

BC_n -symmetric abelian functions

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Abstract

We construct a family of BC_n -symmetric biorthogonal abelian functions generalizing Koornwinder's orthogonal polynomials, and prove a number of their properties, most notably analogues of Macdonald's conjectures. The construction is based on a direct construction for a special case generalizing Okounkov's interpolation polynomials. We show that these interpolation functions satisfy a collection of generalized hypergeometric identities, including new multivariate elliptic analogues of Jackson's summation and Bailey's transformation.

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1 Introduction

In [16], we considered two families of BC_n -symmetric Laurent polynomials: the multivariate orthogonal polynomials introduced by Koornwinder [9], and the “interpolation Macdonald polynomials” of Okounkov [12], related by Okounkov’s “binomial formula”. One of the results of that paper was a multivariate analogue of Jackson’s summation, which had originally been conjectured by Warnaar [29]. In fact, that summation was a limiting case of Warnaar’s conjecture, an identity of elliptic hypergeometric series. This suggested that the theory of [16] should be extensible to the elliptic case; this extension is the topic of the present work.

The main difficulty in constructing this extension is the result of Okounkov [13] which roughly speaking said that the interpolation Macdonald polynomials could not be generalized; more precisely, he defined a general notion of interpolation polynomial, and showed that the Macdonald-type polynomials form a component of the corresponding moduli space. Thus in order to extend our results to the elliptic level, it would be necessary to generalize this notion of interpolation polynomial.

Another indication that an elliptic extension should exist was the existence of elliptic analogues [27, 26], both discrete and continuous, of the Askey-Wilson orthogonal polynomials [2]. The difference from the Askey-Wilson theory (already seen at the trigonometric level in work of Rahman [14] for continuous biorthogonality, and later work by Wilson [30] for discrete biorthogonality) was two-fold: first, rather than being orthogonal, the functions were merely *biorthogonal*, and second, the constraint of being monic of specified degree became a constraint on allowable poles. Extending this prescription directly to the multivariate level ran into the difficulty that divisors in multiple variables are much more complicated than in the univariate case. However, if we first clear the poles, we find that the constraint on poles becomes a requirement that the function vanish at appropriate points.

We thus arrive at the notion of “balanced” interpolation polynomial; this differs from Okounkov’s notion in that, rather than have one collection of vanishing conditions together with an assumption of triangularity on monomials, balanced interpolation polynomials satisfy two complementary collections of vanishing conditions. (This definition was independently discovered by Coskun and Gustafson [4], who also obtained many of the results of Section 4, in particular Theorems 4.1 and 4.9 below.) This allowed the affine symmetry of ordinary interpolation polynomials to be extended to a projective symmetry, and as a result allowed the desired generalization to elliptic interpolation polynomials. Moreover, two new special cases arise: first, by matching up the vanishing conditions, one obtains a collection of delta functions; more subtly, another special case appears in which the interpolation polynomials factor completely. These special cases then play the role that the Macdonald polynomials did in [16], namely that of instances that can be solved directly, thus giving boundary conditions for recurrences.

In particular, we have been able to generalize many of the results of [16], especially concerning the Koornwinder polynomials, which we generalize to a family of biorthogonal abelian functions. A closely related family was studied in [15] using a certain contour integral identity and a related integral operator; in particular, it was shown there that the biorthogonal functions satisfy an analogue of Macdonald’s normalization and evaluation conjectures. The analogue of the remaining conjecture (referred to here as “evaluation symmetry”) requires a more in depth understanding of interpolation polynomials, and will be proved in Theorem 5.4 below. In contrast

to [15], our present approach is essentially algebraic in nature; although to begin with we use the theory of theta functions over \mathbb{C} , we will eventually see that all of our results can be given purely algebraic proofs.

We begin in Section 2 by giving the definition and basic properties of balanced interpolation polynomials, in particular considering four of the main cases satisfying extra vanishing properties (the “perfect” case); the fifth (“elliptic”) case is the subject of the remainder of the paper.

Section 3 begins our treatment of the elliptic case by defining a family of BC_n -symmetric theta functions, parametrized by partitions contained in a rectangle, defined via vanishing conditions. As in [16], the key property of these functions is a difference equation, from which the extra vanishing property follows. We also define a corresponding family of abelian functions, this time indexed by all partitions of length at most n .

Evaluating an interpolation abelian function at a partition gives a generalized binomial coefficient; we study these coefficients in Section 4. In particular, we obtain a pair of identities generalizing Jackson’s summation and Bailey’s transformation respectively (see [6] for the univariate elliptic case); each identity involves a sum over partitions contained in a skew Young diagram of a product of binomial coefficients. Using these, we derive some unexpected symmetries of binomial coefficients, relating a coefficient to coefficients with conjugated or complemented partitions. This then enables us to prove a number of properties of interpolation functions, including a connection coefficient formula, a branching rule, a Pieri identity, and a Cauchy identity.

Section 5 introduces the biorthogonal functions, defined via an expansion in interpolation functions, and shows how they relate to the functions considered in [15]. Using the interpolation function identities, we prove a number of identities for the biorthogonal functions; in addition to the analogue of Macdonald’s conjectures for Koornwinder polynomials, we obtain connection coefficients, a quasi-branching rule, a quasi-Pieri identity, and a Cauchy identity.

In Section 6, we depart from the analytic approach via theta functions, and show that, in fact, the interpolation and biorthogonal functions can be defined purely algebraically (and thus, over the complex numbers, are invariant under transformations of the modular parameter). We sketch a purely algebraic derivation of the identities of Sections 4 and 5; the key step being to construct the algebraic analogue of the difference operator of Section 3.

In Section 7, we use our results on interpolation functions to define a family of perfect bigrids based on elliptic curves. Since this family is an open subset of the space of perfect bigrids, we in particular find that we cannot add any parameters to our elliptic theory, without further extending the notion of interpolation polynomial. We then consider how the bigrids in this family can degenerate, obtaining four general classes of degeneration (corresponding to the Kodaira symbols I_1 , I_2 , II , and III ; these in turn split somewhat further over a non-algebraically-closed field), including ordinary hypergeometric series (II and some instances of III) and basic hypergeometric series (I_1 and some instances of I_2) as well as some apparently unstudied cases.

Section 8 shows how the present results relate to the results of [16]. The identities of that paper involved three different functions of skew Young diagrams: binomial coefficients, “inverse” binomial coefficients, and principal specializations of skew Macdonald polynomials. We show that each of these can be obtained as a limit of elliptic binomial coefficients, and thus in particular prove that the Koornwinder polynomials are a limiting case of the biorthogonal functions.

Finally, in Section 9, we consider some open problems relating to interpolation and biorthogonal functions.

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Notation

As in [16], we use the notations of [11] for partitions, with the following additions. If $\lambda \subset m^n$, then $m^n - \lambda$ denotes the “complementary” partition defined by

$$(m^n - \lambda)_i = m - \lambda_{n+1-i}, \quad 1 \leq i \leq n \quad (1.1)$$

In addition, if $\ell(\lambda) \leq n$, we define $m^n + \lambda$ to be the partition

$$(m^n + \lambda)_i = m + \lambda_i, \quad 1 \leq i \leq n. \quad (1.2)$$

Similarly, if $\lambda_1 \leq m$, we define $m^n \cdot \lambda$ to be the partition such that

$$(m^n \cdot \lambda)_i = \begin{cases} m & 1 \leq i \leq n \\ \lambda_{i-n} & i > n. \end{cases} \quad (1.3)$$

(Note that this is specifically not defined for $\lambda_1 > m$; it is a concatenation operation, not to be confused with a union operator.) We also define a relation \prec_m on partitions such that $\kappa \prec_m \lambda$ if and only if

$$\kappa \subset \lambda \subset m^n + \kappa \quad (1.4)$$

for sufficiently large n ; similarly, $\kappa \prec'_m \lambda$ iff $\kappa' \prec_m \lambda'$. (If $m = 1$ in either case, the subscript will be omitted.)

We will also need the following “elliptic” analogues of q -symbols and generalizations. Let p be a complex number such that $0 < |p| < 1$. Then we define

$$\theta(x; p) := \prod_{0 \leq k} (1 - p^k x)(1 - p^{k+1}/x) \quad (1.5)$$

$$\theta(x; q; p)_m := \prod_{0 \leq j < m} \theta(q^j x; p) \quad (1.6)$$

In each case (and similarly for the C symbols (but *not* the Δ symbols) below), we follow the standard convention that the presence of multiple arguments before the (first) semicolon indicates a product; thus for instance $\theta(ax^{\pm 1}; p) = \theta(ax; p)\theta(a/x; p)$. We also need elliptic analogues of the C symbols of [16]:

$$C_\lambda^0(x; q, t; p) := \prod_{(i,j) \in \lambda} \theta(q^{j-1} t^{1-i} x; p) \quad (1.7)$$

$$C_\lambda^-(x; q, t; p) := \prod_{(i,j) \in \lambda} \theta(q^{\lambda_i - j} t^{\lambda'_j - i} x; p) \quad (1.8)$$

$$C_\lambda^+(x; q, t; p) := \prod_{(i,j) \in \lambda} \theta(q^{\lambda_i + j - 1} t^{2 - \lambda'_j - i} x; p). \quad (1.9)$$

Note that in the limit $p = 0$ we recover the symbols of [16]; similarly, the analogous symbols of [15] can each be expressed as products of two of our present symbols (with q and p switched in one symbol). The transformations of [16] carry over to the present case, via the same arguments.

Two combinations of C symbols are of particular importance, and are thus given their own notations. For any nonzero complex number a and any finite sequence of nonzero complex numbers $\dots b_i \dots$, we define two Δ

symbols as follows.

$$\Delta_\lambda^0(a|\dots b_i \dots; q, t; p) := \prod_i \frac{C_\lambda^0(b_i; q, t; p)}{C_\lambda^0(pqa/b_i; q, t; p)} \quad (1.10)$$

$$\Delta_\lambda(a|\dots b_i \dots; q, t; p) := \Delta_\lambda^0(a|\dots b_i \dots; q, t; p) \frac{C_{2\lambda^2}^0(pqa; q, t; p)}{C_\lambda^-(pq, t; q, t; p) C_\lambda^+(a, pqa/t; q, t; p)}. \quad (1.11)$$

(Here, as in [16], $2\lambda^2$ denotes the partition such that $(2\lambda^2)_i = 2(\lambda_{\lceil i/2 \rceil})$.) Note that although Δ^0 essentially follows the standard multiple argument convention, Δ very much does not. We note the following transformations:

$$\Delta_{\lambda'}(a|\dots b_i \dots; 1/t, 1/q; p) = \Delta_\lambda(a/qt|\dots b_i \dots; q, t; p) \quad (1.12)$$

$$\frac{\Delta_{m^n-\lambda}(a|\dots b_i \dots; q, t; p)}{\Delta_{m^n}(a|\dots b_i \dots; q, t; p)} = \Delta_\lambda\left(\frac{t^{2n-2}}{q^{2m}a} \middle| \dots \frac{t^{n-1}b_i}{q^m a} \dots, t^n, q^{-m}, pqt^{n-1}, pq/q^m t; q, t; p\right) \quad (1.13)$$

$$\frac{\Delta_{m^n+\lambda}(a|\dots b_i \dots; q, t; p)}{\Delta_{m^n}(a|\dots b_i \dots; q, t; p)} = \Delta_\lambda(q^{2m}a|\dots q^m b_i \dots, t^n, pqt^{n-1}, q^m a/t^{n-1}, pqq^m a/t^n; q, t; p) \quad (1.14)$$

$$\frac{\Delta_{m^n \cdot \lambda}(a|\dots b_i \dots; q, t; p)}{\Delta_{m^n}(a|\dots b_i \dots; q, t; p)} = \Delta_\lambda(a/t^{2n}|\dots b_i/t^n \dots, q^{-m}, pq/q^m t, q^m a/t^{n-1}, pqq^m a/t^n; q, t; p), \quad (1.15)$$

with corresponding transformations for Δ^0 ; we also note that

$$\Delta_{m^n}(a|\dots b_i \dots; q, t; p) = \lim_{x \rightarrow 1} \Delta_{m^n}^0(a|\dots b_i \dots, q^m pqa/x, pqa/xt^n, pqa x, pqt^{n-1}x/q^m; q, t; p) \quad (1.16)$$

A *meromorphic (p-)theta function* is a meromorphic function f on \mathbb{C}^* (the multiplicative group of \mathbb{C} ; i.e., f is meromorphic away from 0 and ∞) such that

$$f(x) = ax^n f(px) \quad (1.17)$$

for suitable $a \in \mathbb{C}^*$, $n \in \mathbb{Z}$; without the adjective ‘‘meromorphic’’, theta functions are assumed holomorphic. Thus in particular the function $\theta(x; p)$ is a theta function, satisfying

$$\theta(x; p) = -x\theta(1/x; p) = -x\theta(px; p). \quad (1.18)$$

An *elliptic function* is a meromorphic function on \mathbb{C}^* invariant under multiplication by p ; in other words, a meromorphic function on the quotient group $\mathbb{C}^*/\langle p \rangle$ (which is a compact 1-dimensional complex Lie group, or in other words an elliptic curve).

We also have the following classes of multivariate functions.

Definition 1. A *meromorphic BC_n -symmetric theta function of degree m* is a meromorphic function f on $(\mathbb{C}^*)^n$ such that

$f(x_1, \dots, x_n)$ is invariant under permutations of its arguments.

$f(x_1, \dots, x_n)$ is invariant under $x_i \mapsto 1/x_i$ for each i .

$f(px_1, x_2, \dots, x_n) = (1/px_1^2)^m f(x_1, x_2, \dots, x_n)$.

A BC_n -symmetric (p -)abelian function is a meromorphic BC_n -symmetric theta function of degree 0.

Remark 1. Again, BC_n -symmetric theta functions are assumed holomorphic unless specifically labeled “meromorphic”.

Remark 2. It is not entirely clear to which root system these functions are truly attached. Certainly, the biorthogonal functions considered below generalize the Koornwinder polynomials, traditionally associated to the root system BC_n , but there is also good reason to associate this class of theta functions to the affine root system of type $A_n^{(2)}$ (see, for instance, the work of Looijenga [10] and Saito [22] on the invariant theory of abelian varieties associated to (extended) affine root systems.) In the absence of a general theory, we have chosen the more familiar label.

The canonical example of this is the function $\prod_{1 \leq i \leq n} \theta(ux_i^{\pm 1}; p)$ for $u \in \mathbb{C}^*$; indeed, the graded algebra of BC_n -symmetric theta functions is generated by such functions.

We define a difference operator on (meromorphic) BC_n -symmetric theta functions that will play a crucial role in the sequel; this is the analogue of the difference operator of [16], and also appeared prominently in [15].

Definition 2. Let n be a nonnegative integer, and let $q, t, a, b, c, d \in \mathbb{C}^*$ be arbitrary parameters. Define a difference operator $D^{(n)}(a, b, c, d; q, t; p)$ acting on BC_n -symmetric meromorphic functions on $(\mathbb{C}^*)^n$ by:

$$\begin{aligned} & (D^{(n)}(a, b, c, d; q, t; p)f)(x_1, \dots, x_n) \\ &= \sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq i \leq n} \frac{\theta(ax_i^{\sigma_i}, bx_i^{\sigma_i}, cx_i^{\sigma_i}, dx_i^{\sigma_i}; p)}{\theta(x_i^{2\sigma_i}; p)} \prod_{1 \leq i < j \leq n} \frac{\theta(tx_i^{\sigma_i} x_j^{\sigma_j}; p)}{\theta(x_i^{\sigma_i} x_j^{\sigma_j}; p)} f(\dots q^{\sigma_i/2} x_i \dots). \end{aligned} \quad (1.19)$$

Proposition 1.1. For each nonnegative integer m , if $q^m t^{n-1} abcd = p$, then $D^{(n)}(a, b, c, d; q, t; p)$ maps the space of BC_n -symmetric theta functions of degree m into itself.

Proof. Every term in the sum is a theta function in each variable with the same functional equation, and thus the sum is itself a theta function in each variable. It remains only to show that it is holomorphic, which follows from the usual symmetry argument. \square

Remark. For $n = 1$, the above operators span a four-dimensional space of difference operators acting on BC_1 -symmetric theta functions of degree m . This

its action on theta functions [23, 24]. Moreover, the relations of the Sklyanin algebra are simply given by the quasi-commutation relation of Corollary 3.4 below.

The case $m = 0$ is worth noting:

Corollary 1.2. If $t^{n-1} abcd = p$, then

$$\sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq i \leq n} \frac{\theta(ax_i^{\sigma_i}, bx_i^{\sigma_i}, cx_i^{\sigma_i}, dx_i^{\sigma_i}; p)}{\theta(x_i^{2\sigma_i}; p)} \prod_{1 \leq i < j \leq n} \frac{\theta(tx_i^{\sigma_i} x_j^{\sigma_j}; p)}{\theta(x_i^{\sigma_i} x_j^{\sigma_j}; p)} = \prod_{1 \leq i \leq n} \theta(abt^{n-i}, act^{n-i}, adt^{n-i}; p) \quad (1.20)$$

Proof. By the proposition, the sum is independent of x_i ; setting $x_i = at^{n-i}$ gives the desired result. \square

Remark. This identity is, in fact, equivalent to the identity used in [18, 19] to prove BC_n -type elliptic hypergeometric identities, a special case of Theorem 5.1 of [29]. In particular, there is a determinantal proof of this identity, which turns out to generalize to a determinantal formula for the difference operator, although we will not consider that formula here.

2 Balanced interpolation polynomials

For this section, we fix R to be an arbitrary commutative ring. A point on the projective line $\mathbb{P}^1(R)$ is represented by relatively prime homogeneous coordinates (z, w) (i.e., the ideal generated by z and w contains 1), modulo multiplication by units; a polynomial of degree m on $\mathbb{P}^1(R)$ (a section of $\mathcal{O}(m)$) then corresponds to a homogeneous polynomial of degree m in these coordinates. Similarly, given an n -tuple of nonnegative integers, one has a notion of a polynomial on $\mathbb{P}^1(R)^n$ of degree (m_1, m_2, \dots, m_n) . The value of such a polynomial at a given n -tuple of points on $\mathbb{P}^1(R)$ is only defined modulo units, unless specific homogeneous coordinates are chosen for each point; however, this freedom leaves invariant the zero-locus of the polynomial as well as the ratio of two such polynomials of the same degree. We will thus feel free to write

$$p(x_1, x_2, \dots, x_n) \tag{2.1}$$

for the value of p at the points $x_1, \dots, x_n \in \mathbb{P}^1(R)$, with this understanding.

Of particular importance is the polynomial $x \cdot y$ of degree $(1, 1)$ defined by

$$(z_1, w_1) \cdot (z_2, w_2) = z_1 w_2 - w_1 z_2. \tag{2.2}$$

Note that $x \cdot y$ is invariant under the action of $\mathrm{SL}_2(R)$; since $(1, 0) \cdot (z, w) = w$, we find that $x \cdot y = 0$ if and only if x and y represent the same point. Similarly, $x \cdot y \in R^*$ (the unit group of R) if and only if they represent distinct points in $\mathbb{P}^1(R_0)$ for all homomorphic images R_0 of R .

Definition 3. A symmetric polynomial on $\mathbb{P}^1(R)$ of degree m in n variables is a polynomial on $\mathbb{P}^1(R)^n$ of degree (m, m, \dots, m) invariant under permutations of the n arguments.

The space of symmetric polynomials of degree m in n variables (denoted Λ_n^m) is a free module of rank $\binom{m+n}{n}$; this also counts the number of partitions $\lambda \subset m^n$. An explicit basis of this module is, for instance, given by the homogeneous monomials of degree m in the $n + 1$ multilinear polynomials

$$e_i(z_1, w_1, \dots, z_n, w_n) = \sum_{\substack{I \subset \{1, 2, \dots, n\} \\ |I|=i}} \prod_{j \in I} z_j \prod_{j \notin I} w_j, \tag{2.3}$$

$0 \leq i \leq n$. A polynomial in Λ_n^m will be said to be *primitive* if the ideal generated by its coefficients contains 1; note that it suffices to consider its coefficients in the e_i basis.

The following decomposition will prove useful in the sequel.

Lemma 2.1. *Fix integers $m, n \geq 0$, and a point $x_0 \in \mathbb{P}^1(R)$ (with chosen homogeneous coordinates). Then we have the following short exact sequence:*

$$0 \rightarrow \Lambda_n^{m-1} \xrightarrow{f} \Lambda_n^m \xrightarrow{g} \Lambda_{n-1}^m \rightarrow 0, \tag{2.4}$$

in which the map f is defined by

$$f(p)(x_1, \dots, x_n) = p(x_1, \dots, x_n) \prod_{1 \leq i \leq n} (x_0 \cdot x_i) \quad (2.5)$$

and the map g is defined by

$$g(p)(x_1, \dots, x_{n-1}) = p(x_0, x_1, \dots, x_{n-1}). \quad (2.6)$$

Moreover, this exact sequence splits: there exists a map $h : \Lambda_{n-1}^m \rightarrow \Lambda_n^m$ such that $g \circ h = 1$.

Proof. The claim is clearly invariant under $\mathrm{SL}_2(R)$, so we may assume that $x_0 = (0, 1)$. But then $f(p) = e_n p$, $g(e_i) = e_i$ for $i < n$, and $g(e_n) = 0$; that the sequence is exact follows immediately. The map h is then given by $h(e_i) = e_i$. \square

Definition 4. Let $m, n > 0$ be positive integers. A (R -valued) *extended bigrid of shape m^n* is a function

$$\gamma : \{0, 1\} \times \{1, \dots, n\} \times \{0, 1, \dots, m\} \rightarrow \mathbb{P}^1(R). \quad (2.7)$$

A *bigrig of shape m^n* is the restriction of an extended bigrid of shape m^n to the subset of (α, i, j) such that if $\alpha = 0$ then $j < m$, and if $\alpha = 1$, then $j > 0$. If the extended bigrid γ^+ restricts to γ , then γ^+ is said to be an *extension* of γ ; in considering the extensions of a given bigrid, we allow the extended values to lie in any ring containing R .

Definition 5. Let γ be a bigrid of shape m^n , and let $\lambda \subset m^n$ be a partition. A *balanced interpolation polynomial* of index λ for γ is a symmetric polynomial $P_\lambda^*(; \gamma)$ of degree m on $\mathbb{P}^1(R)^n$ such that for all extensions γ^+ of γ , the following vanishing conditions hold: (1) For all partitions $\mu \subset m^n$, $\lambda \not\subset \mu$,

$$P_\lambda^*(\dots \gamma^+(0, i, \mu_i) \dots; \gamma) = 0, \quad (2.8)$$

and (2) for all partitions $\mu \subset m^n$, $\mu \not\subset \lambda$,

$$P_\lambda^*(\dots \gamma^+(1, i, \mu_i) \dots; \gamma) = 0. \quad (2.9)$$

The bigrid γ is *perfect* if for all $\lambda \subset m^n$, there exists a primitive balanced interpolation polynomial of index λ for γ .

In the sequel, we will omit the word “balanced” except when necessary to avoid confusion with Okounkov’s interpolation polynomials, which we will refer to as “ordinary” interpolation polynomials throughout. (The $*$ to denote interpolation polynomials is inherited from shifted Schur functions via Okounkov’s interpolation polynomials.) The word “balanced” is used for two reasons. First, ordinary interpolation polynomials are defined via vanishing conditions together with a triangularity condition; we have replaced the latter by another, symmetrical, set of vanishing conditions (thus obtaining a more “balanced” definition). Second, the corresponding hypergeometric sums (most notably Theorem 4.1 below) are multivariate analogues of balanced, very-well-poised, hypergeometric series; the corresponding identity for ordinary interpolation polynomials is merely very-well-poised. Moreover, although it is easy to degenerate the sums to remove very-well-poisedness,

there appears to be no straightforward way to obtain a non-balanced sum without degenerating to ordinary interpolation polynomials.

Note that it suffices to consider the generic extension of γ , in which the extended points are all algebraically independent over R . Thus the existence of an interpolation polynomial of specified shape is equivalent to the existence of a primitive solution to a certain finite system of linear equations, or in other words to the simultaneous vanishing of an appropriate collection of determinants. Thus the space of perfect bigrids is a closed, R -rational subscheme of the space of bigrids; moreover, perfection is preserved under base change.

Proposition 2.2. *Let γ be a perfect bigrid. Then the complementary bigrid $\tilde{\gamma}$ defined by $\tilde{\gamma}(\alpha, i, j) = \gamma(1 - \alpha, n + 1 - i, m - j)$ is perfect.*

Proof. Indeed, an interpolation polynomial of index λ for γ is also an interpolation polynomial of index $m^n - \lambda$ for $\tilde{\gamma}$, and vice versa. \square

The simplest example of a perfect bigrid is the following.

Proposition 2.3. *Let γ be an arbitrary bigrid of shape m^1 . Then γ is perfect, with associated family of (univariate) interpolation polynomials given by*

$$P_j^*(x; \gamma) = \prod_{0 \leq i < j} (x \cdot \gamma(0, 1, i)) \prod_{j < i \leq m} (x \cdot \gamma(1, 1, i)). \quad (2.10)$$

Proof. Indeed, the vanishing conditions say precisely that $\gamma(0, 1, i)$ is a zero for $0 \leq i < j$ and that $\gamma(1, 1, i)$ is a zero for $j < i \leq m$. \square

This generalizes in several different ways to the multivariate case. The simplest of these is the case of “monomial” bigrids, in which $\gamma(\alpha, i, j)$ is independent of i . In other words, a monomial bigrid γ is constructed from a bigrid γ_0 of shape m^1 by $\gamma(\alpha, i, j) = \gamma_0(\alpha, 1, j)$.

