{\ell 1}^*) = 0$ by inspection of equation (11). Since $x_{j1}^* \geq \bar{x}_1$, $x_{\ell 1}^* \geq \bar{x}_1$, and

$$g_{j1}'(x_{j1}) = h_1''(x_{j1})[r_1 V_0^I + \sum_{k\neq j} h_1(x_{k1}^*)V_0^I + x_{j1}]$$
$$+ h_1'(x_{j1})[1 - h_1'(x_{\ell 1}^*)V_0^I] < 0$$

for all $x_{j1} > \bar{x}_1$, it follows that $x_{j1}^*(i) = x_{\ell 1}^*(i)$ for all j, $\neq i$. Denote this common value by x_{N1}.

Q.E.D.

<u>Proposition 2.</u> Under assumption 1, at a Nash equilibrium the incumbent invests less than each nonincumbent. That is, $x_{i1}^*(i) = x_{I1} < x_{N1}$.

<u>Proof.</u> Again suppress the argument i. Define the function

$$g_{i1}(x_{i1}) = h_1'(x_{i1})[r_1 V_0^I + \sum_{k\neq i} h_1(x_{k1}^*)V_0^I + x_{i1} - R_1]$$
$$- h_1'(x_{j1}^*)[r_1 V_0^I + \sum_{k\neq i, j} h_1(x_{k1}^*)V_0^I + h_1(x_{i1})V_0^I + x_{j1}] \quad (12)$$

From equations (7) and (8), $g_{i1}(x^*_{i1}) = 0$, while $g_{i1}(x^*_{j1}) = -R_1 h_1'(x^*_{j1}) < 0$. For all $x_{i1} > \bar{x}_1$, equations (9) and (6) imply that

$$g_{i1}'(x_{i1}) = h_1''(x_{i1})[r_1 V^I_0 + \sum_{k\neq i} h_1(x^*_{k1})V^I_0 + x_{i1} - R_1]$$
$$+ h_1'(x_{i1})[1 - h_1'(x^*_{j1})V^I_0] < 0.$$

Since both x^*_{i1} and $x^*_{j1} \geq \bar{x}_1$, it follows that $x^*_{i1}(i) < x^*_{j1}(i)$. Defining $x_{I1} = x^*_{i1}(i)$, we have $x_{I1} < x_{N1}$.

Q.E.D.

<u>Proposition 3.</u> Each firm would prefer to be the incumbent rather than a nonincumbent.

Proof. Let $V^I_1 = V^i_1(i, X^*_1(i))$ and $V^N_1 = V^j_1(i, X^*_1(i))$, $j\neq i$. Then equations (5) and (6) imply that $V^I_1 = [h_1'(x_{I1})V^I_0 - 1]/h_1'(x_{I1})$ and $V^N_1 = [h_1'(x_{N1})V^I_0 - 1]/h_1'(x_{N1})$. Since the function $[h_1'(x)V^I_0 - 1]/h_1'(x)$ is decreasing in x for $x \geq \bar{x}_1$ and since $x_{N1} > x_{I1} \geq \bar{x}_1$, it follows that $V^I_1 > V^N_1$.

Q.E.D.

This completes the analysis of the game with one stage remaining. This game is compared with the original symmetric game of Lee and Wilde in Section III. Section III also discusses comparative static effects. To summarize the results of this section, we note that any Nash equilibrium is characterized by two numbers, (x_{I1}, x_{N1}),

where x_{I1} is the rate of investment by the incumbent and x_{N1} is the rate of investment by each of the remaining $n_1 - 1$ nonincumbents. The incumbent always invests less, $x_{I1} < x_{N1}$, and each firm would prefer to be the incumbent rather than a nonincumbent, $V^I_1 > V^N_1$.

<u>Stage t = 2</u>

Again let the current incumbent be firm i ($I_2 = i$). Assuming that play in the one-stage game is as described in the previous section, the value of the two-stage game to the incumbent firm i is

$$V^i_2(i, X_2) = \int_0^\infty e^{-r_2\tau} e^{-\sum h_2(x_{k2})\tau}[h_2(x_{i2})V^I_1 + \sum_{k\neq i} h_2(x_{k2})V^N_1 + R_2 - x_{i2}]d\tau$$
$$= [h_2(x_{i2})V^I_1 + \sum_{k\neq i} h_2(x_{k2})V^N_1 + R_2 - x_{i2}]/[r_2 + \sum h_2(x_{k2})].$$

Firm i receives V^I_1 at time τ if i succeeds at time τ and no other firm has yet done so; firm i receives V^N_1 at time τ if any other firm succeeds first at time τ; finally, the incumbent firm i receives flow profits of R_2 and incurs R and D costs of x_{i2} so long as no firm has succeeded. Firm j's payoff is again symmetric except for the flow revenue term R_2.

$$V^j_2(i, X_2) = \int_0^\infty e^{-r_2\tau} e^{-\sum h_2(x_{k2})\tau}[h_2(x_{j2})V^I_1 + \sum_{k\neq j} h_2(x_{k2})V^N_1 - x_{j2}]d\tau$$
$$= [h_2(x_{j2})V^I_1 + \sum_{k\neq j} h_2(x_{k2})V^N_1 - x_{j2}]/[r_2 + \sum h_2(x_{k2})].$$

Imposing the property of subgame perfectness (Selten [1975]) implies that, from stage 1 on, firms must follow the Nash equilibrium behavior described in the preceding section. Thus the continuation values V_1^I, V_1^N are independent of actions taken at stage 2.

<u>Definition 4.</u> A <u>strategy</u> for firm k in the game with two innovations remaining is an investment rule $x_{k2}:\{1,2,\ldots,n_2\} \longrightarrow [0,\infty)^{n_2}$. The expression $x_{k2}(i)$ represents firm k's investment rate if i is the current incumbent.

Again, symmetry suggests that $x_{k2}(j) = x_{k2}(i)$ for all $k \neq i,j$. That is, firm k will invest the same amount regardless of the identity of the incumbent so long as firm k itself is not the incumbent.

<u>Definition 5.</u> The <u>payoff to firm i</u> in the game with two innovations remaining when $I_2 = i$ and the strategy vector X_2 is played is $V_2^i(i,X_2)$. The <u>payoff to firm j</u> ($j \neq i$) is $V_2^j(i,X_2)$ as described above.

