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EXISTENCE, LOCAL UNIQUENESS, AND OPTIMALITY OF A MARGINAL COST PRICING EQUILIBRIUM IN AN ECONOMY WITH INCREASING RETURNS

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SOCIAL SCIENCE WORKING PAPER 415

January 1982

ABSTRACT

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This paper proposes a notion of equilibrium for an economy with increasing returns to scale and gives sufficient conditions for its existence and local uniqueness. The optimality properties of this equilibrium notion follows from our previous investigations on economies with increasing returns.

The notion of equilibrium used in this paper, i.e. a marginal cost pricing equilibrium, is a family of consumption plans, production plans, prices and lump sum taxes such that: all the first order conditions are satisfied in equilibrium; the lump sum taxes cover the aggregate losses of firms with increasing returns to scale; all markets for goods and services clear.

The intended model is an economy with a regulated natural monopoly and a large number of unregulated competitive firms.

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I. INTRODUCTION

This paper proposes a notion of equilibrium for an economy with increasing returns to scale and gives sufficient conditions for its existence and local uniqueness.

We offer two proofs of existence. The first is based on an elegant fixed point argument of Mantel. Our proof that regular economies, with increasing returns, have an odd number of locally unique equilibria, which extends Kehoe's theorem on regular economies with production [10], gives an index-theoretic proof of existence. This second argument requires additional assumptions on the technology and preferences.

The intended model is an economy with a regulated natural monopoly and a large number of unregulated competitive firms. The increasing returns to scale technology of the natural monopoly is viewed as a nonconvex production set, where marginal cost pricing may lead to a deficit. The production possibilities of the competitive firms comprise convex sets. All firms price at marginal cost and households are price taking utility maximizers subject to lump-sum taxes which cover the losses incurred by the regulated monopoly.

This is an extensive revision of a discussion paper [1], that we circulated several years ago, in which we proposed a notion of equilibrium for an economy having a single firm with a nonconvex production set. Subsequently, Mantel produced a simpler proof of the existence of such an equilibrium, albeit under stronger assumptions on the set of feasible social production possibilities than we had used. We shall show that his argument can be extended to an economy which includes a decentralized set of competitive firms.

A marginal cost pricing equilibrium is a set of consumption plans, production plans, lump-sum taxes and prices, where the regulated firm is given an efficient production plan and instructed to price at marginal cost, i.e. buy and sell inputs and outputs in the plan at the associated prices. Competitive firms maximize profits at equilibrium prices. All competitive firms are limited, i.e. we assume that shareholdings in firms carry limited liability. Each consumer is subject to a lump-sum tax, and these in aggregate cover the losses of the regulated firm. In addition, we require market clearing and the first order conditions for Pareto optimality to hold in equilibrium. Hence the prices faced by the natural monopoly are market clearing equilibrium prices.

This notion of equilibrium in economies with increasing returns is suggested by Hotelling's classic contributions to the marginal cost pricing literature, see [7] and [8], where he considers an economy in which all products are priced at marginal cost and the difference between marginal and total cost is recovered through lump-sum taxation.

II. MANTEL'S PROOF

In this section, we outline Mantel's proof of existence [11]. Mantel assumes that Y, the set of feasible social production possibilities net the social endowment, is a compact comprehensive subset of R_n^+, the non-negative cone; the efficiency frontier of Y, eff(Y), is a C^2 n-1 dimensional submanifold of R_n^+; p(z), the normal to $\overline{Y}$ at z (the marginal rates of transformation at z), is strictly positive for all $z \varepsilon$ eff(Y). These assumptions imply that eff(Y) is diffeomorphic to the n-1 dimensional simplex [12]. Hence eff(Y) is a fixed point space.

We assume that there are a finite number of consumers each of whom has a continuous strictly quasi-concave locally non-satiated utility function, U_i, on his consumption set, X_i, a closed convex subset of R_n^+, containing o.