Proposition 2.4. *All monomial bigrids are perfect, with associated interpolation polynomials given by*

$$P_\lambda^*(x_1, \dots, x_n; \gamma) \propto \sum_{\pi \in S_n} \prod_{1 \leq i \leq n} P_{\lambda_{\pi(i)}}^*(x_i; \gamma_0). \quad (2.11)$$

Proof. The given polynomial is clearly symmetric and homogeneous, so it remains to verify the vanishing conditions. For this, we observe that if $\mu \subset m^n$ is such that $\mu \not\subset \lambda$, then for any permutation π , $\mu_i < \lambda_{\pi(i)}$ for some $1 \leq i \leq n$. Then the i th factor of the term for π vanishes as required. The vanishing conditions for $\gamma(1, i, j)$ follow by the symmetrical argument. \square

Similarly, the Schur case extends to balanced polynomials. Fix m, n , and let γ_0 be a bigrid of shape $(m + n - 1)^1$. Then we define a Schur bigrid γ by:

$$\gamma(\alpha, i, j) = \gamma_0(\alpha, 1, j + n - i). \quad (2.12)$$

Proposition 2.5. *Schur bigrids are perfect, with associated interpolation polynomials given by the formula*

$$P_\lambda^*(x_1, \dots, x_n; \gamma) = \frac{\det(P_{\lambda_i + n - i}^*(x_j; \gamma_0))_{1 \leq i, j \leq n}}{\det(P_{n - i}^*(x_j; \gamma_0))_{1 \leq i, j \leq n}} \propto \frac{\det(P_{\lambda_i + n - i}^*(x_j; \gamma_0))_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} (x_i \cdot x_j)} \quad (2.13)$$

Proof. We first observe that the numerator vanishes when $x_j = x_i$, so the result is a polynomial; similarly, switching x_i and x_j negates both numerator and denominator, so it is a symmetric polynomial. Finally, the numerator is homogeneous of degree $(m+n-1, m+n-1, \dots, m+n-1)$, while the denominator is homogeneous of degree $(n-1, n-1, \dots, n-1)$. Thus $P_\lambda^*(; \gamma)$ as defined is a symmetric polynomial of degree m on $\mathbb{P}^1(R)^n$.

For the vanishing conditions, we observe that, at least in the generic case, the points at which we must vanish satisfy $x_i \neq x_j$ for all $i \neq j$. Thus it suffices to show that the numerator vanishes as required; this follows by the same argument as in the monomial case.

Thus Schur bigrids are generically perfect with given interpolation polynomials; the desired result is closed, so holds for all Schur bigrids. \square

Corollary 2.6. *Any bigrid of shape 1^n is perfect.*

Proof. Indeed, any bigrid of shape 1^n is a Schur bigrid. \square

There is a third important multivariate extension of the univariate case which is not simply an obvious generalization of a special case for ordinary interpolation polynomials. In the monomial and Schur cases, the polynomials have reasonable formulas, but unlike in the univariate case, do not in general have nice factorizations. It turns out that there is a large class of bigrids for which the corresponding interpolation polynomials do factor completely.

Let η be a function $\eta : \{0, 1, \dots, n\} \times \{0, 1, \dots, m-1\} \rightarrow \mathbb{P}^1(R)$, and construct a bigrid γ by

$$\gamma(0, i, j) = \eta(i, j) \tag{2.14}$$

$$\gamma(1, i, j) = \eta(i-1, j-1) \tag{2.15}$$

A bigrid constructed in this manner will be called a ‘‘Cauchy’’ bigrid.

Proposition 2.7. *Cauchy bigrids are perfect; indeed, we can construct corresponding interpolation polynomials via the product expression:*

$$P_\lambda^*(x_1, \dots, x_n; \gamma) = \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq m} x_i \cdot \eta(\lambda'_j, j-1). \tag{2.16}$$

Proof. Again, the only nonobvious fact is that $P_\lambda^*(; \gamma)$ as defined satisfies the vanishing conditions.

Choose $\mu \subset m^n$ such that $\mu \not\supset \lambda$, and consider

$$P_\lambda^*(\dots \gamma^+(0, i, \mu_i) \dots; \gamma) = \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq m} \gamma^+(0, i, \mu_i) \cdot \eta(\lambda'_j, j-1) \tag{2.17}$$

Since $\mu \not\supset \lambda$, there exists a position $1 \leq k \leq n$ such that $\mu_k < \lambda_k$. If k' is the largest such k , then we note that

$$\lambda_{k'+1} = \mu_{k'+1} \leq \mu_k < \lambda_k \leq n, \tag{2.18}$$

and thus in particular $\lambda'_l = k$ for $\mu_k < l \leq \lambda_k$. But then

$$\gamma^+(0, k, \mu_k) = \gamma(0, k, \mu_k) = \eta(k, \mu_k) = \eta(\lambda'_{\mu_k+1}, (\mu_k+1)-1), \tag{2.19}$$

so the $i = k, j = \mu_k + 1$ factor of the above product vanishes.

The other set of vanishing conditions follow by symmetry. \square

Remark 1. Compare the proof of Lemma 6.3 of [12]. Similarly, interpolation polynomials for Cauchy bigrids will be related to the Cauchy identity for interpolation theta functions, Theorem 4.18 below. In addition, when the Cauchy bigrid is also elliptic (see Section 7), the corresponding interpolation polynomials satisfy a number of symmetries, which we will use to prove corresponding symmetries for the general elliptic case.

Remark 2. This is the only case (at least for a regular (Definition 7) bigrid) for which the associated interpolation polynomials factor completely. Indeed, this holds by direct computation for shape 1^2 , and every equality required of a Cauchy bigrid is supported on some truncation (Definition 6) of shape 1^2 .

The fourth “elementary” case is that of a “delta” bigrid, for which $\gamma(0, i, j) = \gamma(1, i, j)$ whenever both sides are defined. If such a bigrid is perfect, then the corresponding interpolation polynomials must satisfy

$$P_\lambda^*(\dots \gamma_0(i, \mu_i) \dots; \gamma) \propto \delta_{\lambda\mu} \quad (2.20)$$

(here $\gamma_0(i, j) = \gamma(0, i, j)$ or $\gamma(1, i, j)$, whichever is well-defined). Indeed, aside from the fact that the extended bigrid γ^+ need not satisfy $\gamma^+(0, i, j) = \gamma^+(1, i, j)$, the first set of vanishing conditions says that this vanishes unless $\lambda \subset \mu$, while the second requires that it vanish unless $\mu \subset \lambda$. That all delta bigrids are perfect requires some additional machinery, and is shown in Proposition 2.15 below.

Definition 6. Let γ be a perfect bigrid. We define four *truncated* bigrids as follows. The bigrids γ^- , of shape m^{n-1} , and γ_- , of shape $(m-1)^n$, are simply the restrictions of γ to the appropriate domains; the bigrids $\bar{\gamma}$ and $\bar{\gamma}_-$ are then defined by complementing the restrictions of $\tilde{\gamma}$; thus

$$\bar{\gamma}(\alpha, i, j) = \gamma(\alpha, i+1, j) \quad (2.21)$$

$$\bar{\gamma}_-(\alpha, i, j) = \gamma(\alpha, i, j+1) \quad (2.22)$$

The relation of truncation to interpolation polynomials is described by the following propositions.

Proposition 2.8. *Let γ be a bigrid of shape m^n , let $\lambda \subset m^{n-1}$ be a partition, and let $P_\lambda^*(; \gamma)$ be a corresponding interpolation polynomial. Then the specialization*

$$P_\lambda^*(x_1, x_2, \dots, x_{n-1}, \gamma(0, n, 0); \gamma) \quad (2.23)$$

is an interpolation polynomial for γ^- of index λ . Similarly, if $P_{m \cdot \lambda}^(; \gamma)$ is an interpolation polynomial for γ of index $m \cdot \lambda$, then*

$$P_{m \cdot \lambda}^*(\gamma(1, 1, m), x_1, x_2, \dots, x_{n-1}; \gamma) \quad (2.24)$$

is an interpolation polynomial for $\bar{\gamma}$ of index λ .

We cannot quite conclude, however, that if γ is perfect, then so are γ^- and $\bar{\gamma}$; the constructed interpolation polynomials could be trivial even if the original polynomials were primitive.

Proposition 2.9. *Let γ be a bigrid of shape m^n , let $\lambda \subset (m-1)^n$ be a partition, and let $P_\lambda^*(; \gamma_-)$ be a corresponding interpolation polynomial for γ_- . Then*

$$\prod_{1 \leq i \leq n} x_i \cdot \gamma(1, 1, m) P_\lambda^*(x_1, \dots, x_n; \gamma_-) \quad (2.25)$$

is an interpolation polynomial of index λ for γ . Similarly, if $P_\lambda^*(; -\gamma)$ is an interpolation polynomial, then

$$\prod_{1 \leq i \leq n} x_i \cdot \gamma(0, n, 0) P_\lambda^*(x_1, \dots, x_n; -\gamma) \quad (2.26)$$

is an interpolation polynomial of index $1^n + \lambda$ for γ .

With a slight additional hypothesis, we obtain a converse:

Proposition 2.10. *Let γ be a bigrid of shape m^n , and let $\lambda \subset (m-1)^n$. If the point $\gamma(1, 1, m)$ has unit inner product with all points in the image of γ_- , then all interpolation polynomials of index λ for γ are obtained from interpolation polynomials for γ_- via the above proposition. Similarly, if the point $\gamma(0, n, 0)$ has unit inner product with all points in the image of $-\gamma$, then all interpolation polynomials of index $1^n + \lambda$ for γ are obtained via the above proposition.*

Proof. Passing to the complementary bigrid reduces the second claim to the first. Thus consider an interpolation polynomial $P_\lambda^*(; \gamma)$. The partition m is not contained in λ , and thus the polynomial

$$P_\lambda^*(\gamma(1, 1, m), x_2, \dots, x_n; \gamma) = 0. \quad (2.27)$$

It follows that $P_\lambda^*(x_1, \dots, x_n; \gamma)$ is a multiple of $x_1 \cdot \gamma(1, 1, m)$, and thus, by symmetry, that it is a multiple of

$$\prod_{1 \leq i \leq n} x_i \cdot \gamma(1, 1, m). \quad (2.28)$$

We need to show that

$$\frac{P_\lambda^*(\gamma(1, 1, m), x_2, \dots, x_n; \gamma)}{\prod_{1 \leq i \leq n} x_i \cdot \gamma(1, 1, m)} \quad (2.29)$$

satisfies the vanishing identities required of an interpolation polynomial for γ_- ; by the hypothesis, it suffices to show that $P_\lambda^*(; \gamma)$ itself satisfies the identities. But the required identities are a subset of the vanishing identities that P_λ^* already satisfies. \square

Corollary 2.11. *Let γ be a perfect grid of shape m^n . If $\gamma(1, 1, m)$ has unit inner product with the image of γ_- , then γ_- is perfect; if $\gamma(0, n, 0)$ has unit inner product with the image of $-\gamma$, then $-\gamma$ is perfect.*

Naturally, the interesting cases of perfect bigrids are those for which the interpolation polynomials are uniquely determined. This is an open condition (the complement of a closed subscheme again cut out by determinants); unfortunately, the ideal for the complement appears to be prohibitively complicated. We thus need to find a simpler set of sufficient conditions for uniqueness. As in [13, Definition 2.1], the idea is to choose a subset of the equations and add a normalization condition in order to obtain a square system of linear equations with a simple determinant.

Definition 7. A bigrid is *regular* if for all $i \geq i', j < j', \alpha, \beta \in \{0, 1\}$,

$$\gamma(\alpha, i, j) \cdot \gamma(\beta, i', j') \in R^*. \quad (2.30)$$

Note that regularity of a bigrid γ implies that the extra hypotheses of Proposition 2.10 hold for all truncations of γ ; regularity is slightly stronger in that the additional conditions

$$\gamma(1, i, j) \cdot \gamma(0, i', j-1) \in R^* \quad (2.31)$$

for $i \geq i'$ must also hold. (It is unclear whether these extra conditions are necessary for primitive interpolation polynomials to be unique and form a basis; they are needed for our proof below, however.) It follows therefore that if γ is a regular perfect bigrid, then so are γ_- and $-\gamma$.

Given a bigrid γ of shape m^n ($m > 0$), a partition $\lambda \subset m^n$, and an integer $0 \leq l \leq n$, we define a point $\gamma_l(\lambda) \in \mathbb{P}^1(R)^n$ by

$$\gamma_l(\lambda) = (\gamma([1 \leq l], 1, \lambda_1), \gamma([2 \leq l], 2, \lambda_2), \dots, \gamma([n \leq l], n, \lambda_n)), \quad (2.32)$$

where $[i \leq l]$ is 1 if $i \leq l$ and 0 otherwise. For a pair of partitions $\lambda, \mu \subset m^n$, we define $\gamma_\lambda(\mu) = \gamma_l(\mu)$ with l determined as follows. Let l_0 be the largest index such that $\mu_l \neq \lambda_l$ ($l_0 = 0$ if no such index exists), and let l_1 similarly be the largest index such that $\mu_l = m$. If $l_0 = 0$ or $\mu_{l_0} < \lambda_{l_0}$ then take $l = l_1$; otherwise, take $l = l_0$. Thus, for instance, for $m = n = 3$, $\lambda = 21$, we have:

$$\gamma_{21}(0) = (\gamma(0, 1, 0), \gamma(0, 2, 0), \gamma(0, 3, 0)) \quad (2.33)$$

$$\gamma_{21}(1) = (\gamma(0, 1, 1), \gamma(0, 2, 0), \gamma(0, 3, 0)) \quad (2.34)$$

$$\gamma_{21}(2) = (\gamma(1, 1, 2), \gamma(0, 2, 0), \gamma(0, 3, 0)) \quad (2.35)$$

$$\gamma_{21}(11) = (\gamma(0, 1, 1), \gamma(0, 2, 1), \gamma(0, 3, 0)) \quad (2.36)$$

$$\gamma_{21}(21) = (\gamma(1, 1, 2), \gamma(0, 2, 1), \gamma(0, 3, 0)) \quad (2.37)$$

$$\gamma_{21}(22) = (\gamma(1, 1, 2), \gamma(1, 2, 2), \gamma(0, 3, 0)) \quad (2.38)$$

$$\gamma_{21}(111) = (\gamma(1, 1, 1), \gamma(1, 2, 1), \gamma(1, 3, 1)) \quad (2.39)$$

$$\gamma_{21}(211) = (\gamma(1, 1, 2), \gamma(1, 2, 1), \gamma(1, 3, 1)) \quad (2.40)$$

$$\gamma_{21}(221) = (\gamma(1, 1, 2), \gamma(1, 2, 2), \gamma(1, 3, 1)) \quad (2.41)$$

$$\gamma_{21}(222) = (\gamma(1, 1, 2), \gamma(1, 2, 2), \gamma(1, 3, 2)) \quad (2.42)$$

Definition 8. Fix a ring R , positive integers m, n , a R -valued bigrid γ of shape m^n , and a partition $\lambda \subset m^n$. A primitive symmetric polynomial p of degree m and n variables is a *quasi-interpolation polynomial* of index λ and bigrid γ if for all partitions $\mu \subset m^n$, $\mu \neq \lambda$,

$$p(\gamma_\lambda(\mu)) = 0. \quad (2.43)$$

Lemma 2.12. *If the bigrid γ of shape m^n is regular, then for all $\lambda \subset m^n$, there exists a quasi-interpolation polynomial of index λ , unique up to scale. The quasi-interpolation polynomials for γ span the space of symmetric polynomials of degree m on $\mathbb{P}^1(R)^n$.*

Proof. Introduce a total ordering $<_l$ on partitions by first sorting the parts into increasing order, then lexicographically ordering the sorted partitions. We construct a corresponding filtration of Λ_n^m as follows. (That is, a

sequence of submodules $F_\mu(\lambda; \gamma)$ such that $F_\mu(\lambda; \gamma) \supset F_\nu(\lambda; \gamma)$ whenever $\mu <_l \nu$.) Recall from Lemma 2.1 the short exact sequence:

$$0 \rightarrow \Lambda_n^{m-1} \xrightarrow{f} \Lambda_n^m \xrightarrow{g} \Lambda_{n-1}^m \rightarrow 0, \quad (2.44)$$

in which the map f is defined by

$$f(p)(x_1, \dots, x_n) = p(x_1, \dots, x_n) \prod_{1 \leq i \leq n} (\gamma(0, n, 0) \cdot x_i) \quad (2.45)$$

and the map g is defined by

$$g(p)(x_1, \dots, x_{n-1}) = p(x_1, \dots, x_{n-1}, \gamma(0, n, 0)). \quad (2.46)$$

The submodules $F_\mu(\lambda; \gamma) \subset \Lambda_n^m$ are then defined as follows. First, as a base case, $F_0(\lambda; \gamma)$ is always taken to be Λ_n^m ; this in particular covers the case $m = 0$ or $n = 0$. If $\mu_n = \lambda_n = 0$, $m > 0$, then

$$F_\mu(\lambda; \gamma) = g^{-1}(F_\mu(\lambda; \gamma_-)); \quad (2.47)$$

if $\mu_n = 0$ but $\lambda_n > 0$, then

$$F_\mu(\lambda; \gamma) = g^{-1}(F_\mu(\lambda; \gamma')), \quad (2.48)$$

where $\gamma'(\alpha, i, j) := \gamma(0, i, j)$ is a delta bigrid of shape m^{n-1} . If $\mu_n > 0$, $\lambda_n > 0$, then

$$F_\mu(\lambda; \gamma) = f(F_{\mu-1^n}(\lambda - 1^n; -\gamma)). \quad (2.49)$$

Finally, if $\mu_n > 0$ but $\lambda_n = 0$, then

$$F_\mu(\lambda; \gamma) = f(F_{\mu-1^n}(0; \gamma'')), \quad (2.50)$$

where $\gamma''(\alpha, i, j) := \gamma(1, i, j - 1)$ is a delta bigrid of shape $(m - 1)^n$.

Moreover, since the short exact sequences used to construct the filtration are all split, we can construct a corresponding sequence of polynomials $p_\mu(\cdot; \lambda; \gamma)$ such that

$$F_\mu = \langle p_\nu : \mu \leq_l \nu \rangle. \quad (2.51)$$

(When $m = 0$ or $n = 0$, we take $p_0 = 1$) From the construction of this filtration, it follows easily that

$$p_\mu(\gamma_\lambda(\nu); \lambda; \gamma) = 0 \quad (2.52)$$

if $\nu <_l \mu$, while

$$p_\mu(\gamma_\lambda(\mu); \lambda; \gamma) \in R^*, \quad (2.53)$$

using the fact that γ is regular. (Note that if γ is regular, then so are the delta bigrids γ' , γ'' used above) In particular, the determinant of the matrix

$$(p_\mu(\gamma_\lambda(\nu)))_{\mu, \nu \subset m^n} \quad (2.54)$$

is a unit, so for any collection of values $f_\mu \in R$, there is a unique polynomial p such that

$$p(\gamma_\lambda(\mu)) = f_\mu. \quad (2.55)$$

Taking $f_\mu = \delta_{\lambda\mu}$ gives the desired existence and uniqueness claims for the quasi-interpolation polynomials.

For $\nu \leq_l \lambda$, $\gamma_\lambda(\nu)$ is independent of λ ; it follows that

$$P_\lambda^*(\gamma_{m^n}(\nu)) = 0 \tag{2.56}$$

when $\nu <_l \lambda$, while

$$P_\lambda^*(\gamma_{m^n}(\lambda)) \in R^*. \tag{2.57}$$

Thus the corresponding matrix has unit determinant, which immediately implies that the P_λ^* form a basis of Λ_n^m . \square

Remark. Compare the proof of Proposition 2.6 of [13].

Theorem 2.13. *Let γ be a regular, perfect bigrid of shape m^n . Then the corresponding primitive interpolation polynomials are uniquely determined up to scale, and form a basis of the space of symmetric polynomials of degree m on $\mathbb{P}^1(R)^n$. Moreover, for all λ ,*

$$P_\lambda^*(\dots \gamma^+(0, i, \lambda_i) \dots; \gamma) \tag{2.58}$$

is a primitive polynomial in the indeterminates $\gamma^+(0, i, m)$, $1 \leq i \leq n$.

Proof. We find that the vanishing conditions defining the quasi-interpolation polynomials form a subset of the conditions defining the interpolation polynomials, so each interpolation polynomial must be a unit multiple of the corresponding quasi-interpolation polynomial. We then find that

$$P_\lambda^*(\dots \gamma^+(0, i, \lambda_i) \dots; \gamma) \in R^* \tag{2.59}$$

in the special case

$$\gamma^+(0, i, m) = \gamma(1, i, m), \tag{2.60}$$

which implies primitivity in general. \square

Corollary 2.14. *If γ is a regular, perfect bigrid, then the same is true for any truncation of γ .*

Proof. The only thing remaining to prove is that if $\lambda_n = 0$ and $P_\lambda^*(; \gamma)$ is a primitive interpolation polynomial for γ , then substituting $\gamma(0, n, 0)$ for one argument gives another primitive polynomial. But the corresponding statement is true for quasi-interpolation polynomials. \square

In the sequel, we will restrict our attention to regular, perfect bigrids.

Proposition 2.15. *Delta bigrids are perfect.*

Proof. Since regular delta bigrids are Zariski-dense in the space of all delta bigrids, it suffices to consider the case of a regular delta bigrid γ . Let $P_\lambda^*(; \gamma)$ denote the corresponding quasi-interpolation polynomials; thus

$$P_\lambda^*(\dots \gamma(0, i, \mu_i) \dots; \gamma) = \delta_{\lambda\mu}. \tag{2.61}$$

Taking a ring extension as necessary, we can also choose a regular Cauchy bigrid γ' such that $\gamma'(0, i, j) = \gamma(0, i, j)$, with corresponding interpolation polynomials. But then

$$P_\lambda^*(\dots \gamma^+(0, i, \mu_i) \dots; \gamma') = 0 \quad (2.62)$$

unless $\lambda \subseteq \mu$; thus the polynomials corresponding to γ are triangular with respect to the basis $P_\lambda^*(; \gamma')$ under the inclusion ordering. In particular,

$$P_\lambda^*(; \gamma') = \sum_{\mu \supseteq \lambda} c_{\lambda\mu} P_\mu^*(; \gamma) \quad (2.63)$$

for appropriate coefficients $c_{\lambda\mu}$, and therefore

$$P_\lambda^*(; \gamma) = \sum_{\mu \supseteq \lambda} d_{\lambda\mu} P_\mu^*(; \gamma') \quad (2.64)$$

for another set of coefficients $d_{\lambda\mu}$. But then if $\mu \not\supseteq \lambda$,

$$P_\lambda^*(\dots \gamma^+(0, i, \mu_i) \dots; \gamma) = \sum_{\lambda \supseteq \nu \supseteq \mu} d_{\lambda\nu} P_\nu^*(\dots \gamma^+(0, i, \mu_i) \dots; \gamma) = 0. \quad (2.65)$$

The other set of vanishing conditions follow symmetrically; the result follows. \square

Given a pair (C, ϕ) , where C is a curve of genus 1 and ϕ is a quadratic map $C \rightarrow \mathbb{P}^1$, the line bundle $\mathcal{O}(1)$ on \mathbb{P}^1 pulls back to a line bundle on C , on which the Galois group $\mathbb{Z}/2\mathbb{Z}$ acts. Thus symmetric polynomials on P^1 pull back to sections of the corresponding line bundle on C^n , invariant under the action of BC_n . When C is a complex elliptic curve and ϕ preserves the identity, symmetric polynomials of degree m in n variables thus pull back to BC_n -symmetric theta functions of degree m . In the next section, we will construct certain “interpolation theta functions”, pull-backs of interpolation polynomials associated to “elliptic” bigrids. We will consider the case of general curves C in Section 6 below.

3 Interpolation theta functions

Definition 9. Fix integers $m, n \geq 0$, a partition $\lambda \subset m^n$, and generic points $a, b, q, t \in \mathbb{C}^*$; also choose another generic point $v \in \mathbb{C}^*$. The *interpolation theta function*

$$P_\lambda^{*(m,n)}(x_1, \dots, x_n; a, b; q, t; p) \quad (3.1)$$

is the unique function on $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^4$ with the following properties:

- (1) $P_\lambda^{*(m,n)}$ is a BC_n -symmetric theta function of degree m in the variables x_1, \dots, x_n .
- (2) For any partition $\mu \subset m^n$, $\mu \neq \lambda$, let l_0 be the largest index such that $\mu_l \neq \lambda_l$, let l_1 be the largest index such that $\mu_l = m$ (0 if none exists). If $\mu_{l_0} < \lambda_{l_0}$, then set $l = l_1$, otherwise set $l = l_0$. Then

$$P_\lambda^{*(m,n)}(bq^{m-\mu_1}, \dots, bq^{m-\mu_l}t^{l-1}, aq^{\mu_{l+1}}t^{n-l-1}, \dots, aq^{\mu_n}; a, b; q, t; p) = 0. \quad (3.2)$$

(3) $P_\lambda^{*(m,n)}$ is normalized by the specialization

$$P_\lambda^{*(m,n)}(\dots vt^{n-i} \dots; a, b, q, t; p) = C_\lambda^0(t^{n-1}av, a/v; q, t; p)C_{m-\lambda}^0(t^{n-1}bv, b/v; q, t; p) \quad (3.3)$$

Indeed, aside from the normalization, this is simply the pull-back of the definition of quasi-interpolation polynomials relative to the bigrid

$$\gamma(0, i, j) := \phi(aq^j t^{n-i}) \quad \gamma(1, i, j) := \phi(bq^{m-j} t^{i-1}), \quad (3.4)$$

where ϕ maps the elliptic curve $\mathbb{C}^*/\langle p \rangle$ to $\mathbb{P}^1(\mathbb{C})$ via a pair of BC_1 -symmetric theta functions of degree 1; i.e., for some $c, d \in \mathbb{C}^*$,

$$\phi(z) = (\theta(cz^{\pm 1}; p), \theta(dz^{\pm 1}; p)). \quad (3.5)$$

Thus by Lemma 2.12, interpolation theta functions exist and are unique, at least for generic values of a, b, q, t, v . We note in particular that the associated bigrid is regular precisely when

$$q^j t^i \notin abq^{m-1} t^{n-1} \langle p \rangle \quad 0 \leq i < n, 0 \leq j < m \quad (3.6)$$

$$(b/a)q^j t^i \notin abq^{m-1} t^{n-1} \langle p \rangle \quad |i| \leq n-1, |j| \leq m-2 \quad (3.7)$$

$$q^j t^i \notin \langle p \rangle \quad 0 \leq i < n, 1 \leq j < m \quad (3.8)$$

$$(b/a)q^j t^i \notin \langle p \rangle \quad |i| \leq n-1, |j| \leq m-1 \quad (3.9)$$

$$q^j t^i \notin (ab)^{-1} q^{1-m} t^{1-n} \langle p \rangle \quad 0 \leq i < n, 2 \leq j < m \quad (3.10)$$

$$(b/a)q^j t^i \notin (ab)^{-1} q^{1-m} t^{1-n} \langle p \rangle \quad |i| \leq n-1, |j| \leq m-2 \quad (3.11)$$

Moreover, we can also conclude that $P_\lambda^{*(m,n)}(; a, b, q, t; p)$ is a meromorphic theta function in each of a, b, q, t, v . (We will show below that it is independent of v , explaining its omission from the notation)

As one might expect from the fact that we are considering these functions at all, interpolation theta functions indeed satisfy extra vanishing conditions corresponding to the fact that the associated bigrid is perfect. The normalization can thus be justified as follows: viewed as a function of v , the left-hand side must (by extra vanishing) vanish wherever the right-hand side vanishes. By degree considerations and the theta function condition, it follows that the ratio must be independent of v .

