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QUASITRANSITIVE SOCIAL CHOICE WITHOUT THE PARETO PRINCIPLE

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INTRODUCTION

The underlying observation of this paper is that when the Pareto principle fails, the collection X of alternatives may be partitioned into a set X* of unbeatable (against at least one member of X) elements and its complement $X \sim X^*$ on which the Pareto axiom holds. It is then instructive to characterize the decisive, anti-decisive and blocking coalitions for $X \sim X^*$ against $X \sim X^*$, $X \sim X^*$ against X*, X* against $X \sim X^*$, and X* against X*. Now X* itself may contain elements which are unbeatable with respect to alternatives in X*--this is to say that the Pareto axiom fails again, locally on X*. Thus X* may be partitioned into $(X^*)^* = X^{2*}$ and $X^* \sim X^{2*}$, and then the same analysis that was applied in the case of the partition $(X^*, X \sim X^*)$ can be employed again. This process is iterated until $X^{n*} = \phi$ or $X^{n*} = X^{(n+1)*}$, for some n.

It is hoped that a characterization or classification scheme can be developed for sets of alternatives which are invariant under the * operator; such sets are exactly those in which every element of the set is unbeatable against at least one other member of the set.

The main results of this paper are to be found on pages 11 and 15. The last claim on page 11 establishes that the elements of $X \sim X^*$ can never be socially ranked above any of the elements of X*. In a similar vein, the final proposition stated on page 15 establishes that every coalition is weakly antidecisive for members of $X \sim X^*$ over members of X*.

NOTATION

N = a finite set of individuals, $N \subseteq \mathbb{N}$

X = a set of alternatives.

R_i = Mr. i's complete, reflexive, transitive preference relation on X.

I_i, P_i = indifference and strict preference for Mr. i, obtained from R_i.

$\mathcal{R}$ = set of all complete, reflexive, transitive profiles for N on X.

R = some one profile.

Q = set of all complete, reflexive, quasitransitive relations on X.

f = a social choice function; $f : \mathcal{R} \to$ {quasitransitive relations on X}.

F = the family of all such social choice functions as above.

R_f = social preference corresponding to $R \in \mathcal{R}$ under f; $R_f = f(R)$.

I_f, P_f = social indifference and strict preference relations associated with R_f.

P_c, R_c, I_c = $\forall R \in \mathcal{R}$, $\forall x,y \in X$, $\forall C \subseteq N$ ($xP_c y \iff$ $(\forall i \in \mathbb{N} (i \in C \Rightarrow xP_i y)))$;
R_c and I_c are defined similarly.

$D_f(U,V)$ = $\forall f \in \mathcal{F}$, $\forall U,V \subseteq X$, $\forall C \subseteq N$ ($C \in D_f(U,V) \iff$ $(\forall R \in \mathcal{R}, \forall u \in U, \forall v \in V\, [uP_c v \Rightarrow uP_f v]))$. This is the set of coalitions decisive for U over V.

$A_f(U,V)$ = the set of anti-decisive coalitions for U over V; defined exactly as $D_f(U,V)$ above, except the expression in heavy square brackets should be replaced by $[uP_c v \Rightarrow vP_f u]$.

$B_f(U,V)$ = blocking coalitions for U over V. Defined as above but with $[uP_c v \Rightarrow uI_f v]$.

$WD_f(U,V)$ = family of coalitions weakly decisive for U over V; defined as above except for making the replacement: $[uP_c v \Rightarrow \neg vP_f u]$.

$WA_f(U,V)$ = weakly antidecisive coalitions; as before, but with: $[uP_c v \Rightarrow \neg uP_f v]$.

$SWD_f(U,V)$ = strictly weakly decisive coalitions for U over V. $SWD_f(U,V) = WD_f(U,V) \sim D_f(U,V) \sim B_f(U,V)$.

$SWA_f(U,V)$ = family of strictly weakly anti-decisive coalitions for U over V. $SWA_f(U,V) = WA_f(U,V) \sim A_f(U,V) \sim B_f(U,V)$.

$D_f(x,y)$ = If $x,y \in X$, a notational abuse is entertained by writing $D_f(x,y)$ for $D_f(\{x\},\{y\})$. Similarly for $B_f(x,y)$, etc.

PARETO $_f[V]$ = $\forall f \in \mathcal{F}$, $\forall V \subseteq X$ (PARETO $_f[V] \iff$ $(\forall R \in \mathcal{R}, \forall x,y \in V(xP_N y \Rightarrow xP_f y)))$. Unless there is risk of confusion the subscript "f" will be omitted.

X_f^* = $\{x \in X \mid \exists y \in X(x \neq y \wedge D_f(y,x) = \phi\}$. Unless there is risk of confusion the subscript "f" will be omitted. This is the set of all alternatives in X which are unbeatable against at least one (different than itself) alternative in X.

$X_f^{(n+1)*}$ = $\{x \in X_f^{n*} \mid \exists y \in X_f^{n*}(x \neq y \wedge D_f(y,x) = \phi)\}$. Again the subscript will be suppressed. N.B., $X^{1*} = X^*$, and $X^{0*} = X$.

$\#\langle V \rangle$ = the cardinality of V.

IIA = $f \in \mathcal{F}$ is said to satisfy IIA just in case

$$\forall R,R' \in \mathcal{R}, \forall x,y \in X\left(R\Big|_{\{x,y\}} = R'\Big|_{\{x,y\}} \Rightarrow R_f\Big|_{\{x,y\}} = R'_f\Big|_{\{x,y\}}\right).$$

PARETO = $f \in \mathcal{F}$ is said to satisfy PARETO iff PARETO $_f[X]$. Unless otherwise mentioned f is arbitrary $f \in \mathcal{F}$ satisfying IIA.

DETALLS

The following are important, but direct, consequences of the definitions.

