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ON RECONCILING CONFLICTING GOALS: APPLICATIONS OF MULTIOBJECTIVE PROGRAMMING

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ABSTRACT

This paper examines the problem of identifying cost-effective solutions to problems characterized by multiple objectives. The traditional approach has been to minimize costs subject to feasibility constraints and a set of targets. This is compared with a multiobjective programming approach which treats the objectives as choice variables and cost as a parameter. To illustrate the two approaches, the problem of achieving environmental objectives is analyzed. The comparison reveals that the traditional cost-minimizing approach can generate solutions which are inefficient, in the sense that greater emission reductions could have been attained at the same cost. Because the solution sets to the two problems may differ, conditions are derived under which the two approaches yield a similar set of results.

ON RECONCILING CONFLICTING GOALS:

Applications of Multiobjective Programming

I. Introduction

Decision makers typically have several objectives in mind when choosing among different policy alternatives. While these objectives are sometimes associated with target values, it is frequently the case that the objectives are viewed as choice variables which are to be jointly maximized in some manner. There are two basic approaches to such problems. Treating the objectives as targets permits the decision maker to minimize costs over a feasible region. If, instead, the objectives are viewed as control variables, then an alternative approach is to maximize some function of the objectives subject to a set of feasibility constraints which usually includes a limitation on expenditures. This latter approach falls under the general heading of multiobjective programming.

While the two approaches to the problem can yield the same solution, this need not be true, especially for cases in which the tactics available for meeting the proposed objectives have an adverse impact on some subset of those objectives. An example would be the problem of increasing automobile fuel efficiency while decreasing emissions. Several control tactics aimed at reducing emissions can have an adverse impact on fuel economy. This problem is complicated further by the introduction of safety considerations. Lave (1980) analyzes the explicit tradeoffs that result from existing legislation in this area, and provides a cogent analysis of the difficulties inherent in reconciling the objectives of improved safety, better fuel economy and reduced emissions. His conclusion that secondary impacts of automobile regulation may be quite important indicates that this may be a potentially fruitful application for multiobjective programming techniques. The particular problem raised by Lave will be illustrated in greater detail in the conclusion, after the approaches for meeting objectives are analyzed more formally.

The objective of this paper is to compare the two approaches for achieving policy objectives. For illustrative purposes, the problem of meeting environmental objectives is examined in detail. The relative merits of the two approaches for decision making are addressed in the conclusions.

II. Application to Environmental Problems

The traditional approach to the problem of finding cost-effective solutions to environmental problems has been to specify an emissions target and then compute the minimum cost associated with meeting the objective. The choice of an emissions target is usually predicated on some hypothesized relationship between emissions and environmental quality. When the relationship between emissions and air quality is linear, as is assumed in the models developed by Kohn (1971) and Atkinson and Lewis (1974), then the general problem of meeting an environmental quality objective can be solved directly through the use of linear programming. A non-linear relationship between emissions and air quality may mean that the only part of the

problem amenable to solution by linear programming is the relationship between control costs and emissions. Such is the case, for example, in the analysis of the Los Angeles smog problem undertaken by Trijonis (1974).

This analysis specifically focuses on the relationship between costs and emissions. As an alternative to minimizing costs subject to achieving a prescribed reduction in emissions, an approach which treats emissions as the choice variable and cost as a parameter is examined. The analysis reveals two essential points: first, that the alternative approach yields a straightforward method for generating isocost curves and second, that an optimal solution to the traditional cost-minimizing formulation need not coincide with a point on an isocost curve.

III. The Traditional Approach

The problem of selecting a set of control tactics which minimize the cost of meeting a given emissions target is set forth in the following linear program which was applied by Trijonis (1974):

The Cost-Minimizing Approach (CM1)

$$\min_{x} \; cx \qquad (1)$$

Subject to:

$$Bx = E \qquad (1a)$$

$$Ax \leq S \qquad (1b)$$

$$Dx \leq L \qquad (1c)$$

$$x \geq 0 \qquad (1d)$$

where x is the (r x 1) vector of activity levels for the r control methods,

c is a (1 x r) vector of control costs,

B is an (n x r) matrix whose element b_{ij} represents the reduction of pollutant i resulting from one unit of control activity j,

E is the (n x 1) vector indicating the required reduction in emissions,

A is an (s x r) matrix whose element a_{ij} represents the number of units of source i controlled by one unit of control activity j,

S is the (s x 1) vector of source magnitudes,

D is a (p x r) matrix whose element d_{ij} represents the amount of limited supply input i used by one unit of control activity j,

L is the (p x 1) vector specifying the magnitudes of the limited supply inputs.