We assume initially a fixed structure of revenues, i.e. the i^{th} consumer's endowment $w_i = \alpha_i w$, where w is the social endowment, and his share of the j^{th} firms's profits or losses $\theta_{ij} = \alpha_i$, where the α_i are positive real numbers which sum to one. This assumption will guarantee that each consumer's budget set is a nonempty, compact set, with nonempty relative interior, for each pair $\langle z, p(z) \rangle$ where $z \varepsilon$ eff(Y).

Given $z \varepsilon$ eff(Y) and p(z), then the i^{th} consumer maximizes $U_i(x)$ over $x \varepsilon R_n^+$, subject to the constraint : $p(z) \cdot x \leq \alpha_i p(z) \cdot z$. Under our assumptions, this optimization problem is well defined and has a unique solution, which we denote $x_i^*(z,p(z))$. Aggregate demand,

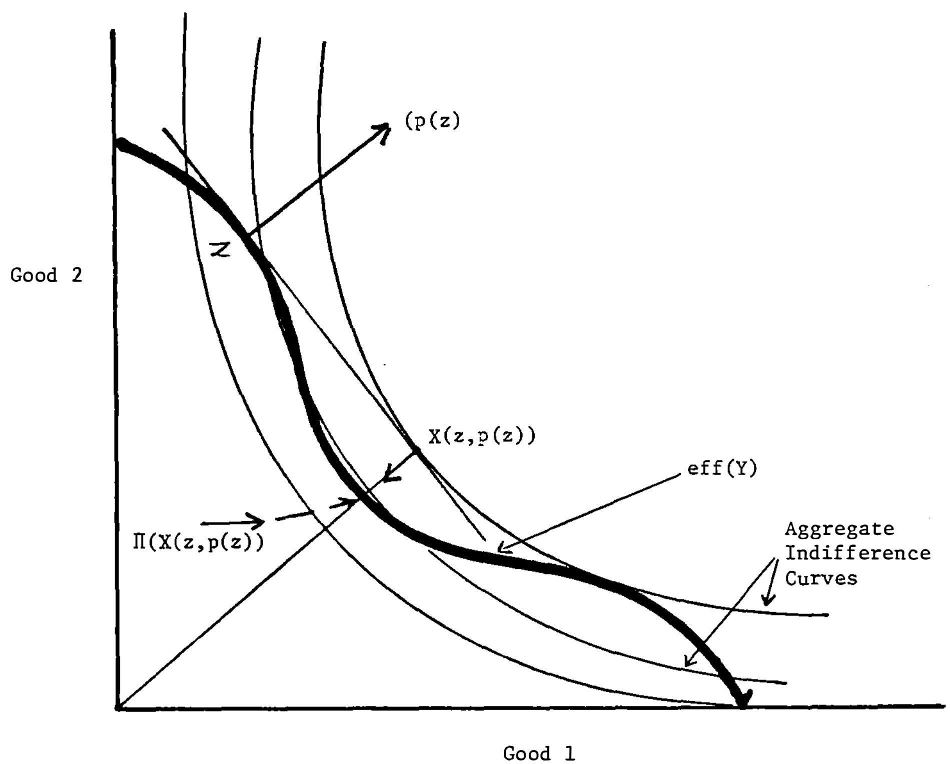
$X(z,p(z))$, is then $\sum_i x_i^*(z,p(z))$.

For any $v \varepsilon R_n^+/\{o\}$, R_n^+ excluding origin, we define $\pi(v)$, the projection of v onto eff(Y), as the intersection of the ray from o through v with eff(Y). Mantel (implicitly) assumes that π is a continuous function.

He now constructs the continuous map Φ : eff(Y) $\rightarrow$ eff(Y), where $z \rightarrow (z,p(z)) \rightarrow X(z,p(z)) \rightarrow \pi(X(z,p(z)))$. Since eff(Y) is a fixed point space, Φ has a fixed point z^* by Brouwer's theorem. It then follows from Walras's law that $(z^* - w)$, $p(z^*)$, and $X(z^*,p(z^*))$ constitute a marginal cost pricing equilibrium. This argument is summarized in Figure 1 for the two-good case.

