DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

NONCOOPERATIVE GAMES, ABSTRACT ECONOMIES AND WALRASIAN EQUILIBRIA

Kim C. Border

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ABSTRACT

The introduction of an additional player to serve as coordinator in an N-person abstract economy leads in a natural way to an N+1-person noncooperative game. Sufficient conditions on the abstract economy are considered which lead to the existence of equilibrium in the resulting game and hence for the abstract economy.

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0. Introduction.

Debreu [1952] introduced the concept of an abstract economy as a generalization of a noncooperative game (Nash [1951]). In a noncooperative game each player has a set of strategies available to him regardless of what strategies the other players select. The choice of strategies by all of the players determine an outcome and the players are all assumed to have preferences over outcomes. These preferences over outcomes then generate preferences over strategy vectors. An equilibrium is a strategy vector such that no player can change his strategy so as to yield a more preferred outcome. In an abstract economy the set of strategies available to a player may be a proper subset of strategies which depends on the choices of the other players. An equilibrium for an abstract economy is a strategy vector which is feasible for all players given everyone's choices and no player can alter his strategy so as to effect a more preferred outcome. Arrow and Debreu [1954] used Debreu's result on the existence of equilibrium for an abstract economy to find sufficient conditions for the existence of Walrasian equilibrium for a market economy. The reduction of a market economy to an abstract economy is accomplished by introducing an "auctioneer" or "market participant", a new player whose strategy set is the set of prices. The auctioneer's choice of price restricts each consumer's choice of consumption through the budget correspondence. Profits of producers also depend on the auctioneer's choice of price, but their set of available productions does not depend on the price.

The preferences of consumers were assumed to be representable by real valued utility functions by Arrow and Debreu. Sonnenschein [1971] showed that transitivity of preferences could be dispensed with for proving the existence of Walrasian equilibrium. Mas-Colell [1974] was further able to dispense with completeness of preferences. Gale and Mas-Colell [1975] present a proof of existence of equilibrium for noncooperative games without complete or transitive preferences and use it to establish existence of Walrasian equilibrium. Shafer and Sonnenschein [1975] present a proof of existence of equilibrium for abstract economies without complete or transitive preferences and Shafer [1976] uses this to prove the existence of Walrasian equilibrium under fairly general conditions.

The theorem of Gale and Mas-Colell on noncooperative games follows from Shafer and Sonnenschein's theorem on abstract economies, but not conversely. This suggests that the Shafer-Sonnenschein theorem is stronger than needed to prove the existence of Walrasian equilibrium. By imposing additional hypotheses on the feasible strategy correspondences, while somewhat weakening assumptions on preferences, we can convert an N-person abstract economy to an N+1-player noncooperative game. These extra assumptions were essentially introduced by Borglin and Keiding [1976], and are automatically satisfied by Walrasian budget correspondences. This is why Gale and

Mas-Colell were able to prove the existence of Walrasian equilibrium with their theorem. Borglin and Keiding reduced an abstract economy to a 1-person game. Our technique of expanding the player set has a neat interpretation in terms of stationary points of a tatônnement procedure.

1. Definitions.

An <u>N-person noncooperative game</u> is a tuple $G = (X_i, P_i)_{i=1}^N$, where for each i,

$$P_i : X \to\to X_i \quad \text{is a possibly empty-valued correspondence,}$$

where $X = \prod_{j=1}^N X_j$. An <u>equilibrium</u> for G is a strategy vector $\bar{x} \in X$ such that for each i

$$P_i(\bar{x}) = \emptyset \quad \text{or} \quad \bar{x}_i \in P_i(\bar{x}).$$

Here X_i is the set of strategies of player i and $P_i(x)$ is interpreted in one of two ways. The set $P_i(x)$ may be the set of strategies which player i strictly prefers to x_i given $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N$. Then $P_i(x) = \emptyset$ means that there are no preferred strategies to x_i given the others' choices. Under this interpretation it never happens that $x_i \in P_i(x)$. The second way we may interpret $P_i(x)$ is as the set of "best replies" to $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N$. Then $P_i(x)$ is never empty and $x_i \in P_i(x)$ is the appropriate equilibrium notion. Our definition of equilibrium allows for P_i to have one interpretation for some players and the other interpretation for the rest.

An <u>N-person abstract economy</u> is a tuple $E = (X_i, S_i, P_i)_{i=1}^N$, where for each i,

$$P_i : \prod_{j=1}^N X_j \to\to X_i \quad \text{is a possibly empty-valued correspondence}$$

and

$$S_i : \prod_{\substack{j=1 \\ j \neq i}} X_j \to\to X_i \quad \text{is a nonempty-valued correspondence.}$$

Put $X = \prod_{j=1}^N X_j$. We will write $S_i : X \to\to X_i$, with the understanding that $S_i(x)$ does not depend on x_i.

