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A NONCOOPERATIVE EQUILIBRIUM CONCEPT WITH AN ENDOGENOUSLY DETERMINED DOMINANT PLAYER: THE CASE OF COURNOT VERSUS STACKELBERG

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ABSTRACT

This paper examines the relationship between Cournot and Stackelberg equilibrium, and whether rational expected profit maximizing behavior of firms can help decide when noncooperative equilibrium might correspond to a Cournot equilibrium and when might it correspond to a Stackelberg equilibrium. An extensive form game is constructed with subgames whose Nash equilibra are Cournot and Stackelberg equilibrium. Equilibria of this larger game are studied. In a particular example with passive consumers, sufficient conditions for an industry to exhibit a Stackelberg structure, are obtained. Systematic changes in the properties of the equilibrium with changes in some specific exogenous characteristics of the market, are examined. This research hopes to help provide insight into circumstances under which one can endogenously determine a dominant player.

1. INTRODUCTION

There have been two classical ways of modeling the behavior of firms in oligopolies. The models differ in their assumptions about firm behaviour and result in different equilibrium outcomes. In one set of models, it is assumed that the firms in a market play a Cournot game with each other. A Cournot game is a noncooperative game in extensive form in which the players are in the same strategic position with respect to each other. That is, the players move simultaneously (or sequentially but unobservably) and their strategy spaces are isomorphic to each other. An example of the Nash equilibrium of such a game is the Cournot equilibrium.

In the other type of models, it is assumed that the firms play a noncooperative game in which some of the players are in a dominant strategic position with respect to some others. Such a game is called a Stackelberg game. Here, the dominant players move first and have strategy spaces that are not isomorphic to those of the other players. Moreover, these are games of perfect information and the payoffs to a player, among other things, also depend upon when a player moves.

Thus, in order to be able to know how to model an industry—Cournot game or Stackelberg game—it would be sufficient to examine timing and information conditions both of which are presumed exogenous. The sizes or technologies of firms, or the characteristics of demand, are in this context, irrelevant. On the other hand there is a "Folk Theorem" that outcomes in oligopolies are best modeled by Cournot equilibria if the firms are of equal size, but by Stackelberg

equilibria if they are not. This suggests the possibility that timing and information conditions could be endogenously determined using among other things, firm sizes or technologies as exogenous.

However, unless one is able to obtain a systematic relation between these exogenous characteristics and the choice between a Cournot game or a Stackelberg game, one has to make an ad hoc assumption about the firms' conduct in the industry. Such would be the case as long as one is unable to discern firm behavior in a systematic way using observable data like firm sizes or demand characteristics. Sometimes this assumption may affect policy decisions. Consider for example, a regulator trying to decide whether or not to regulate a duopoly. Let the regulator's objective function be consumer's surplus. Also, let the firms have zero marginal costs. Let the firms' output quantities be denoted by x1 and x2 respectively. Assume that demand is linear and is given by price = y - x1 - x2. Let R be the cost of regulating this industry. Under a Cournot duopoly assumption, the consumer surplus is $2y^2/9$ and under the assumption of Stackelberg duopoly, the consumer surplus is $9y^2/32$.
If $2y^2/9 < R < 9y^2/32$ then, while it may be worthwhile regulating the industry under the assumption that it is a Cournot duopoly, it is unprofitable to regulate the same industry if it is assumed that it is a Stackelberg duopoly. Note that in general, using output and demand data, one would not be able to infer the type of equilibrium—Cournot or Stackelberg—without complete information on the cost functions of the firms involved.

The basic purpose of this paper is to endogenize timing and information conditions using data on the sizes or technologies of firms and certain characteristics of demand. Thus, we wish to endogenize the choice between a Cournot game and a Stackelberg game. Since in our model, sizes and technologies are exogenous, we will be able to obtain a rigorous formulation and verification of the "Folk Theorems."

One may try to endogenize timing and information conditions (i.e., endogenizing the choice between a Cournot game and a Stackelberg game) by simply developing a framework in which the firms are allowed to decide which game they want to play. This will not work because in general, in a Stackelberg equilibrium, the leader (dominant) firms are better off than the follower firms and in general, could be better off than in a Cournot equilibrium. Hence all the firms might want to play the Stackelberg game expecting to be the leader. In other words, in order to obtain a Stackelberg equilibrium in which there is a leader and a follower, one would be forced to exogenously assign the dominant player. This assignment would be quite as ad hoc as making the assumption that the firms are playing a Cournot or Stackelberg game. What we do therefore, is to describe a game of imperfect information in which ex ante, the players are in the same strategic position with respect to each other. However, when the sequentially rational Nash equilibrium strategies are being played, it would appear as if the firms are playing the equilibrium strategies of a Stackelberg game or a Cournot game.

In a quantity-setting example of the Cournot game, the strategies of all the firms are output levels. In a quantity-setting example of the Stackelberg game, the dominant firms' strategies are output levels while the other firms' strategies are reaction functions (i.e., output levels that are functions of the dominant firm's output level). On the other hand, if demand uncertainty is resolved overtime, then firms may face a trade-off between making quantity decisions early so as to establish a "leadership" position, or waiting until the demand uncertainty has been resolved so as to avoid production decision mistakes. Thus, a larger game is constructed in which there are two logical time periods. There is uncertainity in demand which is revealed between the two time periods. We assume that the firms behave in a sequentially rational way given the information they have in each time period. The firms move simultaneously before the beginning of the first period. The behavior strategy for each firm in the beginning of the game consists of a probability that it would enter in period 1 and the quantity it would produce if it were to enter in period 1. If both firms end up entering in the same period, the sequentially rational Nash equilibrium is Cournot-like, whereas if they end up entering in different periods, it is Stackelberg-like. This will now provide a framework in which we can ask how firm's sizes and technologies, and the nature of demand can determine whether an industry is best modelled as Cournot or Stackelberg. An example of the larger game we are alluding to, with two firms and two levels of production for each firm is shown in

Figure 5. We will describe this in greater detail in section 2. The basic results of this paper are about the nature of the sequentially rational Nash equilibrium of this larger game and are the following:

1. Under some conditions, firms in this equilibrium will not randomize their "times of entry." Every temporally nonrandomized equilibrium corresponds to either a Cournot or a Stackelberg equilibrium in the strong sense that the quantities produced are exactly those predicted by the respective extensive form concepts.[1] This result holds in full generality.
2. With symmetric firms, there is a symmetric equilibrium which is the appropriate generalization of a Cournot equilibrium. (Cf. Section 4 below.) This result assumes a particular technology with linear demand and quadratic costs.
3. With two symmetric firms, under certain conditions, symmetric equilibrium must be temporally randomized. This is proved in the paper using the technology of result 2, but this result holds in full generality.
4. In a market with one large firm and a continuum of small firms, the <u>only</u> equilibrium is a Stackelberg Equilibrium. This result is true in general.
5. With respect to this equilibrium, a market with one large firm and many small atomic firms, behaves nearly like a market with one large firm and a continuum of small firms. This is also true in general.