The major technical differences between our present model and Mantel's model are: firstly, the compactness of Y is not assumed but is derived from assumptions of irreversibility, free disposal, and closeness of the aggregate production set; secondly, the normalized vector of marginal rates of transformation, p(z), is now only required to be transverse to o, i.e. $p(z) \cdot z > 0$ for all $z \varepsilon$ eff(Y); finally, we assume that eff(Y) is a connected C^1 n-1 dimensional submanifold of R_n^+.

FIGURE 1



A diagramatic representation of the proof of existence. $z \in \mathrm{eff}(Y)$ generates prices $p(z)$ and demand $X(z,p(z))$. X is projected by Π onto eff(Y), and the process starts again. A fixed point is a marginal cost pricing equilibrium.

Maintaining our other assumptions on tastes and technology, we show that Y is a compact subset of R^+_n; eff(Y) is diffeomorphic to the n-1 dimensional simplex, which we denote as Δ; and $\pi : R^+_n/\{o\} \to \mathrm{eff}(Y)$ is a C^1 map.

Assuming a fixed structure of revenues, we show that Mantel's map has a fixed point which is a marginal cost pricing equilibrium. Later we demonstrate that the case of general ownership rights can be reduced to a fixed structure of revenues.

III. THE MODEL

In this section, we lay out the assumptions of our model. Consumers are indexed over i, where $i \in \{1,2,\ldots,C\}$. Firms are indexed over j, where $j \in \{1,2,\ldots,F,F+1\}$.

(A1) For each i, X_i is the consumption set of consumer i and X_i is a closed convex subset of R^+_n, which contains o.

(A2) For each i, U_i is the utility function of consumer i and U_i is continuous, strictly quasi-concave, and locally non-satiated.

(A2)' For each i, U_i is the utility function of consumer i and U_i is C^2, $D^2U(x)$ is negative definite on the kernel of $DU(x)$, U_i is monotone, and the closures of the indifference curves of U lie in R^{++}_n, the positive cone. That is, preferences are smooth as defined by Debreu.

(A3) For each i, w_i is the endowment of consumer i and w_i is an element of X_i. $\sum_{j \varepsilon L} w_i(j) \leq T$, where L is set of labor services and T is total time available for consumption.

(A4) $w = \sum_i w_i$ is the social endowment.

(A5) For each j,

(i) Y_j is the production set of producer j

(ii) Y_j is a closed subset of R_n

(iii) $o \ \varepsilon \ Y_j$

(iv) $eff(Y_j)$ is a k_j dimensional C^1 submanifold of R_n.

(A6) Firms indexed by 1 through F have convex production sets.

(A7)

(i) $\sum_j Y_j$ is closed

(ii) $(\sum_j Y_j) \supset (-R_n^+)$, free disposal

(iii) $A(\sum_j Y_j) \cap - A(\sum_j Y_j) = \{0\}$, irreversibility where A(H) denotes the asymptotic cone of H, a subset of R_n.

(A8) The aggregate feasible set is defined as Y, where

$Y = (\sum_j Y_j + w) \cap R_n^+$

(i) eff(Y) is connected

(ii) eff(Y) is a n-1 dimensional C^1 submanifold of R_n^+

(iii) $p(z) \cdot z > o$ for all $z \ \varepsilon$ eff(Y).

(A8)' The aggregate feasible set is defined as Y, where

$Y = (\sum_j Y_j + w) \cap R_n^+$

(i) eff(Y) is a C^2 hypersurface in R_n^+

(ii) $p(z) >> o$ for all $z \ \varepsilon$ eff(Y), i.e. p(z) is positive in every component.

(A9) The endowments w_i and shares θ_{ij} constitute a fixed structure of revenues, i.e.

(i) α_i are positive real numbers which sum to one

(ii) $w_i = \alpha_i w$, for all i

(iii) $\theta_{ij} = \alpha_i$, for all i and j.

