A HAM SANDWICH THEOREM FOR GENERAL MEASURES

Gary W. Cox and Richard D. McKelvey

SOCIAL SCIENCE WORKING PAPER 337

July 1980
ABSTRACT

The "ham sandwich" theorem has been proven only for measures that are absolutely continuous with respect to Lebesgue measure. We prove a generalized version of the ham sandwich theorem which is applicable to arbitrary finite measures, and we give some sufficient conditions for uniqueness of the hyperplane identified by the theorem.

A HAM SANDWICH THEOREM FOR GENERAL MEASURES

Gary W. Cox and Richard D. McKelvey

I. INTRODUCTION

The Ham Sandwich problem was first posed by Ulam [1930], and has since been examined by Borsuk [1933], Steinhaus [1945], Stone and Tukey [1942], Tucker [1945], and Dubins and Spanier [1961]. The problem derives its name from Steinhaus' picturesque formulation of the problem as that of dividing a ham, butter, and bread sandwich by a plane into two parts each containing exactly one half of the ham, one half of the butter, and one half of the bread.

The theorem has an n-dimensional generalization which uses the following definitions: A hyperplane is any set of the form

\[ H = \{ x \in \mathbb{R}^n | x \cdot v > c \} \]

where \( v \in S^{n-1} \), and \( c \in \mathbb{R} \). Here \( S^{n-1} \) is the \( n-1 \) sphere of unit length vectors in \( \mathbb{R}^n \). We use the notation

\[ H^+ = \{ x \in \mathbb{R}^n | x \cdot v > c \} \]

and

\[ H^- = \{ x \in \mathbb{R}^n | x \cdot v < c \} \]
to denote the positive and negative open half spaces defined by $H$.

If $\mu$ is a finite measure on the Borel sets of $\mathbb{R}^n$, we say $H$ bisects $\mu$ iff $\mu(H^+) = \mu(H^-)$. We have the following statement of the theorem:

**Theorem 1.** Given any $n$ finite measures, $\mu_1, \ldots, \mu_n$ defined on the Borel subsets of $n$-dimensional Euclidean space, $\mathbb{R}^n$, if each $\mu_i$ is absolutely continuous with respect to Lebesgue measure, there exists a hyperplane which simultaneously bisects each measure.

The usual proof goes as follows:

1. Consider the measure $\mu_n$. We know from measure-theoretic considerations that, for each unit vector $v$, there exists a real number $c_v$ such that the hyperplane $H_v = \{x \in \mathbb{R}^n \mid x \cdot v = c_v\}$ bisects $\mu_n$.

2. Now define a mapping $f$ from the unit $n - 1$ sphere $S^{n-1}$ to $\mathbb{R}^{n-1}$ as:

   $$f_j(v) = (\mu_j(H^+_v) - \mu_j(H^-_v))$$ for $j = 1, \ldots, n - 1$.

   Note that $f$ is continuous and that $f(v) = -f(-v)$.

3. Use the Borsuk-Ulam theorem to infer that there exists a $v \in S^{n-1}$ which $f$ maps into the origin in $\mathbb{R}^{n-1}$, implying $\mu_j(H^+_v) = \mu_j(H^-_v)$ for $j = 1, \ldots, n$ and proving the theorem.

The Ham sandwich theorem and its proof depend on the absolute continuity of the measures $\mu_i$. Otherwise, the function $f$ defined above need not be continuous. In fact, the theorem as stated is not true for general measures, as is illustrated by the example of Figure 1, where $\mu_1$ is the atomic measure defined by setting $\mu_1(x_i) = \mu_2(y_i) = 1/3$ for $i = 1, 2, 3$, and $\mu_1(x) = \mu_2(y) = 0$ otherwise. Here, any bisecting line for $\mu_1$ must pass thru one and only one $x_i$, with the remaining $x_i$'s lying on either side of the line. A bisecting line for $\mu_2$ must have similar properties. But no line passing thru one $x_i$ and one $y_i$ splits the remaining points in the desired fashion.
II. A GENERALIZED VERSION OF THE HAM SANDWICH THEOREM