As in [16], we first show that the interpolation theta functions satisfy a difference equation, and in the process prove independence of v .

Theorem 3.1. *Let $c, d \in \mathbb{C}^*$ be chosen so that $q^m t^{n-1} abcd = p$. Then*

$$\begin{aligned} D^{(n)}(a, b, c, d; q, t; p) P_\lambda^{*(m,n)}(; \sqrt{q}a, \sqrt{q}b; q, t; p) \\ = \prod_{1 \leq i \leq n} \theta(abq^m t^{n-i}, acq^{\lambda_i} t^{n-i}, bcq^{m-\lambda_i} t^{i-1}; p) P_\lambda^{*(m,n)}(; a, b; q, t; p) \end{aligned} \quad (3.12)$$

Proof. To begin with, we take the v normalizing the interpolation theta function in the left-hand side of the equation to be \sqrt{q} times the v normalizing the interpolation theta function on the right.

We first need to show that the left-hand side satisfies the relevant vanishing conditions. Suppose more generally that we evaluate the left-hand side at a point of the form

$$bq^{m-\mu_1}, \dots, bq^{m-\mu_l}t^{l-1}, aq^{\mu_{l+1}}t^{n-l-1}, \dots, aq^{\mu_n} \quad (3.13)$$

for an arbitrary choice of l , where μ simply satisfies the conditions

$$m \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_l, \quad (3.14)$$

$$\mu_{l+1} \geq \mu_{l+2} \geq \dots \geq \mu_n \geq 0, \quad (3.15)$$

but need not satisfy $\mu_l \geq \mu_{l+1}$. Carrying this point through the difference operator and using the BC_n symmetry of $P^{*(m,n)}$, we find that in each term, $P^{*(m,n)}(; \sqrt{q}a, \sqrt{q}b; q, t; p)$ is evaluated at a point of the same form, with a, b replaced by $q^{1/2}a, q^{1/2}b$, and μ replaced by ν satisfying

$$\mu_i \leq \nu_i \leq \mu_i + 1, \quad 1 \leq i \leq l \quad (3.16)$$

$$\mu_i - 1 \leq \nu_i \leq \mu_i, \quad l + 1 \leq i \leq n \quad (3.17)$$

(Indeed, $\nu_i = \mu_i + 1/2 - \sigma_i$ if $1 \leq i \leq l$, and $\nu_i = \mu_i - 1/2 - \sigma_i$ if $l + 1 \leq i \leq n$.) We furthermore observe that if $m \geq \nu_1$, $\nu_i < \nu_{i+1}$ for $i \neq l$, or $\nu_n < 0$, then that term of the expansion necessarily vanishes. Thus the only surviving ν satisfy

$$m \geq \nu_1 \geq \nu_2 \geq \dots \geq \nu_l, \quad (3.18)$$

$$\nu_{l+1} \geq \nu_{l+2} \geq \dots \geq \nu_n \geq 0. \quad (3.19)$$

In particular, if μ and l are chosen to make the right-hand side vanish, then all terms on the left-hand side will also vanish, as required.

It follows that

$$D^{(n)}(a, b, c, d; q, t; p)P_\lambda^{*(m,n)}(; \sqrt{q}a, \sqrt{q}b; q, t; p) \propto P_\lambda^{*(m,n)}(; a, b; q, t; p), \quad (3.20)$$

for some scale factor independent of x_1, \dots, x_n . Taking $\mu = \lambda$, and l to be the largest index such that $\lambda_l = m$, we find that the interpolation theta function on the right has a nonzero value (by Lemma 2.12), and that only one term on the left survives. The desired dependence of the scale factor on c follows immediately.

If we now take $c = v$ and set $x_i = vt^{n-i}$, again only one term survives on the left, and the difference equation follows. By symmetry, the same difference equation would hold if we had normalized the interpolation theta function on the left using $q^{-1/2}v$ instead, and thus these interpolation theta functions agree. In other words, interpolation theta functions are invariant under $v \mapsto qv$; since q is generic, they must indeed be independent of v . \square

Remark. In particular, we find that the interpolation theta functions are solutions to a ‘‘generalized eigenvalue problem’’ (ala [28]), to wit:

$$\begin{aligned} D^{(n)}(a, b, c, \frac{p}{q^m t^{n-1} abc}; q, t; p)P_\lambda^{*(m,n)}(; \sqrt{q}a, \sqrt{q}b; q, t; p) \\ = \left(\prod_{1 \leq i \leq n} \frac{\theta(acq^{\lambda_i} t^{n-i}, bcq^{m-\lambda_i} t^{i-1}; p)}{\theta(ac'q^{\lambda_i} t^{n-i}, bc'q^{m-\lambda_i} t^{i-1}; p)} \right) D^{(n)}(a, b, c', \frac{p}{q^m t^{n-1} abc'}; q, t; p)P_\lambda^{*(m,n)}(; \sqrt{q}a, \sqrt{q}b; q, t; p) \end{aligned} \quad (3.21)$$

Corollary 3.2. Fix a partition $\lambda \subset m^n$, and let $1 \leq l \leq n$ and a sequence $0 \leq \mu_n \leq \mu_{n-1} \leq \dots \leq \mu_{l+1}$ be chosen such that $\mu_{l+1} < \lambda_{l+1}$. Then

$$P_\lambda^{*(m,n)}(x_1, \dots, x_l, aq^{\mu_{l+1}}t^{n-l-1}, \dots, aq^{\mu_n}; a, b; q, t; p) = 0. \quad (3.22)$$

Similarly, if $m \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_l$ is chosen such that $\mu_l > \lambda_l$, then

$$P_\lambda^{*(m,n)}(bq^{m-\mu_1}, \dots, bq^{m-\mu_l}t^{l-1}, x_{l+1}, \dots, x_n; a, b; q, t; p) = 0. \quad (3.23)$$

Proof. By analytic continuation, it suffices to prove that if μ_1, \dots, μ_n are as above, such that either $\mu_l > \lambda_l$ or $\mu_{l+1} < \lambda_{l+1}$, then

$$P_\lambda^{*(m,n)}(bq^{m-\mu_1}, \dots, bq^{m-\mu_l}t^{l-1}, aq^{\mu_{l+1}}t^{n-l-1}, \dots, aq^{\mu_n}; a, b; q, t; p) = 0. \quad (3.24)$$

We proceed by induction on

$$\sum_{1 \leq i \leq l} (m - \mu_i) + \sum_{l+1 \leq i \leq n} \mu_i. \quad (3.25)$$

Indeed, if we evaluate both sides of the difference equation (3.12) at this point, we find by the inductive assumption that at most one term on the left survives (that for which $\sigma_i = 1$, $1 \leq i \leq n$). If $\mu_l > \lambda_l$, then setting $c = bq^{m-\mu_l}t^{l-1}$ makes this term vanish without annihilating the factor on the right; similarly, if $\mu_{l+1} < \lambda_{l+1}$, we set $c = aq^{\mu_{l+1}}t^{n-l-1}$. The result follows. \square

Corollary 3.3. We have the symmetry property

$$P_{m^n - \lambda}^{*(m,n)}(; a, b; q, t; p) = P_\lambda^{*(m,n)}(; b, a; q, t; p) \quad (3.26)$$

Proof. This follows immediately from Corollary 3.2 and the fact that the normalization has this symmetry. \square

This also gives the following commutation relation for the difference operators:

Corollary 3.4. If $cd = c'd'$, then

$$\begin{aligned} D^{(n)}(a, b, c', d'; q, t; p) D^{(n)}(q^{1/2}a, q^{1/2}b, q^{-1/2}c, q^{-1/2}d; q, t; p) \\ = D^{(n)}(a, b, c, d; q, t; p) D^{(n)}(q^{1/2}a, q^{1/2}b, q^{-1/2}c', q^{-1/2}d'; q, t; p) \end{aligned} \quad (3.27)$$

Proof. Suppose $q^m t^{n-1} abcd = p$. Applying both sides to interpolation theta functions, we find that the relation holds on the full space of BC_n -symmetric theta functions of degree m . Now, applied to a BC_n -symmetric meromorphic function f , both sides can be expressed in the form

$$\sum_{\epsilon \in \{-1, 0, 1\}^n} C_\epsilon(\dots x_i \dots; a, b, c, d, c', d') f(\dots q^{\epsilon_i} x_i \dots) \quad (3.28)$$

for suitable functions C_ϵ ; the claim is that the two resulting families of functions agree. For m sufficiently large, this follows from the action on theta functions; since each C_ϵ is a meromorphic theta function, the result follows by analytic continuation. \square

When $q^m t^n ab = pq$, the associated bigrid is a Cauchy bigrid; we thus obtain the following formula for interpolation theta functions in that case:

Proposition 3.5. *If $q^m t^n ab = pq$, then*

$$P_\lambda^{*(m,n)}(\dots x_i \dots; a, b, q, t; p) = \prod_{1 \leq i \leq n, 1 \leq j \leq m} \theta(at^{n-\lambda'_j} q^{j-1} x_i, at^{n-\lambda'_j} q^{j-1} / x_i; p), \quad (3.29)$$

We will be using this Cauchy specialization quite frequently; the point is that by using the difference equation, one can shift ab arbitrarily, but cannot change a/b . Thus the Cauchy specialization, which includes cases having all values of a/b , gives a suitable base case for inductive arguments. (It thus plays much the same role that Macdonald polynomials did in [16].) For instance:

Proposition 3.6. *Interpolation theta functions satisfy the identity*

$$P_\lambda^{*(m,n)}(\dots aq^{\lambda_i} t^{n-i} \dots; a, b, q, t; p) \quad (3.30)$$

$$= C_{m^n - \lambda}^0\left(\frac{pq}{q^m ab}; q, t; p\right) C_{m^n - \lambda}^+\left(\frac{t^{n-1}b}{q^m a}; q, t; p\right) \frac{C_\lambda^-(pq, t; q, t; p) C_\lambda^+(t^{2n-2}a^2; q, t; p)}{C_\lambda^0(t^n; q, t; p)}$$

Proof. Using the difference equation, it follows immediately that this formula holds, up to a scale factor depending only on a/b (more precisely, independent under $(a, b) \mapsto (\sqrt{q}a, \sqrt{q}b)$, but by analytic continuation, this amounts to the same thing); taking the Cauchy specialization shows that this scale factor is 1. \square

The other important base case is the delta case:

Proposition 3.7. *If $q^m t^{n-1} ab = 1$, then*

$$P_\lambda^{*(m,n)}(\dots aq^{\mu_i} t^{n-i} \dots; a, b, q, t; p)$$

$$= \delta_{\lambda\mu} C_{m^n - \lambda}^0(pqt^{n-1}; q, t; p) C_{m^n - \lambda}^+\left(\frac{1}{q^{2m} a^2}; q, t; p\right) \frac{C_\lambda^-(pq, t; q, t; p) C_\lambda^+(t^{2n-2}a^2; q, t; p)}{C_\lambda^0(t^n; q, t; p)}$$

The dependence of interpolation theta functions on m is fairly simple; we have the following identity.

Proposition 3.8. *If $\lambda \subset m^n$, then*

$$P_\lambda^{*(m+k,n)}(\dots x_i \dots; a, b, q, t; p) = \prod_{1 \leq i \leq n} \theta(bx_i, b/x_i; q; p)_k P_\lambda^{*(m,n)}(\dots x_i \dots; a, bq^k; q, t; p). \quad (3.31)$$

In particular,

$$P_0^{*(m,n)}(\dots x_i \dots; a, b, q, t; p) = \prod_{1 \leq i \leq n} \theta(bx_i, b/x_i; q; p)_m \quad (3.32)$$

In particular, the dependence on m is independent of λ , and thus the following abelian functions are well-defined:

Definition 10. The interpolation abelian function $R_\lambda^{*(n)}(; a, b, q, t; p)$ is defined by

$$R_\lambda^{*(n)}(; a, b, q, t; p) = \frac{P_\lambda^{*(m,n)}(; a, q^{-m}b; q, t; p)}{P_0^{*(m,n)}(; a, q^{-m}b; q, t; p)}, \quad (3.33)$$

for any $m \geq \lambda_1$.

Of course, the price we pay for dealing with these functions is that we cannot evaluate them at points $bq^{\mu_i}t^{n-i}$, and as a result the complementation symmetry becomes more complicated. However, the abelianness, the lack of dependence on m , and the resulting ability to have an infinite family of functions, is often an overall win, and as a result we will tend to use these functions in preference to the interpolation theta functions.

Translating the above identities to the interpolation abelian functions, we obtain:

Proposition 3.9. *If $t^{n-1}abcd = p$, then*

$$\begin{aligned} D^{(n)}(a, b, c, d; q, t; p) R_{\lambda}^{*(n)}(; \sqrt{q}a, \sqrt{q}b; q, t; p) \\ = \prod_{1 \leq i \leq n} \theta(abt^{n-i}, acq^{\lambda_i}t^{n-i}, bcq^{-\lambda_i}t^{i-1}; p) R_{\lambda}^{*(n)}(; a, b; q, t; p) \end{aligned} \quad (3.34)$$

As an abelian function in v ,

$$R_{\lambda}^{*(n)}(vt^{n-i}; a, b; q, t; p) = \Delta_{\lambda}^0(t^{n-1}a/b|t^{n-1}av, a/v; q, t; p) \quad (3.35)$$

If $t^n ab = pq$, then

$$R_{\lambda}^{*(n)}(x; a, b; q, t; p) = \prod_{1 \leq i \leq n, 1 \leq j \leq \lambda_1} \frac{\theta(at^{n-\lambda'_j}q^{j-1}x_i, at^{n-\lambda'_j}q^{j-1}/x_i; p)}{\theta(at^nq^{j-1}x_i, at^nq^{j-1}/x_i; p)} \quad (3.36)$$

Finally,

$$R_{\lambda}^{*(n)}(aq^{\lambda_i}t^{n-i}; a, b; q, t; p) = \frac{C_{\lambda}^+(t^{2n-2}a^2; q, t; p)C_{\lambda}^0(pqt^{n-1}a/b; q, t; p)}{C_{\lambda}^+(t^{n-1}a/b; q, t; p)C_{\lambda}^0(t^{n-1}ab; q, t; p)} \Delta_{\lambda}(t^{n-1}a/b|t^n; q, t; p)^{-1} \quad (3.37)$$

We also note the following symmetries that follow from symmetries of the definition of interpolation theta functions.

Proposition 3.10. *The interpolation abelian functions satisfy the following symmetry identities:*

$$R_{\lambda}^{*(n)}(\dots z_i \dots; a, b; q, t; p) = \left(\frac{q^2 t^{2n-2} a^2}{b^2}\right)^{|\lambda|} t^{-4n(\lambda)} q^{4n(\lambda')} R_{\lambda}^{*(n)}(\dots z_i \dots; 1/a, 1/b; 1/q, 1/t; p) \quad (3.38)$$

$$R_{\lambda}^{*(n)}(\dots z_i \dots; a, b; q, t; p) = R_{\lambda}^{*(n)}(\dots -z_i \dots; -a, -b; q, t; p) \quad (3.39)$$

$$R_{\lambda}^{*(n)}(\dots \sqrt{p}z_i \dots; \sqrt{p}a, \sqrt{p}b; q, t; p) = (bt^{1-n}/aq)^{|\lambda|} t^{2n(\lambda)} q^{-2n(\lambda')} R_{\lambda}^{*(n)}(\dots z_i \dots; a, b; q, t; p) \quad (3.40)$$

$$R_{\lambda}^{*(n)}(\dots \sqrt{p}z_i \dots; \sqrt{p}a, b/\sqrt{p}; q, t; p) = (pq/ab)^{|\lambda|} R_{\lambda}^{*(n)}(\dots z_i \dots; a, b; q, t; p) \quad (3.41)$$

$$R_{m^n+\lambda}^{*(n)}(\dots z_i \dots; a, b; q, t; p) = \prod_{1 \leq i \leq n} \frac{\theta(az_i^{\pm 1}; q; p)_m}{\theta(\frac{pa}{b}z_i^{\pm 1}; q; p)_m} R_{\lambda}^{*(n)}(\dots z_i \dots; aq^m, b/q^m; q, t; p) \quad (3.42)$$

$$R_{\lambda}^{*(n+k)}(\dots z_i \dots, a, at, \dots, at^{k-1}; a, b; q, t; p) = \frac{C_{\lambda}^0(t^n, pqa/bt; q, t; p)}{C_{\lambda}^0(t^{n+k}, pqt^k a/bt; q, t; p)} R_{\lambda}^{*(n)}(\dots z_i \dots; t^k a, b; q, t; p) \quad (3.43)$$

4 Binomial coefficients and hypergeometric identities

Definition 11. The *generalized binomial coefficients* are meromorphic functions of a, b, q, t defined by

$$\binom{\lambda}{\mu}_{[a,b];q,t;p} := \Delta_{\mu}\left(\frac{a}{b}|t^n, 1/b; q, t; p\right) R_{\mu}^{*(n)}(\dots \sqrt{a}q^{\lambda_i}t^{1-i} \dots; t^{1-n}\sqrt{a}, b/\sqrt{a}; q, t; p), \quad (4.1)$$

for any integer $n \geq \ell(\lambda), \ell(\mu)$.

Note that by equation (3.39) this is indeed independent of the choice of \sqrt{a} . While for many purposes the given normalization of the binomial coefficients is the nicest, it does have the significant drawback of being singular at points of the form $b = q^{-k}t^l$ for integers $k, l \geq 0$. With this in mind, we introduce a second normalization:

$$\left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{[a,b](v_1, \dots, v_k); q, t; p} := \frac{\Delta_\lambda^0(a|b, v_1, \dots, v_k; q, t; p)}{\Delta_\mu^0(a/b|1/b, v_1, \dots, v_k; q, t; p)} \binom{\lambda}{\mu}_{[a,b]; q, t; p}. \quad (4.2)$$

(We will omit the parentheses when $k = 0$.) Thus for instance for $b = 1$ we find

$$\left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{[a,1](v_1, \dots, v_k); q, t; p} = \delta_{\lambda\mu}. \quad (4.3)$$

The standard normalization satisfies particularly nice transformation laws:

$$\binom{\lambda}{\mu}_{[pa,b]; q, t; p} = \binom{\lambda}{\mu}_{[a,b]; q, t; p} \quad (4.4)$$

$$\binom{\lambda}{\mu}_{[a,pb]; q, t; p} = \binom{\lambda}{\mu}_{[a,b]; q, t; p} \quad (4.5)$$

$$\binom{\lambda}{\mu}_{[1/a, 1/b]; 1/q, 1/t; p} = \binom{\lambda}{\mu}_{[a,b]; q, t; p}. \quad (4.6)$$

We also note the following special values:

$$\binom{\lambda}{0}_{[a,b]; q, t; p} = 1 \quad (4.7)$$

$$\binom{\lambda}{\lambda}_{[a,b]; q, t; p} = \frac{C_\lambda^+(a; q, t; p) \Delta_\lambda^0(a/b|1/b; q, t; p)}{C_\lambda^+(a/b; q, t; p) \Delta_\lambda^0(a|b; q, t; p)} \quad (4.8)$$

$$\binom{m^n}{\lambda}_{[a,b]; q, t; p} = \Delta_\lambda\left(\frac{a}{b} | t^n, q^{-m}, t^{1-n} q^m a, 1/b; q, t; p\right) \quad (4.9)$$

Furthermore, we readily obtain the following symmetry, from the corresponding symmetry of interpolation functions:

$$\frac{\binom{m^n + \lambda}{m^n + \mu}_{[a,b]; q, t; p}}{\binom{m^n}{m^n}_{[a,b]; q, t; p}} = \frac{\Delta_\lambda^0(q^{2m} a | b, pqaq^m/b, pqt^{n-1} q^m, t^{1-n} q^{2m} a; q, t; p)}{\Delta_\mu^0(q^{2m} a/b | 1/b, pqaq^m, pqt^{n-1} q^m, t^{1-n} q^{2m} a/b; q, t; p)} \binom{\lambda}{\mu}_{[q^{2m} a, b]; q, t; p}. \quad (4.10)$$

As announced in [15], the main summation identity for elliptic generalized binomial coefficients is the following identity; as the proof below suggests, it can be thought of as a ‘‘bulk’’ difference equation, ala the bulk Pieri identities and branching rule of [16].

Theorem 4.1. *For any partitions $\kappa \subset \lambda$, and generic parameters $a, b, c, d, e, q, t \in \mathbb{C}^*$ such that $bcd = apq$,*

$$\binom{\lambda}{\kappa}_{[a,c]; q, t; p} = \frac{\Delta_\kappa^0(a/c | 1/c, bd, be, pqa/b; q, t; p)}{\Delta_\lambda^0(a | c, bd, be, pqa/b; q, t; p)} \sum_{\kappa \subset \mu \subset \lambda} \Delta_\mu^0(a/b | c/b, pqa, d, e; q, t; p) \binom{\lambda}{\mu}_{[a,b]; q, t; p} \binom{\mu}{\kappa}_{[a/b, c/b]; q, t; p}. \quad (4.11)$$

Proof. If we take the difference equation (3.12) for interpolation abelian functions and rewrite it in terms of binomial coefficients, we obtain an identity of the form:

$$\begin{aligned} \binom{\lambda}{\kappa}_{[a^2, ab]; q, t; p} &= \frac{\Delta_{\kappa}^0(a/b | 1/ab, ac, ad, pq^2 a^2; q, t; p)}{\Delta_{\lambda}^0(a^2 | ab, ac, ad; q, t; p)} \\ &\sum_{\mu < \lambda} \Delta_{\mu}^0(qa^2 | qab, qac, qad; q, t; p) F_{\lambda/\mu}(a; q, t; p) \binom{\mu}{\kappa}_{[a^2 q, abq]; q, t; p} \end{aligned} \quad (4.12)$$

with $abcd = p$; the coefficients $F_{\lambda/\mu}(a; q, t; p)$ are complicated, but are in principle explicit. Extending F to be 0 if $\mu \not\leq \lambda$, we can replace the range of summation with $\kappa \subset \mu \subset \lambda$ without changing the sum. Now, in the special case $b = 1/aq$ (and thus $cd = pq$), the binomial coefficient on the right-hand side becomes a delta function, and we thus obtain the identity

$$F_{\lambda/\kappa}(a; q, t; p) = \left\langle \begin{matrix} \lambda \\ \kappa \end{matrix} \right\rangle_{[a^2, 1/q]; q, t; p}. \quad (4.13)$$

Substituting this in gives the special case $b = 1/q$ of the theorem.

We next observe that the values of b for which the theorem holds is closed under multiplication. Indeed, if $b = b_1 b_2$ such that the theorem holds for $b = b_1$ and $b = b_2$, then we in particular find using the case $b = b_1$ of the theorem that

$$\begin{aligned} \binom{\lambda}{\mu}_{[a, b_1 b_2]; q, t; p} &= \frac{\Delta_{\kappa}^0(a/b_1 b_2 | 1/b_1 b_2, pqa/b_1, b_1 d, ce; q, t; p)}{\Delta_{\lambda}^0(a | b_1 b_2, pqa/b_1, b_1 d, ce; q, t; p)} \\ &\sum_{\kappa \subset \mu \subset \lambda} \Delta_{\mu}^0(a/b_1 | b_2, pqa, d, ce/b_1; q, t; p) \binom{\lambda}{\mu}_{[a, b_1]; q, t; p} \binom{\mu}{\kappa}_{[a/b_1, b_2]; q, t; p}. \end{aligned} \quad (4.14)$$

Substituting this in to the right-hand side, we find that the resulting double sum can be summed over μ using the case $b = b_2$ of the theorem, and then summed over ν using the case $b = b_1$ again.

We thus conclude that the theorem holds whenever $b = q^{-k}$ for k a positive integer. Since for a, c, d fixed, both sides are meromorphic theta functions in b with the same multiplier, the result for general b follows by analytic continuation. \square

Remark 1. This identity includes a number of known elliptic hypergeometric identities as special cases. Taking both λ and κ to consist of a single part gives the Frenkel-Turaev summation for univariate elliptic hypergeometric series [6] (the elliptic analogue of the ${}_8\phi_7$ summation due to Jackson). Taking λ to be a rectangle and $\kappa = 0$ gives an identity conjectured by Warnaar [29, Corollary 6.2] (also proved in [18]), which in our notation reads

$$\sum_{\mu \subset m^n} \Delta_{\mu}(a | t^n, q^{-m}, b_0, b_1, b_2, b_3; q, t; p) = \frac{C_{m^n}^0(pqa, pqa/b_0 b_1, pqa/b_0 b_2, pqa/b_1 b_2; q, t; p)}{C_{m^n}^0(pqa/b_0, pqa/b_1, pqa/b_2, pqa/b_0 b_1 b_2; q, t; p)}, \quad (4.15)$$

where $q^{-m} t^n b_0 b_1 b_2 b_3 = a^2 p q t$.