If we assume a fixed schedule of revenues, then the income of the i^{th} consumer, I_i, can be expressed as $p \cdot w_i + \alpha_i \sum_{j=1}^{F} p \cdot y_j + \alpha_i p \cdot y_{F+1}$. Hence $I_i = p \cdot w_i + \alpha_i \sum_{j=1}^{F} p \cdot y_j - T_i$, where T_i is the lump-sum tax $-\alpha_i p \cdot y_{F+1}$. Also $I_i = \alpha_i p \cdot z$, where $z = w + \sum_j y_j$.

We define a marginal cost pricing equilibrium as a 4-tuple $\langle x_i^*, T_i, y_j^*, p \rangle$, where $y^* = \sum_j y_j^*$, $x^* = \sum_i x_i^*$, and $z^* = y^* + w$, such that:

(i) $$U_i(x_i^*) = \max_{x \ \varepsilon \ X_i} \{U_i(x) | p \cdot x \leq p \cdot w_i + \sum_{j=1}^{F} \theta_{ij} p \cdot y_j^* - T_i\}$$

(ii) $x^* = z^*$

(iii) p is normal to the tangent space of $\text{eff}(Y_j)$ at y_j^*, for all j.

(iv) T_i is the lump-sum tax imposed on consumer i, and $\sum_i T_i + p \cdot y_{F+1}^* = o$.

For firms with a convex technology, condition (iii) in the definition of a marginal cost pricing equilibrium implies profit maximization at y_j^*, with respect to prices p. More generally, it implies that the first-order conditions for profit maximization are satisfied, with marginal rates of transformation and substitution equal to price ratios. This seems the obvious generalization of marginal cost pricing beyond a single-output partial equilibrium world.

IV. EXISTENCE THEOREM

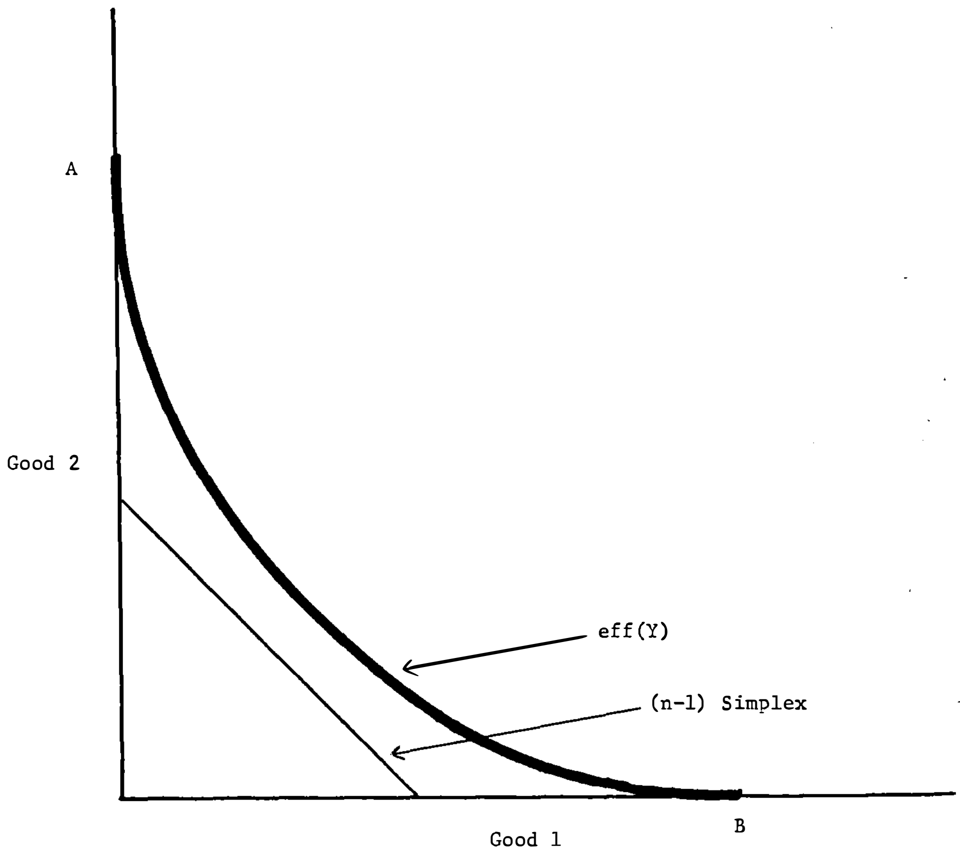
Our proof of existence follows the structure of Debreu's proof [4], i.e. first, we prove existence assuming compactness of the consumption and production sets; second, we show that the attainable sets of consumers and producers are compact, using a theorem of Hurwicz and Reiter [9]; finally, we demonstrate that any marginal cost pricing equilibrium in the economy defined in terms of attainable consumption and production sets is also a marginal cost pricing equilibrium in the original economy. Initially we assume a fixed schedule of revenues.