The CM1 approach minimizes control costs subject to a set of constraints. Equation (1a) states that the vector of emissions be reduced by E units. The second set of constraints (1b) places limitations on the level at which different sources can be controlled. The third set of constraints (1c) places limits on the use of certain fixed inputs in control activities, while (1d) states that all control activities be set at some nonnegative level.

IV. The Multiobjective Formulation

An alternative approach to identifying cost-effective control strategies is to consider the problem of maximizing the reduction in emissions subject to capacity constraints, supply constraints and a budget constraint. Formally the problem can be stated as follows:

The Multiobjective Approach (MO)

$$\max_{x} \; Bx \qquad (2)$$

Subject to:

$$cx \leq \overline{C} \qquad (2a)$$

$$Ax \leq S \qquad (2b)$$

$$Dx \leq L \qquad (2c)$$

$$x \geq 0 \qquad (2d)$$

where $\overline{C}$ is a scalar which fixes the annual expenditure on pollution control at some prescribed value.

The constraints in the CM1 formulation are similar to those contained in the multiobjective formulation; however, there are two important differences. A budget constraint (2a) is added and the constraint on emissions reductions is dropped.[1]

As stated, the MO problem needs some further clarification, since the concept of maximizing a vector may not be clear. The vector x is defined to be an efficient solution to (2) if and only if the following two conditions hold:

1. x must be feasible, i.e., it must satisfy the constraint set, and
2. there does not exist a feasible solution, x′ such that $Bx' \geq Bx$ and $Bx' \neq Bx$.

While the solution of the MO formulation may appear, at first glance, to present a difficult problem, the formulation can be simplified considerably by applying the following lemma which allows the problem to be converted to a linear program.

Lemma 1: The vector x* is an efficient solution of the MO problem if and only if there is a (1 x n) vector q > 0 for which x* optimizes the following linear program:[2,3]

The Corresponding Multiobjective Linear Program (MOLP)

$$\text{Maximize } qBx \qquad (2')$$

Subject to: (2a)–(2d).

Lemma 1 makes it possible to generate isocost curves (or at least very good approximations thereto) by carefully selecting several values for q and solving the MOLP problem.

V. The Relationship Between the Two Approaches

Comparing the CM1 linear programming formulation with the MOLP, one might think that the two are equivalent in some sense, since the former minimizes costs subject to a given level of emissions reductions while the latter takes expenditures as given and maximizes a linear combination of emissions reductions. Surprisingly, the relationship between the two approaches is not obvious. The following two examples will serve to highlight the differences between the two problems. In Example 1, we consider a case where the solution to (1) does not exist, but a solution to (2′) exists for any given level of expenditures.

Example 1: Suppose there is one control strategy x_1 with c_1 = \$1, $b_{11} = 1$ and $b_{21} = 1$, with the constraint set only requiring that x_1 be nonnegative. A graph of this strategy is shown in Figure 1.

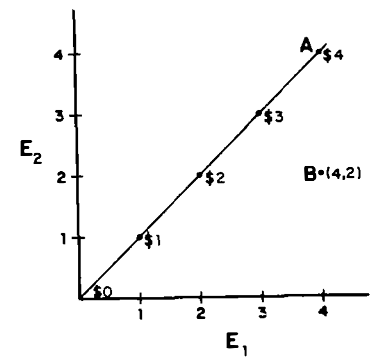

Figure 1. Illustration of Feasible Emissions Reductions and Associated Costs

Let the objective for reducing emissions be given by point B with coordinates (4,2). The 45° line represents the control strategy x_1. Note that as you move down the line towards the origin, the level of costs decrease in a linear fashion. For example, at (4,4), $x_1 = 4$ and the cost (c_1x_1) also equals 4. Reducing both types of emissions by one unit each so that $(E_1,E_2) = (1,1)$ implies $x_1 = 1$ and the cost is \$1.00. Using the original CM1 formulation, the prescribed goal of (4,2) is infeasible. This suggests an extension of the CM1 formulation which would permit reductions greater than or equal to the stated targets.[4] Formally the problem can be stated as follows:

A Revised Cost Minimizing Approach (CM2)

$$\underset{x}{\text{Minimize}}\ cx \quad (3)$$

$$\text{Subject to:}\quad Bx \geq E \quad (3a)$$

$$Ax \leq S \quad (3b)$$

$$Dx \leq L \quad (3c)$$

$$x \geq 0 \quad (3d)$$

Viewing Example 1 in terms of the CM2 approach, the solution set consists of point A in Figure 1. If $\overline{C} = 4$, then point A would also be optimal in terms of the MO formulation.