We now develop some notations in Section 2. In Section 3, a Cournot game, a Stackelberg game and the larger game of our concern are defined. Section 4 examines some properties of the equilibrium of the larger game and Section 5 concludes the paper.

There are three definitions in this paper which we will make in the context of a game depicted in a figure and which will be generalized as specified. These are the definitions of a Cournot equilibrium, a Stackelberg equilibrium and the equilibrium of the larger game that endogenizes the dominant player. We find that this is sufficient for the purposes of this paper set forth in this introduction. One can conceivably make these definitions more general. However, except as noted, our results are valid in general and our approach makes it easier to follow the arguments.

2. DEFINITIONS AND NOTATIONS

We recall the definition of a game in its extensive form as in Kuhn [3]. However, since only a particular game of the form shown in Figure 5 is analyzed, in order to minimize notation, we develop the definitions only with respect to the game depicted in Figure 5.

The game is represented by a tree. The edges that come out of each node represent the alternatives at that node. Nodes which possess alternatives are called moves and those that do not possess alternatives are called terminal nodes. The rank of a node is the number of moves that are on the path from the initial node to itself. The set of moves of a given rank represents a turn for some player.

The turns of player 1, player 2 and nature are denoted by (1), (2), and (N) respectively. There is a path or branch running from the initial node to each terminal node. Each such branch is associated with certain payoffs to the players. Often, in a particular players's turn, the player does not have information about which alternative the previous player has chosen. In such a case, the set of nodes at the end points of all those edges is called an information set. For example, the nodes J_{11}, J_{12}, J_{13} form an information set for player 2.

In any game represented in its extensive form by a tree, we could consider the set of nodes in any information set as the set of initial nodes of another game. This is called the subgame of the original game, and the tree that follows this information set is called the subtree of the original tree.

The game tree could be uncountably infinite—i.e., there could be a continuum of alternatives at some or all of the moves—but of finite play length. Clearly, this might lead to some measurability problems discussed in Aumann [1]. However, in the game tree of our concern, the respective spaces are standard measurable spaces as required by Aumann, and therefore, these problems do not confront us.

N is the number of players. In Figure 5, N = 2.

The set of vertices that are not terminal vertices are partitioned according to the moves that represent each player's turn. This in the player partition $\{P_1, P_2, \ldots P_N\}$. In Figure 5, the player partition is $\{P_1, P_2\}$ where $P_1 = \{I_1 I_2, \ldots I_7\}$ and $P_2 = \{J_{11}, J_{12}, J_{13}, J_2, \ldots J_{61}, J_{62}, J_{71}, J_{72}\}$.

B is the set of branches (denoted by b). Since randomized strategies may be played by all players including nature, a probability is assigned to each branch. It is with respect to these probabilities that players make their expected payoff calculations. $G_b(B)$ is the set of probability measures on B, and g_b is an element of G_b.

The information partition is a refinement of the player partition into information sets U_i for each player i. Again in Figure 5,

$$U_1 = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7\}$$

$$U_2 = \{\{J_{11}, J_{12}, J_{13}\}, J_2, J_3, J_4, J_5, \{J_{61}, J_{62}\}, \{J_{71}, J_{72}\}\}.$$

A_n^i is the set of edges that come out of node n. Alternatively, one could think of A_n^i as the set of nodes at the end of these edges.

Next, M_n^i is the set of probability measures on A_n^i and $m_n^i \in M_n^i$. M_i is the product $\prod_{n \in P_i} M_n^i$.

A behavior strategy for each player is a strategy that consists of randomizing over the alternatives at each move of that player. Further, since in an information set a player cannot distinguish between the nodes, the randomization over the alternatives at each node in the information set should be the same.

Thus for a player i, a behavior strategy at a node n is a probability measure s_n^i on A_n^i such that for every information set u, and all n,n′ ε u, $s_n^i = s_{n'}^i$. For this reason we may ignore the subscript n on s_n^i. Let S_i be a subset of M_i. It is the set of all behavior strategies of player i, each behavior strategy denoted by $s_i = \prod_{n \varepsilon P_i} s_n^i$. Let the payoff function be given by $\Pi : B \to R^N$ and $\Pi_i(b)$ be the i^{th} component of $\Pi(b)$; i.e., the i^{th} player's payoff.

Next, let

$$S = \prod_{i \varepsilon N} S_i$$

and let

$$\mu : S \to G_b(B)$$

be the mapping that induces probability measures on the set of branches, due to behavior strategy N-tuples. In general, μ is derived inductively in the following way. Consider a game in its extensive form that has $(\bar{n} + 1)$ turns numbered from 0 to $\bar{n}$. There may be more than one node in each turn, and the particular information set which is of concern to the player when his turn arrives will depend upon the alternative that was chosen in the previous move. Denote the ith node in the jth turn by $\bar{n}_{ji}$, and the set of nodes in that turn by $\bar{n}_j$. Then define

$$\mu_1 = s_{\bar{n}_o}$$

$$\mu_{k+1}(A^*) = \int_{\bar{n}_k} s_{\bar{n}_{kj}} (A_{\bar{n}_{kj}} \cap A^*) d\mu_k.$$

for every μ_{k+1} — measurable subset A* of $\bar{n}_{k+1}$. Since each terminal node is associated with a unique branch in the game tree, $\mu_{\bar{n}+1}$ defines the function μ mentioned above. E_μ denotes the expectation with respect to μ and $E_{\mu|(\cdot)}$ is the expectation with respect to μ with (˙) given.

3.

In this section, a Cournot game, a Stackelberg game and the larger game of our concern are defined, and results (1) and (3) of the introduction are derived.

First, Cournot and Stackelberg equilibria are defined in their extensive forms. Uncertainty in the market demand is next embedded into the above definitions with players assumed to be Bayesian decision makers whose alternatives at each turn are quantities of production.

In the Stackelberg game, the natural time for the demand uncertainty to be resolved is between the "entry time" of the leader and that of the follower. In Cournot equilibrium, it is possible that uncertainty might be resolved either before of after the time at which firms simultaneously make their quantity decisions.