H, a subset of R_n^+ is said to be comprehensive if $x \in H$, $y \in R_n^+$ and $y \leq x$ implies $y \in H$.

Proposition (1). (H. Samelson) Let Y be a compact comprehensive subset of R_n^+ and eff(Y) a connected C^1 hypersurface in R_n^+ such that for all $z \in \text{eff}(Y)$, $p(z) \cdot z > o$. Then eff(Y) is diffeomorphic to the n-1 dimensional simplex and $\pi : R_n^+/\{o\} \to \text{eff}(Y)$ is a C^1 mapping.

The proof of this proposition is given in the appendix. Note that Mantel's assumption on the continuity of π follows from Samelson's theorem. The role of the condition $p(z) \cdot z > o$ in ensuring that eff(Y) is diffeomorphic to the n-1 dimensional simplex, is indicated in Figure 2. This shows a case where $p(z) \cdot z = o$ at points A and B, and along the vertical and horizontal lines the map that retracts eff(Y) to the simplex is not one to one. In economic terms this case causes problems because at A and B the value of the production z at its associated prices p(z), is zero. Hence consumers may have empty relative interiors to their budget sets.

FIGURE 2



Lemma (1). If each X_i is a compact convex subset of R^+_n, containing o, and $p(z) \cdot z > o$, then the budget correspondence β_i is continuous at z, where $\beta_i(z) = \{x \in X_i | p(z) \cdot x_i \leq \alpha_i p(z) \cdot z\}$.

Proof: See lemma (3) in [4].

Theorem (1). An economy has a marginal cost pricing equilibrium if for every i, X_i is a compact convex subset of R^+_n, containing o; assumptions (A2), (A3) and (A4) hold; Y is a compact comprehensive subset of R^+_n; assumptions (A5), (A6), (A8), and (A9) hold.

Proof: Since we assumed a fixed structure of revenues, the budget correspondence for the i^{th} consumer, $\beta_i(z)$, is defined as $\{x \in X_i | p(z) \cdot x_i \leq \alpha_i p(z) \cdot z\}$ for all z ε eff(Y). $\beta_i(z)$ is a continuous correspondence on eff(Y) by (A8) and lemma (1). Hence by (A2), X(z,p(z)) is a continuous function of eff(Y). Therefore, Mantel's map Φ : eff(Y) $\rightarrow$ eff(Y) is continuous and has a fixed point z^*, by proposition (1). Local non-satiation of the utility functions guarantees the validity of Walras's law, i.e.
$p(z) \cdot X(z,p(z)) = p(z) \cdot z$, for all z ε eff(Y). Consequently, $X(z^*, p(z^*)) = z^*$. Since each eff(Y_j) is a manifold and $y^*_j \in \text{eff}(Y_j)$, where $z^* = \sum_j y^*_j + w$, $p(z^*)$ is normal to the tangent space of eff(Y_j) at y^*_j, for each j.

In a recent paper [9], Hurwicz and Reiter gave sufficient conditions for an economy to have a bounded feasible set without assuming convexity of production or consumption sets. In the next lemma we show that their conditions are met by our economy. It will then follow that the attainable set of each agent is compact and that Y is compact.