<u>Definition 6.</u> A <u>Nash equilibrium</u> for the game with two stages remaining is a strategy $x_{k2}(\cdot)$ for each firm k such that, for all i,

$$V_2^k(i,x_{k2}^*(i),x_{-k2}^*(i)) \geq V_2^k(i,x_{k2},x_{-k2}^*(i))$$

for all $x_{k2} \in [0,\infty)$, $k = 1,2,\ldots,n_2$.

Fix $I_2 = i$ and suppress the dependence of $X_2^*(i)$ on i. Since X_2^* is a Nash equilibrium strategy vector, the following conditions must hold simultaneously (for interior X_2^*):

$$\partial V_2^i(i,X_2^*)/\partial x_{i2} = [r_2 + \sum h_2(x_{k2}^*)][h_2'(x_{i2}^*)V_1^I - 1]$$

$$- [h_2(x_{i2}^*)V_1^I + \sum_{k\neq i} h_2(x_{k2}^*)V_1^N - x_{i2}^* + R_2]h_2'(x_{i2}^*) = 0 \quad (13)$$

and, for all $j \neq i$,

$$\partial V_2^j(i,X_2^*)/\partial x_{j2} = [r_2 + \sum h_2(x_{k2}^*)][h_2'(x_{j2}^*)V_1^I - 1]$$
$$- [h_2(x_{j2}^*)V_1^I + \sum_{k\neq j} h_2(x_{k2}^*)V_1^N - x_{j2}^*]h_2'(x_{j2}^*) = 0. \quad (14)$$

Following the analysis of the preceding section, equations (13) and (14) imply that

$$V_2^i(i,X_2^*) = [h_2'(x_{i2}^*)V_1^I - 1]/h_2'(x_{i2}^*) \quad (15)$$

and

$$V_2^j(i,X_2^*) = [h_2'(x_{j2}^*)V_1^I - 1]/h_2'(x_{j2}^*). \quad (16)$$

Nonnegativity of profits implies that $h_2'(x_{i2}^*)V_1^I - 1 \geq 0$ and $h_2'(x_{j2}^*)V_1^I - 1 \geq 0$. An alternative simplification of (13) and (14) yields

$$h_2'(x_{i2}^*)[r_2V_1^I + \sum_{k\neq i} h_2(x_{k2}^*)(V_1^I - V_1^N) + x_{i2}^* - R_2] = r_2 + \sum h_2(x_{k2}^*) \quad (17)$$

and, for all $j \neq i$,

$$h_2'(x_{j2}^*)[r_2V_1^I + \sum_{k\neq j} h_2(x_{k2}^*)(V_1^I - V_1^N) + x_{j2}^*] = r_2 + \sum h_2(x_{k2}^*). \quad (18)$$

Comparison of (17) and (18) with their one-stage counterparts (7) and (8) shows that (17) and (18) include an extra term -- $-\sum_{k\neq i} h_2(x_{k2}^*)V_1^N$ and $-\sum_{k\neq j} h_2(x_{k2}^*)V_1^N$, respectively -- which reflects the impact of future benefits which are possible even if the firm fails in the current stage.

Second-order necessary conditions for a maximum are

$$\partial^2 V_2^i/\partial x_{i2}^2 = h_2''(x_{i1}^*)[r_2 V_1^I + \sum_{k\neq i} h_2(x_{k2}^*)(V_1^I - V_1^N) + x_{i2}^* - R_2] \leq 0 \quad (19)$$

and, for $j\neq i$,

$$\partial^2 V_2^j/\partial x_{j2}^2 = h_2''(x_{j2}^*)[r_2 V_1^I + \sum_{k\neq j} h_2(x_{k2}^*)(V_1^I - V_1^N) + x_{j2}^*] \leq 0 \quad (20)$$

Since $r_2 + \sum h_2(x_{k2}^*) > 0$, equations (17) and (18) imply that $h_2''(x_{i2}^*) \leq 0$ and $h_2''(x_{j2}^*) \leq 0$. That is, $x_{i2}^* \geq \bar{x}_2$ and $x_{j2}^* \geq \bar{x}_2$; all firms operate in the decreasing returns portion of $h_2(\ )$.

<u>Assumption 2.</u> There exists x_2^o such that $h_2(x_2^o)(V_1^I - V_1^N) - x_2^o \geq r_2 V_1^N$.

<u>Proposition 4.</u> Under assumptions 1 and 2, any Nash equilibrium in the two-stage game is symmetric among the nonincumbents. That is, $x_{j2}^*(i) = x_{\ell 2}^*(i) = x_{N2}$ for all $j,\ell\neq i$.

<u>Proof.</u> Define the function

$$g_{j2}(x_{j2}) = h_2'(x_{j2})[r_2 V_1^I + \sum_{k\neq j} h_2(x_{k2}^*)(V_1^I - V_1^N) + x_{j2}] - h_2'(x_{\ell 2}^*)[r_2 V_1^I + (\sum_{k\neq \ell, j} h_2(x_{k2}^*) + h_2(x_{j2}))(V_1^I - V_1^N) + x_{\ell 2}^* \quad (21)$$

Equation (18) implies that $g_{j2}(x_{j2}^*) = 0$ and $g_{j2}(x_{\ell 2}^*) = 0$ by inspection of (21). For all $x_{j2} > \bar{x}_2$, $h_2''(x_{j2}) < 0$. Differentiating (21)

yields

$$g_{j2}'(x_{j2}) = h_2''(x_{j2})[r_2 V_1^I + \sum_{k\neq j} h_2(x_{k2}^*)(V_1^I - V_1^N) + x_{j2}] + h_2'(x_{j2})[1 - h_2(x_{\ell 2}^*)(V_1^I - V_1^N)]. \quad (22)$$