In the extensive form of the larger game each firm is free to make its quantity decisions either before or after the demand is known. The sequentially rational Nash equilibrium of the subgame that results when both firms decide at the same time (either before or after) is the Cournot equilibrium and the equilibrium of the subgame that results when one firm makes quantity decisions before the information is revealed and the other makes it after, is the Stackelberg equilibrium.

Thus, we want to describe an extensive form game whose Nash equilibria, under certain conditions, corresponds to a Cournot or a Stackelberg equilibrium. In order to do this, we construct a game by combining subgames in which the Nash equilibria are precisely the Cournot or the Stackelberg equilibrium. Therefore, we first define these in their extensive forms, and then embed these trees in an extensive form game whose Nash equilibrium we examine.

<u>Cournot Equilibrium in its Extensive Form</u>

Consider the following game, in which there are two players 1 and 2 and two production levels, high (H) and low (L).

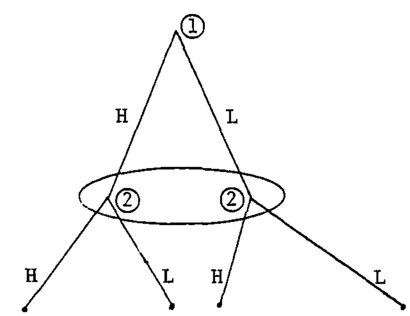
[Figure 1 Here]

In the general multiplayer game with continuously variable production, let $s_i \varepsilon M_i$ be a behavior strategy of player i. A Cournot equilibrium is a vector $s \varepsilon S$ such that $\forall i \varepsilon N$, $\nexists s_i' \varepsilon S_i$ with

$$E_{\mu(s_1,s_2,\ldots,s_i,\ldots)} \Pi_i(b)_{b\varepsilon B} < E_{\mu(s_1,s_2,\ldots,s_i',\ldots)} \Pi_i(b)_{b\varepsilon B} \qquad (1)$$

where E_μ is the expectation over μ.

FIGURE 1



<u>Stackelberg Equilibrium in its Extensive Form</u>

Consider a game in extensive form whose representative tree for two players, and two production levels is:

[Figure 2 Here]

Stackelberg equilibrium is the sequentially rational Nash equilibrium of this game and it is a dominant player equilibrium.

In the general multiplayer case, let $D \in N$ be the dominant player, who moves first. Again, let $\Pi : B \rightarrow R^N$ be the payoff function, and $\mu : S \rightarrow G_b(B)$ be the induced probability measure on the branches.

Then a sequentially rational Nash equilibrium of such a game is a vector $s \in S$ such that,

(a) $\nexists s'_D \in M_D$,

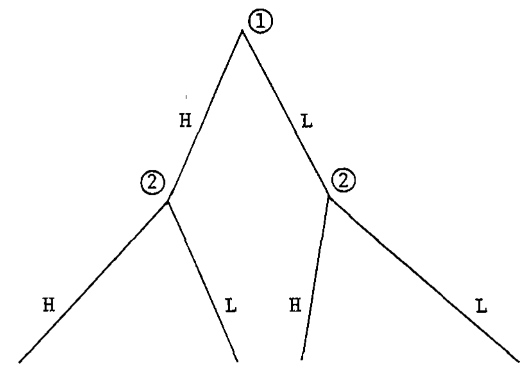
$$E_{\mu|s_D} \Pi_D(b)_{b\in B} < E_{\mu|s'_D} \Pi_D(b)_{b\in B}$$

(b) $\forall s'_D \in M_D$, $\forall i \in N$, $i \neq D$, $\nexists s'_i \in M_i$,

$$E_{\mu|s_i, s'_D} \Pi_i(b)_{b\in B} < E_{\mu|s'_i, s_D} \Pi_i(b)_{b\in B} \quad (2)$$

Condition (b) ensures that the equilibrium is sequentially rational.

FIGURE 2



Next, uncertainty in the market demand is embedded into the above definitions. Let there be two "time periods." Assume that demand is revealed between these periods. The periods are referred to as "Before" (B) and "After" (A). Thus, nature is conceived of as having a distribution over a demand shift parameter y. Recall that the reason for introducing demand uncertainty is to describe the strategic relation between the players, allowing them not only to choose production quantities, but "entry times" as well. In a Cournot equilibrium it is possible that uncertainty might be resolved either before or after the time at which firms simultaneously make their quantity decisions. The players are assumed to be Bayesian decision makers.

For instance, if y had two possible values high (h) and low (l), then a representative tree might look like Figure 3a or Figure 3b.

[Figure 3a and 3b Here]

In general, let nature's strategy be a particular probability measure $g_y(Y)$ on Y which consists of the possible actions nature can take, denoted by y. Y is a subset of R. Let $g_y(Y)$ have a variance γ. The probability measure g_y in an element of $G_y(Y)$, the set of all probability measures on Y. Then if,

$$S = \prod_{i \in N} M_i \times G_y(Y)$$

FIGURE 3a

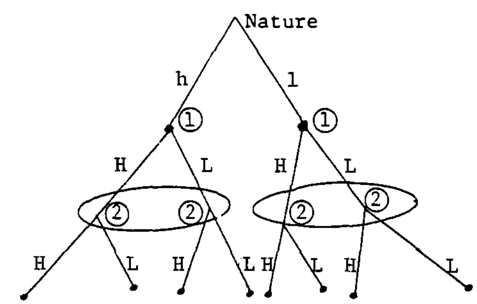
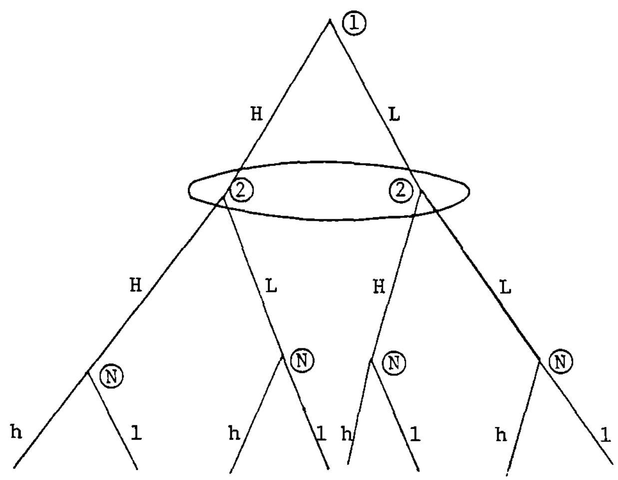


FIGURE 3b



In this figure (N) denotes nature's move.

and $\mu : S \to G_b(B)$ is the induced probability measure on the branches, then a Cournot-Nash equilibrium is a vector $s \in S$ such that,

$$\forall i \in N, \ \nexists s_i' \in M_i, \ E_{\mu|s_i,g_y} \Pi_i(b)_{b\in B} < E_{\mu|s_i',g_y} \Pi_i(b)_{b\in B} \qquad (3)$$

In the Stackelberg game, on the other hand, the natural time for the uncertainty to be resolved is between the "entry time" of the leader and that of the follower. Again, a simple example is given in Figure 4. It is reasonable to assume here that firm 1, which enters before nature's play, is the dominant player.