<u>Claim</u>. $\forall R \in \mathcal{R}, \forall x,y \in X(xP_\phi y)$

<u>Claim</u>. $\forall x \in X(\phi \notin D_f(x,x))$

<u>Claim</u>. $\forall x \in X(D_f(x,x) = 2^N \sim \phi)$

<u>Claim</u>. PARETO $[\phi]$

<u>Claim</u>. $\forall x \in X$(PARETO $[x]$)

<u>Claim</u>. $D_f(\phi,\phi) = D_f(\phi,V) = D_f(V,\phi) = 2^N$, for all $V \subseteq X$

<u>Claim</u>. Let $u \subseteq X$ s.t. $u \neq \phi$; then $\phi \notin D_f(u,u)$. Note, however, that if $U,V \subseteq X$ s.t. $U \cap V = \phi$, then it is possible that $\phi \in D_f(U,V)$. This means every element of U is strictly imposed over every element of V.

<u>Claim</u>. $U,V \subseteq X(D_f(U,V) = \bigcap_{\substack{u \in U \\ v \in V}} D_f(u,v) = \bigcap_{u \in U} D_f(u,V) = \bigcap_{v \in V} D_f(U,v))$

<u>Claim</u>. Let $U,V,W \subseteq X$ s.t. $U \subseteq V$; then $D_f(V,W) \subseteq D_f(U,W)$, and $D_f(W,V) \subseteq D_f(W,U)$.

<u>Claim</u>. Let $U,V,W, \subseteq X$; then $D_f(U,V) \cap D_f(V,W) \subseteq D_f(U,W)$.

<u>Claim</u>. Let $U,V \subseteq X$; then $D_f(U,V)$, $A_f(U,V)$ and $B_f(U,V)$ are closed under superset in N.

<u>$X \sim X^*$, $X \sim X^*$</u>

$X^* \subseteq X$ is the set of alternatives for which the Pareto axiom fails; an alternative lies in X^* just in case it is unbeatable w.r.t. at least one other alternative in X. Thus $X \sim X^*$ may be thought of as having been purged of any unbeatable alternatives. Pareto holds on $X \sim X^*$ and so, therefore, $D_f(X \sim X^*, X \sim X^*)$ is a filter. For the same reasons $A_f(X \sim X^*, X \sim X^*) = \phi$.

The next four claims are direct consequences of the definitions but instructive nevertheless. The fifth is Hanssen's classic result.

<u>Claim</u>. PARETO $[X]$ iff $X^* = \phi$

<u>Claim</u>. PARETO $[X \sim X^*]$

<u>Claim</u>. $X \sim X^* \in \max_{Y \subseteq X} \{Y |$ PARETO $[Y]\}$

<u>Claim</u>. If $X = X^*$, then $\forall n \in \mathbb{N}(X = X^{n^*})$

<u>Claim</u>. (Hanssen) If $\# \langle X \sim X^* \rangle \geqslant 3$, then $D_f(X \sim X^*, X \sim X^*)$ is a filter. If f also satisfies transitive range, the $D_f(X \sim X^*, X \sim X^*)$ is an ultrafilter.

<u>Proof</u>. well known.

<u>As an aside</u>, consider the following problem. Let u,v,w,y be distinct elements of $X \sim X^*$; is it the case that $D_f(u,v) = D_f(w,y)$? In case D_f $(X \sim X^*, X \sim X^*)$ is an ultrafilter one can certainly answer affirmatively. However, if only quasitransitive range is assumed, $D_f(X \sim X^*, X \sim X^*)$ is merely a filter; and the problem is more complex.

One might consider the functions $D_f(\cdot,x) : X \sim X^* \to 2^{2^N} : y \to D_f(y,x)$ and define the relation $\equiv$ on $X \sim X^*$ by $x \sim y$ iff

$$D_f(\cdot,x)\Big|_{X\sim X^*\sim\{x,y\}} = D_f(\cdot,y)\Big|_{X\sim X^*\sim\{x,y\}}.$$

Now the relation $\sim$ would partition $X \sim X^*$ into finitely many equivalence classes. Define an associative, binary operation $*$ on $[X \sim X^*]_{\equiv}$ according to $[x] * [y] = [z]$ iff

$$D_f(\cdot,\{x,y\})\Big|_{X\sim X^*\sim\{x,y,z\}} = D_f(\cdot,z)\Big|_{X\sim X^*\sim\{x,y,z\}}.$$

A problem arises in that $[X \sim X^*]_{\equiv}$ may fail to be $*$-complete; there simply may not be an alternative Z which is equal w.r.t. unbeatability as the set of alternatives $\{x,y\}$. However, $X \sim X^*$ could be augmented so as to make $[X \sim X^*]_{\equiv}$ $*$-complete. Then $([X \sim X^*]_{\equiv},*)$ would form a semigroup. Insofar as $D_f(X \sim X^*, X \sim X^*)$ could be said to have been encoded into its associated semigroup, the large body of theorems addressing the taxonomy, characterization, and complexity of finite semigroups could be brought to bear on the nature of $D_f(X \sim X^*, X \sim X^*)$. Notice, for example, that if $D_f(X \sim X^*, X \sim X^*)$ is an ultrafilter, then $[X \sim X^*]_{\equiv}$ is just a single equivalence class. As another example, in a context more general than $X \sim X^*$, if there is an alternative $x \in X$ which is always strictly imposed so that $D_f(x,X) = 2^N$, then $[x]$ will act as an identity in the semigroup $([x],*)$.

<u>$X^*,X \sim X^*$</u>

One anticipates that $D_f(X^*,X \sim X^*)$ will be a large family; even puny coalitions ought to be able to carry the unbeatable alternatives in X^* over the beatable alternatives in $X \sim X^*$. Once it is established that $N \in D_f(X^*,X \sim X^*)$, closure under superset forces $A_f(X^*,X \sim X^*)$ and $B_f(X^*,X \sim X^*)$ are empty. In this section, unless otherwise stated, X^* and $X \sim X^*$ are assumed to be non-empty.

The following claims all have one line proofs but are nevertheless important.

<u>Claim</u>. $D_f(X^*,X \sim X^*)$, $A_f(X^*,X \sim X^*)$, and $B_f(X^*,X \sim X^*)$ are closed under superset in N.

<u>Claim</u>. $D_f(X^*,X \sim X^*)$, $A_f(X^*,X \sim X^*)$, and $B_f(X^*,X \sim X^*)$ are mutually disjoint.

<u>Claim</u>. $D_f(X^*,X \sim X^*) \neq \phi$ and $\forall x \in X^*, \forall y \in X \sim X^*(D_f(x,y) \neq \phi)$.