The first term is negative for $x_{j2} > \bar{x}_2$, while equation (16) implies that the second term is

$$h_2'(x_{j2})[h_2'(x_{\ell 2}^*)V_1^N - h_2'(x_{\ell 2}^*)V_2^\ell(i,X_2^*)].$$

This term is nonpositive so long as $V_2^\ell(i,X_2^*) \geq V_1^N$. By the definition of $V_2^\ell(i,X_2^*)$ as the

$$\max_{x_{\ell 2}}[h_2(x_{\ell 2})V_1^I + \sum_{k\neq\ell} h_2(x_{k2}^*)V_1^N - x_{\ell 2}]/[r_2 + \sum_{k\neq\ell} h_2(x_{k2}^*) + h_2(x_{\ell 2})],$$

a sufficient condition for $V_2^\ell(i,X_2^*) \geq V_1^N$ is that there exists some x_2^o such that $h_2(x_2^o)(V_1^I - V_1^N) - x_2^o \geq r_2 V_1^N$. But this is the content of Assumption 2. Thus $g_{j2}'(x_{j2}) < 0$ for $x_{j2} > \bar{x}_2$. Therefore, for $x_{j2} > \bar{x}_2$, $g_{j2}(\cdot)$ is monotone decreasing. Since $g_{j2}(x_{j2}^*) = g_{j2}(x_{\ell 2}^*) = 0$ and both $x_{j2}^* \geq \bar{x}_2$ and $x_{\ell 2}^* \geq \bar{x}_2$, we have $x_{j2}^* = x_{\ell 2}^*$. Denote this common value by x_{N2}.

Q.E.D.

<u>Proposition 5.</u> Under assumptions 1 and 2, the incumbent invests less than nonincumbents in the game with two innovations remaining. That is, $x_{i2}^*(i) = x_{I2} < x_{N2}$.

Proof. Define the function

$$g_{i2}(x_{i2}) = h_2'(x_{i2})[r_2V_1^I + \sum_{k\neq i} h_2(x_{k2}^*)(V_1^I - V_1^N) + x_{i2} - R_2]$$
$$- h_2'(x_{j2}^*)[r_2V_1^I + (\sum_{k\neq j,i} h_2(x_{k2}^*) + h_2(x_{i2}))(V_1^I - V_1^N) + x_{j2}^*] \quad (23)$$

Differentiating $g_{i2}(x_{i2})$ yields

$$g_{i2}'(x_{i2}) = h_2''(x_{i2})[r_2V_1^I + \sum_{k\neq i} h_2(x_{k2}^*)(V_1^I - V_1^N) + x_{i2} - R_2]$$
$$+ h_2'(x_{i2})[1 - h_2'(x_{j2}^*)(V_1^I - V_1^N)]. \quad (24)$$

The first term is negative for $x_{i2} > \bar{x}_2$ while the second term is nonpositive under assumption 2 (see the proof of Proposition 5). Since $x_{i2}^* \geq \bar{x}_2$, $x_{j2}^* \geq \bar{x}_2$, $g_{i2}(x_{i2}^*) = 0$, $g_{i2}(x_{j2}^*) = -R_2h_2'(x_{j2}^*) < 0$ and $g_{i2}(\ )$ is monotone decreasing, it follows that $x_{i2}^* > x_{j2}^*$. That is, $x_{I2} < x_{N2}$.

Q.E.D.

The essential content of Assumption 2 is that, under Nash equilibrium play in the second stage, profits to a nonincumbent, $V_2^N = V_2^j(i, X_2^*(i))$, exceed profits to a nonincumbent when only one innovation remains. That is, $V_2^N \geq V_1^N$. While this seems extremely plausible, I have been unable to derive it as a result.

Proposition 6. Under assumptions 1 and 2, each firm would prefer to be the incumbent in stage 2 than a nonincumbent. That is, $V_2^I > V_2^N$.

Proof. $V_2^I = [h_2'(x_{I2})V_1^I - 1]/h_2'(x_{I2})$, while

$V_2^N = [h_2'(x_{N2})V_1^I - 1]/h_2'(x_{N2})$. Since the function $[h_2'(x)V_1^I - 1]/h_2'(x)$ is decreasing in x for $x \geq \bar{x}_2$ and since $x_{N2} > x_{I2} \geq \bar{x}_2$, it follows that $V_2^I > V_2^N$.

Q.E.D.

Stage t

The results of the preceding section in no way depended on the fact that two stages remained. These results are valid (under the analogous assumptions) if we replace ''2'' with ''t'' and ''1'' with ''t-1.''

Proposition 7. Nash equilibrium play in the game with t innovations remaining is characterized by x_{It}, the rate of investment by the incumbent, and x_{Nt}, the rate of investment by each nonincumbent. These rates can be ordered as follows: $x_{Nt} > x_{It} \geq \bar{x}_t$. Thus the nonincumbent invests more than the incumbent, but both firms operate in the decreasing returns portion of $h_t(\cdot)$. Each firm would prefer to be the incumbent in stage t than to be a nonincumbent in stage t.

Proposition 8. $V_t^I < V_{t-1}^I$ for all t. That is, the value of being the incumbent is greater the smaller the number of remaining innovations.

Proof. By equation (15), $V_t^I = V_{t-1}^I - 1/h_t'(x_{It})$.

Q.E.D.

The model described above may be extended backward recursively for an indefinite number of innovations, or closed off with a symmetric stage game which represents the ''first'' innovation. It is clear that this stage game fits the analysis of the t-stage game with R_t set equal to zero. Then Proposition 4 implies that any Nash equilibrium will be symmetric.

III. Comparative Statics

Determining the impact of various parameters upon the values of Nash equilibrium strategies is always difficult, since there are both direct effects and indirect effects through the equilibrium conditions. We will make one additional assumption below in order to sign the comparative statics effects. We will subsequently argue that there are compelling reasons for our particular assumption.