[Figure 4 Here]

In general, using the notation of the earlier discussion of dominant player equilibrium, and letting

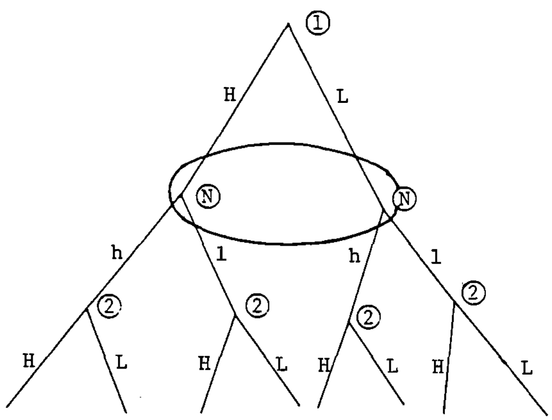
$$S = \prod_{i\in N} M_i \times G_y(Y),$$

a dominant player equilibrium is the sequentially rational Nash equilibrium of this type of game. The equilibrium is a vector $s \in S$, with,

(a) $\nexists s_D' \in M_D$,

$$E_{\mu|s_D} \Pi_D(b)_{b\in B} < E_{\mu|s_D'} \Pi_D(b)_{b\in B}$$

FIGURE 4



In this figure (N) denotes nature's moves.

(b) $\forall s'_D \in M_D,\ \forall i \in N,\ i \neq D,\ \nexists s'_i \in M_i,$

$$E_{\mu|s_i, s'_D, y^*}\, \Pi_i(b) < E_{\mu|s'_i, s'_D, y^*}\, \Pi_i(b).^{2} \qquad (4)$$

The equilibrium which endogenizes the Cournot-Stackelberg choice is now defined as the sequentially rational Nash equilibrium of the extensive form game in which each firm is free to make its quantity decisions either before or after the demand is known. The Nash equilibrium of the subgame that results when both firms decide at the same time (either before or after) is the Cournot equilibrium and the equilibrium of the subgame that results when one firm makes quantity decisions before the information is revealed and the other makes it after, is the Stackelberg equilibrium.

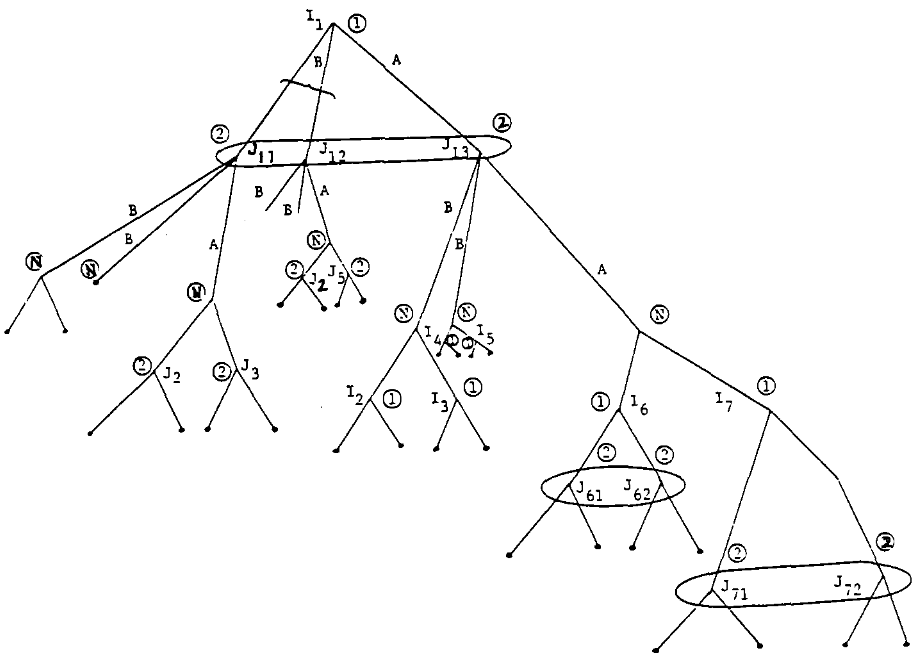
A typical tree when there are two players, two levels of production for each player and two values that y can take is shown in Figure 5. We will refer to this game in general, as the "larger game."

[Figure 5 Here]

To indicate the Nash equilibrium for the general case becomes very complicated and so we will do so for the case of a duopoly. We do this to show the explicit relationship between this equilibrium, the Cournot equilibrium and the Stackelberg equilibrium. Refer to Figure 5.

The vector $(s_1, s_2) \in S_1 \times S_2$ is a sequentially rational Nash equilibrium if,

FIGURE 5



(Similar looking paths have not all been completely drawn)

$$\forall i \neq j,\ \forall s'_j \in M_j,\ \nexists s'_i \in M_i,$$

$$E_{\mu|s_i,s'_j,g_y}\ \Pi_i(b) < E_{\mu|s'_i,s'_j,g_y}\ \Pi_i(b) \tag{5}$$

We are now in a position to compare these equilibria by stating and proving two propositions. The second proposition is proved in the body of the proof of the first proposition.

<u>Proposition 1</u>: For two symmetric firms, every temporally nonrandomized equilibrium of the larger game corresponds to either a Cournot or a Stackelberg equilibrium.

<u>Proof</u>: If an equilibrium of the larger game is temporally nonrandomized, then at each node n, for every player i, s^i_n is such that the probability of an edge which is an action about only <u>when</u> to enter is 0 or 1. Of course if a strategy is such that if the probability of an edge is 0, then $g_b(b) = 0$ for every path that contains that edge.

Thus if $s^1_{I_1}$ is such that the probability of A = 0, and $s^2_{J_{11}}$ is such that the probability of A = 0, then the equilibrium of the larger game is such that from equation (5), the set of paths with nonzero probabilities is the same as the ex ante Cournot game in Figure 3b.

Similarly if $s^1_{I_1}$ is such that the probability of A = 1, and $s^2_{J_{13}}$ is such that the probability of A = 1, then the equilibrium of the larger game corresponds to an ex post Cournot equilibrium. Notice however that this is not a Nash equilibrium of the larger game tree because if temporal randomization is allowed, then at J_{11} (say) player two can find a strategy which will yield him at least as good a payoff as an ex ante Cournot, viz, $s^2_{J_{11}}$ which is such that the probability of A = 1.