Let

$$M_w = \{\langle x_1,\dots,x_c,y_1,\dots,y_{F+1}\rangle : \sum_i x_i = \sum_j y_j + w\}$$

$$A_w = [(\prod_i X_i) \times (\prod_j Y_j)] \cap M_w$$

$$\dot{X} = A(\sum_i X_i)$$

$$\dot{Y} = A(\sum_j Y_j)$$

$\hat{X}_i = \{x \in X_i \mid x_k \in X_k,\ k \neq i,$ and $y_j \in Y_j$ such that $\langle x_1,\dots,x_c,y_1,\dots,y_{F+1}\rangle \in M_w\}$. $\hat{X}_i$ is the attainable set of consumer i.

$\hat{Y}_j = \{y \in Y_j \mid y_k \in Y_k,\ k \neq j,$ and $x_i \in X_i$ such that $\langle x_1,\dots,x_c,y_1,\dots,y_{F+1}\rangle \in M_w\}$. $\hat{Y}_j$ is the attainable set of producer j.

<u>Proposition (2)</u>. (Hurwicz and Reiter)

If (i) $\dot{X} \cap \dot{Y} = \{o\}$

(ii) $\dot{X} \cap (-\dot{X}) = \{o\}$

(iii) $\dot{Y} \cap (-\dot{Y}) = \{o\}$

then A_w is bounded.

<u>Lemma (3)</u>.

If (i) X_i is closed, $o \in X_i$, and $X_i \subseteq R_n^+$ for all i

(ii) Y_j is closed for all j

(iii) $\sum_j Y_j$ is closed

(iv) $(\sum_j Y_j) \supseteq (-R_n^+)$, free disposal

(v) $\dot{Y} \cap (-\dot{Y}) = \{o\}$, irreversibility

then (a) Y is a compact comprehensive subset of R_n^+

(b) $\hat{X}_i$ and $\hat{Y}_j$ are compact sets, for all i and j.

<u>Proof</u>: Assumptions (i) and (ii) imply that A_w is closed, $X_i \subseteq R_n^+$ for all i, hence $\sum_i X_i \subseteq R_n^+$ and $\dot{X} \subseteq R_n^+$. Therefore, $\dot{X} \cap (-\dot{X}) = \{o\}$. By (iv), $\dot{Y} \supseteq -R_n^+$, but $\dot{X} = R_n^+$. Hence $\dot{Y} \supseteq -\dot{X}$. Therefore

$-\dot{Y} \cap \dot{Y} \supseteq -\dot{Y} \cap -\dot{X}$ which implies $\{o\} = \dot{Y} \cap \dot{X}$, by (v). It follows from proposition (2) that A_w is bounded, hence compact. $\hat{X}_i$ and $\hat{Y}_j$ are simply projections of A_w and therefore compact. Since the sum of compact sets is compact, we see that $\sum_j \hat{Y}_j + w$ is compact. $o \in X$ implies that $Y = (\sum_j Y_j + w) \cap R_n^+ \subseteq \sum_j \hat{Y}_j + w$. Moreover, it follows from (iii) that Y is closed and therefore compact.

Theorem (2). If an economy E satisfies assumptions (A1) through (A9), then E has a marginal cost pricing equilibrium.

Proof: The conditions of lemma (3) hold for E, hence $\hat{X}_i$, $\hat{Y}_j$ are compact for all i and j; and Y is a compact, comprehensive subset of R_n^+. Since eff(Y) satisfies the hypotheses of Samelson's theorem, eff(Y) is a fixed point space and Φ is a continuous mapping by proposition (1). Choose a compact, convex set K in R_n^+ containing in its interior all the attainable consumption sets $\hat{X}_i$ and define $X_i' = K$ for all i. Note that o is in each $\hat{X}_i$. Call this economy E'. E' satisfies all the assumptions of theorem (1) and hence has a marginal cost pricing equilibrium $\langle x_1^*, \ldots, x_c^*, \langle x_1^*, \ldots, x_c^*, y_1^*, \ldots, y_{F+1}^*; p\rangle$ $y_1^*, y_{F+1}^*; p\rangle$. Let $x^* = \sum_i x_i^*$, $y^* = \sum_j y_j^*$, and $z^* = y^* + w$, then $p = p(z^*)$, and $x^* = z^*$. Hence each x_i^* is attainable. A routine argument shows that each x_i^* is optimal in $\{x_i \in X_i | p(z^*) \cdot x_i \leq \alpha_i p(z^*) \cdot z^*\}$, which completes the proof.