The pair (x_{It}, x_{Nt}) are jointly determined as functions of the parameters $r_t, R_t, V^I_{t-1}, V^N_{t-1}$ and n_t by the t-period analogs of the first-order conditions (17) and (18).

$$h_t'(x_{It})[r_t V^I_{t-1} + (n_t - 1)h_t(x_{Nt})(V^I_{t-1} - V^N_{t-1}) + x_{It} - R_t] - r_t - h_t(x_{It}) - (n_t - 1)h_t(x_{Nt}) = 0 \tag{25}$$

and

$$h_t'(x_{Nt})[r_t V^I_{t-1} + (h_t(x_{It}) + (n_t - 2)h_t(x_{Nt}))(V^I_{t-1} - V^N_{t-1}) + x_{Nt}] - r_t - h_t(x_{It}) - (n_t - 1)h_t(x_{Nt}) = 0. \tag{26}$$

Totally differentiating equations (25) and (26) and solving

yields the matrix of comparative static effects.

$$\begin{bmatrix} dx_{It} \\ dx_{Nt} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{M_t} \end{bmatrix} \begin{bmatrix} d_t & -b_t \\ -c_t & a_t \end{bmatrix} \begin{bmatrix} \alpha_{It}dr_t + \beta_{It}dR_t + \delta_{IT}dV^I_{t-1} + \gamma_{It}dn_t + \varepsilon_{It}dV^N_{t-1} \\ \alpha_{Nt}dr_t + \beta_{Nt}dR_t + \delta_{Nt}dV^I_{t-1} + \gamma_{Nt}dn_t + \varepsilon_{Nt}dV^N_{t-1} \end{bmatrix}$$

where $M_t = a_t d_t - b_t c_t$, and

$$a_t = h_t''(x_{It})[r_t V^I_{t-1} + (n_t - 1)h_t(x_{Nt})(V^I_{t-1} - V^N_{t-1}) + x_{It} - R_t] < 0$$

$$b_t = h_t'(x_{It})(n_t - 1)[h_t'(x_{Nt})(V^I_{t-1} - V^N_{t-1}) - 1] \geq 0$$

$$c_t = h_t'(x_{It})[h_t'(x_{Nt})(V^I_{t-1} - V^N_{t-1}) - 1] \geq 0$$

$$d_t = h_t''(x_{Nt})[r_t V^I_{t-1} + (h_t(x_{It}) + (n_t - 2)h_t(x_{Nt}))(V^I_{t-1} - V^N_{t-1}) + x_{Nt}] + h_t'(x_{Nt})(n_t - 2)[h_t'(x_{Nt})(V^I_{t-1} - V^N_{t-1}) - 1].$$

The inequalities $a_t < 0$, $b_t \geq 0$ and $c_t \geq 0$ follow from equation (19), assumption 2 and assumption 2, respectively. Expression d_t is not signable from previous assumptions, since the first term is negative by equation (20), while the second term is nonnegative under assumption 2. The remaining terms are

$$\alpha_{It} = h_t'(x_{It})V^I_{t-1} - 1 \geq 0 \text{ by equation (15)}$$

$$\alpha_{Nt} = h_t'(x_{Nt})V^I_{t-1} - 1 \geq 0 \text{ by equation (16)}$$

$$\beta_{It} = -h_t'(x_{It}) < 0, \; \beta_{Nt} = 0$$

$$\delta_{It} = h_t'(x_{It})[r_t + (n_t - 1)h_t(x_{Nt})] > 0$$

$$\delta_{Nt} = h_t'(x_{Nt})[r_t + (n_t - 2)h_t(x_{Nt}) + h_t(x_{It})] > 0$$

$$\gamma_{It} = h_t(x_{Nt})[h_t'(x_{It})(V^I_{t-1} - V^N_{t-1}) - 1] \geq 0 \text{ by assumption 2}$$

$$\gamma_{Nt} = h_t(x_{Nt})[h_t'(x_{Nt})(V^I_{t-1} - V^N_{t-1}) - 1] \geq 0 \text{ by assumption 2}$$

$$\varepsilon_{It} = -h_t'(x_{It})(n_t - 1)h_t(x_{Nt}) < 0,$$

and

$$\varepsilon_{Nt} = -h_t'(x_{Nt})[(n_t - 2)h_t(x_{Nt}) + h_t(x_{It})] < 0.$$

Consider first the comparative statics of x_{Nt}.

$$\partial x_{Nt}/\partial r_t = (c_t\alpha_{It} - a_t\alpha_{Nt})/M_t; \text{ thus sgn } \partial x_{Nt}/\partial r_t = \text{sgn } M_t.$$
$$\partial x_{Nt}/\partial R_t = (c_t\beta_{It} - a_t\beta_{Nt})/M_t; \text{ thus sgn } \partial x_{Nt}/\partial R_t = -\text{sgn } M_t.$$
$$\partial x_{Nt}/\partial V^I_{t-1} = (c_t\delta_{It} - a_t\delta_{Nt})/M_t; \text{ thus sgn } \partial x_{Nt}/\partial V^I_{t-1} = \text{sgn } M_t.$$
$$\partial x_{Nt}/\partial n_t = (c_t\gamma_{It} - a_t\gamma_{Nt})/M_t; \text{ thus sgn } \partial x_{Nt}/\partial n_t = \text{sgn } M_t.$$
$$\partial x_{Nt}/\partial V^N_{t-1} = (c_t\varepsilon_{It} - a_t\varepsilon_{Nt})/M_t; \text{ thus sgn } \partial x_{Nt}/\partial V^N_{t-1} = -\text{sgn } M_t.$$

The expression M_t is also not signable from previous assumptions. However, from the comparative statics expressions for x_{Nt} above, it seems only reasonable that $M_t > 0$. In particular, it seems almost a requirement for stability that $\partial x_{Nt}/\partial V^I_{t-1} > 0$ and $\partial x_{Nt}/\partial V^N_{t-1} < 0$. That is, an increase in the value of winning (losing) the current race results in an increase (a decrease) in the nonincumbent's rate of investment. In addition, the fact that $\partial x_{Nt}/\partial n_t > 0$ agrees with the Lee and Wilde result for their symmetric game. While the opposite sign pattern cannot be ruled out on the basis of the previous analysis, it is not likely to be pervasive.

Assumption 3. $M_t > 0$.

Proposition 9. Under assumptions 1,2 and 3, the investment rate of nonincumbents, x_{Nt}, increases with an increase in the discount rate (r_t), the value of being the incumbent next period (V^I_{t-1}), and the number of firms in the industry (n_t). The investment rate of the nonincumbents decreases in response to an increase in the flow profit associated with the current innovation (R_t) or the value of being a nonincumbent next period (V^N_{t-1}).

Since $M_t > 0$ implies that $d_t \leq 0$, we can also sign the comparative static effects of parameter changes on x_{It}.