Remark 2. When $t = q$, the binomial coefficient is related to Schur-type interpolation functions, and can thus be expressed as a determinant; to be precise, we have:

$$\left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{[a, b]; q, q; p} = \prod_{1 \leq i < j \leq n} \frac{q^{-\mu_i} \theta(q^{\mu_i - i - \mu_j + j}, q^{\mu_i + \mu_j + 2 - i - j} a/b; p)}{q^{-\lambda_i} \theta(q^{\lambda_i - i - \lambda_j + j}, q^{\lambda_i + \lambda_j + 2 - i - j} a; p)} \det_{1 \leq i, j \leq n} \left(\left\langle \begin{matrix} \lambda_i + n - i \\ \mu_j + n - j \end{matrix} \right\rangle_{[\frac{a}{q^{2n-2}}, b]; q, q; p} \right) \quad (4.16)$$

(The given scale factors can be derived via Warnaar’s determinant identity, Lemma 5.3 of [29]; also note that setting $\lambda = \mu$ makes the determinant triangular, so trivial to evaluate.) This gives rise to Warnaar’s Schlosser-type identity (Theorem 5.1 of [29]), via the case $t = q$, $b = q^{-k}$, $\kappa = 0$ above, in the following way. When thus specialized, the binomial coefficients with lower index κ are ratios of C^0 symbols, while the remaining binomial coefficient can be expressed as a determinant. Now, for this coefficient not to vanish, we must have $\lambda_i - k \leq \mu_i \leq \lambda_i$ for each i ; if in addition $\lambda_i - \lambda_{i+1} \geq k$, the relevant matrix becomes diagonal. Since the sum is terminating for generic λ , we can analytically continue in q^{λ_i} , $1 \leq i \leq n$ (summing over $\lambda_i - \mu_i$); the desired identity results.

Corollary 4.2. *The binomial coefficient $\langle \lambda \rangle_{[a,1/q];q,t;p}^\mu$ vanishes unless $\mu \prec \lambda$, in which case*

$$\begin{aligned} \langle \lambda \rangle_{[a,1/q];q,t;p}^\mu &= \prod_{\substack{(i,j) \in \lambda \\ \lambda_i = \mu_i}} \frac{\theta(q^{\lambda_i+j-1}t^{2-\lambda'_j-i}a;p)}{\theta(q^{\mu_i-j}t^{\mu'_j-i}t;p)} \prod_{\substack{(i,j) \in \lambda \\ \lambda_i \neq \mu_i}} \frac{\theta(q^{\lambda_i-j}t^{\lambda'_j-i}pq;p)}{\theta(q^{\mu_i+j-1}t^{2-\mu'_j-i}apq^2/t;p)} \\ &\quad \prod_{\substack{(i,j) \in \mu \\ \lambda_i = \mu_i}} \frac{\theta(q^{\lambda_i-j}t^{\lambda'_j-i}t;p)}{\theta(q^{\mu_i+j-1}t^{2-\mu'_j-i}aq;p)} \prod_{\substack{(i,j) \in \mu \\ \lambda_i \neq \mu_i}} \frac{\theta(q^{\lambda_i+j-1}t^{2-\lambda'_j-i}pqa/t;p)}{\theta(q^{\mu_i-j}t^{\mu'_j-i}pq;p)} \end{aligned} \quad (4.17)$$

Proof. The proof of Theorem 4.1 gives a formula for the binomial coefficient, which in particular implies that it vanishes when required. Simplifying this as in the proof of Theorem 4.12 of [16] gives the desired result. \square

Remark. Similarly, for any integer $k \geq 0$, $\langle \lambda \rangle_{[a,q^{-k}];q,t;p}^\mu$ vanishes unless $\mu \prec_k \lambda$.

When $c = 1$ in the bulk difference equation (4.11), the left-hand side becomes a delta function, and we thus conclude:

Corollary 4.3. *Elliptic generalized binomial coefficients satisfy the inversion identity*

$$\sum_{\kappa \subset \mu \subset \lambda} \binom{\lambda}{\mu}_{[a,b];q,t;p} \binom{\mu}{\kappa}_{[a/b,1/b];q,t;p} = \delta_{\lambda\kappa} \quad (4.18)$$

From the form of Cauchy interpolation polynomials, the following symmetry of interpolation theta functions is essentially automatic:

$$P_\mu^{*(m,n)}(\lambda; a, pq/q^m t^n a; q, t; p) = P_{n^m - \mu'}^{*(n,m)}(n^m - \mu'; \sqrt{p}/a, \sqrt{p}t^{n-1}q^m a; 1/t, 1/q; p); \quad (4.19)$$

note in particular that the roles of argument and index are interchanged via this symmetry. The special form of the bigrid in our case gives rise to another symmetry as well, which follows from a computation along the lines of Lemma 2.1 of [16]:

$$\begin{aligned} P_\mu^{*(m,n)}(\lambda; a, pq/q^m t^n a; q, t; p) &= \frac{C_{m^n}^0(t^n; q, t; p)}{C_{m^n}^0(q^{-m}; q, t; p)} \frac{C_\lambda^0(q^m t^{2n-1} a^2, (pq/t)t^n; q, t; p)}{C_\lambda^0(t^{n-1} a^2, (pq/t)q^{-m}; q, t; p)} \frac{C_\mu^0(t^{n-1} a^2, q^{-m}; q, t; p)}{C_\mu^0(q^m t^{2n-1} a^2, t^n; q, t; p)} \\ &\quad P_{\mu'}^{*(n,m)}(\lambda'; t^{n-1/2} q^{m-1/2} a, pq^{3/2} t^{1/2}/a; 1/t, 1/q; p) \end{aligned} \quad (4.20)$$

We thus obtain an action of the group Z_2^2 on Cauchy interpolation theta functions; this symmetry, it turns out, actually extends to all interpolation theta functions. In terms of binomial coefficients, we obtain the following two symmetries: “duality” and “complementation”.

Corollary 4.4. (*Duality*)

$$\binom{\lambda}{\mu}_{[a,b];q,t;p} = \binom{\lambda'}{\mu'}_{[aqt,b];1/t,1/q;p} \quad (4.21)$$

Proof. When $b = pq/t$, the relevant interpolation functions are of Cauchy type. We thus find that

$$\binom{\lambda}{\mu}_{[a,pq/t];q,t;p} = F_\lambda(a; q, t; p) G_\mu(a; q, t; p) \binom{\lambda'}{\mu'}_{[aqt,pq/t];1/t,1/q;p} \quad (4.22)$$

for appropriate functions F_λ , G_μ , with $G_0 = 1$. Setting $\mu = 0$ shows that $F_\lambda = 1$; setting $\mu = \lambda$ then shows that $G_\mu = 1$. (These could, of course, also be computed directly.)

The bulk difference equation (4.11) is preserved by this symmetry; it follows that the set of b for which this duality holds is closed under multiplication, and thus the result follows by analytic continuation. \square

Corollary 4.5. *The binomial coefficient $\langle \mu \rangle_{[a,t];q,t;p}^\lambda$ vanishes unless $\mu \prec' \lambda$, in which case*

$$\begin{aligned} \langle \mu \rangle_{[a,t];q,t;p}^\lambda &= \prod_{\substack{(i,j) \in \lambda \\ \lambda'_j = \mu'_j}} \frac{\theta(q^{\lambda_i+j-1} t^{2-\lambda'_j-i} a; p)}{\theta(q^{\mu_i-j} t^{\mu'_j-i} pq; p)} \prod_{\substack{(i,j) \in \lambda \\ \lambda'_j \neq \mu'_j}} \frac{\theta(q^{\lambda_i-j} t^{\lambda'_j-i} t; p)}{\theta(q^{\mu_i+j-1} t^{2-\mu'_j-i} apq/t^2; p)} \\ &\quad \prod_{\substack{(i,j) \in \mu \\ \lambda'_j = \mu'_j}} \frac{\theta(q^{\lambda_i-j} t^{\lambda'_j-i} pq; p)}{\theta(q^{\mu_i+j-1} t^{2-\mu'_j-i} a/t; p)} \prod_{\substack{(i,j) \in \mu \\ \lambda'_j \neq \mu'_j}} \frac{\theta(q^{\lambda_i+j-1} t^{2-\lambda'_j-i} pqa/t; p)}{\theta(q^{\mu_i-j} t^{\mu'_j-i} t; p)} \end{aligned} \quad (4.23)$$

Remark. Similarly, $\langle \mu \rangle_{[a,t^k];q,t;p}^\lambda$ vanishes unless $\mu \prec'_k \lambda$.

Corollary 4.6. *If $\lambda_1, \mu_1 \leq m$, the binomial coefficients satisfy the identity*

$$\frac{\binom{m^n \cdot \lambda}{m^n \cdot \mu}_{[a,b];q,t;p}}{\binom{m^n}{m^n}_{[a,b];q,t;p}} = \frac{\Delta_\lambda^0(\frac{a}{t^{2n}} | b, \frac{pqa}{t^n b}, \frac{pq}{q^m t^{n+1}}, \frac{q^m a}{t^{2n-1}}; q, t; p)}{\Delta_\mu^0(\frac{a}{t^{2n} b} | \frac{1}{b}, \frac{pqa}{t^n}, \frac{pq}{q^m t^{n+1}}, \frac{q^m a}{t^{2n-1} b}; q, t; p)} \binom{\lambda}{\mu}_{[t^{-2n} a, b];q,t;p}. \quad (4.24)$$

Similarly,

Corollary 4.7. (*Complementation*)

$$\frac{\binom{\lambda}{\mu}_{[a,b];q,t;p}}{\Delta_{m^n}(a|t^n, q^{-m}, b, q^m t^{1-n} a/b; q, t; p)} = \frac{\Delta_\mu(a/b|t^n, q^{-m}, 1/b, q^m t^{1-n} a; q, t; p)}{\Delta_\lambda(a|t^n, q^{-m}, b, q^m t^{1-n} a/b; q, t; p)} \binom{m^n - \mu}{m^n - \lambda}_{[q^{-2m} t^{2n-2} b/a, b];q,t;p}. \quad (4.25)$$

If $b = q^{-k}$ or $b = t^k$ there is a further symmetry:

Corollary 4.8. *If $\ell(\lambda) \leq n$,*

$$\left\langle \begin{matrix} k^n + \lambda \\ \mu \end{matrix} \right\rangle_{[a, q^{-k}];q,t;p} = \frac{C_{k^n}^0(q^{-k}; q, t; p)}{C_{k^n}^0(pqaq^k; q, t; p)} \frac{\Delta_\mu(q^k a | t^n, t^{1-n} q^k a; q, t; p)}{\Delta_\lambda(q^{2k} a | t^n, t^{1-n} q^k a; q, t; p)} \left\langle \mu \right\rangle_{[q^k a, q^{-k}];q,t;p}. \quad (4.26)$$

If $\lambda_1 \leq m$,

$$\left\langle \begin{matrix} m^k \cdot \lambda \\ \mu \end{matrix} \right\rangle_{[a, t^k];q,t;p} = \frac{C_{m^k}^0(t^k; q, t; p)}{C_{m^k}^0(pqat^{-k}; q, t; p)} \frac{\Delta_\mu(t^{-k} a | q^{-m}, q^m t^{1-k} a; q, t; p)}{\Delta_\lambda(t^{-2k} a | q^{-m}, q^m t^{1-k} a; q, t; p)} \left\langle \mu \right\rangle_{[t^{-k} a, t^k];q,t;p}. \quad (4.27)$$

Proof. If we express the binomial coefficient on the left of the first equation in terms of interpolation theta functions, we find that complementing μ gives an interpolation theta function that is still evaluated at a partition, namely $m^n - \lambda$. Re-expressing the result as a binomial coefficient and applying complementation symmetry gives the desired result. The second equation follows by duality. \square

Since the bulk difference equation (4.11) can be viewed as a multivariate Jackson summation indexed over a skew Young diagram, we also expect there to be a corresponding analogue of the Bailey transformation.

Theorem 4.9. *The sum*

$$\frac{\Delta_\lambda^0(a|b, apq/bf; q, t; p)}{\Delta_\kappa^0(a/c|b/c, apq/bd; q, t; p)} \sum_{\kappa \subset \mu \subset \lambda} \frac{\Delta_\mu^0(a/b|c/b, f, g; q, t; p)}{\Delta_\mu^0(a/b|1/b, d, e; q, t; p)} \binom{\lambda}{\mu}_{[a, b]; q, t; p} \binom{\mu}{\kappa}_{[a/b, c/b]; q, t; p} \quad (4.28)$$

is symmetric in b and b' , where $bb'de = capq$, $bb'fg = apq$.

Proof. Summing

$$\frac{\Delta_\lambda^0(a|b', apq/b'f; q, t; p)}{\Delta_\kappa^0(a/c|b/c, apq/bd; q, t; p)} \sum_{\mu, \nu} \frac{\binom{\lambda}{\nu}_{[a, b']; q, t; p} \binom{\nu}{\mu}_{[a/b', b/b']; q, t; p} \binom{\mu}{\kappa}_{[a/b, c/b]; q, t; p}}{\Delta_\nu^0(a/b'|1/b', apq/b, apq/b'f, apq/b'g; q, t; p) \Delta_\mu^0(a/b|b'/b, pqa/c, d, e; q, t; p)} \quad (4.29)$$

over ν gives (4.28); summing over μ gives the same sum with b replaced by b' . \square

Remark 1. The bulk difference equation (4.11) is the special case $c = b$ above.

Remark 2. The rectangular case $\lambda = m^n$, $\mu = 0$ gives Warnaar's conjectured multivariate elliptic Bailey transformation, Conjecture 6.1 of [29], as a corollary. One can also obtain a transformation of Schlosser-type sums by setting $t = q$, $b = q^{-N_1}$, $b' = q^{-N_2}$, and $\kappa = 0$.

This induces a symmetry under the Weyl group D_4 .

Theorem 4.10. *Define a function $\Omega_{\lambda/\kappa}(a, b; v_0, v_1, v_2, v_3; q, t; p)$ by*

$$\begin{aligned} & \Omega_{\lambda/\kappa}(a, b; v_0, v_1, v_2, v_3; q, t; p) \\ & := \sum_{\kappa \subset \mu \subset \lambda} \frac{C_{\lambda/\mu}^0(pqa/v_0, pqa/v_1, pqa/v_2, pqa/v_3; q, t; p) \langle \lambda \rangle}{C_{\mu/\kappa}^0(v_0/c, v_1/c, v_2/c, v_3/c; q, t; p)^{-1} \langle \mu \rangle_{[pqa^2, pqa c]; q, t; p} \langle \kappa \rangle_{[a/c, b/c]; q, t; p}}, \end{aligned} \quad (4.30)$$

where $c = \sqrt{bv_0v_1v_2v_3/apq}$. Then $\Omega_{\lambda/\kappa}$ is invariant under the natural action of D_4 on (v_0, v_1, v_2, v_3) ; in particular,

$$\Omega_{\lambda/\kappa}(a, b; v_0, v_1, v_2, v_3; q, t; p) = \Omega_{\lambda/\kappa}(a, b; v_0, v_1, 1/v_2, 1/v_3; q, t; p). \quad (4.31)$$

Proof. Indeed, the symmetry

$$\Omega_{\lambda/\kappa}(a, b; v_0, v_1, v_2, v_3; q, t; p) = \Omega_{\lambda/\kappa}(a, b; v_0, v_1, 1/v_2, 1/v_3; q, t; p) \quad (4.32)$$

is simply Theorem 4.9, up to a change of variables; since $\Omega_{\lambda/\kappa}$ has a manifest S_4 symmetry, the D_4 symmetry follows immediately. \square

From the center of D_4 , we obtain the following transformation, generalizing the commutation relation of Corollary 3.4 (which corresponds to the case $b = b' = 1/q$ when evaluated at a partition).

Corollary 4.11. *If $bcd = b'c'd'$, then*

$$\frac{\Delta_\lambda^0(a|b, c, d; q, t; p)}{\Delta_\kappa^0\left(\frac{a}{bb'} \middle| \frac{pq}{b'}, \frac{pq}{bc}, \frac{pq}{bd}; q, t; p\right)} \sum_\mu \frac{\binom{\lambda}{[\mu]_{[a,b];q,t;p}} \binom{\mu}{[\mu]_{[a/b,b'];q,t;p}}}{\Delta_\mu^0\left(a/b \middle| \frac{pq}{bb'}, \frac{pq}{c'}, \frac{pq}{d'}, 1/b, c, d; q, t; p\right)} \quad (4.33)$$

is invariant under $(b, c, d, b', c', d') \mapsto (b', c', d', b, c, d)$.

The case $b = q^{-1}$ gives rise to the following difference equation:

Theorem 4.12. *Define a difference operator $D_q^{+(n)}(u_0:u_1:u_2, u_3, u_4; t; p)$ by*

$$\begin{aligned} & (D_q^{+(n)}(u_0:u_1:u_2, u_3, u_4; t, p)f)(\dots z_i \dots) \\ & := \prod_{1 \leq i \leq n} \frac{\theta(pqt^{n-i}u_1/u_0; p)}{\prod_{2 \leq r \leq 5} \theta(u_r t^{n-i}u_1; p)} \\ & \quad \prod_{1 \leq i \leq n} (1 + R_{z_i}) \prod_{1 \leq i \leq n} \frac{\prod_{1 \leq r \leq 5} \theta(u_r z_i; p)}{\theta(pqz_i/u_0; p)\theta(z_i^2; p)} \prod_{1 \leq i < j \leq n} \frac{\theta(tz_i z_j; p)}{\theta(z_i z_j; p)} f(\dots q^{1/2} z_i \dots), \end{aligned} \quad (4.34)$$

where $u_5 = p^2 q/t^{n-1} u_0 u_1 u_2 u_3 u_4$, and R_{z_i} acts on functions by replacing z_i by its reciprocal. Then

$$D_q^{+(n)}(u_0:u_1:u_2, u_3, u_4; t; p) R_\lambda^{*(n)}(; q^{1/2}u_1, q^{-1/2}u_0; q, t; p) = \sum_{\lambda \subset \mu \subset \lambda + 1^n} c_{\lambda\mu} R_\mu^{*(n)}(; u_1, u_0; q, t; p) \quad (4.35)$$

where

$$c_{\lambda\mu} = \frac{\Delta_\mu(t^{n-1}u_1/u_0 | t^n, \frac{pq}{u_0 u_2}, \frac{pq}{u_0 u_3}, \frac{pq}{u_0 u_4}, \frac{pq}{u_0 u_5}; q, t; p)}{\Delta_\lambda(qt^{n-1}u_1/u_0 | t^n, \frac{pq}{u_0 u_2}, \frac{pq}{u_0 u_3}, \frac{pq}{u_0 u_4}, \frac{pq}{u_0 u_5}; q, t; p)} \left\langle \mu \right\rangle_{[t^{n-1}u_1/u_0, 1/q]; q, t; p} \left\langle \lambda \right\rangle \quad (4.36)$$

One can obtain other identities from our main identities (4.11) and Theorem (4.9) by first applying some combination of duality and complementation symmetry, specializing the result so that the sum terminates regardless of λ and/or κ , then analytically continuing.

For instance:

Theorem 4.13. *The interpolation theta functions satisfy the following connection coefficient identity:*

$$\begin{aligned} & [P_\mu^{*(m,n)}(; a, b'; q, t; p)] P_\lambda^{*(m,n)}(; a, b; q, t; p) \\ & = \frac{C_{m^n}^0(t^{n-1}ab, \frac{b}{a}; q, t; p)}{C_{m^n}^0(t^{n-1}ab', \frac{b'}{a}; q, t; p)} \frac{\Delta_\mu\left(\frac{t^{n-1}a}{q^m b'} \middle| \frac{b}{b'}, t^n, q^{-m}, \frac{pq}{q^m b b'}; q, t; p\right)}{\Delta_\lambda\left(\frac{t^{n-1}a}{q^m b} \middle| \frac{b'}{b}, t^n, q^{-m}, \frac{pq}{q^m b b'}; q, t; p\right)} \binom{\mu}{[\frac{t^{n-1}a}{q^m b'}, \frac{b}{b'}]; q, t; p} \end{aligned} \quad (4.37)$$

Proof. If we set $e = q^{-m}$ in the bulk difference equation (4.11), the resulting sum will vanish for $\mu \not\subset m^n$, regardless of λ , and we may thus replace the range of summation with $\mu \subset m^n$. If we express the binomial coefficients with upper index λ in terms of interpolation theta functions, we find that the remaining factors involving λ are the same on both sides; dividing by those factors and analytically continuing gives the above result. \square

Applying Corollary 3.3 gives the following.

Corollary 4.14. *We have the following connection coefficient identity:*

$$[P_\mu^{*(m,n)}(; a', b, q, t, p)]P_\lambda^{*(m,n)}(; a, b, q, t, p) = \left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{\left[\frac{t^{n-1}a}{q^m b}, \frac{a}{a'} \right] (t^{n-1}aa'); q, t, p} \quad (4.38)$$

Similarly,

$$[R_\mu^{*(n)}(; a', b, q, t, p)]R_\lambda^{*(n)}(; a, b, q, t, p) = \left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{\left[\frac{t^{n-1}a}{b}, \frac{a}{a'} \right] (t^{n-1}aa'); q, t, p} \quad (4.39)$$

Combining the two identities for interpolation theta functions gives the following result.

Theorem 4.15. *General connection coefficients for interpolation theta functions are given by the sum*

$$\begin{aligned} [P_\kappa^{*(m,n)}(; a', b'; q, t, p)]P_\lambda^{*(m,n)}(; a, b, q, t, p) &= \frac{C_{m^n}^0(t^{n-1}a'b, \frac{b}{a'}; q, t, p)}{C_{m^n}^0(t^{n-1}a'b', \frac{b'}{a'}; q, t, p)} \frac{\Delta_\kappa(\frac{t^{n-1}a'}{q^m b'} | t^n, q^{-m}, \frac{pq}{q^m b b'}; q, t, p)}{\Delta_\lambda^0(\frac{t^{n-1}a}{q^m b} | \frac{pq}{q^m a' b}; q, t, p)} \\ &= \sum_{\mu \subset \lambda, \kappa} \frac{\left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{\left[\frac{t^{n-1}a}{q^m b}, \frac{a}{a'} \right] ; q, t, p} \left\langle \begin{matrix} \kappa \\ \mu \end{matrix} \right\rangle_{\left[\frac{t^{n-1}a'}{q^m b'}, \frac{b'}{b'} \right] ; q, t, p}}{\Delta_\mu(\frac{t^{n-1}a'}{q^m b} | t^n, q^{-m}, t^{n-1}aa', \frac{pq}{q^m b b'}; q, t, p)} \end{aligned}$$

Remark. These connection coefficients generalize Rosengren's construction of elliptic 6-j symbols in [17].

Using the connection coefficient formula (4.39), we can extend the special branching rule (3.43) to a general branching rule.

Theorem 4.16. *Interpolation functions satisfy the following "bulk" branching rule.*

$$R_\lambda^{*(n+k)}(\dots z_i \dots, v, vt, \dots vt^{k-1}; a, b, q, t, p) = \sum_{\kappa} c_{\lambda\kappa} R_\kappa^{*(n)}(\dots z_i \dots; a, b, q, t, p), \quad (4.40)$$

where

$$c_{\lambda\kappa} = \left\langle \begin{matrix} \lambda \\ \kappa \end{matrix} \right\rangle_{\left[t^{n+k-1}a/b, t^k \right] (t^{n+k-1}av, t^n a/v, pqa/tb); q, t, p}. \quad (4.41)$$

Proof. Expand the left-hand side in interpolation functions with parameters (v, b) , rewrite those as n -variable interpolation functions, and convert the parameters back. This gives the coefficients of the branching rule as a sum over partitions, which can be summed via the bulk difference equation (4.11). \square

Similarly:

Theorem 4.17. *The interpolation functions satisfy the following generalized Pieri identity:*

$$\prod_{1 \leq i \leq n} \frac{\theta(vz_i^{\pm 1}; q, p)_m}{\theta((pq/b)z_i^{\pm 1}; q, p)_m} R_\lambda^{*(n)}(\dots z_i \dots; a, q^{-m}b, q, t, p) = \sum_{\kappa} c_{\kappa\lambda} R_\kappa^{*(n)}(\dots z_i \dots; a, b, q, t, p), \quad (4.42)$$

where

$$c_{\kappa\lambda} = \Delta_{m^n}^0(t^{n-1}v/b | t^{n-1}va, v/a; q, t, p) \frac{\Delta_\kappa(t^{n-1}a/b | t^n, pq/vb, q^m v/b; q, t, p)}{\Delta_\lambda(q^m t^{n-1}a/b | t^n, pq/vb, q^m v/b; q, t, p)} \left\langle \begin{matrix} \kappa \\ \lambda \end{matrix} \right\rangle_{\left[t^{n-1}a/b, q^{-m} \right]; q, t, p}. \quad (4.43)$$

If we expand a Cauchy-type interpolation theta function in other interpolation theta functions, the result can be analytically continued again; we obtain the following result.