Next if $s^1_{I_1}$ is such that the probability of A = 0, and $s^2_{J_{11}}$ is such that the probability of A = 1, then in the equilibrium of the larger game, for player 1:

$$\nexists s_1' \in M_1 \ni$$

$$E_{\mu|s_1, g_y} \Pi_1(b) < E_{\mu|s_1', g_y} \Pi_1(b)$$

Similarly this condition can be reinterpreted for player 2, and the set of paths of nonzero probabilities is the same as that of the Stackelberg game in Figure 4.

Notice here that this is an asymmetric equilibrium of the larger game even when we allow temporal randomization. For person 2 can do no better against person 1's equilibrium strategy of $s^1_{I_1}$ for which the probability of A = 0.

Of course another equilibrium strategy would be with person 2 having a strategy with $s^2_{J_{11}}$ resulting in the probability of A = 0 and $s^1_{I_1}$ is such that the probability of A = 1.

Also, it is easy to see that a symmetric equilibrium with the probability of A = 1 in both s_1 and s_2 is not an equilibrium of the larger game if we assume that $\gamma \in (0,\gamma')$, where γ' is some finite value of the variance of nature's distribution such that the gain to being a Stackelberg leader is greater than playing an ex post Cournot game.

Thus $\forall\gamma \in (0,\gamma')$, there can be no symmetric nonrandomized equilibrium of the larger game.

Hence we have the next proposition.

<u>Proposition 2</u>: With two symmetric firms, there is an equilibrium of the larger game corresponding to a Stackelberg equilibrium. There is also a symmetric (in both timing and information contingent output) equilibrium. If the uncertainty is nontrivial but sufficiently small so that being a Stackelberg leader is more profitable than being an ex post Cournot firm, then the symmetric equilibrium must be temporally randomized.

4.

In this section, results (2), (4) and (5) in the Introduction are proved. To recapitulate, they are stated below.

The equilibrium of the larger game has two forms, one of which is symmetric and the other asymmetric. It will be shown that the symmetric form depends on the nature of the uncertainty. Thus, at the extremes of risk (i.e., zero variance of the demand distribution or a "too diffuse" distribution), it is temporally nonrandomized and corresponds to a Cournot equilibrium, and at intermediate values of the variance it is temporally randomized. A continuous function contained in the equilibrium correspondence links these Cournot end points and all symmetric equilibria lying on this path. It is in this sense remarked to be an appropriate extension of a Cournot equilibrium. This is result (2) of the Introduction.

Next, we will study how this equilibrium depends on relative firm sizes. It will be demonstrated that in a market with one large firm and a continuum of small firms, the only equilibrium is a Stackelberg equilibrium where the large firm moves before the demand is revealed as a leader, and the small firms enter the market as followers after the demand is revealed. This is result (4) of the Introduction.

Finally, the following question is addressed: Is the above asymmetric equilibrium of a mixed market the limit of the equilibria of markets with one large firm and several small atomic firms? The answer is in the affirmative. This is result (5) of the Introduction. A very complicated model would be needed to derive these results in complete generality. However, since what is important is the nature of the game tree, rather than the technology of the individual players, the results are proved in the context of a particular technology where the demand is linear and firms have quadratic costs. It must be remarked again, however, that results (4) and (5) hold in full generality.

The following notation is used in this section:

γ – is the variance of nature's distribution function.

x_i – is a production level of firm i.

x_{iB} – is a value of the production level when firm i decides to enter "Before."

x_{iA} – is a value of the production level when firm i decides to enter "After."

$b|x_{iB},y$ – are the paths that result when x_{iB} and y, the outcome of the random variable are fixed, and x_{jB} or x_{jA} are allowed to vary, for all j, $j \neq i$.

Let the demand be given by $p_r = y - \sum_i x_i$, where p_r is the price and y is the random shift parameter. Let N_f be the set of firms that want to enter the market. At first we shall consider the case of duopoly.

The market demand is given by $p_r = y - x_1 - x_2$. The cost function for both firms is $C(x_i) = (x_i^2)$ for an output level x_i.

Denote by $\Pi(b|x_{iB},y)$, the set of values that Π takes for the different branches represented by $b|x_{iB},y$. Similarly, $\Pi(b|x_{i(\cdot)},x_{j(\cdot)},y)$ is the payoff associated with the particular branch containing $x_{i(\cdot)}$, $x_{j(\cdot)}$ and y. From the tree in Figure 5, it is easy to see that there is always at most one such branch.

The Cournot equilibrium points are then easily seen to be

$$x_{1B} = x_{2B} = \frac{E_y(y)}{5} \tag{6}$$

and

$$x_{1A} = x_{2A} = \frac{y}{5} \tag{7}$$

While the Stackelberg equilibrium points are:

$$x_{1B} = \frac{3E_y(y)}{14} \tag{8}$$

$$x_{2A} = \frac{y - \frac{3E_y(y)}{14}}{4} \tag{9}$$

The equilibrium of the larger game is obtained as follows.

Denote $E_y(y)$ by E, $E_y(y^2)$ by E_2

and $(E_y(y))^2$ by E^2.

Further, let firm i's probability of entering in period B be ν_i and the quantity it decides to produce when entering in period B be x_{iB}.

In the larger game, in the first information set of every player, the player has to decide on a probability of entering in period B and the quantity it would produce if it were to enter in period B. In order to decide on the quantity it would produce if it were to enter in period B, the firm has to choose a quantity so that its ante expected payoff of entering in period B is maximized given

certain beliefs about nature's actions and the other player's actions. Thus, player i will maximize over his choice variable x_{iB},

$$E_y(\nu_j \cdot \pi_i(x_{iB}, x_{jB}, y) + (1 - \nu_j)\pi_i(x_{iB}, x_{jA}, y))\ i \neq j$$

noting that x_{jA} is the best response function given x_{iB} and y. Thus it obtains an optimal $\hat{x}_{iB}$ as a function of x_{jB}, ν_2 and E. On the other hand, the probability that the i^{th} player will enter in period B is calculated in the following way: Given that it is going to randomize between entering in period A and entering in period B with the optimal amount $\hat{x}_{iB}$, it must be indifferent (since we are considering only sequentially rational Nash equilibria) between entering in period A and entering in period B.