Proposition 10. Under assumptions 1,2 and 3, the incumbent's rate of investment increases with an increase in the discount rate (r_t), the value of being the incumbent next period (V^I_{t-1}), and the number of firms in the industry (n_t). The incumbent's rate of investment decreases in response to an increase in the flow revenue associated with the current innovation (R_t) or an increase in the value to being a nonincumbent in the next stage (V^N_{t-1}). That is,

$$\partial x_{It}/\partial r_t = -(d_t\alpha_{It} - b_t\alpha_{Nt})/M_t > 0,$$
$$\partial x_{It}/\partial R_t = -(d_t\beta_{It} - b_t\beta_{Nt})/M_t < 0,$$
$$\partial x_{It}/\partial V^I_{t-1} = -(d_t\delta_{It} - b_t\delta_{Nt})/M_t > 0,$$
$$\partial x_{It}/\partial n_t = -(d_t\gamma_{It} - b_t\gamma_{Nt})/M_t > 0, \text{ and}$$
$$\partial x_{It}/\partial V^N_{t-1} = -(d_t\varepsilon_{It} - b_t\varepsilon_{Nt})/M_t < 0.$$

With these comparative statics results in hand, we can compare the game with one stage remaining (in which there are one incumbent and $n_1 - 1$ nonincumbents) to the original Lee and Wilde model in which $R_1 = 0$ and all firms are identical. Since $\partial x_{I1}/\partial R_1 < 0$ and $\partial x_{N1}/\partial R_1 < 0$, both x_{I1} and x_{N1} are less than the symmetric equilibrium rate of investment in the Lee and Wilde case.

Another question of interest is whether firms invest more or less on a <u>given</u> innovation when there are more innovations remaining. That is, we want to <u>fix an innovation,</u> which is characterized by the profit flow R and the technology production function $h(\cdot)$, and fix n, and ask whether x_{It} and x_{Nt} are greater, smaller or equal to $x_{I(t-1)}$ and $x_{N(t-1)}$. Since $V^I_{t-1} > V^I_t$ and $V^N_{t-1} < V^N_t$, the comparative statics results above imply that firms would invest less on the <u>same innovation</u> if there were more innovations remaining.

One can accomodate entry and exit in this model by allowing n_t to be determined by a zero profit condition. Since firms can only enter as nonincumbents (and the incumbent would be the last to exit), we need only consider V^N_t. Define n^o_t to be the value of n_t such that $V^N_t = 0$. Assuming entry and exit in future stages keeps $V^N_{t-1} = 0,\ldots,V^N_1 = 0$, the expressions $x^o_{Nt} = x_{Nt}(n^o_t)$ and $x^o_{It} = x_{It}(n^o_t)$ are implicitly defined by

$$V^N_t = [h_t(x^o_{Nt})V^I_{t-1} - x^o_{Nt}]/[r_t + (n^o_t-1)h_t(x^o_{Nt}) + h_t(x^o_{It})] = 0.$$

Then x^o_{Nt} is defined by $h_t(x^o_{Nt})V^I_{t-1} - x^o_{Nt} = 0$. From equation (16), it must also be that $h_t'(x^o_{Nt})V^I_{t-1} = 1$. Combining these two equalities gives $h_t(x^o_{Nt}) - x^o_{Nt}h_t'(x^o_{Nt}) = 0$. But this is true only when $x = \tilde{x}_t$. Thus $x^o_{Nt} = \tilde{x}_t$.

<u>Proposition 11.</u> In a Nash equilibrium with initial increasing returns and free entry and exit, the nonincumbents invest at efficient scale (i.e., maximum average product); the incumbent invests at less than efficient scale.

It is interesting to compare this with the results of Loury [1979] and Lee and Wilde [1980]. These two models are identical except for the specification of R and D costs. Assuming that R and D costs are contractual, Loury showed that in a symmetric Nash equilibrium with free entry all firms invest at less than efficient scale. Assuming that firms pay R and D costs only until some firm is successful, Lee and Wilde showed that in a symmetric Nash equilibrium with free entry all firms invest precisely at efficient scale. This model also assumes that R and D investment is made only until some firm succeeds with the next innovation. We find the hybrid result that while nonincumbents invest at efficient scale, the incumbent invests at less than efficient scale.

Although the equilibrium number of firms will not be exactly n^o_t due to the integer requirement, Proposition 11 will be approximately true, and V^N_t will be approximately zero. Under these circumstances, Assumption 2 is not particularly restrictive when there is free entry and exit.

IV. Existence of Nash Equilibrium

<u>Proposition 12.</u> Suppose that $R_t < r_tV^I_{t-1}$ and

$h_t'(0) \geq \max\{1/(V^I_{t-1} - V^N_{t-1}), 1/(V^I_{t-1} - R_t/r_t)\}$. Then there exists a Nash equilibrium in pure strategies for the t-stage game. Moreover, the Nash equilibrium satisfies the first-order conditions.

The proof will proceed via a series of lemmas.

<u>Lemma 1.</u> Under the hypotheses of Proposition 12, $x_{jt} \leq x^m_{jt} < \infty$, regardless of x_{-jt}, $j = 1,2,\ldots,n_t$.

<u>Proof.</u> First let j = i. Recall that sgn $\partial V^i_t/\partial x_{it}$ = sgn $f_{it}(x_t)$, where

$$f_{it}(x_t) = h_t'(x_{it})[r_tV^I_{t-1} - R_t] - r_t + h_t'(x_{it})x_{it} - h_t(x_{it}) + \sum_{k\neq i} h_t(x_{kt})[h_t'(x_{it})(V^I_{t-1} - V^N_{t-1}) - 1].$$