Theorem 4.18. *Interpolation theta functions satisfy the following ‘‘Cauchy’’ identity.*

$$\prod_{1 \leq i \leq n, 1 \leq j \leq m} \theta(y_j x_i, y_j / x_i; p) = \sum_{\mu \subset m^n} \frac{\Delta_\mu(\frac{t^{n-1}a}{q^m b} | t^n, q^{-m}; q, t; p)}{C_{m^n}^0(t^{n-1}ab, b/a; q, t; p)} P_\mu^{*(m,n)}(\dots x_i \dots; a, b; q, t; p) \quad (4.44)$$

$$P_{n^m - \mu}^{*(n,m)}(\dots \sqrt{p}/y_i \dots; \sqrt{p}/a, \sqrt{p}/b; 1/t, 1/q; p)$$

5 Biorthogonal functions

Definition 12. Let $t_0, t_1, t_2, t_3, u_0, u_1, q, t$ be parameters such that $t^{2n-2}t_0t_1t_2t_3u_0u_1 = pq$. Then define

$$\tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p) := \sum_{\mu \subset \lambda} \frac{\binom{\lambda}{\mu}_{[\frac{1}{u_0 u_1}, \frac{1}{t^{n-1} t_0 u_1}]; q, t; p} R_\mu^{*(n)}(; t_0, u_0; q, t; p)}{\Delta_\mu^0(\frac{t^{n-1}t_0}{u_0} | t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, t^{n-1}t_0u_1; q, t; p)}. \quad (5.1)$$

Remark. This is an analogue of the ‘‘binomial formula’’ of [12].

The specific form of the above expansion is effectively determined by the requirement that $\tilde{R}_\lambda^{(n)}$ be symmetrical in t_0 through t_3 ; more precisely:

Theorem 5.1. *The function $\tilde{R}_\lambda^{(n)}$ satisfies the symmetry*

$$\tilde{R}_\lambda^{(n)}(; t_1:t_0, t_2, t_3; u_0, u_1; q, t; p) = \frac{\tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p)}{\tilde{R}_\lambda^{(n)}(\dots t^{n-i}t_1 \dots; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p)}, \quad (5.2)$$

where

$$\tilde{R}_\lambda^{(n)}(\dots t^{n-i}t_1 \dots; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p) = \Delta_\lambda^0(1/u_0u_1 | t^{n-1}t_1t_2, t^{n-1}t_1t_3, t^{1-n}/t_1u_1, pqt^{n-1}t_0/u_0; q, t; p). \quad (5.3)$$

Proof. If we expand

$$\tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p) \quad (5.4)$$

in terms of $R_\lambda^{*(n)}(; t_1, u_0; q, t; p)$ using the connection coefficient formula (4.39), the result simplifies via the bulk difference equation (4.11) to give the desired identity. \square

Remark. Alternatively, one can apply the same steps used above to derive the connection coefficient identity, but starting with Theorem 4.9 instead of (4.11). The resulting computation is somewhat more complicated, but does have the merit of *deriving* the expansion above.

In particular, we have:

$$\tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, \frac{1}{t^{n-1}t_1}; q, t; p) = \frac{R_\lambda^{*(n)}(; t_1, u_0; q, t; p)}{\Delta_\lambda^0(t^{n-1}t_1/u_0 | t^{n-1}t_0t_1, t_1/t_0; q, t; p)}. \quad (5.5)$$

In addition, from the difference equation (3.34) for interpolation abelian functions, we directly obtain the following difference equation for $\tilde{R}_\lambda^{(n)}$.

Lemma 5.2. *The function $\tilde{R}_\lambda^{(n)}$ satisfies the difference equation*

$$\begin{aligned} D^{(n)}(u_0, t_0, t_1, t^{1-n}p/u_0t_0t_1; q, t; p)\tilde{R}_\lambda^{(n)}(; q^{1/2}t_0; q^{1/2}t_1, q^{-1/2}t_2, q^{-1/2}t_3; q^{1/2}u_0, q^{-1/2}u_1; q, t; p) \\ = \prod_{1 \leq i \leq n} \theta(t^{n-i}u_0t_0, t^{n-i}u_0t_1, t^{n-i}t_0t_1; p)\tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p) \end{aligned} \quad (5.6)$$

Theorem 5.3. *The functions $\tilde{R}_\lambda^{(n)}$ agree with the biorthogonal functions of [15]. To be precise, in the notation of that paper,*

$$\tilde{\mathcal{R}}_{\lambda, \mu}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) = \tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; p, t; q)\tilde{R}_\mu^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p). \quad (5.7)$$

Proof. We need to show

$$\tilde{\mathcal{R}}_{0, \lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) = \tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p). \quad (5.8)$$

This equation is preserved by the action of the difference operators

$$D^{(n)}(u_0, t_r, t_s, t^{1-n}p/u_0t_r t_s; q, t; p), \quad 0 \leq r < s \leq 3; \quad (5.9)$$

in particular, both sides satisfy the same generalized eigenvalue equation with respect to the compositions

$$D^{(n)}(u_0, t_0, t_1; q, t; p)D^{(n)}(q^{1/2}u_0, q^{-1/2}t_2, q^{-1/2}t_3; q, t; p), \quad (5.10)$$

$$D^{(n)}(u_0, t_0, t_2; q, t; p)D^{(n)}(q^{1/2}u_0, q^{-1/2}t_1, q^{-1/2}t_3; q, t; p), \dots \quad (5.11)$$

and thus agree up to scalar multiples. Since both sides evaluate to 1 at $\dots t_0 t^{n-i} \dots$, they agree everywhere. \square

Remark. Similarly, one can show using Theorem 4.12 that the difference operator defined there acts as a raising operator on the biorthogonal functions, just as in [15].

In [15], it was shown that these functions are biorthogonal with respect to an appropriate contour integral, and a number of other properties were given. A few remaining properties were outside the scope of that paper; we are now in a position to prove these. Most striking of these is evaluation symmetry, generalizing the analogous property of Macdonald [11] and Koornwinder [9, 5, 21] polynomials (see also [16] for a proof for Koornwinder polynomials along the present lines).

Theorem 5.4. *For otherwise generic parameters satisfying $t^{2n-2}t_0t_1t_2t_3u_0u_1 = pq$,*

$$\tilde{R}_\lambda^{(n)}(\dots t_0 t^{n-i} q^{\kappa_i} \dots; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p) = \tilde{R}_{\kappa}^{(n)}(\dots \hat{t}_0 t^{n-i} q^{\lambda_i} \dots; \hat{t}_0:\hat{t}_1, \hat{t}_2, \hat{t}_3; \hat{u}_0, \hat{u}_1; q, t; p), \quad (5.12)$$

where

$$\hat{t}_0 = \sqrt{t_0 t_1 t_2 t_3 / pq} \quad \hat{t}_0 \hat{t}_1 = t_0 t_1 \quad \hat{t}_0 \hat{t}_2 = t_0 t_2 \quad \hat{t}_0 \hat{t}_3 = t_0 t_3 \quad \frac{\hat{u}_0}{\hat{t}_0} = \frac{u_0}{t_0} \quad \frac{\hat{u}_1}{\hat{t}_0} = \frac{u_1}{t_0}. \quad (5.13)$$

Proof. Upon specializing the variables as required, the result can be expressed as a sum over binomial coefficients; changing from the original parameters to the ‘‘hatted’’ parameters interchanges the binomial coefficients. \square

Remark 1. Aside from some simple factors, this sum over binomial coefficients is the analytic continuation of the “multivariate 6-j symbol” of Theorem 4.15.

Remark 2. When the biorthogonal function is specialized to an interpolation function, we obtain the identity

$$\frac{R_\mu^{*(n)}(\dots (v/a)t^{n-i}q^{\lambda_i} \dots; a, b/a; q, t; p)}{R_\mu^{*(n)}(\dots (v/a)t^{n-i} \dots; a, b/a; q, t; p)} = \frac{R_\lambda^{*(n)}(\dots (v/a')t^{n-i}q^{\mu_i} \dots; a', b/a'; q, t; p)}{R_\lambda^{*(n)}(\dots (v/a')t^{n-i} \dots; a', b/a'; q, t; p)}, \quad (5.14)$$

where

$$a' = \sqrt{t^{n-1}bv/a}. \quad (5.15)$$

If $v = q^{-m}t^{1-n}$ here, we recover the complementation symmetry of binomial coefficients; by analytic continuation, the two results are equivalent.

Remark 3. In the univariate case, this was proved in Section 9 of [27].

Using inversion of binomial coefficients, one can expand interpolation functions in terms of biorthogonal functions.

Theorem 5.5. *Biorthogonal functions satisfy the following “inverse binomial formula”.*

$$[\tilde{R}_\mu^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p)]R_\lambda^{*(n)}(; t_0, u_0; q, t; p) = \frac{\binom{\lambda}{\mu}_{[t^{n-1}t_0/u_0, t^{n-1}t_0u_1]; q, t; p}}{\Delta_\lambda^0(\frac{t^{n-1}t_0}{u_0} | \frac{pq}{u_0t_1}, \frac{pq}{u_0t_2}, \frac{pq}{u_0t_3}, \frac{pq}{u_0u_1}; q, t; p)} \quad (5.16)$$

Combined with the binomial formula (5.1), we obtain connection coefficient formulas, analogous to connection coefficients for Askey-Wilson polynomials [2].

Theorem 5.6. *If $t^{2n-2}t_0t_1t_2t_3u_0u_1 = pq$ and $t'_1t'_2t'_3u'_1 = t_1t_2t_3u_1$, then*

$$\begin{aligned} & [\tilde{R}_\kappa^{(n)}(; t_0:t'_1, t'_2, t'_3; u_0, u'_1; q, t; p)]\tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p) \\ &= \sum_{\kappa \subset \mu \subset \lambda} \frac{\Delta_\mu^0(\frac{t^{n-1}t_0}{u_0} | t^{n-1}t_0t'_1, t^{n-1}t_0t'_2, t^{n-1}t_0t'_3, t^{n-1}t_0u'_1; q, t; p)}{\Delta_\mu^0(\frac{t^{n-1}t_0}{u_0} | t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, t^{n-1}t_0u_1; q, t; p)} \\ & \quad \binom{\lambda}{\mu}_{[\frac{1}{u_0u_1}, \frac{1}{t^{n-1}t_0u_1}]; q, t; p} \binom{\mu}{\kappa}_{[t^{n-1}t_0/u_0, t^{n-1}t_0u'_1]; q, t; p}. \end{aligned} \quad (5.17)$$

If $t'_3 = t_3$, the same connection coefficients can be computed via $R^{*(n)}(; t_3, u_0; q, t; p)$; the result is precisely our generalized Bailey transformation, Theorem 4.9. If also $t'_2 = t_2$, the bulk difference equation (4.11) applies, giving the following result announced in [15].

Corollary 5.7.

$$\begin{aligned} & [\tilde{R}_\mu^{(n)}(; t_0:t_1v, t_2, t_3; u_0, u_1/v; q, t; p)]\tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; p) \\ &= \left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{[1/u_0u_1, 1/v](t^{n-1}t_2t_3, pqt^{n-1}t_0/u_0, t_1v/u_1); q, t; p} \end{aligned} \quad (5.18)$$

We also have discrete biorthogonality, which was derived in [15] via residue calculus, but can also be derived via the present “hypergeometric” methods.

Theorem 5.8. For any partitions $\lambda, \kappa \subset m^n$, and for otherwise generic parameters satisfying $t_0 t_1 = q^{-m} t^{1-n}$, $t^{n-1} t_2 t_3 u_0 u_1 = pq^{m+1}$

$$\sum_{\mu \subset m^n} \tilde{R}_\lambda^{(n)}(\dots t_0 t^{n-i} q^{\mu_i} \dots; t_0: t_1, t_2, t_3; u_0, u_1; q, t; p) \tilde{R}_\kappa^{(n)}(\dots t_0 t^{n-i} q^{\mu_i} \dots; t_0: t_1, t_2, t_3; u_1, u_0; q, t; p) \quad (5.19)$$

$$\Delta_\mu(t^{2n-2} t_0^2 | t^n, t^{n-1} t_0 t_1, t^{n-1} t_0 t_2, t^{n-1} t_0 t_3, t^{n-1} t_0 u_0, t^{n-1} t_0 u_1; q, t; p) = 0$$

unless $\lambda = \kappa$, when the sum is

$$\frac{\Delta_{m^n}^0\left(\frac{t^{n-1} t_1}{u_0} \middle| \frac{t_1}{t_0}, \frac{pq}{u_0 t_2}, \frac{pq}{u_0 t_3}, \frac{pq}{u_0 u_1}; q, t; p\right)}{\Delta_\lambda(1/u_0 u_1 | t^n, t^{n-1} t_0 t_1, t^{n-1} t_0 t_2, t^{n-1} t_0 t_3, \frac{1}{t^{n-1} t_0 u_0}, \frac{1}{t^{n-1} t_0 u_1}; q, t; p)} \quad (5.20)$$

Proof. The argument of [16] carries over essentially verbatim; alternatively, the argument of [17] generalizes, using Theorem 4.15. \square

Remark. Note that the above inner product is normalized by equation (4.15).

We also have a special quasi-branching rule (having the branching rule for interpolation functions (Theorem 4.16) as a special case):

Theorem 5.9. The biorthogonal functions satisfy the expansion

$$\begin{aligned} \tilde{R}_\lambda^{(n+k)}(\dots z_i \dots, t_0, t_0 t, \dots t_0 t^{k-1}; t_0: t_1, t_2, t_3; u_0, u_1; q, t; p) \\ = \sum_{\kappa \subset \lambda} c_{\lambda \kappa} \tilde{R}_\kappa^{(n)}(\dots z_i \dots; t_0 t^k: t_1, t_2, t_3; u_0, u_1 t^k; q, t; p) \end{aligned} \quad (5.21)$$

where

$$c_{\lambda \kappa} = \left\langle \lambda \middle| \kappa \right\rangle_{\left[\frac{1}{u_0 u_1}, t^k\right] \left(\frac{pq}{t^{n+k} u_0 u_1}, \frac{1}{t^{k-1} t_0 u_1}, \frac{pq t^{n+k-1} t_0}{u_0}\right); q, t; p} \quad (5.22)$$

Proof. Expand the left-hand side in interpolation functions, apply equation (3.43), and expand back into biorthogonal functions; again the resulting sum can be simplified via the bulk difference equation (4.11). \square

Similarly, we obtain a special quasi-Pieri identity.

Theorem 5.10. The biorthogonal functions satisfy the expansion

$$\begin{aligned} \prod_{1 \leq i \leq n} \frac{\theta(t_0 z_i, t_0/z_i; q; p)_m}{\theta((pq/u_0) z_i, (pq/u_0)/z_i; q; p)_m} \tilde{R}_\lambda^{(n)}(\dots z_i \dots; q^m t_0: t_1, t_2, t_3; q^{-m} u_0, u_1; q, t; p) \\ = \sum_{\kappa} c_{\kappa \lambda} \tilde{R}_\kappa^{(n)}(\dots z_i \dots; t_0: t_1, t_2, t_3; u_0, u_1; q, t; p), \end{aligned} \quad (5.23)$$

where

$$\begin{aligned} c_{\kappa \lambda} = \Delta_{m^n}^0\left(\frac{t^{n-1} t_0}{u_0} \middle| t^{n-1} t_0 t_1, t^{n-1} t_0 t_2, t^{n-1} t_0 t_3, t^{n-1} t_0 u_1; q, t; p\right) \\ \left\langle \kappa \middle| \lambda \right\rangle_{\left[\frac{1}{u_0 u_1}, q^{-m}\right] \left(t^n, \frac{1}{t^{n-1} t_0 u_1}, \frac{q^m t_0}{u_0}\right); q, t; p}. \end{aligned} \quad (5.24)$$

Finally, we have a Cauchy identity for biorthogonal functions.

Theorem 5.11. *The function*

$$F(x_1, \dots, x_n; y_1, \dots, y_m) := \frac{\prod_{1 \leq i \leq n, 1 \leq j \leq m} \theta(y_j x_i, y_j / x_i; p)}{\prod_{1 \leq i \leq n} \theta(q^{-m} u_0 x_i, q^{-m} u_0 / x_i; q; p)_m \prod_{1 \leq j \leq m} \theta(p q^m / u_0 y_j, q^m y_j / u_0; 1/t; p)_n} \quad (5.25)$$

admits an expansion

$$\begin{aligned} \sum_{\mu \subset m^n} c_\mu \tilde{R}_\mu^{(n)}(\dots x_i \dots; t_0: t_1, t_2, t_3; u_0, u_1; q, t; p) \tilde{R}_{n^m - \mu'}^{(m)}(\dots y_i \dots; t_0: t_1, t_2, t_3; \frac{t^n u_0}{q^m}, \frac{t^{n-1} u_1}{q^{m-1}}; t, q; p), \\ = \frac{C_{m^n}^0(pq/u_0 t_0, q^m t_0 / u_0; q, t; p) F(x_1, \dots, x_n; y_1, \dots, y_m)}{\Delta_{m^n}^0(t^{n-1} t_0 / u_0 | t^{n-1} t_0 t_1, t^{n-1} t_0 t_2, t^{n-1} t_0 t_3, t^{n-1} t_0 u_1; q, t; p)} \end{aligned} \quad (5.26)$$

where

$$c_\mu = \Delta_\mu \left(\frac{1}{u_0 u_1} | t^n, q^{-m}, \frac{1}{t^{n-1} t_0 u_1}, \frac{q^m t_0}{u_0}; q, t; p \right). \quad (5.27)$$

6 Algebraic modularity and rationality

The purpose of the present section is to give a purely algebraic definition of the biorthogonal functions; as a consequence, it will follow that the biorthogonal abelian functions are in addition modular functions. This could of course be shown directly by determining how the interpolation functions and C^* symbols behave under modular transformation; the approach used here has the advantage of being more conceptual in nature. In addition, we also obtain analogues of interpolation functions, binomial coefficients, and biorthogonal functions over arbitrary fields (including those of positive characteristic).

In addition to making sure the construction is sensible geometrically, we also want things to be reasonable *arithmetically*; that is, in such a way that the functions depend rationally on the parameters. In particular, the difference equation as given above is problematical in this respect, as it requires us to choose a square root of q , despite the fact that the interpolation functions themselves are independent of that choice. The simplest way to fix that is to relax the notion of elliptic curve slightly, by forgetting which point represents the identity; this allows us to absorb the freedom in choosing the square root. We thus obtain the following fundamental definition. Recall that for a (smooth) curve C , the Picard group $\text{Pic}(C)$ is the group of divisors modulo principal divisors; for an integer n , $\text{Pic}^n(C)$ is then the preimage of n under the natural degree map. For each n , $\text{Pic}^n(C)$ has a natural structure of algebraic variety, a principal homogeneous space over the algebraic group $\text{Pic}^0(C)$. We will use multiplicative notation for divisors (and the group law in $\text{Pic}(C)$), and will denote the divisor associated to a point $p \in C$ by $\langle p \rangle$, and to a function f on C by $\langle f \rangle$.

Definition 13. A *genus 1 hyperelliptic curve* is a pair (C, τ) , where C is a genus 1 curve, and $\tau \in \text{Pic}^2(C)$ is a divisor class of degree 2 (which thus induces an involutory automorphism $\iota_\tau : x \mapsto \tau/x$ of C with genus 0 quotient). A $BC_n(\tau)$ -symmetric function on C is a function f on C^n invariant under permutations of the variables and replacements $x_i \mapsto \tau/x_i$.

Remark 1. Note in particular that $\text{Pic}^0(C)$ is an elliptic curve, and $\text{Pic}^1(C)$ can be canonically identified with C . Each point $x \in C$ thus induces a map $\text{Pic}(C) \rightarrow \text{Pic}^0(C)$ defined by $y \mapsto yx^{-\deg(y)}$, and in particular gives an identification of C with the elliptic curve $\text{Pic}^0(C)$.

Remark 2. The action of ι_τ on C extends to an automorphism of the Picard group $\text{Pic}(C)$, which we will denote by $\iota_\tau(x) := \tau^{\deg(x)}/x$. Similarly, its action on the i th copy of C in C^n will be denoted by $\iota_{i,\tau}$.

Remark 3. For $x \in \text{Pic}^0(C)$, there is a natural map from the genus 1 hyperelliptic curve (C, τ) to the curve $(C, x^2\tau)$ given by $p \mapsto xp$. Since all of the functions we will define below are canonically defined, they in particular will transform nicely under this isomorphism; in general, replacing each element $y \in \text{Pic}(\tau)$ by $x^{\deg(y)}y$ will leave the function invariant.

While there is in general no canonical choice of divisor representing τ , there is in fact a canonical choice of representative for τ^2 , namely the ramification divisor of the quotient map $C \mapsto C/\iota_\tau$. In odd characteristic, this is simply the product of the fixed points of ι_τ in $\text{Pic}^1(C)$; in even characteristic, it is the square or fourth power of that product (depending on whether the curve is ordinary or supersingular). We will denote this divisor by $\langle \tau^2 \rangle$.

Given a divisor D , let $\mathcal{L}(D)$ denote the space of functions f on C such that $\langle f \rangle D$ is effective; by the Riemann-Roch theorem, this space has dimension $\max(\deg(D), 0)$, unless $\deg(D) = 0$ and D is principal (when the dimension is 1). The following proposition is key to proving the difference equation in arbitrary characteristic.

Proposition 6.1. *Let D be a divisor of degree m . Then $\iota_\tau^* \mathcal{L}(D\iota_\tau(D)) = \mathcal{L}(D\iota_\tau(D))$, and the dimension of the space of ι_τ^* -invariants is $\max(m+1, 0)$. Moreover,*

$$(1 - \iota_\tau^*) \mathcal{L}(D\iota_\tau(D)) \subset \mathcal{L}(D\iota_\tau(D)/\langle \tau^2 \rangle), \quad (6.1)$$

$$(1 + \iota_\tau^*) \mathcal{L}(D\iota_\tau(D)\langle \tau^2 \rangle) \subset \mathcal{L}(D\iota_\tau(D)), \quad (6.2)$$

with images of dimension $\max(m-1, 0)$, $\max(m+1, 0)$, respectively.

Proof. The statement about the space of ι_τ^* -invariants follows immediately from Riemann-Roch, since ι_τ^* -invariants are simply functions on the quotient \mathbb{P}^1 . For the remaining statements, multiplication by a ι_τ^* -invariant function allows us to replace D by any other divisor of degree m ; in particular, we can assume that D is supported away from the support of $\langle \tau^2 \rangle$. That

$$(1 - \iota_\tau^*) \mathcal{L}(D\iota_\tau(D)) \subset \mathcal{L}(D\iota_\tau(D)/\langle \tau^2 \rangle) \quad (6.3)$$

then follows from the definition of ramification divisor; the dimension of the image can be computed from the dimension of the kernel. The remaining claims follow upon multiplication by a ι_τ^* -anti-invariant function. \square

Fix a genus 1 hyperelliptic curve, as well as two points $q, t \in \text{Pic}^0(C)$. Given four generic points $u_0, u_1, u_2, u_3 \in C$ such that $t^{n-1}u_0u_1u_2u_3 = \tau^2$, we define a difference operator $\tilde{D}^{(n)}(; u_0, u_1, u_2; q, t; C, \tau)$ acting on $BC_n(\tau/q)$ -symmetric functions as follows:

$$(\tilde{D}^{(n)}(; u_0, u_1, u_2; q, t; C, \tau)f)(\dots x_i \dots) = \prod_{1 \leq i \leq n} (1 + \iota_{i,\tau}^*) \delta^{(n)}(\dots x_i \dots; u_0, u_1, u_2, u_3; t; C, \tau) f(\dots x_i \dots) \quad (6.4)$$

for a function $\delta^{(n)}$ defined inductively as follows:

$$(0) \delta^0(u_0, u_1, u_2, u_3; t; C, \tau) = 1.$$

(1) As a function of x_n , $\delta^{(n)}(\dots x_i \dots; u_0, u_1, u_2, u_3; t; C, \tau)$ has divisor

$$\frac{\langle \frac{\tau}{u_0} \rangle \langle \frac{\tau}{u_1} \rangle \langle \frac{\tau}{u_2} \rangle \langle \frac{\tau}{u_3} \rangle \prod_{1 \leq i < n} \langle \frac{\tau}{tx_i} \rangle}{\langle \tau^2 \rangle \prod_{1 \leq i < n} \langle \frac{\tau}{x_i} \rangle} \quad (6.5)$$

(2) Setting $x_n = u_0$ gives

$$\delta^{(n)}(x_1, \dots, x_{n-1}, u_0; u_0, u_1, u_2, u_3; t) = \delta^{(n-1)}(x_1, \dots, x_{n-1}; tu_0, u_1, u_2, u_3; t) \quad (6.6)$$

Proposition 6.2. For fixed (C, τ) and t , $\delta^{(n)}$ is a well-defined rational function of all variables and parameters. It is invariant under permutations of x_1 through x_n and under permutations of u_0 through u_3 .

Proof. For $n = 1$, we find from Proposition 6.1 that

$$(1 + \iota_\tau^*) \delta^{(1)}(x; u_0, u_1, u_2, u_3; t; C, \tau) \quad (6.7)$$

is a constant; evaluating this constant at $x = u_0$ shows that it is 1. Evaluating at $x = u_1$ shows that

$$\delta^{(1)}(x; u_0, u_1, u_2, u_3; t; C, \tau) = \delta^{(1)}(x; u_1, u_0, u_2, u_3; t; C, \tau) \quad (6.8)$$

as required, and thus the proposition holds when $n = 1$.