Thus,

$$x_{1B} = \underset{x}{\text{argmax}}\ E_y\{\nu_2 \quad \Pi(b|x,x_{2B},y) + (1 - \nu_2) \quad \Pi(b|x,x_{2A},y)\}$$

noting that x_{2A} is a measurable function of x and y. It is thus a random variable.

Similarly, we get an expression for x_{2B}. In equilibrium

$\hat{x}_{iB}$ and $\hat{\nu}_i$ satisfy (i = 1,2)

(a) $\hat{x}_{iB} = \hat{x}_{jB} = x_B$ and $\hat{\nu}_i = \hat{\nu}_j = \nu$, $i \neq j$

and, given that we are now interested in temporally randomized equilibria,

$$\text{(b)} \quad E_y\{\nu \cdot \Pi(b|x_B,y) + (1-\nu) \cdot \Pi(b|x_B,x_A,y)\} \tag{10}$$

$$= E_y\{\nu \cdot \Pi(b|x_A,x_B,y) + (1-\nu) \cdot \Pi(b|x_A,y)\}$$

It is easy to verify that the values of x_B and of ν thus obtained are the equilibrium values. Solving the maximization problem using (10a), we find

$$x_B = \frac{3E + E\nu}{14 + 6\nu} \tag{11}$$

Finally, substituting for x_B in (10b) above, we obtain

$$\nu^3(650E^2 - 648E_2) + \nu^2(4200E^2 - 4176E_2)$$

$$+ \nu(8850E^2 - 8904E_2) + (6300E^2 - 6272E_2) = 0. \tag{12}$$

A solution to the above equation assuming y is normally distributed with mean 1 and variance γ, yields ν as a function of γ, the variance of y. We deduce from equations (10a), (11) and (12) that the equilibrium is symmetric across players.

Also, from equation (12), $\nu(0) = 1$, and with $E = 1$, $\nu(.0044) = 0$. Thus, when the uncertainty is nontrivial ($\gamma \neq 0$), but sufficiently small ($\gamma < 0.0044$), temporal randomization occurs i.e., $\nu \neq 0,1$. Also, for $\gamma > .0044$, any nonzero ν is not a Nash equilibrium.

Hence there are two equilibria. An asymmetric one corresponding to a Stackelberg equilibrium and a symmetric one, with the probability of entering "Before" given by the solution to equation (12) and the quantity to be produced a function of that probability as given in (11).

Further, note that in expression (11) $\nu = 1$ yields $x_B = \frac{E}{5}$ which is indeed the Cournot level of production (see equation (6)) should both firms decide to go "Before."

We are now in a position to prove the following theorem.

Theorem 2: In the above duopoly game parameterized by variance as a risk parameter, the equilibrium corresponding to Cournot equilibria (which occur at both extremes of risk) are connected by a continuous path in the graph of the equilibrium correspondence, and the equilibria along this path are symmetric.

Proof: Clearly from Proposition 1, if $\gamma = 0$, then $\nu = 1$ for both players, an ex ante Cournot equilibrium results. Further for $\gamma = 0.0044$, (if $E = 1$) $\nu = 0$, and an ex post Cournot equilibrium results. Consider the correspondence $\phi_i : \{\gamma\} \rightrightarrows [0,1]$, with $\phi_i(\gamma) = \nu_i$ obtained from equation (12).

Let us first look at the nature of the correspondence knowing the following facts.

(a) From Proposition 2, $d_i(\gamma) \neq 0$ or 1 when $\gamma \neq 0$ or $< .0044$.

(b) The cubic (12) can be written in terms of γ, the variance, i.e.,

$$f(\nu,\gamma) = \nu^3(648(.003086 - \gamma)) + \nu^2(4176(.005 - \gamma))$$
$$+ \nu(8904(-.005 - \gamma)) + (62722(.004 - 1)) = 0.$$

(c) At $\gamma = 0$, by Descartes' Rule of Signs there are 2 or 0 positive roots, 1 negative root and if the discriminant is 0, then there are two identical roots.

(d) At $\gamma = 0.0044$, there are 2 or 0 positive roots, no negative roots, no identical roots, or 1 real root and 2 imaginary roots, and the discriminant is negative.

(e) At $\gamma = 0.003086$, there are 2 or 0 positive roots, no negative roots, and the discriminant is positive; i.e., there are two distinct roots. This is because the polynomial becomes a quadratic at this point.

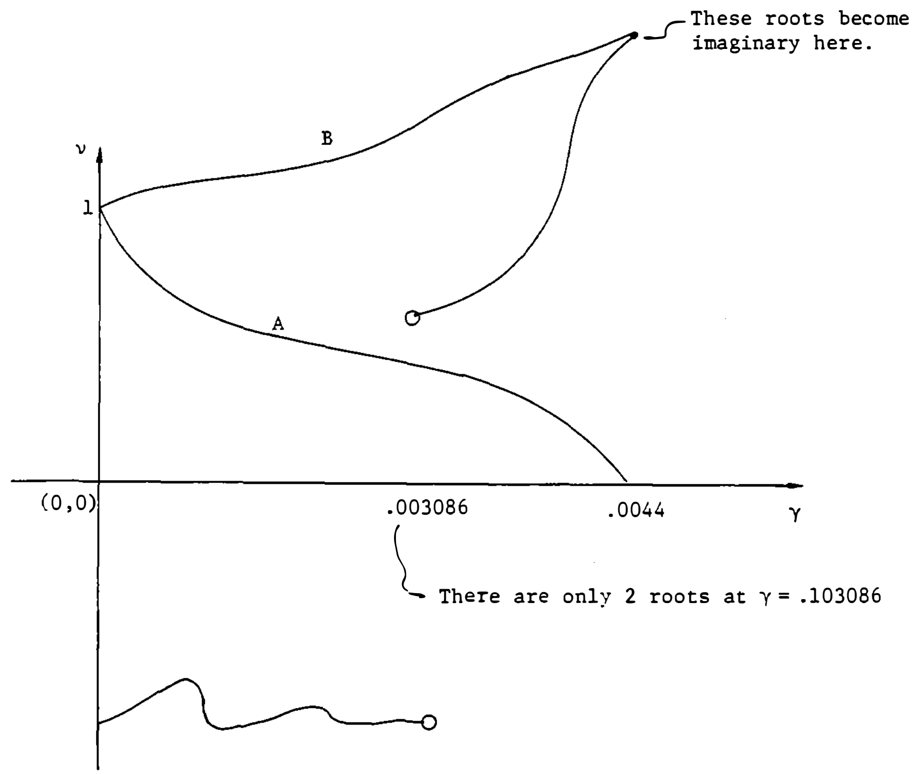
(f) The partial derivative of the polynomial $\frac{\partial f(\nu,\gamma)}{\partial \gamma}$ is never 0 for $\nu \in [0,1]$ and $\frac{\partial f(\nu,\gamma)}{\partial \nu} \neq 0$ for all γ. Thus, the Jacobian of the polynomial is never 0 in the range of concern.