Define $x^m_{it} = \max\{\tilde{x}_t, \hat{x}_t, \hat{\hat{x}}_t\}$ where $\tilde{x}_t$ is as in assumption 1(iii); $\hat{x}_t = \min\{x \geq \bar{x}_t \mid h_t'(x) \leq 1/(V^I_{t-1} - R_t/r_t)\}$ and $\hat{\hat{x}}_t = \min\{x \geq \bar{x}_t \mid h_t'(x) \leq 1/(V^I_{t-1} - V^N_{t-1})\}$. The values $\hat{x}_t$ and $\hat{\hat{x}}_t$ exist and are unique because $h_t'(x)$ is continuous and $\lim_{x_{it} \to \infty} h_t'(x_{it}) = 0$. Then we claim that i would never choose $x_{it} > x^m_{it}$. To see this, note that $f_{it}(x^m_{it}) \leq 0$ and

$$\partial f_{it}/\partial x_{it} = h_t''(x_{it})[r_tV^I_{t-1} + \sum_{k\neq i} h_t(x_{kt})(V^I_{t-1} - V^N_{t-1}) + x_{it} - R_t] < 0$$

for all $x_{it} > x^m_{it}$ because $h_t''(x) < 0$ for all $x > \bar{x}_t$, and $r_tV^I_{t-1} > R_t$ by hypothesis. Thus if $x_{it} > x^m_{it}$, then $\partial V^i_t/\partial x_{it} < 0$ for all x_{-it}. Therefore, any x_{it} which would be chosen would be no greater than x^m_{it}

regardless of x_{-it}. A similar argument regarding x_{jt} yields $x^m_{jt} = \max\{\tilde{x}_t, \hat{\hat{x}}_t\}$, but requires no additional assumptions.

Q.E.D.

<u>Lemma 2.</u> $V^j_t(i, x_{jt}, x_{-jt})$ is single-peaked in x_{jt} for all x_{-jt}, j = $1,2,\ldots,n_t$.

<u>Proof.</u> First let j = i. Recall that

$$\partial f_{it}/\partial x_{it} = h_t''(x_{it})[r_tV^I_{t-1} + \sum_{k\neq i} h_t(x_{kt})(V^I_{t-1} - V^N_{t-1}) + x_{it} - R_t].$$

Since $r_tV^I_{t-1} > R_t$ by hypothesis, f_{it} is first increasing, then decreasing in x_{it} (for all x_{-it}) under the assumption that $h_t''(x) \geq (\leq)$ 0 as $x \geq (\leq) \bar{x}_t$. Since $f_{it}(0, x_{-it}) \geq 0$ and $f_{it}(x^m_{it}, x_{-it}) \leq 0$ for all x_{-it}, f_{it} is first positive, then negative. Since sgn $\partial V^i_t/\partial x_{it}$ = sgn f_{it}, V^i_t is first increasing, then decreasing in x_{it} for all x_{-it}. Under the hypotheses of Proposition 11, $\partial V^i_t(i, 0, x_{-it})/\partial x_{it} \geq 0$, for all x_{-it}. From Lemma 1 we know that $\partial V^i_t(i, x^m_{it}, x_{-it})/\partial x_{it} \leq 0$ for all x_{-it}. Thus V^i_t is single-peaked in x_{it} (for all x_{-it}). A similar argument establishes the result for $j \neq i$.

Q.E.D.

Figure 2

Note that V_t^j will typically <u>not</u> be concave in x_{jt}. If it were, we could simply apply the existence result for concave games in Rosen [1965].

<u>Proof of Proposition 12.</u> Lemmas 1 and 2 imply that V_t^j is single-peaked in x_{jt} and has a unique maximizer $\phi_{jt}(x_{-jt})$ which always lies in the interval $[0,x_{jt}^m]$ for each $x_{-jt} \varepsilon [0,\infty)^{n_t-1}$. Thus we can, without loss of generality, restrict x_{-jt} to $\underset{k \neq j}{X} [0,x_{kt}^m]$. This maximizer ϕ_{jt} is a continuous function of x_{-jt} by the theorem of the maximum. Then the vector mapping $\phi_t = (\phi_{kt})_{k=1}^{n_t}$ maps the compact, convex, nonempty set $\overset{n_t}{\underset{k=1}{X}} [0,x_{kt}^m]$ into itself continuously. Brouwer's theorem implies the existence of a fixed point for ϕ_t. This fixed point is a Nash equilibrium for the game with t stages remaining, and it satisfies the first-order conditions.

Q.E.D.

V. Conclusions and Related Literature

We have developed a model of patent races with a sequence of innovations. Each innovative success initiates a new stage; within each stage firms engage in a continuous-time patent race. The game with t stages remaining is built up recursively from shorter horizon games under the assumption of subgame perfect Nash equilibrium play.

We have shown that, under reasonable conditions, a Nash equilibrium exists for the t-stage game. The value of the t-stage game to the incumbent exceeds the value of the t-stage game to the nonincumbents. A Nash equilibrium is symmetric among the nonincumbents while the incumbent always invests less than each nonincumbent. Thus the incumbency will tend to change hands more frequently than by random selection -- at each stage t the incumbent has less than a $1/n_t$ chance of being the incumbent in the next stage. Thus instead of the industry becoming dominated by a single innovator, this model suggests an industry which, in equilibrium, is characterized by frequent upsets of the current monopolist.

In addition to the single-innovator models mentioned earlier, there are two other multi-stage models which are related to this work. The first of these is Futia [1980]. Futia presents a discrete-time model with at most one successful innovation per period. The R and D model is not central to the purpose of Futia's paper, but is used only to provide a choice-theoretic basis for the analysis of the stochastic process which is assumed to summarize aggregate inventive activity over time. The model is symmetric in the sense that all firms are incumbents -- any innovation is adopted by all firms who remain in the industry. Under the assumption that

$$\Pr\{\text{firm } i \text{ succeeds with } t^{th} \text{ innovation}\} = x_{it}/\sum x_{kt},$$

the Nash equilibrium for each stage game is symmetric. As Futia remarks, the assumption of rapid imitation ''prevents this theory from describing a world in which there is a perfect patent.''