Although there is no bisecting hyperplane for the example of the previous section, note that the line $L$ is a "median" hyperplane for both measures. A median hyperplane for $\mu_1$ is defined as a hyperplane $H$ for which $\mu_1(H^+) \leq \frac{\mu_1(\mathbb{R}^n)}{2}$ and $\mu_1(H^-) \leq \frac{\mu_1(\mathbb{R}^n)}{2}$. We prove that with this modification of the notion of bisection, the ham sandwich theorem is true whether or not the measures are absolutely continuous. Specifically, we prove the following theorem:

**Theorem 2** Given any $n$ finite measures $\mu_1, \ldots, \mu_n$ defined on the Borel sets of $\mathbb{R}^n$, there exists a hyperplane $H = \{x \in \mathbb{R}^n | x \cdot v = c\}$ with $v \in S^{n-1}$, $c \in \mathbb{R}$, such that for all $1 \leq i \leq n$,

$$\mu_i(H^+) \leq \frac{\mu_i(\mathbb{R}^n)}{2} \quad \text{and} \quad \mu_i(H^-) \leq \frac{\mu_i(\mathbb{R}^n)}{2}$$

**Proof:** For each $\delta > 0$, and $1 \leq i \leq n$, we define the derived measure $\mu_1^\delta$ by,

$$\mu_1^\delta(A) = \int A \frac{\chi_{B_\delta}^\delta(x)}{m(B_\delta)} \, dx$$

where,

$$B_\delta = \{x \in \mathbb{R}^n | \|x\| \leq \delta\}$$

and $m(B_\delta)$ is the Lebesgue measure of $B_\delta$.

Now $\mu_1^\delta$ is absolutely continuous with respect to Lebesgue measure, $m$, because if $m(A) = 0$, then $\mu_1^\delta(A + x) = 0$ for almost every $x = \mu_1^\delta(A)$. Hence by Theorem 1 for each $\delta$,

$$H_\delta^\delta = \{x \in \mathbb{R}^n | x \cdot v_\delta = c_\delta\} \text{ with } v_\delta \in S^{n-1}, c_\delta \in \mathbb{R} \text{ such that}$$

$$\mu_1^\delta(H_\delta^\delta) = \mu_1^\delta(H_\delta^-) = \frac{\mu_1(\mathbb{R}^n)}{2}$$

for $i = 1, \ldots, n$.

For each $\delta$, define $g(\delta) = c_\delta \cdot v_\delta$ and pick $r \in \mathbb{R}$ so that

$$\mu_1^\delta(B_r) > \frac{1}{2} \mu_1^\delta(\mathbb{R}^n) \text{ for } i = 1, 2, \ldots, n.$$  Then for $0 < \delta < r$, $g$ associates with $\delta$ a vector $g(\delta)$ in the compact set $B_r$. Now if $(\delta_k)$ is a monotone decreasing infinite sequence converging to zero, then $g(\delta_k)$ is an infinite sequence in the compact set $B_r$. Hence, there is a subsequence $n(k)$ such that $g(\delta_{n(k)})$ converges to a point in $B_r$. For notational convenience, we will assume that the original sequence is such that $g(\delta_k)$ converges, and we write

$$\lim_{k \to \infty} g(\delta_k) = c^* \cdot v^*,$$

where $\|v^*\| = 1$, and $c^* \leq r$. We will show that the hyperplane

$$H^\delta = \{x \in \mathbb{R}^n | x \cdot v^* = c^*\}$$

satisfies the conditions of the theorem. Suppose, to the contrary that $H^\delta$ does not satisfy the conditions of the theorem. Then for some $i$, either

$$\mu_i(H^+) > \frac{\mu_i(\mathbb{R}^n)}{2} \quad \text{or} \quad \mu_i(H^-) > \frac{\mu_i(\mathbb{R}^n)}{2}$$
We assume, without loss of generality that

$$\mu_1(H^+) > \frac{\mu_1(R^n)}{2}.$$  

Now, we define a sequence of sets, $A_k \subset \mathbb{R}^n$ as follows:

$$A_k = H_+^k + (\delta_k \cdot v_k)$$

It is easily verified that

$$H^+ \subseteq \liminf_k A_k$$

Thus, from Fatou's Lemma, (see e.g., Kingman and Taylor [1966] p. 20), we have the following relation