The graph of the correspondence then would look like Figure 6.

[Figure 6 Here]

Now, it is sufficient to show that there is at least one continuous path from the point (0,1) to the point (0.0044, 0).

FIGURE 6



Let $f(\nu,\gamma) = 0$ be the polynomial equation under consideration. We know that $\forall(\gamma,\nu) \in [0, 0.0044] \times [0,1]$, the Jacobian of partial derivatives of first order, $J(f(\nu,\gamma)) \neq 0$.

Then consider any point (γ_0,ν_0), with $\gamma_0 \in (0, 0.0044]$, and $\nu_0 < 1$ such that $f(\nu_0,\gamma_0) = 0$. Then by the implicit function theorem, $\exists$ a smooth function g and a neighborhood $N(\gamma_0)$ such that $\forall\gamma \in N(\gamma_0)$, $J(f(\nu,\gamma)) \neq 0$.

$$g(\gamma_0) = \nu_0$$

and

$$\forall\gamma \in N(\gamma_0),\ f(g(\gamma),\gamma) = 0$$

From observation (e) above, the neighborhood $N(\gamma_0) = [0, 0.0044]$. Thus, the continuous path that is required is the graph of g.

Q.E.D.

Two simple but interesting corollaries follow from theorem 2 above. The first corollary states that the continuous path connecting the Cournot extremes is monotone decreasing in the graph of the correspondence ϕ described above. This means that for both firms, in the symmetric equilibrium, the probability of entering "Before" keeps getting smaller as the variance of nature's distribution increases. That is, as the uncertainty in the demand increases, they are less likely to enter the market before the demand information is revealed.

Corollary 1: $\forall\gamma \geq 0$, $\forall\nu \in [0,1]$, $\frac{d\nu}{d\gamma} < 0$.

Proof: Writing $\frac{d\nu}{d\gamma}$ as $\frac{\frac{-\partial f}{\partial\gamma}}{\frac{\partial f}{\partial\nu}}$ using the implicit function theorem, the proof is obvious.

The second corollary states that given one firm is more likely to go "After" as demand uncertainty increases, the other firm will want to produce more in the period "Before." Further, this desire to produce more is continuous in the probability of entry "Before," until the other firm will want to produce the Stackelberg leader's quantity when the first firm wants to enter "After" for certain.

Corollary 2: $\forall\nu \in [0,1]$, $\frac{dx_B}{d\gamma} > 0$.

Proof: Obvious from expression (11) and corollary 1 above.

We next consider mixed markets. The concept of a mixed market introduced by Shitovitz (1973) embodies the fact that often, markets are composed of a few large firms and several small firms.

In our context, we will see that if a mixed set of firms (one large atomic firm and a nonatomic continuum of firms), are contemplating entry into a market with uncertain demand, it is a Nash Equilibrium for the atomic firm to go "Before" as a Stackelberg leader, and for the nonatomic firms to go "After" as followers. The intuitive reason is that each nonatomic firms is so small that it can have no incentive effect on the atomic firm or other nonatomic firms. Furthermore, we will show that moving "After" is the dominant strategy for the nonatomic firms, and therefore, this is the unique Nash equilibrium.

Denote the large firm's production level by x_1 and the small firms' production level by x_2.

Let the set of "small" firms be indexed by the unit interval $I = [0,1]$, endowed with Lebesgue measure σ. Thus for $S \subseteq I$, $\sigma(s)$, is the proportion of firms belonging to the subset s. Let x(i) denote the amount produced by each firm $i \varepsilon I$. The profit associated with x(i) is denoted by $\Pi^i(x(i))$. Let the "large" atomic firm be referred to as the firm of type 1 with a cost function $C_1(x)$. In terms of cost functions, we say that firm i is of efficiency α, $0 \leq \alpha \leq 1$ if and only if it has a cost function $C_i(\cdot)$ such that,

$$C_i(\alpha x) = \alpha \cdot C_1(\alpha \cdot x)$$

i.e., cost for the little firm to produce αx is α times the cost for the atomic firm to produce αx. However, since in this section, $\alpha = 0$, in order to have any meaningful cost functions, we let the cost function for the atomic firm be $F_1 + C_1x_1^2$ and for the nonatomic firm be $F_2 + C_2x_2^2$. $F_i, C_i \varepsilon R$, $i = 1,2$. We are now in a position to state and prove result (4) of the Introduction.

<u>Theorem 3</u>: With one large firm and a continuum of nonatomic firms, with the technology given above, the small firms will enter in period A in the equilibrium of the larger game if there is nontrivial uncertainty. If this uncertainty is sufficiently small, then the only equilibrium corresponds to a Stackelberg equilibrium.

<u>Proof</u>: We will show that a Stackelberg equilibrium in mixed markets with the atomic firm entering in period B as leader and all the nonatomic firms entering in period A as followers, is the only equilibrium of the larger game provided there is sufficiently small uncertainty in the demand parameter.

Should it decide to enter early as a Stackelberg leader, firm 1 decides on its production level, as follows:

$$X_1 = \text{argmax } E_y(y - (x_1 + \int_0^1 x(\gamma,x_1)d\mu))x_1 - C_1x_1^2 - F_1),$$

where $x(\gamma,x_1)$ is the follower firm's reaction function.

For a follower, nonatomic firm, x_2 maximizes ex post profits and is the solution to

$$\max_{x_2} (y - x_1 - \int_0^1 x(\gamma,x_1)d\mu)x_2 - C_2x_2^2 - F_2$$

yielding $x_2 = \dfrac{y - x_1 - x(x_1)}{2C_2}$ where $x(x_1) = \int_0^1 x(\gamma,x_1)d\mu$.