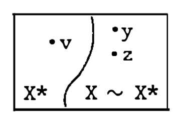
<u>Proof</u>. Suppose not. Then for some $x \in X^*$, $y \in X \sim X^*$, $D_f(x,y) = \phi$. But this gives $y \in X^*$, a contradiction. Local claim is now obvious.

<u>Claim</u>. $A_f(X^*,X \sim X^*) = B_f(X^*,X \sim X^*) = \phi$ and $\forall x \in X^*, \forall y \in X \sim X^*(A_f(x,y) = B_f(x,y) = \phi)$.

The next several claims establish a lower bound for $D_f(X^*,X \sim X^*)$.

<u>Claim</u>. Suppose $X^* \neq \phi$ and $\# < X \sim X^* > \geqslant 2$. Let $c \in D_f(X^*,X \sim X^*)$ and $d \in D_f(X \sim X^*,X \sim X^*)$, then $c \cap d \in D_f(X^*,X \sim X^*)$.

<u>Proof</u>.



Choose $v \in X^*$ and distinct $y,z \in X \sim X^*$. Let $R \in \mathcal{R}$ be given s.t. $vP_{c\cap d}y$; it is required to show that vP_fy obtains. Construct R' s.t. the following conditions hold:

1) $R\big|_{\{v,y\}} = R'\big|_{\{v,y\}}$ and

2)

R			R'		
$c \sim d$	$c \cap d$	$d \sim c$	$c \sim d$	$c \cap d$	$d \sim c$
[v,y]	v	[v,y]	$[v,y]_R$	v	z
	y		z	z	$[v,y]_R$
				y	

N.B. Notation [v,y] means the relation between v and y under R among the members of the coalition $c \sim d$ is arbitrary.

N.B. The subscript R indicates that the relation between v and y under the Profile R' among the members of the coalition shown at the column heading agrees with R.

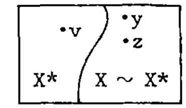
Now $vP_c'z$ so that $vP_f'z$ since $c \in D_f(X^*,X \sim X^*)$ and
$zP_d'y$ so that $zP_f'y$ since $d \in D_f(X \sim X^*,X \sim X^*)$.
Thus quasitransitivity gives $vP_f'y$, and by IIA vP_fy obtains.
Hence $c \cap d \in D_f(X^*,X \sim X^*)$, as desired.

Claim. $D_f(X \sim X^*,X \sim X^*) \subseteq D_f(X^*,X \sim X^*)$ where $X^* \neq \phi$ and $\# < X \sim X^* > \geqslant 2$.

Proof. In the above claim choose C = N (recall $N \in D_f(X^*,X \sim X^*)$) and let d range over the members of $D_f(X \sim X^*,X \sim X^*)$. In fact, the sharper but more obscure local result is true.

Claim. Let $X^* \neq \phi$ and $\# < X \sim X^* > \geqslant 2$. Suppose $v \in X^*$ and distinct $y,z \in X \sim X^*$. Then $D_f(Z,y) \subseteq D_f(v,z)$.

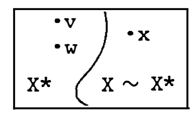
Proof. Apply the technique immediately above.



A stronger condition gives $D_f(X^*,X^*)$ as a lower bound for $D_f(X^*,X \sim X^*)$.

Claim. Let $\# < X^* > \geqslant 2$ and $X \sim X^* \neq \phi$. Suppose there is at least one alternative $w \in X^*$ s.t. $D_f(X^*,w) \neq \phi$. If $c \in D_f(X^*,X^*)$ and $d \in D_f(X^*,X \sim X^*)$, then $c \cap d \in D_f(X^*,X \sim X^*)$.

Proof.



Let $v,w \in X^*$ and $x \in X \sim X^*$. Suppose $R \in \mathcal{R}$ is given s.t. $vP_{c \cap d}x$. Construct $R' \in \mathcal{R}$ s.t.

1) $R'\big|_{\{v,x\}} = R\big|_{\{v,x\}}$ and

2)

R			R'		
$c \sim d$	$c \cap d$	$d \sim c$	$c \sim d$	$c \cap d$	$d \sim c$
[v,x]	v	[v,x]	$[v,x]_R$	v	w
	x			w	$[v,x]_R$
			w	x	

Thus $vP_c'w$ so $vP_f'w$ since assumed $c \in D_f(X^*,X^*)$ and
$wP_d'x$ so $wP_f'x$ since assumed $d \in D_f(X^*,X \sim X^*)$.
$vP_f'x$ follows by quasitransitivity and IIA gives vP_fx. Hence $c \cap d \in D_f(v,x)$ and arbitrariness of v,x allow $c \cap d \in D_f(X^*,X \sim X^*)$.

N.B. Could write a more localized version of the above claim.

Claim. Suppose the conditions of the preceding claim obtain. Then

1) $D_f(X^*,X^*) \subseteq D_f(X^*,X \sim X^*)$.

2) If every member of $D_f(X^*,X^*)$ does not have nonempty intersection with every member of $D_f(X^*,X \sim X^*)$, then $D_f(X^*,X \sim X^*) = 2^N$.

<u>Proof</u>. 1) In the above claim choose d = N and let c range over the members of $D_f(X^*,X^*)$.

2) If $c \cap d = \phi \in D_f(X^*,X \sim X^*)$, closure under superset gives that $D_f(X^*,X \sim X^*) = 2^N$.

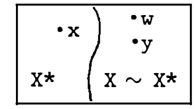
In the above note that the condition $\exists w \in X^*$ s.t. $D_f(X^*,w) \neq \phi$ obtains if PARETO [X*] or ir $X^* \neq X^{2*}$.