Note that we have actually proved the existence of a marginal cost pricing equilibrium that is socially efficient. Of course, if

all firms have a convex technology then a marginal cost pricing equilibrium is a competitive equilibrium in the sense of Arrow-Debreu. Obviously, our analysis can be extended to a regulated monopolistic sector having several firms.

We now remove the restrictive assumption of a fixed schedule of revenues by defining the after tax income of the i^{th} consumer I_i, for given $z \in$ eff(Y), as $p \cdot w_i + \sum_{j=1}^{F} \theta_{ij} p \cdot y - T_i$ where $p = p(z)$, $T_i = p \cdot w_i + \sum_{j=1}^{F} \theta_{ij} p \cdot y_j - \alpha_i p \cdot z$, and α_i are some fixed set of positive real numbers which sum to one. Thus whatever are the pre-tax incomes, taxes may be chosen so that the after tax incomes constitute a fixed structure of revenues and the previous proof suffices. The equilibrium lump-sum taxes are just $p \cdot w_i + \sum_{j=1}^{F} \theta_{ij} p \cdot y_j^* - \alpha_i p \cdot z^*$.

V. LOCAL UNIQUENESS

In this section, we make assumptions that imply that all goods in the economy are final goods, i.e. that there are no goods which are exclusively intermediate goods. This is an unrealistic assumption in an economy with production, but technically it allows us to impose the following condition:

(C1) There are no equilibria on the boundary of eff(Y).

The analogous assumption for exchange economies was first proposed by Nishimura (13).

Note that (C1) follows from (A2)′ and (A8)′. In addition, these assumptions imply that Φ is C^1.

We define a smooth economy with increasing returns, E, as one satisfying assumptions (A1), (A2)′, (A3), (A4), (A5), (A6), (A7), (A8)′, and (A9).

E is said to be regular if o is a regular value of $\Phi - I$.

Theorem (2). If E is a regular smooth economy with increasing returns, then E has an odd number of locally unique marginal cost pricing equilibria.

Proof: By hypotheses, E has no equilibria on the boundary of eff(Y). Moreover, each equilibrium is isolated by the inverse function theorem. Hence E has a finite number of equilibria, since eff(Y) is compact.

If E is regular, then Φ is a Lefschetz map—see Guillemin and Pollack [4] for a lucid discussion of Lefschetz fixed-point theory. The global Lefschetz number of Φ denoted $L(\Phi)$ is equal to $\sum_{\Phi(x)=x} L_x(\Phi)$ where $L_x(\Phi)$ = sign of the determinant of $D\Phi(x) - I$. Since eff(Y) is diffeomorphic to the n-1 dimensional simplex, we see that $L(\Phi)$ equals the Euler characteristic of the simplex, i.e. $L(\Phi) = 1$. Hence Φ has an odd number of fixed points. We complete the proof by noting the one-to-one correspondence between the fixed points of Φ and the marginal cost pricing equilibria of E,—assuming that the underlying exchange economy is regular; generically, we can choose the technology such that no fixed point of Φ is an exchange equilibrium.

VII. OPTIMALITY

In [2], we considered an economy with increasing returns, where there is one firm and two consumers; the firm is owned by a single consumer. The notion of equilibrum considered in that paper is marginal cost pricing. Hence the results of that paper apply here. That is, there exists economies with increasing returns to scale where no marginal cost pricing equilibrium is Pareto optimal. In other words, the first welfare theorem does not hold for economies with increasing returns, if the equilibrium notion is marginal cost pricing.