Then the profits of a follower are

$$-\frac{(y - x_1)^2(2C_2 - 1)}{(2C_2 + 1)^2} - F_2 \qquad (13)$$

Since $\int_0^1 x_2 d\mu = x(x_1)$, we have

$$x(x_1) = \int_0^1 \frac{y - x_1 - x(x_1)}{2C_2} d\mu$$

therefore

$$x(x_1) = \frac{y - x_1}{2C_2 + 1} = x_2 \qquad (14)$$

Substituting this into the first order condition for firm 1 yields

$$x_1 = \frac{C_2 E}{C_1 + 2C_1C_2 + C_2}. \qquad (15)$$

It is easy to show that, when the uncertainity is not too large, the large firm's profits are lower if it decides to enter in period A. On the other hand, the firm contemplating moving "Before" (i.e., the deviant firm), decides on its production level by maximizing its ex ante profits. Thus,

$$x_2^{de} = \frac{E - x_1 - Ey(x(x_1))}{2C_2}$$

$$= \frac{E - x_1}{2C_2 + 1} \qquad (16)$$

so that its ex ante profits are,

$$-\frac{(E - x_1)^2(2C_2 - 1)}{(2C_2 + 1)^2} - F_2 \qquad (17)$$

Therefore, ex ante, if the deviant firm wants to compare profits, it sees that

$$E_y\left\{\frac{(y - x_1)^2(2C_2 - 1)}{(2C_2 + 1)^2}\right\} \geq \frac{(E - x_1)^2(2C_2 - 1)}{(2C_2 + 1)^2}$$

since $E_2 - E^2 = var \geq 0$.

So for the nonatomic firm it is dominant to be a follower and enter "After" for all γ. It can be shown further, that if the amount of uncertainity as measured by γ is larger than a certain value, depending upon the cost characteristics of the large firm, all firm will enter in period A. The proof of this in the case of linear demand—quadratic cost is easy to see. A general proof appears in [6].

Thus, we observe that in the case of two identical firms a symmetric equilibrium results. While in the case of one atomic firm of measure one and a nonatomic continua of firms, with non-trivial but small uncertainity, the only equilibrium is a Stackelberg equilibrium. A natural question would be: Is it true that, as we increase the cardinality of one set of firms while decreasing the measure of every firm in it, the resulting respective equilibria converge to the case of an asymmetric equilibrium of a mixed market?

The next theorem, which is result (5), in the Introduction, addresses this question.

Theorem 4: With respect to the larger game, a market with one large firm and several small atomic firms can be approximated by a market with one large firm and a continuum of small firms.

The proof of this theorem will make use of Theorem 1 of Green [2], and is a direct consequence of that theorem. Details of the proof will be provided in a subsequent paper.

5. CONCLUSION

In this paper, we set out to answer the question: under what circumstances might noncooperative equilibrium take a Cournot form and when might it take a Stackelberg form? In a Cournot game, the players are in the same strategic position with respect to each other and they are assumed to be moving simultaneously (or sequentially but unobservably). In a Stackelberg game, there are some players who are dominant, who move first and who are in a different strategic position with respect to the other players. Thus, a classical way to try and answer the question was to examine timing and information conditions, both of which were presumed exogenous. If these were unobservable, then one was guided by the "Folk Theorem" that outcomes in oligopolies are best modeled by a Cournot equilibrium if the firms are of equal

size and by a Stackelberg equilibrium, if they were not. The technologies of the firms and demand characteristics were irrelevant.

On the other hand, we answered the question by defining a game in which ex ante all the players were in the same strategic position with respect to each other, while demand characteristics and sizes and firm technologies were exogenous. The basic idea was that if demand uncertainty was resolved over time, then firms may face a trade-off between making quantity decisions early so as to establish a "leadership" position, or waiting until the demand uncertainty is resolved so as to avoid production mistakes. A sequentially rational Nash equilibrium of the resulting game was Cournot-like if all firms produced at the same time, whereas it was Stackelberg-like if some produced before, and others after, the demand uncertainity was resolved. Equilibrium with respect to this game was studied and it was shown that there are two classes of equilibria, one of which directly corresponded to a Stackelberg equilibrium and the other represented a natural generalization of Cournot equilibrium. We also showed that in a market with one "large" firm and a continuum of "small" firms facing a set of passive consumers, the only equilibrium was the Stackelberg equilibrium with the "large" firm as the leader. There were also some comparative static results on the symmetric form of the equilibrium and how it changed with the amount of uncertainty in demand. This confirmed one part of the Folk Theorem: namely, that when there are firms of different sizes in an industry, it is best modeled by a Stackelberg equilibrium.

On the other hand, we showed through results 1 and 2, that even when an industry has identical firms, a Stackelberg equilibrium is <u>an</u> endogenously determined Nash equilibrium. This refutes the other part of the Folk Theorem: namely, that when an industry has firms of identical sizes, it is best modelled as a Cournot equilibrium.

Further research could adopt the model developed in this paper to the framework of a model of noncooperative exchange where all agents are treated symmetrically, i.e., they are in the same strategic position (such as the noncooperative general exchange model of Shapley). This way one would be able to obtain an endogenously determined price-setting monopolist as an equilibrium of a noncooperative game. Finally, this model can be used to examine advertising and timing of technological innovations as strategic market activities.

FOOTNOTES

1. This result closely resembles an observation made by Guasch and Weiss [5].

2. We could have more than 1 dominant player. In general let $D \subset N$ be the set of dominant players the dominant players move together but before the other players. Then, the dominant players' equilibrium would be a vector $v \in V$ with:

(a) $d \in D,\ s'_d \in M_d$,

$$E_{\mu|s_d, s_j j \in D}\, \Pi_d(b) < E_{\mu|s'_d, s_j j \in D}\, \Pi_d(b)$$

(b) for every $d \in D,\ s'_d \in M_d,\ i \in N,\ i \notin D,\ s'_i \in M_i$,

$$E_{\mu|s_i, s'_j j \in D, y*}\, \Pi_i(b) < E_{\mu|s'_i, s'_j j \in D, y*}\, \Pi_i(b) \qquad (4')$$

REFERENCES

1. Aumann, R. J. "Mixed and Behaviour Strategies in Infinite Extensive Games," in Advances in Game Theory, edited by Dresher, L.S. Shapley and A. W. Tucker, Princeton University Press, Princeton, 1964, pp. 627–650.

2. Green, E. "Continuum and Finite-Player Noncooperative Models of Competition." Social Science Working Paper no. 418, California Institute of Technology, March 1982, forthcoming in Econometrica.

3. Kuhn, H. W. "Extensive Games and the Problem of Information," Contributions to the Theory of Games, vol. 2, Annals of Mathematical Studies, 28, pp. 193–216, Princeton 1953.

4. Shitovitz, B. "Oligopoly in Markets with a Continuum of Traders," Econometrica, 41 (3), pp. 467–502, May 1973.

5. Guasch, L. J., and Weiss, A. "Adverse Selection by Markets and the Advantage of Being Late," Quarterly Journal of Economics 94 (3), pp. 453–466, May 1980.

6. Sadanand, V. "Endogenously Determined Price-Setting Monopolist in an Exchange Economy." Working Paper number 915, Faculty of Commerce and Business Administration, University of British Columbia, November 1982.