For $n > 1$, we first observe by induction that $\delta^{(n)}$ is symmetric in x_1 through x_{n-1} and in u_1, u_2, u_3 . Now, if we set $x_n = u_0$, the divisor in x_{n-1} will be

$$\frac{\langle \frac{\tau}{tu_0} \rangle \langle \frac{\tau}{u_1} \rangle \langle \frac{\tau}{u_2} \rangle \langle \frac{\tau}{u_3} \rangle \prod_{1 \leq i < n-1} \langle \frac{\tau}{tx_i} \rangle}{\langle \tau^2 \rangle \prod_{1 \leq i < n-1} \langle \frac{\tau}{x_i} \rangle}; \quad (6.9)$$

from the dependence on x_{n-1} of the divisor in x_n , we deduce the dependence on x_n of the divisor in x_{n-1} , and thus conclude that the divisor in x_{n-1} of $\delta^{(n)}$ is

$$\frac{\langle \frac{\tau}{u_0} \rangle \langle \frac{\tau}{u_1} \rangle \langle \frac{\tau}{u_2} \rangle \langle \frac{\tau}{u_3} \rangle \prod_{i \neq n-1} \langle \frac{\tau}{tx_i} \rangle}{\langle \tau^2 \rangle \prod_{1 \leq i < n} \langle \frac{\tau}{x_i} \rangle} \quad (6.10)$$

We claim that setting $x_{n-1} = u_1$ gives $\delta^{(n-1)}(; u_0, tu_1, u_2, u_3) = \delta^{(n-1)}(; tu_1, u_0, u_2, u_3)$; indeed, setting $x_n = u_0$, $x_{n-1} = u_1$ gives $\delta^{(n-2)}(; tu_0, tu_1, u_2, u_3)$, and the divisor in x_n is also correct. It follows that

$$\delta^{(n)}(x_1, \dots, x_n; u_0, u_1, u_2, u_3; t; C, \tau) = \delta^{(n)}(x_1, \dots, x_{n-2}, x_n, x_{n-1}; u_1, u_0, u_2, u_3; t; C, \tau), \quad (6.11)$$

at which point the proposition follows. \square

Remark. If C is a complex elliptic curve of the form $\mathbb{C}^*/\langle p \rangle$, then $\delta^{(n)}$ can be expressed in theta functions as

$$\begin{aligned} & \delta^{(n)}(\dots x_i \dots; u_0, u_1, u_2, u_3; q, t; C, \tau) \\ &= \prod_{1 \leq i \leq n} \frac{\theta(u_0 x_i / \tau, u_1 x_i / \tau, u_2 x_i / \tau, u_3 x_i / \tau; p)}{\theta(x_i^2 / \tau) \theta(u_0 u_1 t^{n-i} / \tau, u_0 u_2 t^{n-i} / \tau, u_0 u_3 t^{n-i} / \tau; p)} \prod_{1 \leq i < j \leq n} \frac{\theta(tx_i x_j / \tau; p)}{\theta(x_i x_j / \tau; p)}, \end{aligned} \quad (6.12)$$

where the parameters are lifted to \mathbb{C}^* so that $u_0 u_1 u_2 u_3 = p\tau^2$; thus the difference operator defined above agrees (up to a scale factor) with the earlier definition.

For an integer $m \geq 0$, let $A_m(u_0; q; C, \tau) \subset \mathcal{L}(\prod_{1 \leq i \leq n} \langle q^{-i} u_0 \rangle \langle \frac{\tau}{q^{-i} u_0} \rangle)$ denote the subspace of ι_τ -invariant functions, and let $A_m^{(n)}(u_0; q; C, \tau)$ be the n -th symmetric power of that space, viewed as a space of functions on C^n ; in particular, $A_m^{(n)}(u_0; q; C, \tau)$ consists of $BC_n(\tau)$ -symmetric functions. We also write $A^{(n)}(u_0; q; C, \tau)$ for the union $\lim_{m \rightarrow \infty} A_m^{(n)}(u_0; q; C, \tau)$.

Theorem 6.3. *For any integer $m \geq 0$,*

$$\tilde{D}^{(n)}(u_0, u_1, u_2; q, t; C, \tau) A_m^{(n)}(u_0; q; C, \tau/q) \subset A_m^{(n)}(u_0; q; C, \tau). \quad (6.13)$$

Moreover,

$$\begin{aligned} (\tilde{D}^{(n+k)}(u_0, u_1, u_2; q, t; C, \tau) f)(z_1, \dots, z_n, u_0, \dots, t^{k-1} u_0) \\ = (\tilde{D}^{(n)}(u_0, u_1, u_2; q, t; C, \tau) f(\underline{\quad}, u_0, \dots, t^{k-1} u_0))(z_1, \dots, z_n). \end{aligned} \quad (6.14)$$

Proof. The second claim follows readily from the definition of the difference operator and of $\delta^{(n)}$, so we need only consider the first claim. In particular, it suffices to show that as a function of z_n , the only poles of

$$\tilde{D}^{(n)}(u_0, u_1, u_2; q, t; C, \tau) f \quad (6.15)$$

for $f \in A_m^{(n)}(u_0; q; C, \tau/q)$ are of the form $q^{-i} u_0$ or $q^i \tau / u_0$ for $1 \leq i \leq m$, of multiplicity at most 1. Now, as a function of z_n ,

$$\delta^{(n)}(\dots z_i \dots; u_0, u_1, u_2; q, t; C, \tau) f(\dots z_i \dots) \quad (6.16)$$

has polar divisor at most

$$\langle \tau^2 \rangle \prod_{1 \leq i < n} \langle \tau / z_i \rangle \prod_{1 \leq i \leq m} \langle q^{-i} u_0 \rangle \langle q^{i-1} \tau / u_0 \rangle / \langle \tau / u_0 \rangle, \quad (6.17)$$

with the last factor coming from the numerator of $\delta^{(n)}$. In particular, upon symmetrization by ι_τ , the result clearly has polar divisor at most

$$\langle \tau^2 \rangle \prod_{1 \leq i < n} \langle z_i \rangle \langle \tau / z_i \rangle \prod_{1 \leq i \leq m} \langle q^{-i} u_0 \rangle \langle q^i \tau / u_0 \rangle. \quad (6.18)$$

We thus need only show that the potential poles corresponding to the first two sets of factors disappear upon symmetrization; we may assume (since it is true generically) that the three sets of factors have disjoint support.

That the singularities corresponding to $\langle \tau^2 \rangle$ disappear follows immediately from Proposition 6.1, so it remains by symmetry to consider the potential singularity at $z_n = z_{n-1}$. Now, in the sum

$$\begin{aligned} \prod_{1 \leq i \leq n} (1 + \iota_{i, \tau}^*) \delta^{(n)}(\dots z_i \dots; u_0, u_1, u_2; q, t; C, \tau) f(\dots z_i \dots) \\ = \prod_{1 \leq i \leq n-2} (1 + \iota_{i, \tau}^*) [(1 + \iota_{n-1, \tau}^* + \iota_{n, \tau}^* + \iota_{n-1, \tau}^* \iota_{n, \tau}^*) \delta^{(n)}(\dots z_i \dots; u_0, u_1, u_2; q, t; C, \tau) f(\dots z_i \dots)] \end{aligned} \quad (6.19)$$

the factors corresponding to 1 and $\iota_{n-1, \tau}^* \iota_{n, \tau}^*$ are already nonsingular at $z_n = z_{n-1}$, so it suffices to show that

$$(\iota_{n-1, \tau}^* + \iota_{n, \tau}^*) \delta^{(n)}(\dots z_i \dots; u_0, u_1, u_2; q, t; C, \tau) f(\dots z_i \dots) \quad (6.20)$$

has no pole at $z_{n-1} = z_n$. Now, since $\delta^{(n)}$ and f are symmetric under permutations of z_1, \dots, z_n , this sum can be written as the symmetrization of

$$\iota_{n,\tau}^* \delta^{(n)}(\dots z_i \dots; u_0, u_1, u_2; q, t; C, \tau) f(\dots z_i \dots) \quad (6.21)$$

under exchanging z_{n-1} and z_n ; the disappearance of the pole then follows from local considerations. \square

We will also need to construct algebraic analogues of certain products of C^* -symbols. First, we need the following family of univariate functions. For a nonnegative integer m , let $u_0, \dots, u_{2m-1} \in C$ be a collection of $2m$ points such that $u_0 u_1 \cdots u_{2m-1} = \tau^m$, and let D be a divisor of degree m . Then there exists up to scale a unique function f with divisor

$$\frac{\prod_{0 \leq i < 2m} \langle u_i \rangle}{(D \iota_\tau(D))}. \quad (6.22)$$

Thus the function

$$\omega(x; u_0, \dots, u_{2m-1}; C, \tau) := \frac{f(x)}{f(\tau/x)} \quad (6.23)$$

is well-defined; multiplication of f by an ι_τ^* -invariant function shows that ω is independent of the choice of D . Note that

Lemma 6.4. *The function ω has divisor*

$$\frac{\prod_{0 \leq i < 2m} \langle u_i \rangle}{\prod_{0 \leq i < 2m} \langle \tau/u_i \rangle}, \quad (6.24)$$

and satisfies the normalization that for any point $x_0 \in C$ such that $x_0^2 = \tau$,

$$\omega(x_0; u_0, \dots, u_{2m-1}; C, \tau) = 1. \quad (6.25)$$

Moreover,

$$\omega(x; u_0, \dots, u_{2m-1}, v, \tau/v; C, \tau) = \omega(x; u_0, \dots, u_{2m-1}; C, \tau), \quad (6.26)$$

and

$$\omega(x; u_0, \dots, u_{2m-1}; C, \tau) \omega(x; v_0, \dots, v_{2n-1}; C, \tau) = \omega(x; u_0, \dots, u_{2m-1}, v_0, \dots, v_{2n-1}; C, \tau). \quad (6.27)$$

Proof. The first claim is straightforward; for the second claim, we find that

$$\omega(x; u_0, \dots, u_{2m-1}; C, \tau) - 1 = \frac{f(x) - f(\tau/x)}{f(\tau/x)} \in \mathcal{L}(\langle \tau^2 \rangle^{-1}), \quad (6.28)$$

and thus it vanishes as required. The final claims follow immediately from the original definition of ω . \square

Another univariate function can be obtained from the observation that since τ induces a map to \mathbb{P}^1 , any four points induce a uniquely defined scalar, namely the cross-ratio of their images. We denote this by

$$\chi(a, b, c, d; C, \tau), \quad (6.29)$$

which can also be defined as the unique ι_τ^* -invariant function of a such that

$$\chi(b, b, c, d; C, \tau) = 0, \quad \chi(c, b, c, d; C, \tau) = 1, \quad \chi(d, b, c, d; C, \tau) = \infty. \quad (6.30)$$

We can also define this by

$$\chi(a, b, c, d; C, \tau) = \omega(x: bax/\tau, bx/a, dx/c, dcx/\tau; C, x^2bd/\tau), \quad (6.31)$$

for any $x \in C$. Note also that since χ is defined as a cross-ratio, all of the usual transformations of cross ratios apply; note in particular the symmetry

$$\chi(a, b, c, d; C, \tau)\chi(a', b, c', d; C, \tau) = \chi(a, b, c', d; C, \tau)\chi(a', b, c, d; C, \tau). \quad (6.32)$$

In addition, the identification with ω , with its different natural symmetry, gives rise to the transformations

$$\chi(a, b, c, d; C, \tau) = \chi(a, b, c, d; C, abcd/\tau) \quad (6.33)$$

$$\omega(x:v_0, v_1, v_2, v_3; C, \tau) = \omega(x:v_0, v_1, x^2/v_2, x^2/v_3; C, \tau x^2/v_2v_3). \quad (6.34)$$

Definition 14. Let $a, v_0, \dots, v_{2m-1} \in \text{Pic}^0(C)$ be such that

$$v_0v_1 \cdots v_{2m-1} = q^m a^m. \quad (6.35)$$

Then for any partition λ , we define

$$\Delta_\lambda^0(a|v_0, \dots, v_{2m-1}; q, t; C) = \prod_{(i,j) \in \lambda} \omega(q^{1-j}t^{i-1}x:v_0x, \dots, v_{2m-1}x; C, qax^2), \quad (6.36)$$

for a generic point $x \in C$ (on which the value does not depend).

Definition 15. Let $a, b_0, \dots, b_{2l-1} \in \text{Pic}^0(C)$ be such that $\prod_{0 \leq r < 2l} b_r = (t/q)(qa)^{l-1}$. Then we define a symbol $\Delta_\lambda(a|b_0, \dots, b_{2l-1}; q, t; C)$ for all partitions λ via the recurrence

$$\begin{aligned} \Delta_{m \cdot \lambda}(a|b_0, \dots, b_{2l-1}; q, t; C) &= \lim_{x \rightarrow 1} \Delta_m^0(a|b_0, \dots, b_{2l-1}, q^{m+1}a/x, qa/xt, qax, qx/q^m; q, t; C) \\ &\Delta_\lambda(t^{-2}a|b_0/t, \dots, b_{2l-1}/t, q^{1-m}/t, q^{-m}, q^{m+1}a/t, q^m a; q, t; C) \end{aligned} \quad (6.37)$$

with

$$\Delta_0(a|b_0, \dots, b_{2l-1}; q, t; C) = 1; \quad (6.38)$$

here, of course, the limit is to be taken in the algebraic sense: the evaluation at the appropriate point of the stated rational function in the limit variable. Similarly, we define $\binom{\lambda}{[\lambda]_{[a,b];q,t;C}}$ via the recurrence

$$\begin{aligned} \binom{m \cdot \lambda}{m \cdot \lambda}_{[a,b];q,t;C} &= \lim_{x \rightarrow 1} \Delta_m^0(a/b|q^m a/x, 1/bx, qax/b, qx/q^m; q, t; C) \\ &\frac{\Delta_\lambda^0(a/t^2|b, qa/tb, q^{1-m}/t^2, q^m a/t; q, t; C)}{\Delta_\lambda^0(a/bt^2|1/b, qa/t, q^{1-m}/t^2, q^m a/tb; q, t; C)} \binom{\lambda}{[\lambda]_{[a/t^2,b];q,t;C}}, \end{aligned} \quad (6.39)$$

with

$$\binom{0}{0}_{[a,b];q,t;C} = 1. \quad (6.40)$$

Remark. For a complex elliptic curve, the symbols Δ_λ^0 and Δ_λ are precisely those defined above; the parameters must be lifted so that the constraints on $\prod_i b_i$ are satisfied as given, except with q replaced by pq . The remaining symbol corresponds to the product

$$\frac{C_\lambda^+(a; q, t; p)C_\lambda^0(1/b, pq a/b; q, t; p)}{C_\lambda^+(a/b; q, t; p)C_\lambda^0(b, pq a; q, t; p)} = \binom{\lambda}{\lambda}_{[a, b]; q, t; p}. \quad (6.41)$$

The algebraic analogue of interpolation functions are defined as follows.

Definition 16. For generic parameters $a, b, v \in C$, $q, t \in \text{Pic}^0(C)$, we define the algebraic interpolation function $R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau)$ as follows.

- (1) $R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau) \in A_{\lambda_1}^{(n)}(b; q; C, \tau)$.
- (2) For each integer $m > \lambda_1$, and every partition $\mu \subset m^n$ with $\mu \neq \lambda$, let l be as in the definition of $P^{*(m, n)}$ above. Then for generic $c \in C$,

$$\left(\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \chi(z_i, q^{-j} b, t^{n-i} v, c; C, \tau) R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau) \right) (bq^{-\mu_1}, \dots, bq^{-\mu_l} t^{l-1}, aq^{\mu_{l+1}} t^{n-l-1}, \dots, aq^{\mu_n}) = 0. \quad (6.42)$$

- (3) $R_\lambda^{*(n)}$ satisfies the normalization

$$R_\lambda^{*(n)}(\dots vt^{n-i} \dots; a, b(v); q, t; C, \tau) = 1. \quad (6.43)$$

As before, these functions are generically uniquely determined by the conditions; moreover, they satisfy a difference equation, the proof of which is a direct analogue of the corresponding proof for theta functions.

Theorem 6.5. *The functions $R_\lambda^{*(n)}$ satisfy the difference equation*

$$\begin{aligned} \tilde{D}^{(n)}(a, b, c; q, t; C, \tau) R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau/q) \\ = \Delta_\lambda^0\left(\frac{t^{n-1}a}{b} \mid \frac{avt^{n-1}}{\tau}, \frac{acqt^{n-1}}{\tau}, \frac{\tau}{bv}, \frac{q\tau}{bc}; q, t; C\right) R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau) \end{aligned} \quad (6.44)$$

and the normalization condition

$$R_\lambda^{*(n)}(\dots v' t^{n-i} \dots; a, b(v); q, t; C, \tau) = \Delta_\lambda^0(t^{n-1} a/b \mid t^{n-1} av'/\tau, a/v', q\tau/vb, t^{n-1} qv/b; q, t; C). \quad (6.45)$$

Proof. As mentioned, the proof follows that of Theorem 3.1; the only difference is the treatment of the normalization. We find as before that

$$\tilde{D}^{(n)}(a, b, c; q, t; C, \tau) R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau/q) \propto R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau); \quad (6.46)$$

for $c = v$, we find immediately that

$$\tilde{D}^{(n)}(a, b, v; q, t; C, \tau) R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau/q) = R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau). \quad (6.47)$$

Now, if we evaluate at the point $(\dots at^{n-i}q^{\lambda_i} \dots)$, we find that only one term survives, and thus the constant of proportionality in the difference equation can be computed as

$$\frac{\delta^{(n)}(\dots at^{n-i}q^{\lambda_i} \dots; a, b, c, \tau^2/abc; t; C, \tau)}{\delta^{(n)}(\dots at^{n-i}q^{\lambda_i} \dots; a, b, v, \tau^2/abv; t; C, \tau)}. \quad (6.48)$$

We readily verify from divisor conditions that the function

$$\frac{\delta^{(n)}(\dots x_i \dots; a, b, c, t^{1-n}\tau^2/abc; t; C, \tau)}{\delta^{(n)}(\dots x_i \dots; a, b, v, t^{1-n}\tau^2/abv; t; C, \tau)} = \prod_{1 \leq i \leq n} \chi(x_i, \tau/c, at^{n-i}, \tau/v; C, t^{n-1}ab), \quad (6.49)$$

and we thus obtain

$$\frac{\delta^{(n)}(\dots at^{n-i}q^{\lambda_i} \dots; a, b, c, \tau^2/abc; t; C, \tau)}{\delta^{(n)}(\dots at^{n-i}q^{\lambda_i} \dots; a, b, v, \tau^2/abv; t; C, \tau)} = \prod_{1 \leq i \leq n} \chi(at^{n-i}q^{\lambda_i}, \tau/c, at^{n-i}, \tau/v; C, t^{n-1}ab) \quad (6.50)$$

On the other hand, we find

$$\Delta_\lambda^0(t^{n-1}a/b|avt^{n-1}/\tau, acqt^{n-1}/\tau, \tau/bv, q\tau/bc; q, t; C) = \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq \lambda_i} \chi(at^{n-i}q^j, \tau/c, at^{n-i}q^{j-1}, \tau/v; C, t^{n-1}ab) \quad (6.51)$$

$$= \prod_{1 \leq i \leq n} \chi(at^{n-i}q^{\lambda_i}, \tau/c, at^{n-i}, \tau/v; C, t^{n-1}ab) \quad (6.52)$$

as required.

The formula for changing the normalization follows by applying the difference equation to both sides of the equation

$$R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau) = R_\lambda^{*(n)}(; a, b(t^{1-n}\tau/v); q, t; C, \tau), \quad (6.53)$$

which gives a recurrence for the relevant scale factor, with the stated solution. \square

In particular, we obtain the extra vanishing conditions just as before. The various identities of Section 3 for the analytic interpolation functions all carry over to the algebraic interpolation functions. In particular, we note the Cauchy case

$$R_\lambda^{*(n)}(\dots z_i \dots; a, b(v); q, t; C, t^n ab/q) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j}} \chi(z_i, at^{n-\lambda'_j}q^{j-1}, vt^{n-i}, at^n q^{j-1}; C, t^n ab/q) \quad (6.54)$$

Proposition 6.6. *We have the identities*

$$R_{m^n - \lambda}^{*(n)}(\dots z_i \dots; a, b(v); q, t; C, \tau) = \frac{R_\lambda^{*(n)}(\dots z_i \dots; q^{-m}b, q^m a(v); q, t; C, \tau)}{\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \chi(z_i, q^{-j}b, vt^{n-i}, q^{j-1}a; C, \tau)}, \quad (6.55)$$

$$R_{m^n + \lambda}^{*(n)}(\dots z_i \dots; a, b(v); q, t; C, \tau) = \frac{R_\lambda^{*(n)}(\dots z_i \dots; aq^m, b/q^m(v); q, t; C, \tau)}{\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \chi(z_i, q^{-j}b, vt^{n-i}, q^{j-1}a; C, \tau)}, \quad (6.56)$$

$$R_\lambda^{*(n+k)}(\dots z_i \dots, a, at, \dots, at^{k-1}; a, b(v); q, t; C, \tau) = \Delta_\lambda^0(t^{n+k-1}a/b|t^n, qa/bt, t^k a/v, qt^{n+k-1}v/b; q, t; C) R_\lambda^{*(n)}(\dots z_i \dots; t^k a, b(v); q, t; C, \tau), \quad (6.57)$$

and thus

$$R_\lambda^{*(n)}(\dots at^{n-i}q^{\lambda_i} \dots; a, b(v); q, t; C, \tau) = \frac{\binom{\lambda}{\mu}_{[t^{2n-2}a^2/\tau, t^{n-1}ab/\tau]; q, t; C}}{\Delta_\lambda(\frac{t^{n-1}a}{b}|t^n, \tau/t^{n-1}ab, t^{n-1}av/\tau, a/v; q, t; C)}. \quad (6.58)$$

Proof. The first two identities are straightforward, as both sides satisfy the same vanishing conditions and normalization. For the second identity, the vanishing conditions are trivial to verify, and thus both sides are proportional. From the difference equation, it follows that the scale factor is independent of τ , and we may thus reduce to the Cauchy case, for which the verification is straightforward. The final equation follows immediately. \square

We also have the following symmetries which are trivial consequences of the isomorphism invariance of our definitions:

$$R_\lambda^{*(n)}(\dots z_i \dots; a, b(v); q, t; C, \tau) = R_\lambda^{*(n)}(\dots (x/a)z_i \dots; x, xb/a(xv/a); q, t; C, x^2\tau/a^2) \quad (6.59)$$

$$= R_\lambda^{*(n)}(\dots z_i \dots; \tau/a, \tau/b(\tau/v); 1/q, 1/t; C, \tau) \quad (6.60)$$

$$= R_\lambda^{*(n)}(\dots z_i \dots; a, b(\tau/t^{n-1}v); q, t; C, \tau) \quad (6.61)$$

With the above in mind, we define the algebraic binomial coefficients as follows. For a genus 1 curve C and $a, b, q, t \in \text{Pic}^0(C)$, we define

$$\binom{\lambda}{\mu}_{[a, b]; q, t; C} := \Delta_\mu\left(\frac{a}{b}|t^n, 1/b, v, t^{1-n}a/v; q, t; C\right) R_\mu^{*(n)}(xq^{\lambda_i}t^{1-i}; t^{1-n}x, xb/a(xv/a); q, t; C, x^2/a), \quad (6.62)$$

where the right-hand side is independent of n , $x \in C$, $v \in \text{Pic}^0(C)$. (In particular, the binomial coefficients really are elliptic in nature.) When $\mu = \lambda$, this is of course consistent with our previous notation $\binom{\lambda}{\lambda}_{[a, b]; q, t; C}$. If C is a complex elliptic curve, then we obtain the analytic binomial coefficients:

$$\binom{\lambda}{\mu}_{[a, b]; q, t; \mathbb{C}/\langle p \rangle} = \binom{\lambda}{\mu}_{[a, b]; q, t; p}. \quad (6.63)$$

Similarly, if $v_0v_1v_2 = q^2a^2/b$, we define

$$\left\langle \frac{\lambda}{\mu} \right\rangle_{[a, b](v_0, v_1, v_2); q, t; C} := \frac{\Delta_\lambda^0(a|b, v_0, v_1, v_2; q, t; C)}{\Delta_\mu^0(a/b|1/b, v_0, v_1, v_2; q, t; C)} \binom{\lambda}{\mu}_{[a, b]; q, t; C}. \quad (6.64)$$

In particular, the same proof as in the analytic case gives the algebraic bulk difference equation.

Theorem 6.7. *For otherwise generic parameters on $\text{Pic}^0(C)$ satisfying $bcde = aq$,*

$$\binom{\lambda}{\kappa}_{[a, c]; q, t; C} = \frac{\Delta_\kappa^0(a/c|1/c, bd, be, aq/b; q, t; C)}{\Delta_\lambda^0(a|c, bd, be, aq/b; q, t; C)} \sum_{\kappa \subset \mu \subset \lambda} \Delta_\mu^0(a/b|c/b, qa, d, e; q, t; C) \binom{\lambda}{\mu}_{[a, b]; q, t; C} \binom{\mu}{\kappa}_{[a/b, c/b]; q, t; C} \quad (6.65)$$

In particular,

$$\sum_{\mu} \binom{\lambda}{\mu}_{[a, b]; q, t; C} \binom{\mu}{\kappa}_{[a/b, 1/b]; q, t; C} = \delta_{\lambda\kappa}. \quad (6.66)$$

Remark. One immediate consequence is Spiridonov's observation [25] that the various more traditionally hypergeometric special cases discussed following Theorem 4.1 above are modular and abelian in all parameters. (This includes the BC_n -type sum of [19], since as observed there, that sum can be obtained by specializing Warnaar's Schlosser-type identity.)

Theorem 6.8. *The algebraic interpolation functions satisfy the connection coefficient identity*

$$[R_\mu^{*(n)}(; a', b(v); q, t; C, \tau)] R_\lambda^{*(n)}(; a, b(v); q, t; C, \tau) = \left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{[\frac{t^{n-1}a}{b}, \frac{a}{a'}] (\frac{t^{n-1}aa'}{\tau}, \frac{q\tau}{bv}, \frac{qt^{n-1}v}{b}); q, t; C}. \quad (6.67)$$

We define algebraic biorthogonal functions by:

$$\tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; C, \tau) := \sum_\mu \frac{\binom{\lambda}{\mu}_{[\frac{\tau}{u_0u_1}, \frac{\tau}{t^{n-1}t_0u_1}]; q, t; C} R_\mu^{*(n)}(; t_0, u_0(t_1); q, t; C, \tau)}{\Delta_\mu^0(\frac{t^{n-1}t_0}{u_0} | \frac{t^{n-1}t_0t_2}{\tau}, \frac{t^{n-1}t_0t_3}{\tau}, \frac{t^{n-1}t_0u_1}{\tau}, \frac{qt^{n-1}t_1}{u_0}; q, t; C)}, \quad (6.68)$$

where $t_0, t_1, t_2, t_3, u_0, u_1 \in C$ such that $t^{2n-2}t_0t_1t_2t_3u_0u_1 = q\tau^3$. In particular, when C is complex, we exactly recover the analytic biorthogonal functions, which must therefore be modular and abelian in all parameters.