An <u>equilibrium</u> for E is a vector $\bar{x} \in X$ such that for each i,

$$\bar{x}_i \in S_i(\bar{x})$$

and

$$P_i(\bar{x}) \cap S_i(\bar{x}) = \emptyset.$$

Again X_i is player i's strategy set and $S_i(x)$ is the set of feasible strategies for i given the choices $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N$. The set $P_i(x)$ is the set of strategies preferred to x_i given everyone else's choice. The equilibrium condition is that the vector be feasible for everyone and no one has a strategy which is both preferred and feasible.

2. Theorems.

The following theorem is a slight variation of a theorem of Gale and Mas-Colell [1975]. Denote by co P_i, the correspondence whose value at x is the convex hull of $P_i(x)$.

Theorem 1. Let $G = (X_i, P_i)_{i=1}$ be a noncooperative game satisfying for each i,

(i) X_i is a nonempty compact convex subset of a finite dimensional euclidean space.

(ii) Either

(a) co P_i has an open graph in $X \times X_i$, and for each $x \in X$,

$x_i \notin \text{co } P_i(x)$.

or

(b) P_i is a continuous singleton-valued correspondence, i.e., a continuous function.

Then G has an equilibrium.

Proof. The proof is virtually identical to that of Gale and Mas-Colell [1975, p.10] and so we only sketch the proof. If P_i satisfies (ii.a) let $U_i = \{x : \text{co } P_i(x) \neq \emptyset\}$ and let f_i be a continuous selection from co $P_i \,|_{U_i}$. Define $\gamma_i : X \to\to X_i$ via

$$\gamma_i(x) = \begin{cases} \{f_i(x)\} & x \in U_i \\ X_i & x \notin U_i \end{cases} .$$

If P_i satisfies (ii.b) set $\gamma_i = P_i$.

Then $\gamma = \prod \gamma^i$ satisfies the hypotheses of Kakutani's theorem and has a fixed point $\bar{x}$. By construction, if P_i satisfies (ii.a) then co $P_i(\bar{x}) = \emptyset$ so $P_i(\bar{x}) = \emptyset$ and if P_i satisfies (ii.b) then $\bar{x}_i \in P_i(\bar{x})$.

Q.E.D.

Theorem 2 below is a modification of a theorem of Shafer and Sonnenschein [1975]. The assumptions on preferences have been slightly weakened and those on the feasibility correspondences strengthened, but in a way so as to remain useful for proving the nonemptiness of Walrasian equilibrium. The proof proceeds by converting the abstract economy to an N+1 person noncooperative game.

Theorem 2. Let $E = (X_i, S_i, P_i)_{i=1}^N$ be an abstract economy satisfying, for each i,

(i) X_i is a nonempty compact convex subset of a finite dimensional Euclidean space.

(ii) S_i has a closed graph in $X \times X_i$, the correspondence $x \mapsto\to \text{int } S_i(x)$ has an open graph in $X \times X_i$, and for each $x \in X$, $S_i(x)$ is compact, convex, and has nonempty interior (relative to X_i).

(iii) co P_i has an open graph in $X \times X_i$.

(iv) For each $x \in X$, $x_i \notin \text{co } P_i(x)$.

Then E has an equilibrium.

Remark. Shafer and Sonnenschein assume that S_i is lower hemi-continuous as well as closed. We have strengthened this in (ii.a) to assuming int S_i is a nonempty-valued correspondence with open graph. Note that since $S_i(x)$ is compact and convex with nonempty interior that $S_i(x) = \text{cl } (\text{int } S_i(x))$, i.e., $S_i(x)$ is topologically regular. Note that interior here is relative to X_i, not the underlying Euclidean space.

Shafer and Sonnenschein assume that P_i has an open graph. We have weakened this to assuming co P_i has an open graph. Assuming that P_i has open sections (see Bergstrom, Parks and Rader [1976]), then co P_i will have open graph. (In this case co P_i will be convex-valued with open sections. See Bergstrom, Parks and Rader [1976] and Shafer [1974].) The assumption of open sections for P_i is strictly weaker than the assumption of open graph as long as P_i is not convex-valued.

Proof of Theorem 2. We define an N+1 person game as follows.

Put $Z_0 = \prod_{i=1}^{N} X_i$, and for i=1,...,N put $Z_i = X_i$. Set $Z = \prod_{i=0}^{N} Z_i$. A typical element of Z will be denoted (x,y), where $x \in Z_0$ and $y \in \prod_{i=1}^{N} Z_i$. Define preference correspondences $\mu_i : Z \to\to Z_i$ as follows.

Define μ_0 by

$$\mu_0(x,y) = \{y\},$$

and for i = 1,...,N set

$$\mu_i(x,y) = \begin{cases} \text{int } S_i(x) & \text{if } y_i \notin S_i(x) \\ \text{co } P_i(y) \cap \text{int } S_i(x) & \text{if } y_i \in S_i(x). \end{cases}$$

Note that μ_0 is continuous and singleton-valued and that for i=1,...,N the correspondence μ_i is convex-valued and satisfies $y_i \notin \mu_i(x,y)$. Also for i=1,...,N, the graph of μ_i is open. To see this for i = 1,...,N set

$$A_i = \{(x,y,z_i) : z_i \in \text{int } S_i(x)\},$$

$$B_i = \{(x,y,z_i) : y_i \notin S_i(x)\},$$

and

$$C_i = \{(x,y,z_i) : z_i \in \text{co } P_i(y)\},$$

and note that

$$\text{Gr } \mu_i = (A_i \cap B_i) \cup (A_i \cap C_i).$$

The set A_i is open because int S_i has open graph and C_i is open by hypothesis (iii). The set B_i is also open, for if $y_i \notin S_i(x)$ then there is a closed neighborhood F of y_i such that $S_i(x) \subset F^c$ and upper hemicontinuity of S_i then gives the desired result.