Let $U,V \subseteq X$; recall that a coalition $c \subseteq N$ is said to be weakly decisive for U over V, written $c \in WD_f(U,V)$ just in case $\forall R \in \mathcal{R}$, $\forall u \in U$, $\forall v \in V(uP_c v \Rightarrow (uP_f v \vee uI_f v))$. And c is said to be strictly weakly decisive for U over V iff $c \in WD_f(U,V) \sim D_f(U,V) \sim B_f(U,V)$. Thus a coalition c in $SWD_f(U,V)$ will never suffer the imposition of any $v \in V$ over any $u \in U$ contrary to the coalition's strict preference. However, for at least one choice of $u \in U$, $v \in V$, and $R \in \mathcal{R}$ it must be the case that $uP_c v$ and $uI_f v$ so that c is not decisive--$c \notin D_f(U,V)$ and also, for at least one choice of $u' \in U$, $v' \in V$, and $R' \in \mathcal{R}$, $u'P'_c v'$ and $u'P_f v'$ so that c is not merely blocking--$c \notin B_f(U,V)$.

<u>Claim</u>. Let $X^* \neq \phi$, $\# \langle X \sim X^* \rangle \geqslant 2$ and $c \subseteq N$. If $c \in D_f(X^*,X \sim X^*)$, then $c \in SWD_f(X^*,X \sim X^*)$.

<u>Proof</u>. It was previously shown that $N \in D_f(X^*,X \sim X^*)$. Let $c \subset N$ s.t. $c \in D_f(X^*,X \sim X^*)$ be given. Then $\exists R \in \mathcal{R}$, $\exists x \in X^*$, $\exists y \in X \sim X^*$

$(xP_c y \wedge \neg xP_f y)$. Now $\neg xP_f y$ allows for two cases: i) $xI_f y$ and ii) $yP_f x$. Observing that $B_f(X^*,X \sim X^*) = \phi$, if i) obtains, nothing remains to prove. Hence suppose ii), $yP_f x$, is the case. Since assumed $\# \langle X \sim X^* \rangle \geqslant 2$ choose $w \in X \sim X^*$ distinct from y above. Construct $R' \in \mathcal{R}$ such that



1) $R'|_{\{x,y\}} = R|_{\{x,y\}}$, 2) $wP'_N y$, and 3) $wP'_{c^c}x$. Pictorially these conditions are:

R		R'	
<u>c</u>	<u>c^c</u>	<u>c</u>	<u>c^c</u>
x	[x,y]	[x,w]	w
y		y	$[x,y]_R$

Now since $yP_f x$, IIA yields $yP'_f x$. Notice next that $w,y \in X \sim X^*$, and as previously demonstrated Pareto holds on this set. Thus $xP'_N y$ gives $wP'_f y$. Quasitransitivity serves to establish that $yP'_f x$ and $wP'_f y$ yield $wP'_f x$. It will now be shown that $c^c \in D_f(w,x)$, which is a contradiction since $w \in X \sim X^*$, $x \in X^*$ and so as shown earlier $D_f(w,x) = \phi$. Let $R'' \in \mathcal{R}$ be given s.t. $wP''_{c^c}x$. Observe that $wP''_{c^c}x$ and $wP'_{c^c}x$, and also that the relation between x and w under R' was left unspecified w.r.t. the members of c. Thus R' can be constructed to satisfy not only 1), 2), and 3) above but also the condition $R'|_{\{x,w\}} = R''|_{\{x,w\}}$.

Then having established $wP'_f x$, IIA gives $wP''_f x$. Hence $c^c \in D_f(w,x)$, which is the desired contradiction.

<u>Claim</u>. Suppose $X^* \neq \phi$ and $\# < X \sim X^* > \geqslant 2$. Then

$$D_f(X^*, X \sim X^*) \cup SWD_f(X^*, X \sim X^*) = 2^N.$$

<u>Proof</u>. This is just a restatement of the previous claim.

Notice that $SWD_f(X^*, X \sim X^*)$ is contained within an ideal on N. For $D_f(X \sim X^*, X \sim X^*)$ is a filter and is contained within $D_f(X^*, X \sim X^*)$. $N \notin SWD_f(X^*, X \sim X^*)$ and if $D_f(X^*, X \sim X^*) \neq 2^N$, then $\phi \in SWD(X^*, X \sim X^*)$. However, unless $D_f(X^*, X \sim X^*)$ is a filter, closure under union will fail.

<u>Conjecture</u>--Let Y be some set of alternatives. If given $D_f(Y,Y)$ and $SWD_f(Y,Y)$ which are in fact dual it is possible to solve the backwards problem: Does the f which generated $D_f(Y,Y)$ and $SWD_f(Y,Y)$ satisfy transitive range, quasitransitive range, Pareto, IIA?

<u>$(X \sim X^*, X^*)$</u>

One expects that $D_f(X \sim X^*, X^*)$ ought to be small; few if any coalitions ought to be able to best an unbeatable element of X^* with a beatable alternative in $X \sim X^*$. On the other hand, it is plausible that one of disjoint $A_f(X \sim X^*, X^*)$ or $B_f(X \sim X^*, X^*)$ is large.

The sense of the next claim is that if an alternative lies in X^*, it is unbeatable against at least one alternative in $X \sim X^*$. The members of X^* are not unbeatable simply by virtue of taking in each other's laundry.

<u>Claim</u>. Suppose $X \sim X^*, X^* = \phi$ and let $y \in X^*$. Then $\exists x \in X \sim X^*$ s.t. $D_f(x,y) = \phi$. (Note first condition says $X \neq X^*$.)

<u>Proof</u>. Trivial if X^* is a singleton. Hence suppose $\# < X^* > \geqslant 2$. If the claim is false, then $\forall x \in X \sim X^*(D_f(x,y) \neq \phi)$. Now since $y \in X^*$ by definition $\exists z \in X(D_f(z,y) = \phi \wedge y \neq z)$. Consider any $v \in X \sim X^*$ and observe $D_f(z,v) \neq \phi$ for otherwise $v \in X^*$. Since $v \in X \sim X^*$, assuming the claim false gives that $D_f(v,y) \neq \phi$. Closure of decisive families under superset allows in particular that $N \in D_f(z,v)$, $N \in D_f(v,y)$. Thus $N \in D_f(z,v) \cap D_f(v,y) \subseteq D_f(z,y) = \phi$, a contradiction.

This claim can be universalized as below:

<u>Claim</u>. Suppose $X \sim X^*, X^* \neq \phi$. Then $\forall x \in X \sim X^*$, $\forall y \in X^*(D_f(x,y) = \phi)$.