Theorem 6.9. *The algebraic biorthogonal functions satisfy the symmetry identity*

$$\tilde{R}_\lambda^{(n)}(; t_1:t_0, t_2, t_3; u_0, u_1; q, t; C, \tau) = \frac{\tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; C, \tau)}{\tilde{R}_\lambda^{(n)}(\dots t^{n-i}t_1 \dots; t_0:t_1, t_2, t_3; u_0, u_1; q, t; C, \tau)}, \quad (6.69)$$

where

$$\tilde{R}_\lambda^{(n)}(\dots t^{n-i}t_1 \dots; t_0:t_1, t_2, t_3; u_0, u_1; q, t; C, \tau) = \Delta_\lambda^0(\frac{\tau}{u_0u_1} | \frac{t^{n-1}t_1t_2}{\tau}, \frac{t^{n-1}t_1t_3}{\tau}, \frac{\tau}{t^{n-1}t_1u_1}, \frac{qt^{n-1}t_0}{u_0}; q, t; C), \quad (6.70)$$

and the difference equation

$$\tilde{D}^{(n)}(u_0, t_0, t_1; q, t; C, \tau) \tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2/q, t_3/q; u_0, u_1/q; q, t; C, \tau/q) = \tilde{R}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; q, t; C, \tau) \quad (6.71)$$

From isomorphism invariance, we obtain the identities

$$\tilde{R}_\lambda^{(n)}(\dots z_i \dots; t_0:t_1, t_2, t_3; u_0, u_1; q, t; C, \tau) = \tilde{R}_\lambda^{(n)}(\dots xz_i \dots; xt_0:xt_1, xt_2, xt_3; xu_0, xu_1; q, t; C, x^2\tau) \quad (6.72)$$

$$= \tilde{R}_\lambda^{(n)}(\dots z_i \dots; \tau/t_0:\tau/t_1, \tau/t_2, \tau/t_3; \tau/u_0, \tau/u_1; q, t; C, \tau). \quad (6.73)$$

The statement of evaluation symmetry for the algebraic biorthogonal functions requires a certain amount of care, since the usual statement involves a square root; again, this can be absorbed into the hyperelliptic structure, but there does not seem to be as natural a way of doing so. What we find is that the hatted parameters (including a $\hat{\tau}$) must be related to the original parameters by

$$\frac{\hat{t}_1}{\hat{t}_0} = \frac{q\tau}{t_2t_3}, \quad \frac{\hat{t}_2}{\hat{t}_0} = \frac{q\tau}{t_1t_3}, \quad \frac{\hat{t}_3}{\hat{t}_0} = \frac{q\tau}{t_1t_2}, \quad \frac{\hat{u}_0}{\hat{t}_0} = \frac{u_0}{t_0}, \quad \frac{\hat{u}_1}{\hat{t}_0} = \frac{u_1}{t_0}, \quad \frac{\hat{t}_0^2}{\hat{\tau}} = \frac{t_0t_1t_2t_3}{q\tau^2}. \quad (6.74)$$

In particular, we can choose \hat{t}_0 arbitrarily, at which point the remaining hatted parameters are determined. This freedom of \hat{t}_0 comes from the translation symmetries of C , and corresponds to the fact that the evaluation of a biorthogonal function at a partition is really a function on $\text{Pic}^0(C)^7$.

The remaining identities satisfied by the analytic interpolation and biorthogonal functions all carry over to the algebraic case straightforwardly; for instance, we have the connection coefficient

$$\begin{aligned} [\tilde{R}_\mu^{(n)}(;t_0:t_1v, t_2, t_3; u_0, u_1/v; q, t; C, \tau)] \tilde{R}_\lambda^{(n)}(;t_0:t_1, t_2, t_3; u_0, u_1; q, t; C, \tau) \\ = \left\langle \begin{matrix} \lambda \\ \mu \end{matrix} \right\rangle_{\left[\frac{\tau}{u_0u_1}, \frac{1}{v} \right] \left(\frac{t^{n-1}t_2t_3}{\tau}, \frac{qt^{n-1}t_0}{u_0}, \frac{vt_1}{u_1} \right); q, t; C}, \end{aligned} \quad (6.75)$$

and so forth.

7 Elliptic bigrids and degenerations

As we mentioned above, there is a fifth class of perfect bigrids, namely elliptic bigrids. In contrast to the earlier classes, the space of elliptic bigrids is of bounded dimension (8, that is) as $m, n \rightarrow \infty$, but as we have seen, in return for the decreased freedom, we gain a large number of new properties.

Elliptic bigrids of a given shape are parametrized by sextuples $(a, b; q, t; C, \phi)$, where C is a genus 1 curve over a field k , $\phi : C \rightarrow \mathbb{P}^1(k)$ is a degree 2 function (with associated divisor class $\tau \in \text{Pic}^2(C)$), $q, t \in \text{Pic}^0(C)$, and $a, b \in C$. The corresponding elliptic bigrid of shape m^n is then defined by the formula

$$\gamma(0, i, j) = \phi(aq^j t^{n-i}) \quad (7.1)$$

$$\gamma(1, i, j) = \phi(bq^{-j} t^{i-1}). \quad (7.2)$$

Theorem 7.1. *Elliptic bigrids are perfect.*

Proof. Fix a pair (C, ϕ) , and let the other parameters be generic; also choose a generic point $v \in C$. For each partition $\lambda \subset m^n$, define a $BC_n(\tau)$ -symmetric function f_λ by:

$$f_\lambda(\dots x_i \dots) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \frac{\phi(x_i) - \phi(q^j \tau/b)}{\phi(vt^{n-i}) - \phi(q^j \tau/b)} R_\lambda^{*(n)}(\dots x_i \dots; a, b(v); q, t; C, \tau). \quad (7.3)$$

Since f_λ is $BC_n(\tau)$ -symmetric, it factors through a symmetric rational function on $\mathbb{P}^1(k)$; consideration of poles shows that this rational function is in fact a polynomial. Now, from the vanishing properties of $R_\lambda^{*(n)}$, it follows that

$$f_\lambda(\dots at^{n-i} q^{\mu_i} \dots) = 0 \quad (7.4)$$

for $\mu \not\subset \lambda$. In particular, for $\nu \subset m^n$ with $\nu \not\subset \lambda$, we can replace the parts equal to m by any partition with parts $\geq m$. Thus

$$f_\lambda(\dots \gamma^+(0, i, \nu_i) \dots) = 0 \quad (7.5)$$

for a Zariski-dense collection of extensions of γ . Complementation symmetry of $R^{*(n)}$ gives the complementary vanishing conditions. In other words, the polynomial associated to f_λ is an interpolation polynomial, and thus the generic elliptic bigrid is perfect. Since perfection is closed, the theorem follows. \square

If two sextuples $(a, b; q, t; C, \phi)$, $(a', b'; q', t'; C', \phi')$ are equivalent in the sense that there exists an isomorphism $\psi : C \rightarrow C'$ such that

$$\psi(a) = a', \quad \psi(b) = b', \quad \psi(q) = q', \quad \psi(t) = t', \quad \phi' \circ \psi = \phi, \quad (7.6)$$

the associated elliptic bigrid is the same. In other words, we obtain a map from \mathcal{M} to the space of perfect bigrids, where \mathcal{M} is the moduli problem classifying sextuples modulo equivalence.

Theorem 7.2. *If $m, n \geq 2$, $(m, n) \neq (2, 2)$, the map from \mathcal{M} to the space of bigrids is birational; that is, an elliptic bigrid generically determines a unique (up to equivalence) sextuple $(a, b; q, t; C, \tau)$. Moreover, the closure of the space of elliptic bigrids of shape m^n is a rational variety of dimension 8.*

Proof. Let γ be a generic elliptic bigrid; we need to prove that it determines a unique sextuple. It suffices to consider the cases $(m, n) \in \{(3, 2), (2, 3)\}$, since a larger elliptic bigrid necessarily contains one of the two smaller bigrids. Consider the first case. We claim that the sextuple, and thus the full bigrid, is uniquely determined by the eight points

$$\gamma(0, 1, 0), \gamma(0, 1, 1), \gamma(0, 1, 2), \gamma(0, 2, 0), \gamma(0, 2, 1), \gamma(0, 2, 2), \gamma(0, 1, 1), \gamma(0, 1, 2). \quad (7.7)$$

To see this, consider the five pairs of points

$$(\gamma(0, 1, 0), \gamma(0, 1, 1)), (\gamma(0, 1, 1), \gamma(0, 1, 2)), (\gamma(0, 2, 0), \gamma(0, 2, 1)), (\gamma(0, 2, 1), \gamma(0, 2, 2)), (\gamma(1, 1, 1), \gamma(1, 1, 2)). \quad (7.8)$$

Each of these is of the form $(\phi(x), \phi(qx))$ for $x \in C$.

Lemma 7.3. *Let C be a genus 1 curve, with $\phi : C \rightarrow \mathbb{P}^1$ a degree 2 function, and $q \in \text{Pic}^0$ a generic point. Then there exists a unique (up to scale) homogeneous polynomial p on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 2)$ that vanishes on precisely those points of the form $(\phi(x), \phi(qx))$; the resulting subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to C . Moreover, this polynomial is symmetric under exchanging the factors \mathbb{P}^1 .*

Proof. Let ϕ have polar divisor $\langle z_1 \rangle \langle z_2 \rangle$. Then evaluation at $(\phi(x), \phi(qx))$ maps the space of polynomials of bidegree $(2, 2)$, of dimension 9, to the space $\mathcal{L}(\langle z_1 \rangle \langle z_2 \rangle \langle z_1/q \rangle \langle z_2/q \rangle^2)$, of dimension 8; existence of p follows. Any such polynomial cuts out a one-dimensional subscheme of arithmetic genus $(2-1)(2-1) = 1$, which is therefore the full collection of points $(\phi(x), \phi(qx))$, a curve isomorphic to C . In particular, uniqueness of p follows from the connectedness of the subscheme. Since

$$(\phi(x), \phi(qx)) = (\phi(\tau/x), \phi(\tau/qx)), \quad (7.9)$$

the subscheme is symmetric, and thus the symmetry of p follows. \square

Remark. Such ‘‘symmetric biquadratic relations’’ appear in the work of Baxter on the eight-vertex model, and other related statistical mechanical models [3]; they have also been mentioned in the context of univariate elliptic biorthogonal functions [27, 28]. In particular, the latter reference derives such a relation from the existence of an appropriate difference operator.

Now, the space of symmetric polynomials of bidegree $(2, 2)$ is six-dimensional, and thus five points in $\mathbb{P}^1 \times \mathbb{P}^1$ generically determine a unique (up to scale) such polynomial; a simple computation of minors shows that this remains true for our somewhat constrained points. We thus obtain a genus 1 curve C_0 , together with a degree 2 map ϕ_0 given by projection onto the first factor. The given five points are then of the form $a_0 t_0, a_0 q_0 t_0, a_0, a_0 q_0, b_0/q_0^2$ for uniquely determined points $a_0, b_0 \in C_0, q_0, t_0 \in \text{Pic}^0(C_0)$. But then γ is precisely the elliptic bigrid corresponding to the sextuple $(a_0, b_0; q_0, t_0; C_0, \phi_0)$, and the theorem follows for $(m, n) = (3, 2)$; the other case follows symmetrically. \square

Remark. In particular, it follows that an elliptic bigrid takes values over a field k if and only if its parameters can be defined over that field.

A similar argument shows that for $(m, n) \in \{(l, 1), (1, l)\}$ with $l \geq 4$, the corresponding map from quintuples (forgetting t or q as appropriate) is birational, with rational image of dimension 7; for $l < 4$, the map is surjective, and the range too small for birationality. In the remaining case $(m, n) = (2, 2)$, although the space of bigrids is 8-dimensional, the map fails to be birational.

Lemma 7.4. *The closure of the space of elliptic bigrids of shape 2^2 is given by the equation*

$$\chi(\gamma(0, 1, 0), \gamma(0, 2, 1), \gamma(1, 1, 2), \gamma(1, 2, 1)) = \chi(\gamma(0, 1, 1), \gamma(0, 2, 0), \gamma(1, 1, 1), \gamma(1, 2, 2)) \quad (7.10)$$

where χ is the cross-ratio function on \mathbb{P}^1 . In particular, this closure is a seven-dimensional rational variety.

Proof. The four points $at, \tau/aq, b/q^2, q\tau/bt \in C$ multiply to τ^2/q^2 ; it follows that the four points

$$(\gamma(0, 1, 0), \gamma(0, 1, 1)), (\gamma(0, 2, 1), \gamma(0, 2, 0)), (\gamma(1, 1, 2), \gamma(1, 1, 1)), (\gamma(1, 2, 1), \gamma(1, 2, 2)) \in \mathbb{P}^1 \times \mathbb{P}^1 \quad (7.11)$$

satisfy a common bilinear identity (the space of bilinear polynomials and the space $\mathcal{L}(D)$ with $\langle D \rangle = \tau^2/q^2$ are both 4-dimensional). The given identity follows. That the image of the space of elliptic bigrids is seven-dimensional (and thus agrees with the subscheme cut out by the equation) follows from a direct computation for the I_1 degeneration (see below). \square

While we have so far been unable to derive a full classification result for perfect bigrids—that is, a description of its decomposition into irreducible components—we can prove the following partial result.

Theorem 7.5. *If $m, n \geq 2$, then the space of elliptic bigrids is a component of the space of perfect bigrids; that is, the perfect bigrids in a sufficiently small neighborhood of a generic elliptic bigrid are all elliptic.*

Proof. It suffices to show that the tangent space to the space of perfect bigrids at a generic elliptic bigrid has the same dimension as the space of elliptic bigrids; that is, that it is seven-dimensional for $(m, n) = (2, 2)$, and otherwise eight-dimensional. For the small cases $(2, 2), (2, 3), (2, 4), (3, 2), (4, 2), (3, 3)$, we can verify this as follows (again a computation facilitated by using the I_1 degeneration). First, choose a random instance defined over \mathbb{Q} , and verify that the tangent space has the required dimension. Moreover, the ideal generated by appropriate minors determines a finite set of primes; reducing modulo a prime outside that set verifies the result in that characteristic. Then for each remaining characteristic, we can choose a random instance over a field of that characteristic, and again verify the dimension. This proves the result in those cases.

We now proceed by induction; let (m, n) be a pair not on the above list such that the theorem holds for all smaller pairs. If $m \geq 4$, the two natural sub-bigrids of shape $(m-1)^n$ overlap in a sub-bigrid of shape $(m-2)^n$, and thus by induction, for any bigrid in a small neighborhood of the original bigrid, these sub-bigrids are elliptic. By assumption, either $m \geq 5$ or $n \geq 3$, and thus a generic elliptic bigrid of shape $(m-2)^n$ uniquely determines its parameters. It follows that the sub-bigrids of shape $(m-1)^n$ have compatible parameters, and thus join to form an elliptic bigrid of shape m^n as required. The proof for $n \geq 4$ is analogous, and thus the theorem follows. \square

In particular, it follows that, just as we needed to generalize Okounkov's notion of interpolation polynomial to obtain elliptic special functions, any attempt to further generalize the elliptic theory must necessarily involve a further generalization of balanced interpolation polynomials; there is no room in the space of perfect bigrids to add more parameters to elliptic bigrids.

Now that we have ruled out generalization, it is natural to turn to specialization. The space of elliptic bigrids is clearly not closed; it is therefore of interest to describe the various degenerations of elliptic bigrids, i.e., to give explicit descriptions of the points in the closure of the space. Again, we will be content with a partial result, a description of the four (over the algebraic closure) particular classes of degenerations that can produce arbitrarily large regular bigrids.

The key to understanding these degenerations is the curve $C_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ constructed in proving Theorem 7.2. If this curve is smooth, then the bigrid is elliptic, so assume it is singular; we also assume it is reduced (since the nonreduced case can only be regular if $m = 2$, and corresponds to the case $q^2 = 1$). Thus C_0 has isolated singular points, and since it has arithmetic genus one, consists of a collection of \mathbb{P}^1 s.

Suppose first that C_0 is irreducible, and thus has only one singular point over \bar{k} . The singular point must lie on the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$, and by simultaneous (k -rational) linear fractional transformation can be taken to be (∞, ∞) . We thus obtain a curve on $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ of the form

$$a_{20}(x^2 + y^2) + a_{11}xy + a_{10}(x + y) + a_{00}. \quad (7.12)$$

If $a_{20} = 0$, any sequence of points $x_0, x_1, x_2, x_3 \in \mathbb{P}^1$ with $(x_0, x_1), (x_1, x_2), (x_2, x_3) \in C$ will contain the same point twice; it again follows that this case cannot arise from large regular bigrids. We may thus assume $a_{20} = 1$. If $a_{11} \neq -2$, then we can eliminate the linear term; then $a_{00} \neq 0$, lest $(0, 0)$ be another singular point. There are then two cases, depending on whether the quadratic term has roots over k . If it does, let q be such that $q^2 + a_{11}q + 1 = 0$. Then we find that the smooth k -rational points on C are precisely those points of the form

$$\left(\frac{u + a_{00}/u}{q - 1/q}, \frac{qu + a_{00}/qu}{q - 1/q} \right) \quad (7.13)$$

with $u \in k^*$. In other words, the smooth points of C_0 can be identified with the multiplicative group in such a way that translation by q_0 is multiplication by q . Similarly, the translation by t_0 induced by the bigrid is multiplication by some element $t \in k^*$, as is the translation by b/a . We thus obtain the following class of (perfect) bigrids, upon restoring the linear fractional transformation freedom.

Definition 17. A degenerate elliptic bigrid of type I_1 is a bigrid given by the formula

$$\gamma(0, i, j) = \phi(aq^j t^{n-i}) \quad \gamma(1, i, j) = \phi(bq^{-j} t^{i-1}), \quad (7.14)$$

where $a, b, q, t \in k^*$, and ϕ is a k -rational degree 2 function taking the same values at 0 and ∞ .

Remark 1. In particular, it is quite straightforward to compute the tangent space to a generic such bigrid of shape 2^2 , and thus show that that space is 7-dimensional; since the closure of the space of elliptic bigrids contains this space, its dimension must be at least as large.

Remark 2. The notation I_1 is motivated by the fact that the curve C_0 is a degenerate elliptic curve of Kodaira symbol I_1 , and similarly for the labels we use for the other untwisted classes.

We similarly obtain a twisted version of this, when $x^2 + a_{11}xy + y^2$ is irreducible over k . Let l be the splitting field of this polynomial.

Definition 18. A degenerate elliptic bigrid of type I'_1 is a bigrid given by the formula

$$\gamma(0, i, j) = \phi(aq^j t^{n-i}) \quad \gamma(1, i, j) = \phi(bq^{-j} t^{i-1}), \quad (7.15)$$

where a, b, q, t are in the subgroup of l^* of norm 1 and ϕ is a nonconstant function on this subgroup of the form

$$\phi(x) = \frac{\alpha \operatorname{Tr}(x) + \beta}{\gamma \operatorname{Tr}(x) + \delta}, \quad (7.16)$$

$\alpha, \beta, \gamma, \delta \in k$. Note that ϕ takes values in $k \cup \{\infty\}$.

The final case with a single singular point is when the quadratic term has a multiple root; i.e., the singular point is a cusp. In finite characteristic p , we find that the translation map has period p (period 4 in characteristic 2), and thus only leads to bigrids of bounded size; we may thus assume k of characteristic 0. We obtain the following.

Definition 19. A degenerate elliptic bigrid of type II is a bigrid given by the formula

$$\gamma(0, i, j) = \phi(a + jq + (n - i)t) \quad \gamma(1, i, j) = \phi(b - jq + (i - 1)t), \quad (7.17)$$

where $a, b, q, t \in k$, and ϕ is a degree two function on k of the form

$$\phi(x) = \frac{\alpha x(\tau - x) + \beta}{\gamma x(\tau - x) + \delta}, \quad (7.18)$$

with $\alpha, \beta, \gamma, \delta, \tau \in k$.

Remark. This case can also be thought of as a limit of the I_1 case, by taking all of the points to 1 at comparable rates, including the zeros and poles of ϕ . It corresponds (for sufficiently large m, n) to a 6-dimensional subvariety of the space of elliptic grids.

There are two cases to consider when C_0 is reducible. Again, to produce regular bigrids, each factor must have bidegree at least $(1, 1)$, so there are two factors, both of bidegree $(1, 1)$. Either the individual factors are symmetrical, and distinct, or the two factors form an orbit under the exchange map. Note that since the curve possesses a large number of rational points, each component must in fact be rational.

In the second case, we find that either there are two points of intersection (on the diagonal), possibly defined over a quadratic extension, or there is one rational point of intersection where the two components are tangent.

(I.e., the curve has Kodaira symbol I_2 or III respectively.) For I_2 , we can transform to obtain the curve $(x - qy)(y - qx)$, while for III we can transform to $(x - y)^2 - q^2$.

Now, suppose the curve associated to j translation is reducible, and consider the curve associated to i translation. We find that it must also be reducible, and further that there are compatibility constraints. Further analysis reveals the following possibilities for bigrids, covering all of the reducible cases.

Choose an element $\kappa \in k^*$, and construct a group $k^*.2(\kappa)$ on the set $k^* \times \mathbb{Z}_2$ by the rule

$$(x, 0)(y, 0) = (xy, 0) \quad (x, 0)(y, 1) = (xy, 1) \quad (x, 1)(y, 0) = (xy, 1) \quad (x, 1)(y, 1) = (xy/\kappa, 0). \quad (7.19)$$

Definition 20. A degenerate elliptic bigrid of type I_2 is a bigrid given by the formula

$$\gamma(0, i, j) = \phi(aq^j t^{n-i}) \quad \gamma(1, i, j) = \phi(bq^{-j} t^{i-1}), \quad (7.20)$$

where $a, b, q, t \in k^*.2(\kappa)$, and ϕ is defined by

$$\phi(x, 0) = \phi(\tau/x, 1) = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad (7.21)$$

for some nonconstant, rational, linear fractional transformation and some $\tau \in k^*$.

Remark 1. Similarly, there is a twisted version with k^* replaced by the norm 1 subgroup of a quadratic extension, and ϕ replaced by a nonconstant $k \cup \{\infty\}$ -valued map

$$\phi(x, 0) = \phi(\tau/x, 1) = \frac{\alpha x + \bar{\alpha}}{\gamma x + \bar{\gamma}}. \quad (7.22)$$

Remark 2. In truth, this is comprised of a total of 8 different cases, depending on in which coset of k^* each of $a/b, q, t$ lies. The result, for sufficiently large m, n , is a 7-dimensional space of bigrids, unless $q, t \in k^*$, in which case the space is 6-dimensional. In particular, only the latter two cases (in which the full group structure is not being used) can be obtained by degeneration of I_1 . (The corresponding biorthogonal functions are multivariate analogues of the biorthogonal functions of [1].) Apparently the other I_2 cases have never been studied, even at the univariate level (where for $q \notin k^*$ we obtain series such that the term ratios t_{2m}/t_{2m-1} and t_{2m+1}/t_{2m} are *different* rational functions of q^{2m}). One can thus view such a series as a sum of two basic hypergeometric series (by separating out the terms of even index from those of odd index). For instance, one such identity (a limiting case of the Frenkel-Turaev summation identity [6]) is that

$$\begin{aligned} & {}_6\phi_5 \left(\begin{matrix} a, b_1, b_2, b_3, b_4, a^2/b_1 b_2 b_3 b_4 \\ a/b_1, a/b_2, a/b_3, a/b_4, q^2 b_1 b_2 b_3 b_4/a \end{matrix} \middle| q^2, q^2 \right) \\ & - \frac{a^2(1-b_1)(1-b_2)(1-b_3)(1-b_4)}{(a-b_1)(a-b_2)(a-b_3)(a-b_4)} {}_6\phi_5 \left(\begin{matrix} a, q^2 b_1, q^2 b_2, q^2 b_3, q^2 b_4, a^2/b_1 b_2 b_3 b_4 \\ q^2 a/b_1, q^2 a/b_2, q^2 a/b_3, q^2 a/b_4, q^2 b_1 b_2 b_3 b_4/a \end{matrix} \middle| q^2, q^2 \right) \\ & = \frac{(a, a/b_1 b_2, a/b_1 b_3, a/b_1 b_4, a/b_2 b_3, a/b_2 b_4, a/b_3 b_4, a/b_1 b_2 b_3 b_4; q^2)}{(a/b_1, a/b_2, a/b_3, a/b_4, a/b_1 b_2 b_3, a/b_1 b_2 b_4, a/b_1 b_3 b_4, a/b_2 b_3 b_4; q^2)} \end{aligned}$$

as long as both sums terminate. Note that the above hypergeometric sums are balanced but not *quite* well-poised.

Remark 3. This is superficially similar to the degeneration of ordinary interpolation polynomials denoted IV in [13], which exists only for $n = 2$; the latter is not, however, a degeneration of our I_2 , but rather corresponds to a degenerate curve of Kodaira symbol I_4 (which never produces a regular bigrid).

Finally, we have the additive analogue of type I_2 ; we define $k.2(\kappa)$ analogously.

Definition 21. A degenerate elliptic bigrid of type *III* is a bigrid given by the formula

$$\gamma(0, i, j) = \phi(a + jq + (n - i)t) \quad \gamma(1, i, j) = \phi(b - jq + (i - 1)t), \quad (7.23)$$

where $a, b, q, t \in k.2(\kappa)$, and ϕ is given by

$$\phi(x, 0) = \phi(\tau - x, 1) = \frac{\alpha x + \beta}{\gamma x + \delta} \quad (7.24)$$

for some nonconstant linear fractional transformation.

Remark. Again, aside from degenerations of type *II*, and aside from a brief mention of the version of this degeneration for ordinary interpolation polynomials in [13] (there denoted by III(abc)), this case does not appear to have been studied. Each *III* case is one dimension smaller than the corresponding I_2 case.