Thus the hypotheses of Theorem 1 are satisfied and so there exists $(\bar{x},\bar{y}) \in Z$ such that

(a) $\bar{x} \in \mu_0(\bar{x},\bar{y})$

and for i=1,...,N

(b) $\mu_i(\bar{x},\bar{y}) = \emptyset$.

Now (a) implies $\bar{x} = \bar{y}$ and since $S_i(x)$ is never empty (b) becomes

$$\text{co } P_i(\bar{x}) \cap \text{int } S_i(\bar{x}) = \emptyset \qquad \text{for } i=1,\ldots,N.$$

Thus $P_i(\bar{x}) \cap \text{int } S_i(\bar{x}) = \emptyset$, but $S_i(\bar{x}) = \text{cl [int } S_i(\bar{x})]$ and $P_i(\bar{x})$ is open so $P_i(\bar{x}) \cap S_i(\bar{x}) = \emptyset$, i.e., $\bar{x}$ is an equilibrium.

<u>Remark.</u> We can view the N+1 game as a formalization of a tatônnement process. The N+1st player, call him the coordinator, performs in the abstract what a Walrasian auctioneer does in a market economy. The coordinator suggests a strategy vector. The players look at the choices they have made to see whether they are feasible given the coordinator's suggestion. If not, they choose a new strategy which is feasible. If the choice is feasible they try to improve upon it. Otherwise they don't change their strategy. The coordinator changes his suggestion to what the players were doing originally. A stationary point of this process is an equilibrium for the abstract economy. The question of under what circumstances this process is

convergent remains to be investigated.

<u>Note.</u> We could have defined μ_0 in an alternative fashion, namely $\mu_0(x,y) = \{z \in Z_0 : |z - y| < |x - y|\}$. Then μ_0 would have an open graph and be convex valued with $x \notin \mu(x,y)$ so Theorem 1 would still apply.

<u>Remark.</u> In the application of Theorem 2 to the problem of existence of Walrasian equilibrium we have the following interpretations. Call the auctioneer player 1. Except for the auctioneer, each X_i is a subset of the commodity space R^k. Compactness is achieved by suitably truncating consumption and production sets. For the auctioneer X_1 is a subset of price space, also R^k. For producers S_i is a constant correspondence equal to their truncated convex production set. This satisfies (ii) as all interiors are relative to X_i. Likewise for the auctioneer $S_1 \equiv X_1$. For consumers, $S_i(p,x_2\ldots,x_N) = \{y_i \in X_i : p \cdot y_i \leq p \cdot \omega_i + \sum_j \theta_{ij}(p \cdot x_j)^+\}$, where ω_i is his endowment and θ_{ij} is his share of firm j. If player j is not a firm then $\theta_{ij} = 0$. Under the typical assumption (Shafer [1976], Gale and Mas-Colell [1975], Debreu [1959]) that $\omega_i \in \text{int } X_i$ (or something like this), then S_i will satisfy the hypotheses of Theorem 2.

REFERENCES

Arrow, K. J. and G. Debreu. 1954. "Existence of an Equilibrium for a Competitive Economy." Econometrica 22:265-290.

Borglin, A. and H. Keiding. 1976. "Existence of Equilibrium Actions and of Equilibrium: A Note on the 'New' Existence Theorems." Journal of Mathematical Economics 3:313-316.

Bergstrom, T. C., R. P. Parks, and T. Rader. 1976. "Preferences which have open graphs." Journal of Mathematical Economics 3(3):265-268.

Debreu, G. 1952. "A Social Equilibrium Existence Theorem." Proceedings of the National Academy of Sciences 38:886-893.

——————. 1959. Theory of Value. New York: Wiley.

Gale, D. and A. Mas-Colell. 1975. "An Equilibrium Existence Theorem for a General Model without Ordered Preferences." Journal of Mathematical Economics 2:9-15.

Mas-Colell, A. 1974. "An Equilibrium Existence Theorem Without Complete or Transitive Preferences." Journal of Mathematical Economics 1(3):237-246.

Nash, J. 1951. "Non-cooperative games." Annals of Mathematics 54(2):286-295.

Shafer, W. J. 1974. "The Nontransitive Consumer." Econometrica 42:913-919.

Shafer, W. and H. Sonnenschein. 1975. "Equilibrium in Abstract Economics without Ordered Preferences." Journal of Mathematical Economics 2:345-348.

Shafer, W. J. 1976. "Equilibrium in Economies Without Ordered Preferences or Free Disposal." Journal of Mathematical Economics 3(2):135-137.