<u>Proof</u>. Suppose not. As above, since $y \in X^*$, $\exists v \in X \sim X^*$ such that $D_f(v,y) = \phi$. The negation of the claim allows some $x \in X \sim X^*$ s.t. $D_f(x,y) \neq \phi$. Since $v,x \in X \sim X^*$ and Pareto $[X \sim X^*]$, $D_f(v,x) \neq \phi$. In particular $N \in D_f(v,x) \cap D_f(x,y) \subseteq D_f(v,y) = \phi$, a contradiction.

And these local claims can be trivially globalized to give:

<u>Claim</u>. $D_f(X \sim X^*, X^*) = \phi$, where $X \sim X^*, X^* \neq \phi$.

<u>Proof</u>. $D_f(X \sim X^*, X^*) = \bigcap_{\substack{x \in X \sim X^* \\ y \in X^*}} D_f(x,y) = \bigcap_{\substack{x \in X \sim X^* \\ y \in X^*}} \phi = \phi$, as desired.

In what follows, suppose that $X^* \neq \phi$ and $\# < X \sim X^* > \geqslant 2$. We now seek to characterize $B_f(X \sim X^*, X^*)$ and $A_f(X \sim X^*, X^*)$. An important unresolved problem is to develop interesting conditions which guarantee

$N \in B_f(X \sim X^*,X^*)$ or $N \in A_f(X \sim X^*,X^*)$, these two sets being disjoint.

Claim. $B_f(X \sim X^*,X^*) \cap A_f(X \sim X^*,X^*) = \phi$

Proof. Obvious; suppose not.

Claim. $\forall v,w \in X(A_f(v,w) = \phi \Rightarrow (N \in D_f(v,w) \vee N \in B_f(v,w)))$

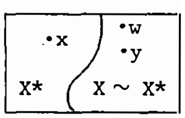
Proof. Suppose not. Then $A_f(v,w) = \phi$ but $N \notin D_f(v,w)$ and $N \in B_f(v,w)$. Now $A_f(v,w) = \phi \Rightarrow N \notin A_f(v,w) \underset{IIA}{\overset{by}{\Longrightarrow}} \forall R \in \mathcal{R}(vP_N w \wedge \neg wP_f v)$. $N \notin D_f(v,w)$ and $N \notin B_f(v,w)$ give, again by IIA, that $\forall R \in \mathcal{R}(vP_N w \wedge \neg vP_f w \wedge \neg vI_f w)$. But this violates completeness of social preference.

Claim. $A_f(X \sim X^*,X^*) = \phi \Rightarrow N \in B_f(X \sim X^*,X^*)$

Proof. Recall $D_f(X \sim X^*,X^*) = \phi$ and apply the above claim.

Claim. Let $c \subseteq N$, if $c \notin A_f(X \sim X^*,X^*)$; then $c \in WA_f(X \sim X^*,X^*)$.

Proof.



If $c \notin A_f(X \quad X^*,X^*)$, then $\exists R \in \mathcal{R}, \exists x \in X^*, \exists y \in X \sim X^*(yP_c x \wedge \neg xP_f y)$. Thus either i) $xI_f y$ or ii) $yP_f x$. Should i) obtain, there is nothing to prove; so suppose ii) is the case. Construct $R' \in \mathcal{R}$ s.t. 1) $R\big|_{\{x,y\}} = R'\big|_{\{x,y\}}$ and

2)

R		R'	
c	c^c	c	c^c
y	[x,y]	w	w
x		y	[x,y]
		x	

Since $yP_f x$, IIA gives $yP'_f x$. Recall that Pareto holds on $X \sim X^*$ so that $wP'_f y$. By quasitransitivity then $wP'_f x$. Now $xP'_N x$ so by IIA $\forall R'' \in \mathcal{R}(wP''_N x \Rightarrow xP''_f x)$. Thus $N \in D_f(w,x)$ which is a contradiction since $w \in X \sim X^*$, $x \in X^*$ require $D_f(w,x) = \phi$, as was previously shown.

The preceding development will allow the following rather weak characterization of $B_f(X \sim X^*,X^*)$. Note in the quasitransitive case $B_f(X \sim X^*,X^*)$ is not in general a filter.

Claim. Let $c \subseteq N$. If $c \in B_f(X \sim X^*,X^*)$, then $c^c \in WA_f(X \sim X^*,X^*)$ and $c^c \in SWD_f(X^*,X \sim X^*)$.

Proof. Shown previously:

i) If $c \in B_f(X \sim X^*,X^*)$, then $c^c \notin A_f(X \sim X^*,X^*)$; and

ii) If $c^c \notin A_f(X \sim X^*,X^*)$, then $c^c \in WA_f(X \sim X^*,X^*)$;

∴ If $c \in B_f(X \sim X^*,X^*)$, then $c^c \in WA_f(X \sim X^*,X^*)$.

Also have shown

iii) If $c \in B_f(X \sim X^*,X^*)$, then $c^c \notin D_f(X^*,X \sim X^*)$; and

iv) If $c^c \notin D_f(X^*,X \sim X^*)$, then $c^c \in SWD_f(X^*,X \sim X^*)$;

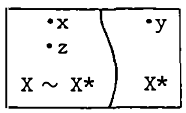
∴ If $c \in B_f(X \sim X^*,X^*)$, then $c^c \in SWD_f(X^*,X \sim X^*)$.

The two conclusions above form the conjunction claimed.

The following claims bear upon the characterization of $A_f(X \sim X^*,X^*)$. Note since $X \sim X^* \cap X^* = \phi$, there is no necessary contradiction if $\phi \in A_f(X \sim X^*,X^*)$.

Claim. Suppose $\# < X \sim X^* > \geqslant 2$ and $X^* \neq \phi$, then $A_f(X \sim X^*,X^*)$ is closed under intersection.

<u>Proof</u>. Trivial if $A_f(X \sim X^*, X^*) = \phi$, so suppose it's not empty.