Example 1 illustrates a case where no feasible solution exists to the original CM1 problem and the solution to the CM2 and MO problems are identical. Next, we consider a problem which has an infinite number of solutions for the CM2 program, only one of which is optimal for the multiobjective program.

Example 2: Suppose there are two control strategies x_1 and x_2 with the following data:

$c_1 = 3$ $b_{11} = 1$ $b_{12} = 3$

$c_2 = 1$ $b_{21} = 3$ $b_{22} = 1$

$E_1 = 12$ $E_2 = 12$

The problem is to reduce each type of emissions by at least 12 units, subject to the constraint that nonnegative levels of x_1 and x_2 be chosen. Since there is only one feasible solution to the original

CM1 formulation, it must be optimal. The solution is $(x_1,x_2) = (3,3)$, which results in a cost C = \$12.00. Setting $C = \overline{C}$, and considering the multiobjective program, it is easily seen that sole use of the control activity x_2 will result in a higher value for E_2, leaving E_1 unchanged, thus showing the solution to the CM1 problem is not optimal for the multiobjective problem. The problem is illustrated in Figure 2.

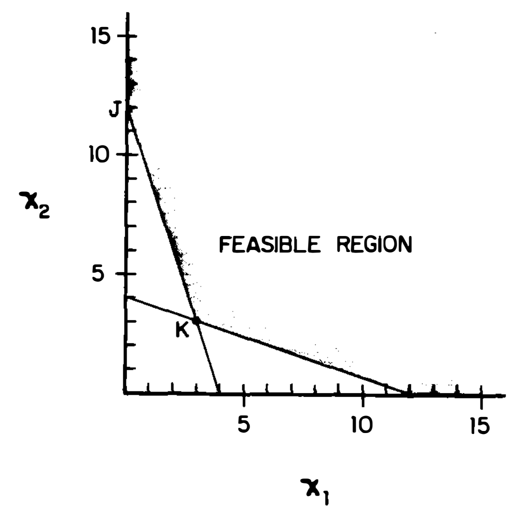

Figure 2. A Graphical Comparison of Approaches for Finding Efficient Environmental Controls

The feasible region in Figure 2 corresponds to the revised cost minimizing problem. For the original cost minimizing problem, the feasible region reduces to the point K with $(x_1, x_2) = (3,3)$. There are an infinite number of solutions to the revised cost minimizing problem characterized by segment $\overline{JK}$; however, of these solutions, only point J is optimal for the multiobjective problem when $\overline{C} = 12$.

The conclusion to be drawn from this analysis is that the CM1 formulation can generate points which are inefficient in the sense that lower emissions may be attainable at the same cost. The CM2 program poses similar problems; however, because the CM2 approach covers a larger feasible region, we are assured that if the solution set to the CM2 formulation is not empty, it contains at least one point which will be an optimal solution to the multiobjective program.[5]

While a solution to the original or revised cost minimizing problem need not be optimal for the multiobjective program, it is possible to develop a sufficient condition under which a solution for the cost minimizing formulations will also solve the multiobjective program.

In order to economize on notation, the source and supply constraints are merged. Without loss of generality, let

$$Fx \geq P \qquad (4)$$

represent constraints (2b) and (2c) or (3b) and (3c).

Because the theory of duality plays a central role in subsequent results, it will be useful to consider the dual formulations of the revised cost minimizing and the multiobjective linear programming problems. The dual to the CM2 problem is:

$$\underset{y^1, y^2}{\text{Maximize}} \quad y^1E + y^2P \qquad (5)$$

$$\text{Subject to:} \quad y \begin{bmatrix} B \\ F \end{bmatrix} \leq c \qquad (5a)$$

$$y \geq 0. \qquad (5b)$$

The solution to the problem is given by the dual row vector $y = [y^1, y^2]$.