Another discrete-time model, due to Rogerson [1979], describes a stochastic game where at each stage firms compete for the right to a monopoly franchise in the subsequent stage. This could be interpreted as a game of R and D competition with patents (though Rogerson does not propose this interpretation). There are one incumbent and a number of nonincumbents. The incumbent is assumed to have an advantage which arises solely because of its incumbency. It is assumed (in our notation) that

$$\Pr\{\text{incumbent } i \text{ wins at } t\} = \beta x_{it}/[\beta x_{it} + \sum_{k \neq i} x_{kt}],$$

while

$$\Pr\{\text{nonincumbent } j \text{ wins at } t\} = x_{jt}/[\beta x_{it} + \sum_{k \neq i} x_{kt}],$$

where $\beta \geq 1$ represents the incumbent's advantage. For $\beta > 1$, there exists a Nash equilibrium in which the nonincumbents invest the same amount, while the incumbent invests more than each nonincumbent. This is because the marginal effectiveness at producing likelihood of winning is higher for the incumbent than for a nonincumbent when $\beta > 1$. If $\beta = 1$, then there is no incumbent advantage and the equilibrium is symmetric. This still contrasts with our results. The reason is as follows. In Rogerson's model (with $\beta = 1$), the length of each stage is fixed at one unit of (discrete) time. The reward at stage t is independent of investment at stage t, which only affects the expected reward at the next stage. In our model, the length of

stage t — and hence the reward to the incumbent over that length of time -- is affected by investment by the incumbent in stage t. Greater investment tends to shorten the length of the t^{th} stage because it hastens the discovery of the next innovation and hence the beginning of the next stage. In this case, we get our result that the incumbent invests less than the nonincumbents, since the incumbent has relatively less incentive to shorten the length of its t^{th}-stage incumbency.

Finally, recent work by Kamien and Schwartz [1980] suggests that these results may be extendable to a more general class of distribution functions than the exponential distribution. This would seem to be an appropriate avenue for further research.

VI. References

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APPENDIX

Cooperative Investment

In this section, we examine the rates of investment which would be selected if firms coordinated their investment decisions so as to maximize joint profits, denoted $V_t^s(x_t)$. Again we define the t-stage problems recursively, beginning with t = 0.

Stage t = 0

When no innovations remain, joint profit is simply monopoly profit on the last innovation: $V_0^S = V_0^I = R_0/r_0$.

Stage t = 1

When one innovation remains, joint profits are simply the sum of individual profits, since V_0^S is received as soon as any firm succeeds.

$$V_1^s(X_1) = \int_0^\infty e^{-r_1\tau} e^{-\sum h_1(x_{k1})\tau}[\sum h_1(x_{k1})V_0^S + R_1 - \sum x_{k1}]d\tau$$
$$= [\sum h_1(x_{k1})V_0^S + R_1 - \sum x_{k1}]/[r_1 + \sum h_1(x_{k1})].$$

The objective of the cooperative firms is to maximize $V_1^s(X_1)$ by their choice of $X_1 \varepsilon [0,\infty)^{n_1}$. An interior maximizer X_1^s is implicitly defined by the first-order conditions: for $i = 1,2,\ldots,n_1$,

$$\partial V_1^s(X_1^s)/\partial x_{i1} = [r_1 + \sum h_1(x_{k1}^s)][h_1'(x_{i1}^s)V_0^S - 1]$$

$$- [r_1V_0^S + R_1 - \sum x_{k1}^s]h_1'(x_{i1}^s) = 0. \qquad (25)$$

Alternatively, (25) can be solved to yield

$$V_1^s(X_1^s) = [h_1'(x_{i1}^s)V_0^S - 1]/h_1'(x_{i1}^s) \qquad (26)$$

and

$$h_1'(x_{i1}^s)[r_1V_0^S - R_1 + \sum x_{k1}^s] = r_1 + \sum h_1(x_{k1}^s). \qquad (27)$$

Equation (26) implies that $h_1'(x_{i1}^s)V_0^S - 1 \geq 0$ while (27) implies $r_1V_0^S - R_1 + \sum x_{k1}^s > 0$. A second-order necessary condition is

$$\partial^2V_1^s(X_1^s)/\partial x_{i1}^2 = h_1''(x_{i1}^s)[r_1V_0^S - R_1 + \sum x_{k1}^s] \leq 0.$$

Hence $h_1''(x_{i1}^s) \leq 0$; that is, $x_{i1}^s \geq \bar{x}_1$.

In view of the symmetry of the firms, it seems clear that any jointly optimal investment decision with one innovation remaining will be symmetric. That is, $x_{i1}^s = x_{j1}^s$ for all i,j. Denote this common value x_{S1} and let $V_1^S = V_1^s(X_1^s)$.

<u>Proposition 13.</u> With one innovation remaining, the investment rate of the noncooperative incumbent is no less than the joint profit maximizing rate. A fortiori, the noncooperative noncinumbents invest more than the joint profit maximizing rate. Formally, $x_{N1} > x_{I1} \geq x_{S1}$.

<u>Proof.</u> From equation (5), $V_1^I = [h_1'(x_{I1})V_0^I - 1]/h_1'(x_{I1})$, while (26) implies $V_1^S = [h_1'(x_{S1})V_0^S - 1]/h_1'(x_{S1})$. Since $V_0^S = V_0^I$, and since it

must be that $V_1^S \geq V_1^I$ (the incumbent can do no better than if it also had the cooperation of its rivals)

$$[h_1'(x_{S1})V_0^I - 1]/h_1'(x_{S1}) \geq [h_1'(x_{I1})V_0^I - 1]/h_1'(x_{I1}). \qquad (30)$$

The function $[h_1'(x)V_0^I - 1]/h_1'(x)$ is decreasing in x for $x \geq \bar{x}_1$. Since both x_{S1} and x_{I1} are at least $\bar{x}_1$, it follows that $x_{I1} \geq x_{S1}$.

Q.E.D.

Thus even though the incumbent invests less than the nonincumbents, it still invests too much relative to the joint profit maximum.

Stage t = 2

When two innovations remain, joint profits are

$$V_2^s(X_2) = \int_0^\infty e^{-r_2\tau} e^{-\sum h_2(x_{k2})\tau}[\sum h_2(x_{k2})V_1^S + R_2 - \sum x_{k2}]d\tau$$
$$= [\sum h_2(x_{k2})V_1^S + R_2 - \sum x_{k2}]/[r_2 + \sum h_2(x_{k2})].$$

That is, the firms (jointly) receive revenue at the flow rate R_2, pay out R and D costs at the rate $\sum x_{k2}$, and experience the constant aggregate hazard rate of success $\sum h_2(x_{k2})$ at τ so long as no one has succeeded before time τ.