Let $c,d \in A_f(X \sim X^*, X^*)$, $x \in X \sim X^*$, $y \in X^*$, and $R \in \mathcal{R}$ s.t. $xP_{c\ d}y$. Then construct $R' \in \mathcal{R}$ such that

R			R'		
$c \sim d$	$c \cap d$	$d \sim c$	$c \sim d$	$c \cap d$	$d \sim c$
[x,y]	x	[x,y]	z	z	z
	y		$[x,y]_R$	x	$[x,y]_R$
				y	

Now zP'_cy so yP'_fz since $c \in A_f(X \sim X^*, X^*)$; and zP'_Nx so zP_fx since PARETO $[X \sim X^*]$. Quasitransitivity gives therefore yP'_fx and by IIA yP_fx obtains. Thus $c \cap d \in A_f(X \sim X^*, X^*)$ as desired.

<u>Claim</u>. Suppose $\# < X \sim X^* > \geqslant 2$ and $X^* = \phi$, then $A_f(X \sim X^*, X^*)$ is closed w.r.t. complementation in N.

<u>Proof</u>. Trivial if $A_f(X \sim X^*, X^*) = \phi$; hence suppose nonempty. Referring to the diagram of the previous proof, let $c \in A_f(X \sim X^*, X^*)$, $x \in X \sim X^*$, $y \in X^*$, and $R \in \mathcal{R}$ s.t. $xP_{c^c}y$. Then construct $R' \in \mathcal{R}$ such that

R		R'	
c	c^c	c	c^c
[x,y]	x	z	z
	y	$[x,y]_R$	x
			y

Now zP'_cy so yP'_fz since $c \in A_f(X \sim X^*, X^*)$;

zP'_Nx so zP'_fx since PARETO $[X \sim X^*]$.

Quasitransitivity gives yP'_fx, and IIA allows yP_fx. Thus $c^c \in A_f(X \sim X^*, X^*)$, as desired.

<u>Claim</u>. $A_f(X \sim X^*, X^*) = \phi$ or 2^N, where $\# < X \sim X^* > \geqslant 2$ and $X^* \neq \phi$.

<u>Proof</u>. If $A_f(X \sim X^*, X^*) \neq \phi$, then closure under superset in N gives $N \in A_f(X \sim X^*, X^*)$. Closure under complementation gives $\phi \in A_f(X \sim X^*, X^*)$ so closure under superset gives $A_f(X \sim X^*, X^*) = 2^N$, since every set is a superset of ϕ.

<u>X*,X*</u>

Having culled the unbeatable alternatives into X*, one can consider this set not only against the background of X but also as a collection of alternatives in its own right. Such a viewpoint immediately leads one to ask: Does Pareto hold among the members of X*, and what do A_f, B_f, and D_f look like on X*? In order to answer these questions one can apply the * operator to X* itself and partition the set into $(X^*)^* = X^{2*}$ and $X^* \sim X^{2*}$. The analysis of the previous pages can then be employed anew; however, it will still remain to analyze the decisive, blocking, and antidecisive coalitions on X^{2*}. But why not put off this misery yet again? Apply * to X^{2*}, partition it, and so on. How does this process end?

<u>Claim</u>. $X \supseteq X^* \supseteq X^{2*} \supseteq \ldots \supseteq X^{n*} \supseteq \ldots$

<u>Claim</u>. PARETO $[X^{n*}] \Rightarrow X^{(n+1)*} = \phi$

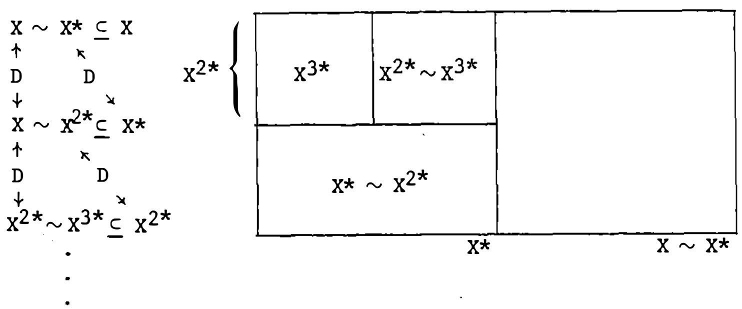
<u>Claim</u>. $\# < X^{n*} > = 1 \Rightarrow X^{(n+1)*} = \phi$

<u>Proof</u>. Let $X^{n*} = \{x\}$. $\nexists y \in x$ s.t. $D_f(y,x) = \phi$.

<u>Claim</u>. Suppose $X^{n*} = X^{(n+1)*}$, then $\forall m \in \mathbb{N}(m \geqslant n \Rightarrow X^{n*} = X^{m*})$.

<u>Proof</u>. $X^{(n+2)*} = (X^{(n+1)*})* = (X^{n*})* = X^{(n+1)*} = X^{n*}$. Proceed by induction.

<u>Claim</u>.



Where $U \leftarrow D \rightarrow V$ means $U \cap V = \phi$

<u>Claim</u>. Suppose $\# < X^{n*} \sim X^{(n+1)*} > \geqslant 2$ and $X^{(n+1)*} \neq \phi$. Then $D_f(X^{n*} \sim X^{(n+1)*}, X^{n*} \sim X^{(n+1)*}) \subseteq D_f(X^{(n+1)*}, X^{n*} \sim X^{(n+1)*})$.

<u>Proof</u>. By induction. Shown previously that $D_f(X \sim X^*, X \sim X^*) \subseteq D_f(X^*, X \sim X^*)$.

<u>Claim</u>.

$$\begin{array}{lll}
D_f(X,X) \subseteq D_f(X \sim X^*, X \sim X^*) & & \\
\cap| \qquad\qquad \subseteq^1 \ \ D_f(X^*, X \sim X^*) \subseteq D_f(X^{2*}, X \sim X^*) \subseteq D_f(X^{3*}, X \sim X^*) \ldots. & & \\
D_f(X^*,X^*) \subseteq^2 \subseteq D_f(X^* \sim X^{2*}, X^* \sim X^{2*}) & & \\
\cap| \qquad\qquad \subseteq^1 \ \ D_f(X^{2*}, X^* \sim X^{2*}) \subseteq D_f(X^{3*}, X^* \sim X^{2*}) \subseteq D_f(X^{4*}, X^* \sim X^{2*}) \ldots. & & \\
D_f(X^{2*},X^{2*}) \subseteq^2 \subseteq D_f(X^{2*} \sim X^{3*}, X^{2*} \sim X^{3*}) & & \\
\vdots \qquad\qquad \subseteq^1 \ \ D_f(X^{3*}, X^{2*} \sim X^{3*}) \subseteq D_f(X^{4*}, X^{2*} \sim X^{3*}) \subseteq D_f(X^{5*}, X^{2*} \sim X^{3*}) \ldots & & \\
\qquad\qquad \vdots & &
\end{array}$$

Where the $\subseteq^1$ and $\subseteq^2$ containments are subject to the following provisos: If $\# < X^{(n+1)*} > \geqslant 2$ and $X^{n*} \sim X^{(n+1)*} \neq \phi$ and also $\exists w \in X^{(n+1)*}$ such that $D_f(X^{(n+1)*}, w) \neq \phi$, then $D_f(X^{(n+1)*}, X^{(n+1)*}) \subseteq^2 D_f(X^{(n+1)*}, X^{n*} \sim X^{(n+1)*})$.