The dual to the MOLP problem is constructed in a similar manner, yielding the following expression:

$$\underset{z^1, z^2}{\text{Minimize}} \quad z^1\overline{C} - z^2P \qquad (6)$$

$$\text{Subject to:} \quad z \begin{bmatrix} -c \\ F \end{bmatrix} \leq -qB \qquad (6a)$$

$$z \geq 0 \qquad (6b)$$

In this case, the solution to the problem is given by the dual vector $z = [z^1, z^2]$.

Two theorems will be developed. The first provides a basis for checking whether a solution to the cost minimizing problem is necessarily a solution to the multiobjective formulation. The second theorem turns the question around, identifying when a solution to the multiobjective problem will necessarily be optimal for the cost minimizing approach.

Theorem 1: Suppose CM2 has an optimal solution x^* with an associated dual solution y^*. Consider the MO problem with $\overline{C} = cx^*$. Then x^* is <u>efficient</u> for the MO problem if $y^{1*} > 0$.

Proof: Suppose that x^* is not efficient. Then there exists an x such that $Bx \geq Bx^*$ and $Bx \neq Bx^*$. This implies:

$$cx \geq [y^{1*}, y^{2*}] \begin{bmatrix} B \\ F \end{bmatrix} x$$

$$= y^{1*}Bx + y^{2*}Fx$$

$$> y^{1*}Bx^* + y^{2*}Fx$$

$$\geq y^{1*}E + y^{2*}P$$

$$= cx^*.$$

The first inequality is obtained from (5a) by postmultiplying by x. This expression is simplified in the next step. The strict inequality is based on the supposition. Expressions (3a) and (4) are used in the subsequent inequality. Finally, the equilibrium theorem of linear programming is applied to obtain the desired result.

Two comments are in order. First, note that the proof also works for the CM1 formulation (i.e., with $Bx = E$). Second, note that the result has a straightforward intepretation when the dual variables are viewed as shadow prices. In short, the theorem says that as long

as it costs more to get a reduction in all types of emissions (at the optimum) the cost minimizing solution will be efficient.

The next problem is to identify when an efficient solution will be cost minimizing. This problem is resolved in the following theorem:

Theorem 2: Let x^* be an efficient solution to the MO problem and set $Bx^* = E$. Then, x^* is optimal for the CM2 problem if $z^{1*} > 0$.

Proof: By contradiction: suppose there exists an x such that $cx < cx^*$ which also satisfies (3a)-(3d). Then,

$$-qBx \geq [z^{1*}, z^{2*}] \begin{bmatrix} -c \\ F \end{bmatrix} x$$

$$= -z^{1*}c\,x + z^{2*}Fx$$

$$> -z^{1*}cx^* + z^{2*}Fx$$

$$\geq -z^{1*}\overline{C} + z^{2*}P$$

$$= -qBx^*.$$

The first inequality is obtained from (6a) by postmultiplying by x. Simplifying the expression and applying the supposition yields the strict inequality. This is followed by a substitution using expressions (2a) and (4). Applying the equilibrium theorem of linear programming yields the desired result.

This result holds for the original cost minimizing problem as well. It shows that an efficient solution to the MO problem will be optimal for CM1 and CM2 provided that, at the margin, an extra dollar will increase qBx. This in turn, implies that at at least one type of emissions can be further reduced at the optimum. Note also that $z^{1*} > 0$ implies that the budget constraint is effective at the optimum, i.e., $cx^* = \overline{C}$.

It is not obvious that the above results will always obtain. In particular, there are several pollution control activities which lead to decreases in one type of emissions at the expense of increasing other types. A case in point were the automobile exhaust emission controls for reactive hydrocarbons and carbon monoxide introduced in California in 1966 and in the remainder of the country in 1968. Unfortunately, the technological modifications adopted by American car manufacturers produced higher engine combustion temperatures which in turn dramatically increased the emissions of another pollutant--nitric oxide. While this problem has been corrected, it highlights the need to understand the likely impact of any new control technique when formulating the mathematical programming problem.

Fortunately, it is a simple matter to check whether, in fact, the above relationships do obtain by generating the appropriate dual variables. Of course, since the conditions are sufficient and not necessary, if they are not satisfied, one may have to resort to a direct computational method by substituting the proposed solution into the problem and checking to see if it works. This can be done in moving from the MO to the CM formulation, but I am not aware of

any simple way to move in the reverse direction if the assumptions of Theorem 1 do not hold.