At the joint profit maximizing vector X_2^s, the following equalities obtain.

$$\partial V_2^s(X_2^s)/\partial x_{i2} = [r_2 + \sum h_2(x_{k2})][h_2'(x_{i2}^s)V_1^S - 1]$$
$$- [\sum h_2(x_{k2}^s)V_1^S + R_2 - \sum x_{k2}^s]h_2'(x_{i2}^s) = 0. \qquad (30)$$

Alternatively,

$$V_2^s(X_2^s) = [h_2'(x_{i2}^s)V_1^S - 1]/h_2'(x_{i2}^s) \qquad (31)$$

and

$$h_2'(x_{i2}^s)[r_2V_1^S - R_2 + \sum x_{k2}^s] = r_2 + \sum h_2(x_{k2}^s). \qquad (32)$$

Equation (31) implies $h_2'(x_{i2}^s)V_1^S - 1 \geq 0$ and (32) implies $r_2V_1^S - R_2 + \sum x_{k2}^s > 0$. Finally, the second-order necessary condition is

$$\partial^2 V_2^s(X_2^s)/\partial x_{i2}^2 = h_2''(x_{i2}^s)[r_2V_1^S - R_2 + \sum x_{k2}^s] \leq 0, \qquad (33)$$

implying that $x_{i2}^s \geq \bar{x}_2$.

Again, it seems clear that $x_{i2}^2 = x_{j2}^s = x_{S2}$ for all i,j. Define $V_2^S = V_2^s(X_2^s)$. Unfortunately, we cannot follow Proposition 13 to prove that $x_{I2} \geq x_{S2}$, since now $V_1^S \geq V_1^I$. That is, for any value V, $x_{S2}(V) \leq x_{I2}(V)$ by the same argument as in the proof of Proposition 13. But since $V_1^S \geq V_1^I$, it is not clear whether or not $x_{S2}(V_1^S) \geq x_{I2}(V_1^I)$.

Stage t

The t-stage analysis follows that of t = 2 exactly. The first-order condition

$$h_t'(x_{St})[r_tV^S_{t-1} - R_t + n_tx_{St}] - r_t - n_th_t(x_{St}) = 0$$

can be differentiated totally and solved for the comparative statics of $x_{St}(r_t,R_t,V^S_{t-1},n_t)$.

<u>Proposition 14.</u> The joint profit maximizing rate of investment increases with an increase in the discount rate (r_t) and the maximum joint profit available in the next period (V^S_{t-1}), and decreases with an increase in the current revenue flow (R_t). The expression $\partial x_{St}/\partial n_t$ has the same sign as $h_t'(x_{St})x_{St} - h_t(x_{St})$.

<u>Proposition 15.</u> Suppose that $R_t < r_tV^S_{t-1}$ and $h_t'(0) \geq 1/(V^S_{t-1} - R_t/r_t)$. Then there exists a unique $x_{St} < \infty$ which maximizes joint profits. Moreover, x_{St} satisfies the first-order condition (30).

The proof follows that of Proposition 12. Under the hypotheses of Proposition 15, $\partial V^S_t/\partial x_{St}$ starts out positive, changes sign exactly once and becomes negative for finite x_{St}.

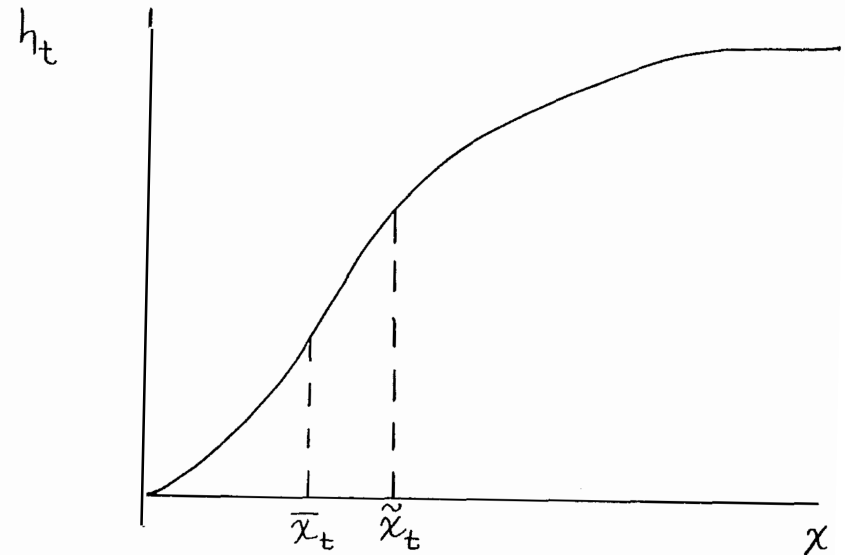

Figure 1

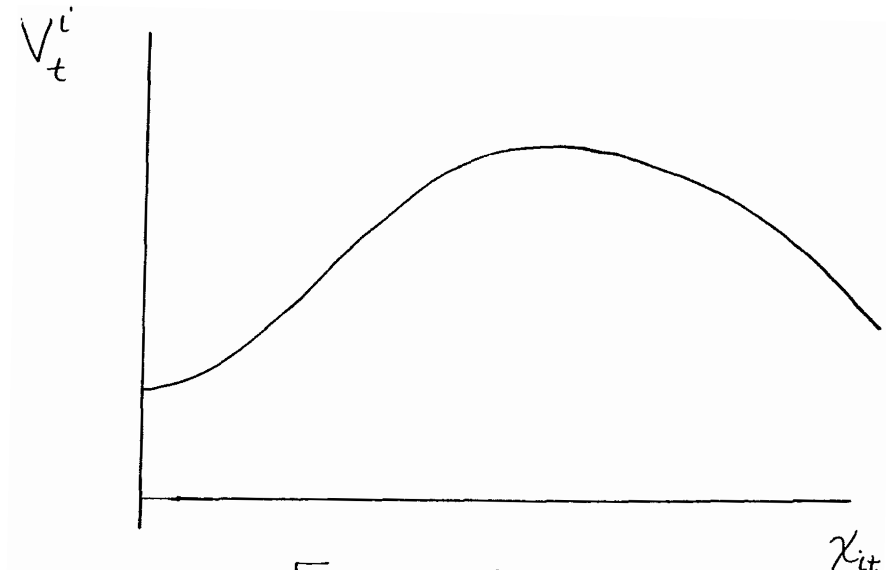

Figure 2