In [3], we showed that the second welfare theorem does hold for our equilibrium notion. That is, every Pareto optimal allocation can be supported by a marginal cost pricing equilibrium after a suitable redistribution of ownership rights.

Acknowledgments: This research was supported in part by grants from the NSF to Yale University. This paper was completed during Brown's tenure as a Sherman Fairchild Distinguished Scholar at Caltech.

We wish to thank Graciela Chichilnisky and our colleagues at Yale, Essex, and Caltech for their useful comments and lively discussions on increasing returns. In addition, we thank H. Samelson for allowing us to include his theorem on diffeomorphic images of smooth compact convex sets, Proposition (1).

APPENDIX: PROOF OF PROPOSITION (1)

Without loss of generality, we assume that eff(Y) is contained in the interior of the unit disk, D_n. Let S_n be the boundary of D_n, i.e. $S_n = \{x \varepsilon R_n | \; ||x|| = 1$, where $|| \cdot ||$, is the Euclidean norm$\}$. Define $g : R_n/\{o\} \to S_n$ as $g(x) = x/||x||$ and let $f : eff(Y) \to S_n^+$ be the restriction of g to eff(Y), where $S_n^+ = S_n \cap R_n^+$. We shall show that f is a diffeomorphism.

g(x) is homogeneous of degree o and hence by Euler's theorem: $Dg(x)v = o$ for all $x \varepsilon R_n/\{o\}$ and for all v of the form $\alpha x/||x||$, where α is some real number. We now wish to show that f is a local diffeomorphism, i.e. that Df(x), the restriction of Dg(x), is nonsingular on T_x, the tangent plane to eff(Y) at x, for all $x \varepsilon$ eff(Y).

Let N_x be the one dimensional subspace spanned by $x/||x||$, where $x \varepsilon$ eff(Y), and V_x be the n-1 dimensional subspace which is orthogonal to N_x, i.e. $V_x = \{y \varepsilon R_n | y \cdot x = o\}$.

Suppose $Dg(x)v = o$ for some $v \varepsilon R_n$ and $x \varepsilon$ eff(Y). Express v as $v_1 + v_2$ where $v_1 \varepsilon N_x$ and $v_2 \varepsilon V_x$. $Dg(x)v = Dg(x)v_1 + Dg(x)v_2 = Dg(x)v_2 = o$. Dg(x) is nonsingular on V_x, since g(x) is the Gauss map for the sphere of radius $||x||$ with center at o. Therefore, $v_2 = o$, and we have shown that $Dg(x)v = o$ iff $v = \alpha x||x||$, for some real number α.

Let $v \varepsilon T_x$, the tangent space to eff(y) at x. If $Df(x)v = o$, then $Dg(x)v = o$ since Df(x) is simply the restriction of Dg(x). Hence $Df(x)v = o$ implies that $v \varepsilon N_x$, but by hypothesis $p(x) \cdot x > o$, i.e.

$v \varepsilon T_x$ implies $v \neq \alpha x$ for any non-zero real number α. Therefore $v = o$ and Df(x) is nonsinglar on T_x.

We now need two propositions from [3].

<u>Proposition (3)</u>. Let $\theta : \beta \to \beta$ be a local homeomorphism, β compact and β connected. Then θ is a covering map.

<u>Proposition (4)</u>. Let $\theta : \beta \to \beta$ be a covering map, β arcwise connected and β simply connected. Then θ is a homeomorphism.

Since f is a local diffeomorphism, f is a covering map by proposition (3). S_n^+ is simply connected and therefore f is a homeomorphism of eff(Y) onto S_n^+ by proposition (4). Since f is a local diffeomorphism, its inverse map is differentiable and is therefore a diffeomorphism.

Finally, $\pi : R_n^+/\{o\} \Rightarrow$ eff(Y) can be expressed as the composition of the C^1 maps g and f^{-1}, i.e. $\pi(x) = f^{-1}og(x)$, hence π is C^1.

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