VI. Conclusions

The analysis in the foregoing paper focuses on the problem of achieving a cost-effective solution to the problem of reducing emissions. The formal comparison of the multiobjective and cost-minimizing approaches has served to illustrate that the traditional cost minimizing solution generated by a linear program will not necessarily be efficient. That is to say, it may be possible to achieve greater emissions reductions than specified in the cost-minimizing formulation at the same cost. The multiobjective approach solves this problem by directly minimizing emissions subject to a budget constraint.

One potential application where the multiobjective approach may yield different solutions than the cost minimizing approach can be illustrated for the case of automobile regulation, which was introduced in Section I. Figure 3 provides a stylized representation of the tradeoffs between air quality, fuel economy and safety.

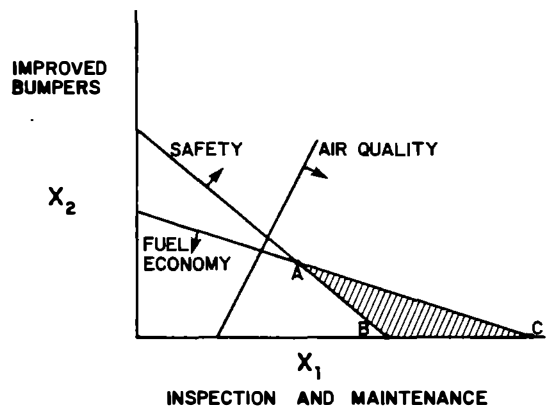

Figure 3. Illustration of the Tradeoffs between Safety, Fuel Economy and Air Quality

There are two control activities, x_1 and x_2. The first activity corresponds to an inspection and maintenance program aimed at improving the safety and reducing emissions of vehicles currently in use. The second activity corresponds to installing improved bumpers on new and/or used cars. The effects of these two activities on the objectives can be seen by noting that the line segments in Figure 3 represent constant levels of safety, air quality and fuel economy for the fleet as a whole. The direction of improvement is given by the vector perpendicular to each of the segments. Thus, for example, safety can be improved by increasing x_1 and/or x_2.

Suppose this problem were cast in terms of a cost minimization where the objective is to find the minimum cost of achieving or exceeding the constant levels of safety, fuel economy and air quality shown in the diagram. The feasible region would then correspond to triangle ABC. Now, suppose further that the isocost curves were

parallel to segment $\overline{AB}$, which means that the set of cost minimizing solutions corresponds to the segment. It should be clear that any point on the segment other than B is dominated in the sense that better fuel economy and improved air quality can be achieved at the same cost without sacrificing safety considerations. Unfortunately, there is no guarantee that the program will yield point B as the solution. This potential pitfall can be overcome simply by reformulating the problem as a multiobjective program.

Given the potential for differences between the solution sets to the two approaches, the question naturally arises as to which approach would be more useful to the policy maker. The answer is that it depends. If the policy maker has already decided on target levels for the objectives, then the cost minimizing approach is tailor made for this problem. If, on the other hand, the policy maker is less certain of the overall objectives, then the multiobjective programming approach would probably be more appropriate since it is designed to identify the range of options available at a given level of expenditures.

FOOTNOTES

1. This formulation does not explicitly preclude the possibility of a new level of emissions with some negative components. This situation can be handled by identifying a baseline level of emissions, say E^o, and then constraining E^o-E to be nonnegative. Introduction of this constraint does not substantively affect the analysis and is rarely, if ever, binding in actual applications.

2. An asterisk will be used to denote an optimal solution to a given program. $q > 0$ implies each element of q is positive.

3. A proof of this lemma for the case of equality constraints is presented in Franklin (1980). The extension to inequality constraints follows immediately upon introducing slack variables.

4. This is the basic approach taken by Kohn (1971).

5. The proof is straightforward. Let x* be a solution to the CM2 formulation, and define $\bar{C}=cx^*$. This implies x* is feasible for the MO problem. If x* is optimal for the MO problem we are done. Suppose x* is not optimal. Then there exists a solution x′ such that $Bx' \geq Bx^*$ and $Bx' \neq Bx^*$. But, by construction x′ would also be a solution to CM2.

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