8 The trigonometric degeneration

Of course, the classification of degenerate elliptic bigrids is just a first step towards an understanding of the various possible limit cases of our results. The “trigonometric” limit (corresponding to bigrids of type I_1) is of particular interest, as this is where the Koornwinder polynomials [9] and Okounkov’s interpolation polynomials [12] (and thus the results of [16]) are to be found. Although we will only be considering those degenerations necessary to connect our results to those of [16], it is striking how many different limits arise there. We thus second Rosengren’s call in [17] for a more systematic classification of limits of elliptic hypergeometric structures.

The only subtle point in extending the algebraic construction to the I_1 case is the construction of the difference operator; the point is that the divisor $\langle \tau^2 \rangle$ on a smooth C degenerates to a divisor that hits the node. As a result, a simple condition on poles is not quite enough to specify $\delta^{(n)}$; we must add a condition comparing the asymptotics along the two branches of the node. We thus obtain the following expression for the degenerate $\delta^{(n)}$, where $t^{n-1}u_0u_1u_2u_3 = \tau^2$, and all parameters lie in k^* .

$$\begin{aligned} & \delta^{(n)}(\dots x_i \dots; u_0, u_1, u_2, u_3; t; G_m, \tau) \\ &= \prod_{1 \leq i \leq n} \frac{u_0 t^{n-i} (1 - u_0 x_i / \tau) (1 - u_1 x_i / \tau) (1 - u_2 x_i / \tau) (1 - u_3 x_i / \tau)}{x_i (1 - x_i^2 / \tau) (1 - u_1 u_0 t^{n-i} / \tau) (1 - u_2 u_0 t^{n-i} / \tau) (1 - u_3 u_0 t^{n-i} / \tau)} \prod_{1 \leq i < j \leq n} \frac{1 - t x_i x_j / \tau}{1 - x_i x_j / \tau}. \end{aligned} \quad (8.1)$$

(Here G_m refers to the multiplicative group scheme, or rather its natural compactification to a nodal \mathbb{P}^1 .) Then, if we define $A_m^{(n)}(u_0; q; G_m, \tau)$ analogously to the elliptic case, we have the following analogue of Theorem 6.3.

Theorem 8.1. Define a difference operator on $BC_n(\tau)$ -symmetric rational functions by

$$\begin{aligned} & (\tilde{D}^{(n)}(u_0, u_1, u_2; q, t; G_m, \tau)f)(\dots x_i \dots) \\ &= \prod_{1 \leq i \leq n} (1 + t_{i, \tau}^*) \delta^{(n)}(\dots x_i \dots; u_0, u_1, u_2, \frac{\tau^2}{t^{n-1}u_0u_1u_2}; t; G_m, \tau)f(\dots x_i \dots), \end{aligned} \quad (8.2)$$

where $\iota_\tau(x) = \tau/x$. Then for any integer $m \geq 0$,

$$\tilde{D}^{(n)}(u_0, u_1, u_2; q, t; G_m, \tau) A_m^{(n)}(u_0; q; G_m, \tau/q) \subset A_m^{(n)}(u_0; q; G_m, \tau). \quad (8.3)$$

Moreover,

$$\begin{aligned} (\tilde{D}^{(n+k)}(u_0, u_1, u_2; q, t; G_m, \tau) f)(z_1, \dots, z_n, u_0, \dots, t^{k-1} u_0) \\ = (\tilde{D}^{(n)}(u_0, u_1, u_2; q, t; G_m, \tau) f(\underline{\quad}, u_0, \dots, t^{k-1} u_0))(z_1, \dots, z_n). \end{aligned} \quad (8.4)$$

Proof. The only way in which the proof differs is that we need to show that the image function is smooth at the node. This is true by assumption for f , as is the condition (by ι_τ -invariance) that either branch of the node gives the same limit. But then multiplying by $\delta^{(n)}$ gives a function with linear growth at the node, in opposite directions along the two branches. Thus symmetrizing kills off this linear growth, giving smoothness as required. \square

Similarly, the Δ^0 , Δ and diagonal binomial coefficient symbols are all straightforward to define. We find, if $u_0 u_1 \cdots u_{2m-1} = \tau^m$,

$$\omega(x; u_0, \dots, u_{2m-1}; G_m, \tau) = \prod_{0 \leq r < 2m} \frac{1 - x/u_r}{1 - u_r x/\tau}, \quad (8.5)$$

and thus if $v_0 v_1 \cdots v_{2m-1} = q^m a^m$,

$$\Delta_\lambda^0(a | v_0, \dots, v_{2m-1}; q, t; G_m) = \prod_{(i,j) \in \lambda} \prod_{0 \leq r < 2m} \frac{1 - q^{1-j} t^{i-1} / v_r}{1 - v_r q^{-j} t^{i-1} / a} \quad (8.6)$$

$$= \frac{C_\lambda^0(v_0, \dots, v_{2m-1}; q, t)}{C_\lambda^0(qa/v_0, \dots, qa/v_{2m-1}; q, t)} \quad (8.7)$$

Similarly, if $v_0 v_1 \cdots v_{2m-1} = (t/q)(qa)^{m-1}$

$$\Delta_\lambda(a | v_0, \dots, v_{2m-1}; q, t; G_m) = \frac{q^{|\lambda|} t^{2n(\lambda)} C_{2\lambda^2}^0(qa; q, t) C_\lambda^0(v_0, \dots, v_{2m-1}; q, t)}{C_\lambda^-(q, t; q, t) C_\lambda^+(a, qa/t; q, t) C_\lambda^0(qa/v_0, \dots, qa/v_{2m-1}; q, t)} \quad (8.8)$$

Finally,

$$\binom{\lambda}{\lambda}_{[a,b];q,t;G_m} = \frac{b^{|\lambda|} C_\lambda^+(a; q, t) C_\lambda^0(1/b, qa/b; q, t)}{C_\lambda^+(a/b; q, t) C_\lambda^0(b, qa; q, t)} \quad (8.9)$$

(Here we use the C^* notations of [16], which are simply the limits $p \rightarrow 0$ of the elliptic notations.)

It is then trivial to state the analogues of the elliptic results for this most general trigonometric case; simply replace C by G_m in each identity of Section 6. This in particular gives multivariate analogues of the ${}_{10}\phi_9$ biorthogonal rational functions of Rahman [14] (or, more precisely, since we are considering only discrete biorthogonality here, they are multivariate analogues of the biorthogonal rational functions of Wilson [30]; in any event, continuous biorthogonality *is* preserved in the trigonometric limit). The new thing that appears is that we can take limits as various parameters approach 0 or infinity, i.e., the node of the curve. There are, it turns out, three different limits of interpolation functions and binomial coefficients that need to be considered for a full understanding of [16] from our current perspective.

Of course, one of these limits is simply the ordinary interpolation polynomials $\bar{P}_\lambda^{*(n)}(; q, t, s)$. More precisely, we have the following limit theorem.

Theorem 8.2. *The interpolation functions and ordinary interpolation polynomials are related by the identity*

$$\bar{P}_\lambda^{*(n)}(\dots x_i \dots; q, t, a) = \frac{t^{2n(\lambda)} C_\lambda^0(t^n, t^{n-1}av, a/v; q, t)}{(-t^{n-1}a)^{|\lambda|} q^{n(\lambda')} C_\lambda^-(t; q, t)} \lim_{b \rightarrow \infty} R_\lambda^{*(n)}(\dots x_i \dots; a, b(v); q, t; G_m, \langle 1 \rangle^2). \quad (8.10)$$

Moreover, replacing the limit $b \rightarrow \infty$ by $b \rightarrow 0$ gives the same result.

Proof. Fix a, q, t , and let

$$f_\lambda^{(n)}(\dots x_i \dots; v) \quad (8.11)$$

denote the corresponding right-hand side. We claim that for each n , $f_\lambda^{(n)}(; v)$ is a well-defined BC_n -symmetric Laurent polynomial of degree at most $|\lambda|$. Indeed, from the branching rule, we find

$$f_\lambda^{(n)}(\dots x_i \dots; v) = \sum_{\kappa} \frac{C_{\lambda/\kappa}^0(t^{n-1}ax_n^{\pm 1}; q, t)}{C_{\lambda/\kappa}^0(t^{n-1}av^{\pm 1}; q, t)} \lim_{b \rightarrow \infty} \left\langle \lambda \right\rangle_{[t^{n-1}a/b, t](qa/tb, qt^{n-1}v/b, t^{n-1}a/v); q, t; G_m} f_\kappa^{(n-1)}(\dots x_i \dots; vt), \quad (8.12)$$

so by induction it suffices to show that the binomial coefficient has a well-defined limit. This then follows readily from the algebraic analogue of Corollary 4.8.

Thus $f_\lambda^{(n)}(; v)$ is a BC_n -symmetric Laurent polynomial of at most the correct degree, vanishing at points of the form $\dots at^{n-i}q^{\mu_i} \dots$ for $\mu \not\subset \lambda$, and is thus a multiple of the ordinary interpolation polynomial as required. That this multiple is 1 follows by comparing values at the point $\dots vt^{n-i} \dots$.

The argument for $b \rightarrow 0$ is analogous. □

Remark 1. One can also use the point $\dots at^{n-i}q^{\lambda_i} \dots$ to set the normalization; the resulting computation is somewhat more complicated, but there is the slight advantage that the proof in [16] of the latter normalization does not use any properties of Macdonald polynomials. As a result, we see that we can deduce such properties from the elliptic theory.

Remark 2. It ought to be possible to prove directly, using only the vanishing conditions, that the interpolation polynomials are limits of interpolation functions. More precisely, given any partial order on partitions refining the inclusion partial order, the complementary vanishing conditions define a corresponding filtration on the space of BC_n -symmetric rational functions; the main question is then to show that one of these filtrations has the correct limit as $b \rightarrow 0, \infty$. (Two cases of particular interest are the dominance order, for which the limiting filtration is simply that given by dominance of monomials, so in particular is independent of q and t , and the inclusion order itself, for which the limiting filtration is, by results of [16], that induced by the Macdonald polynomials.)

Plugging into the respective definitions of binomial coefficients gives the following result (in which the right-hand side uses the notation of Section 4 of [16]).

Corollary 8.3. *We have the following identity of binomial coefficients.*

$$\lim_{b \rightarrow 0, \infty} \binom{\lambda}{\mu}_{[a, b]; q, t; G_m} = \frac{(-1)^{|\mu|} t^{n(\mu)} C_\mu^+(a; q, t)}{q^{n(\mu')} C_\mu^0(qa; q, t)} \left[\begin{matrix} \lambda \\ \mu \end{matrix} \right]_{q, t, \sqrt{a}} \quad (8.13)$$

Now, it then follows that we have a corresponding limit for the inverse binomial coefficients of [16].

Corollary 8.4. *We have the following identity of binomial coefficients.*

$$\lim_{b \rightarrow 0, \infty} \binom{\lambda}{\mu}_{[a/b, 1/b]; q, t; G_m} = \frac{(-1)^{|\lambda|} q^{n(\lambda')} C_{\lambda}^0(qa; q, t)}{t^{n(\lambda)} C_{\lambda}^+(a; q, t)} \left\{ \begin{matrix} \lambda \\ \mu \end{matrix} \right\}_{q, t, \sqrt{a}} \quad (8.14)$$

Now, the limiting bigrid on the left is in fact a regular bigrid of type I_2 ; we thus obtain a direct description of the inverse binomial coefficients in terms of vanishing conditions. In particular, this explains what happened to the evaluation symmetry condition when extending from shifted Macdonald polynomials [20, 8] to ordinary interpolation polynomials [12]: rather than relate two functions of the same type, the symmetry relates functions from two different degenerations. The resulting family of degenerate interpolation functions is simply the limit as $\tau \rightarrow 0$ or ∞ of the general trigonometric interpolation functions. In particular, for these functions, the generalized eigenvalue problem simplifies to an eigenvalue problem, relative to the family of difference operators (parametrized by c with a, b, q, t fixed):

$$\sum_{I \subset \{1, 2, \dots, n\}} \frac{t^{|I|(|I|-1)/2} \prod_{i \in I} (x_i(1 - a/x_i)(1 - b/x_i)) \prod_{i \notin I} (c(1 - x_i/c)(1 - t^{n-1}ab/cx_i))}{\prod_{1 \leq i \leq n} (c(1 - at^{n-i}/c)(1 - bt^{n-i}/c))} \prod_{i \in I, j \notin I} \frac{x_i - tx_j}{x_i - x_j} f(\dots q^{-[i \in I]} x_i \dots), \quad (8.15)$$

with eigenvalues

$$\frac{q^{-|\lambda|} C_{\lambda}^0(aqt^{n-1}/c, qc/b; q, t)}{C_{\lambda}^0(t^{n-1}a/c, c/b; q, t; p)}. \quad (8.16)$$

(Replace c by τ/c in the difference equation for trigonometric interpolation functions, then take $\tau \rightarrow 0, \infty$. Note that the further limit $c \rightarrow 0, \infty$ makes this the identity operator.)

If we attempt to obtain the bulk branching rule for ordinary interpolation polynomials (Theorem 3.9 of [16]) as a limit of our theory, we find a third limit of binomial coefficients appearing.

Theorem 8.5. *We have the following identity of binomial coefficients.*

$$\lim_{a \rightarrow 0, \infty} \binom{\lambda}{\mu}_{[a, b]; q, t; G_m} = \frac{b^{|\mu|} t^{n(\mu)} C_{\lambda}^-(t; q, t) C_{\mu}^0(1/b; q, t)}{t^{n(\lambda)} C_{\mu}^-(t; q, t) C_{\lambda}^0(b; q, t)} P_{\lambda/\mu}([\frac{1-b^k}{1-t^k}]; q, t) \quad (8.17)$$

Proof. Take the appropriate limit of the bulk branching rule for ordinary interpolation functions to obtain a bulk branching rule for ordinary interpolation polynomials. Taking a further limit produces the Macdonald polynomials (i.e., polynomials satisfying Macdonald's difference equation); the result then follows essentially by the definition of $P_{\lambda/\mu}$. \square

Remark 1. If we take a further limit $b \rightarrow 0$, the binomial coefficient can be expressed in shifted Macdonald polynomials; we thus obtain Theorem I.4 of [7], which expresses the corresponding binomial coefficients via the plethystic expression

$$P_{\lambda/\mu}([\frac{1}{1-t^k}]; q, t). \quad (8.18)$$

Remark 2. Of our three types of degenerate trigonometric binomial coefficients, this is the only one that includes the four important special cases

$$\binom{\lambda}{\mu}_{[a, 1]; q, t}, \binom{\lambda}{\mu}_{[a, 1/q]; q, t}, \binom{\lambda}{\mu}_{[a, t]; q, t}, \binom{\lambda}{\mu}_{[a, q/t]; q, t}. \quad (8.19)$$

That the first three specializations of skew Macdonald polynomials have nice factorizations is well known (the first from the fact that $P_{\lambda/\mu}$ has positive degree, so constant term 0, the second and third from the coefficients of the Pieri identities and branching rule). That

$$P_{\lambda/\mu}\left(\left[\frac{1-(q/t)^k}{1-t^k}\right]; q, t\right) \quad (8.20)$$

can be expressed as a product of binomials does not appear to have been observed before.

These also correspond to a degenerate elliptic bigrid of type I_2 , this time with $a/b \notin k^*$. The difference operators are the same as for the previous I_2 case, except that now the family has a, c, q, t fixed and b variable; in particular, we return to a generalized eigenvalue problem scenario.

We thus find that all of the ordinary interpolation polynomial identities of [16] are indeed limits of identities satisfied by our interpolation functions. Similarly, the Koornwinder polynomials are limiting cases of our biorthogonal functions:

$$\lim_{u_0 \rightarrow 0, \infty} \tilde{R}^{(n)}(\dots x_i \dots; t_0: t_1, t_2, t_3; u_0, q/t^{2n-2} t_0 t_1 t_2 t_3 u_0; q, t; G_m, \langle 1 \rangle^2) = \frac{K_\lambda^{(n)}(\dots x_i \dots; q, t; t_0, t_1, t_2, t_3)}{K_\lambda^{(n)}(\dots t_0 t^{n-i} \dots; q, t; t_0, t_1, t_2, t_3)}, \quad (8.21)$$

where the normalization factor on the right can be deduced by taking coefficients of $P_\lambda^{*(n)}(q, t, t_0)$ on both sides. In particular, we note that Theorems 5.6, 5.9, and 5.10 give rise to new results at the Koornwinder level, generalizing similar results from [16] in which the relevant trigonometric binomial coefficient has second parameter $\in \{1/q, t\}$ (i.e., corresponds to a difference or integral operator).

As an indication of some of the subtle issues that can arise when degenerating the elliptic theory, we consider the following ‘‘rational’’ limit. Here we take $q, t \rightarrow 1$ along a one-parameter family in such a way that the corresponding maps of tangent spaces have ratio α at the limit point. In other words, we blow up $\text{Pic}^0(C)^2$ at the point $(1, 1)$ and consider the interpolation function at a point on the exceptional divisor. Note that the result is essentially independent of every parameter *except* α , including the choice of C itself.

Theorem 8.6. *Let (C, τ) be an arbitrary genus 1 hyperelliptic curve over a field of characteristic 0. Then we have the following identity of interpolation functions, for $a, b, v \in C$, $\alpha \in k^*$ generic.*

$$\lim_{\substack{q, t \rightarrow 1 \\ dq/dt \rightarrow \alpha}} R_\lambda^{*(n)}(\dots z_i \dots; a, b(v); q, t; C, \tau) = \frac{J_\lambda(\dots \chi(z_i, a, v, b; C, \tau) \dots; \alpha)}{J_\lambda(\dots 1 \dots; \alpha)}, \quad (8.22)$$

where $J_\lambda(\dots; \alpha)$ is a Jack polynomial.

Proof. It suffices to prove the result for (C, τ) of the form $(\mathbb{C}^*/p, \langle 1 \rangle^2)$, in which case taking a limit in the branching rule gives the desired result. \square

There are several points to observe here. The first is that, unlike in the case of bigrids, we obtained a nontrivial degeneration without degenerating the curve itself. In fact, while the corresponding elliptic bigrid has a well-defined limit, that limit is very far from being regular; indeed, the limiting bigrid is actually independent of i and j , having two constant values independent of α . In other words, the map from elliptic bigrids to

interpolation polynomials is merely birational; a full classification of degenerate cases of the one is insufficient to give a classification the other. The second point is that, compared to the Macdonald polynomial case, it is much harder to obtain a useful limit from the difference equation. Indeed, the difference operator actually becomes the identity in the limit, and even the generalized eigenvalue problem, while it has a nontrivial limit when properly scaled, simply gives (after a nontrivial calculation) the equation

$$\sum_i \frac{\chi(z_i, a, v, b; C, \tau) \partial f(z_i)}{f(z_i) \partial \chi(z_i, a, v, b; C, \tau)} = |\lambda|, \quad (8.23)$$

corresponding to the fact that J_λ is homogeneous of degree $|\lambda|$. The third point is that we can similarly obtain limits of binomial coefficients analogous to the three trigonometric limits we obtained above; we find again that those limits are essentially independent of those parameters not approaching 1. We finally note that using the binomial formula, we can obtain an analogous limit for the biorthogonal functions; if $t_0 t_1, t_2 t_3, u_0 u_1$ all tend to 1 simultaneously with q, t , then the biorthogonal function becomes a multivariate Jacobi polynomial, up to an analogous change of variables. In the case $t_0 \sim 1, t_2 \sim -1$, the continuous biorthogonality density of [15] easily simplifies to the usual Jacobi ensemble (just as with the analogous limit for the Koornwinder polynomials); more generally, if $|t_0| = 1 = |t_2|$, we can at least formally obtain the appropriate ensemble, using the remark following Lemma 3.3 of [15].

9 Open problems

It seems fitting to conclude by discussing several open problems arising from the above theory. We have already discussed partial results for one such problem in the previous section, namely the question of classifying degenerations of interpolation and biorthogonal functions; along these lines, we also note the question of finding a direct proof that the interpolation polynomials are limits of the interpolation functions. For the remaining problems we consider, we have even less in the way of explicit results or even conjectures.

The most popular approach to the theory of Koornwinder polynomials involves the so-called “double affine Hecke algebra” [21], along with the associated representation of the affine Hecke algebra. A major open question, therefore, is how to generalize this theory to the elliptic level, both to understand how the Koornwinder theory relates to the elliptic theory, and ideally to allow the elliptic theory to be extended to other root systems. Via evaluation symmetry, this should be related to the question of finding a general Pieri identity for the biorthogonal functions. We note the following partial result.

Proposition 9.1. *For otherwise generic parameters satisfying $t^{2n-2} t_0 t_1 t_2 t_3 u_0 u_1 = q\tau^3$, and generic $u, v \in C$, let $c_{\lambda\kappa}$ be the coefficient of*

$$\tilde{R}_\kappa^{(n)}(\dots z_i \dots; t_0: t_1, t_2, t_3; u_0 q, u_1/q; q, t; C, \tau) \quad (9.1)$$

in the product

$$\prod_{1 \leq i \leq n} \chi(z_i, u, vt^{n-i}, u_0; C, \tau) \tilde{R}_\lambda^{(n)}(\dots z_i \dots; t_0: t_1, t_2, t_3; u_0, u_1; C, \tau). \quad (9.2)$$

Then $c_{\lambda\kappa} = 0$ unless $\lambda \subset 1^n + \kappa \subset 2^n + \lambda$.

Proof. That $\kappa \subset 1^n + \lambda$ follows from the Pieri identity for interpolation functions and the fact that the biorthogonal functions are triangular in the appropriate basis of interpolation functions. That $\lambda \subset 1^n + \kappa$ follows by the fact that for $t^{n-1}q^m t_0 t_1 = \tau$, $c_{\lambda\kappa}$ can be computed via biorthogonality, and thus satisfies a symmetry switching λ and κ . \square

In the cases $u \in \{t_0, t_1, t_2, t_3, t_0/q, t_1/q, t_2/q, t_3/q\}$, we can combine the quasi-Pieri identity, Theorem 5.10, with the connection coefficient identity, to express the above coefficient as a sum over binomial coefficients with $b = 1/q$, which is thus explicitly supported on the stated set of partitions. The question is thus whether a corresponding formula exists for general u . (An expression in a double sum is straightforward, but it is then nonobvious that the coefficients have the support they do.) This would give a family difference operators via evaluation symmetry, which (by consideration of the Koornwinder case) would give the analogue of the degree 1 subspace of center of the affine Hecke algebra. (See also the remark following Theorem 8.9 of [15].)

Similar considerations hold for a general branching rule.

Proposition 9.2. *For otherwise generic parameters satisfying $t^{2n-2}t_0 t_1 t_2 t_3 u_0 u_1 = q\tau^3$, and generic $u \in C$, let $c_{\lambda\kappa}$ be the coefficient of*

$$\tilde{R}_\kappa^{(n-1)}(\dots z_i \dots; t_0: t_1, t_2, t_3; u_0, u_1 t^2; q, t; C, \tau) \quad (9.3)$$

in the specialization

$$\tilde{R}_\lambda^{(n)}(\dots z_i \dots, u; t_0: t_1, t_2, t_3; u_0, u_1; C, \tau). \quad (9.4)$$

Then $c_{\lambda\kappa} = 0$ unless $\kappa \prec'_2 \lambda$.

Proof. Using the Cauchy identity, Theorem 5.11, one can express the coefficients of the branching rule in terms of the coefficients of the Pieri identity (and vice versa); the stated vanishing property follows. \square

In particular, we see that it is hopeless to expect anything better than a sum for these coefficients, as even in the case $n = 1$ we obtain a nontrivial hypergeometric sum. We also note that such an expression for the branching rule would be a new result even for the Koornwinder case.

Another question, which is less important, but likely to have interesting consequences, concerns the classification of regular perfect bigrids. The four non-elliptic components we have identified above all correspond to important special cases of the interpolation functions, and thus of the corresponding hypergeometric identities. It thus seems likely that other components would give similarly interesting special cases. One such component would appear to correspond to the following.

Conjecture 1. *Any bigrid such that $\gamma(1, i, j) = \gamma(0, i + 1, j)$ whenever both sides are defined is perfect.*

Note that this would give a $((n + 1)(m + 1) - 2)$ -dimensional space of perfect grids, and would thus be larger even than the $m(n + 1)$ -dimensional Cauchy case. When $m = n = 2$ (in which case the conjecture is straightforward to verify), we obtain a 7-dimensional space, distinct from the 7-dimensional space of elliptic bigrids; in contrast, a bigrid of shape 2^2 coming from any of the four known cases is necessarily elliptic. In the elliptic case, this corresponds to binomial coefficients with $b = 1/t$, and thus should correspond in some sense to an inverse of the integral operator of [15].

The other conjectured component we have found (by computing tangent spaces to random elliptic bigrids over small finite fields) is the following.

Conjecture 2. *Let τ be a linear fractional transformation of order 2. Then any bigrid satisfying*

$$\gamma(0, i, j) = \tau(\gamma(1, i, j + 1)) = \tau(\gamma(0, i + 1, j + 1)) \quad (9.5)$$

whenever both sides are defined is perfect.

Finally, we observe that we have only extended about half of the results of [16] to the elliptic case. The remaining results fall into two main classes. The first of these is the construction of families of symmetric functions algebraically continuing interpolation and Koornwinder polynomials in the number of variables (via the quantity t^n). Although it seems unlikely that these results extend cleanly to the elliptic level (given the absence of an appropriate theory of symmetric functions), there should still be some weak version; after all, the connection coefficients that arise above depend on the number of variables only via t^n , and similarly for the bulk branching rule. The other class of results concerns the “vanishing integrals” conjectured in [16], which generalize quadratic transformations of univariate hypergeometric series; elliptic analogues of these would likely be interesting even at the univariate level. Note in particular that there are two potential kinds of quadratic transformation at the elliptic level, depending on whether the relevant isogeny is a multiplication map (squaring, in our notation), or simply an isogeny of degree 2.

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