Consider what happens as * is iterated on X; suppose nonempty X^{n*} has been generated. The next action by * will purge X^{n*} of those alternatives which are unbeatable only against some alternative(s) outside of X^{n*}, in $X^{(n-1)*}$. The elements surviving the purge are exactly those which are unbeatable against some alternative which is itself a member of X^{n*}. This new elite collection forms $X^{(n+1)*}$.

<u>Claim</u>. $D_f(X^{n*}, X^{(n+1)*}) = \phi$ and $N \in D_f(X^{(n+1)*}, X^{n*})$

<u>Proof</u>. Exactly analogous to the case n = 0. The first equation holds since every member of $X^{(n+1)*}$ is unbeatable against some member X^{n*}, and so $X^{(n+1)*} \subseteq X^{n*}$. In case of the second statement, let $x \in X^{(n+1)*}$ and $y \in X^{n*}$; if $D_f(x,y) = \phi$, then $y \in X^{(n+1)*}$-- a contradiction. Since x and y were arbitrary, $N \in D_f(X^{(n+1)*}, X^{n*})$.

<u>Claim</u>. Let $i,n \in \mathbb{N}$ and $i \geqslant 1$; then $N \in D_f(X^{(n+2)*}, X^{n*})$.

<u>Proof</u>. Shown above the claim holds for i = 1. Suppose as an inductive hypothesis that the result is true for i = m. Consider the consequences if the claim fails for i = m + 1:
$N \notin D_f(X^{(n+m+1)*}, X^{n*}) \Rightarrow \exists x \in X^{(n+m+1)*}, \exists y \in X^{n*} (N \notin D_f(x,y))$.
Closure under superset of decisive families then requires $D_f(x,y) = \phi$. By definition, $X^{(n+m+1)*} \subseteq X^{(n+m)*}$ so that $x \in X^{(n+m)*}$, but then $N \in D_f(X^{(n+m)*}, X^{n*})$ requires $N \in D_f(x,y)$,

a contradiction.

As $*$ is repeatedly iterated, several scenarios might eventually ensue:

1) Pareto holds on some X^{n*} so that $\phi = X^{(n+1)*} = X^{(n+2)*} = \ldots$, where $\# \langle X^{n*} \rangle \geq 2$.

2) For some n, $X^{n*} = \{x\}$ so $X^{(n+1)*} = \phi$, $n \geq 1$.

3) $X^{n*} = X^{(n+1)*} = \ldots$ and X^{n*} is nonempty.

4) $X^{n*} \supsetneq X^{(n+1)*} \supsetneq X^{(n+2)*} \supsetneq \ldots$. Pareto never holds, and the chain continues nontrivially, forever.

A moment's reflection serves to establish that the four cases above are exhaustive. For $X \subseteq X^* \subseteq X^{2*} \subseteq \ldots$. As the elements of the chain get smaller, either the empty set must appear, two adjacent elements must be equal in which case all succeeding elements are identical, or else the chain must continue nontrivially, forever.

<u>Claim</u>. In cases 3), 4) Pareto does not hold on any X^{n*}, $n \geq 0$ so that $\phi = D_f(X,X) = D_f(X^*,X^*) = D_f(X^{2*},X^{2*}) = \ldots$ $= D_f(X^{n*},X^{n*}) \ldots$

Case 3) places a limit on the usefulness of the $*$ construction; in this case X contains a set of elements each of which is unbeatable against some other alternative in the set. The most drastic version of this scenario occurs if every member of X^{n*} is unbeatable against every other member of X^{n*}. These appear to be quite complex situations, and no analysis of them has been undertaken here. Case 2) is actually just the degenerate instance of 3). The reason that $X^{(n+1)*} = \phi$ instead of

$X^{(n+1)*} = X^{n*}$ as in 2) is that unbeatability of an alternative, as defined herein, requires unbeatability against a distinct second alternative. Hence a singleton cannot contain any unbeatable alternatives.

Recall earlier that the conditions $\# \langle X^* \rangle \geq 2$ and $\exists w \in X^*(D_f)X^*,w) \neq \phi)$ were employed to show $D_f(X^*,X^*) \subseteq D_f(X^*,X \sim X^*)$ where $X \sim X^* \neq \phi$. Trivially in case 3) and 4) $D_f(X^*,X^*) = \phi$ so the claim obtains regardless of the conditions. In case 2), where $X^{n*} = \{x\}$, if $n > 1$, then $D_f(X^*,X^*) = \phi$ and the claim again holds trivially; if $n = 1$, $X^* = \{x\}$ so $D_f(X^*,X^*) = 2^N \sim \phi$ and it cannot be asserted from the previous development that $2^N \sim \phi \subseteq D_f(X^*,X \sim X^*)$. In case 1) there is a collection of two or more alternatives in X^* such that Pareto holds among them. Then, just as it was shown $D_f(X^*,X^*) \subseteq D_f(X^*,X \sim X^*)$, it can be shown that $D_f(X^{n*},X^{n*}) \subseteq D_f(X^*,X \sim X^*)$. If $\# \langle X^{n*} \rangle \geq 3$ then $D_f(X^{n*},X^{n*})$ is a filter; and the collegium composed of its generating set can not only dictate social preference on X^{n*} but also between X^* and $X \sim X^*$. In fact $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X)$, which will give that $D_f(X^{n*},X^{n*}) = D_f(X^{n*},X)$, as developed below.

<u>Claim</u>. Suppose case 1) holds for X^{n*}, that is to say $\# \langle X^{n*} \rangle \geq 2$ and Pareto $[X^{n*}]$, then the following hold:

i) $D_f(X^{n*},X^{n*}) \subseteq D_f(X^*,X \sim X^*) \subseteq D_f(X^{n*},X \sim X^*)$

ii) $D_f(X^{n*},X^{n*}) = D_f(X^{n*},X^*)$

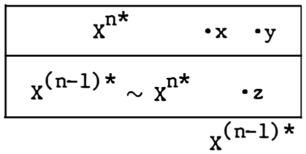
iii) $D_f(X^{n*},X^{n*}) = D_f(X^{n*},X)$

<u>Proof</u>. i) The first containment was shown for $n = 1$ previously; the extension to Pareto holding on X^{n*} instead is trivial. The second containment holds since $X^{n*} \subseteq X^*$.

ii) $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X^{n*})$ since $X^{n*} \subseteq X^{n*}$.

$D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X^*)$ will be shown by induction.

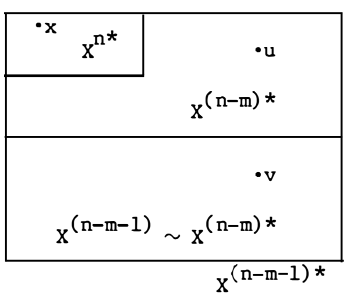
First we will establish that $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X^{(n-1)*})$.



Choose any $c \in D_f(X^{n*},X^{n*})$ and let $x \in X^{n*}$, $z \in X^{(n-1)*}$. Trivially $c \in D_f(x,z)$ if $z \in X^{n*}$ so take $z \in X^{(n-1)*} \sim X^{n*}$. Let $R \in \mathcal{R}$ be given such that xP_cz, and construct $R' \in \mathcal{R}$ such that

R: $\underline{c}$	R: $\underline{c^c}$	R': $\underline{c}$	R': $\underline{c^c}$
x	[x,z]	x	y
z		y	$[x,z]_R$
		z	

Now xP'_cy so that xP'_fy since $c \in D_f(X^{n*},X^{n*})$ by assumption, and yP'_Nz so that yP'_fz since $N \in D_f(X^{n*},X^{(n-1)*})$, as previously shown. Therefore by quasitransitivity xP'_fz and by IIA $c \in D_f(x,z)$. Arbitrariness then gives $c \in D_f(X^{n*},X^{(n-1)*})$. Suppose as an inductive hypothesis that $c \in D_f(X^{n*},X^{(n-m)*})$, for $m < n$. Required now to show that $c \in D_f(X^{n*},X^{(n-m-1)*})$. To this end let $x \in X^{n*}$, $v \in X^{(n-m-1)*}$, and $u \in X^{(n-m)*}$. If



$v \in X^{(n-m)*}$, $c \in D_f(x,v)$ directly from the inductive hypothesis. Hence suppose that $v \in X^{(n-m-1)*} \sim X^{(n-m)*}$. Let $\tilde{R} \in \mathcal{R}$ be given s.t. xP_cv and construct $\tilde{R}' \in \mathcal{R}$ such that

$\tilde{R}$: $\underline{c}$	$\tilde{R}$: $\underline{c^c}$	$\tilde{R}'$: $\underline{c}$	$\tilde{R}'$: $\underline{c^c}$
x	[x,v]	x	u
v		u	$[x,v]_{\tilde{R}}$
		v	

Now $x\tilde{P}'_cu$ gives that $x\tilde{P}'_fu$ since $c \in D_f(X^{n*},X^{(n-m)*})$ by the inductive hypothesis. $u\tilde{P}'_Nv$ gives that $u\tilde{P}'_fv$ since $N \in D_f(X^{(n-m)*},X^{(n-m-1)*})$, as demonstrated earlier. By quasitransitivity $x\tilde{P}'_fv$ obtains, and IIA gives $x\tilde{P}_fv$ so that $c \in D_f(x,v)$; and again arbitrariness of x,v allows $c \in D_f(X^{n*},X^{(n-m-1)*})$. Since c was arbitrary, the conclusion can be universalized to give $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*}, X^{(n-m-1)*})$, where $m < n$. In particular this is true for $m = n - z$, in which case $D_f(X^{n*},X^{n*}) \subseteq D_f(X^{n*},X^*)$.

iii) As above but choose $m = n - 1$, and recall $X = X^{0*}$.

Lastly, antidecisive and blocking families are considered on X^*.

<u>Claim</u>. Suppose case 1) obtains. Then $A_f(X^*,X^*) = B_f(X^*,X^*) = \phi$.

<u>Proof</u>. Suppose $A_f(X^*,X^*) \neq \phi$, then $N \in A_f(X^*,X^*)$. Let $x,y \in X^{n*}$, then $\forall R \in \mathcal{R}$, $xP_Ny \Rightarrow yP_fx$. But $N \in D_f(X^{n*},X^{n*}) \Rightarrow N \in D_f(x,y) \Rightarrow (xP_Ny \Rightarrow xP_fy)$, which is a contradiction. Similarly for $B_f(X^*,X^*)$.

It clearly will be important in this line of inquiry to investigate $B_f(X^{n*},X^{n*})$, $A_f(X^{n*},X^{n*})$ in case 3). It might be easiest to start with the simplest version of case 3), that is, where $\forall x,y \in X^{n*}(D_f(x,y) = \phi)$. In this case then $\forall x,y \in X^{n*}(N \in A_f(x,y) \vee N \in B_f(x,y))$. The set of ordered pairs $X^{n*} \times X^{n*}$ can then be partitioned according as $N \in A_f(x,y)$ or $N \in B_f(x,y)$--both cannot occur. This will hardly be an equivalence relation in the quasitransitive case, but a few pleasant properties do obtain. The A half of the relation is its own transitive closure for $N \in A_f(w,x)$ and $N \in A_f(x,z)$ imply $N \in A_f(w,z)$. Things are not so pleasant regarding symmetry or transitivity. This is the approach of Wilson; but without transitive range, the analysis of "Social Choice without the Pareto Principle" is less penetrating.