$$\liminf_k \mu_1(A_k) = \liminf_k \int \chi_{A_k}(x) d\mu_1(x)$$

$$\geq \int \liminf_k \chi_{A_k}(x) d\mu_1(x)$$

$$\geq \int \chi_{H^+}(x) d\mu_1(x) = \mu_1(H^+)$$

So we can pick $k^*$ such that for $k > k^*$,

$$\mu_1(A_k) > \frac{\mu_1(R^n)}{2} \quad (\ast)$$

Further, by construction, for all $x \in B_{\delta_k}$, $A_k = H_+^k + \delta_k v_k \subset H_+^k + x$, so for $x \in B_{\delta_k}$,

$$\mu_1(H_+^k + x) \geq \mu_1(A_k) \quad (\ast\ast)$$

But now using ($\ast$) and ($\ast\ast$) together with the definition of $\mu_1^\delta_k$, we have

$$\mu_1^\delta_k(H_+^k + x) = \int_{B_{\delta_k}} \frac{\mu_1(H_+^k + x)}{m(B_{\delta_k})} dx \geq \int_{B_{\delta_k}} \frac{\mu_1(A_k)}{m(B_{\delta_k})} dx$$

$$= \mu_1(A_k) \int_{B_{\delta_k}} \frac{dx}{m(B_{\delta_k})} = \mu_1(A_k) > \frac{\mu_1(R^n)}{2}$$

But this implies that $H_+^k$ does not bisect $\mu_1^\delta_k$, a contradiction. So the hyperplane $H$ must satisfy the conditions of the theorem.

Q.E.D.

III. UNIQUENESS

In this section we give some sufficient conditions for the uniqueness of the bisecting (or median) hyperplane identified by the ham sandwich theorem. That the bisecting hyperplane need not in general be unique be easily illustrated in Figure 2, using the two atomic measures $\mu_1$ and $\mu_2$ in $\mathbb{R}^2$ defined by setting $\nu_1(x_1) = \nu_2(y_1) = 1/3$.
for \( i = 1, 2, 3 \). Here we have three distinct bisecting lines. Similar examples can clearly be constructed for absolutely continuous measures.

For this section, we let \( A_i \) be the support set for the measure \( \mu_i \), i.e., \( x \in A_i \iff \mu_i(B_r(x)) \neq 0 \) for every open \( r \) ball \( B_r(x) \) around \( x \), and we let \( \text{co}(A_i) \) denote the convex hull of \( A_i \). The measure \( \mu_i \) has a unique median in the direction \( v \in S^{n-1} \) iff there is a unique \( c \in \mathbb{R} \) for which \( H = \{ x | x \cdot v = c \} \) is a median hyperplane for \( \mu_i \). We deal here only with measures which have a unique median in every direction. This would include any absolutely continuous measure. It would also include any measure \( \mu_i \) whose support set cannot be partitioned into two separable sets (see below for definition of separability) each containing half the measure. For example, a completely atomic measure with \( \frac{1}{k} \) of the measure at each of \( k \) points has a unique median in every direction, as long as \( k \) is odd.

**Lemma 1** Let \( \mu_i \) have a unique median in every direction, and let \( H_1 \) and \( H_2 \) be two distinct median hyperplanes for \( \mu_i \) then

\[
H_1 \cap H_2 \cap \text{co}(A_i) \neq \emptyset.
\]

**Proof:** Suppose that \( H_1 \cap H_2 \cap \text{co}(A_i) = \emptyset \). Then, since \( H_1 \cap H_2 \) is convex, and \( \text{co}(A_i) \) is convex, there is a separating hyperplane, say \( H_0 = \{ x \in \mathbb{R}^n | x \cdot v_0 = c_0 \} \) such that

\[
H_1 \cap H_2 \subset H_0^+,
\]

\[
\text{co}(A_i) \subset H_0^-.
\]

Write \( H_1 = \{ x \in \mathbb{R}^n | x \cdot v_1 = c_1 \} \), and \( H_2 = \{ x \in \mathbb{R}^n | x \cdot v_2 = c_2 \} \), where \( v_1, v_2 \in S^{n-1} \), and \( c_1, c_2 \in \mathbb{R} \). Also pick \( v_1 \) and \( v_2 \) so that \( v_1, v_2 \) and \( v_0 \) positively span 0. This is possible since \( v_1, v_2 \) and \( v_0 \) are
linearly dependent, any pair is linearly independent, and \( v_1 \) and \( v_2 \) can either one or both be replaced by \(-v_1\), or \(-v_2\) (with an appropriate change to \( c_1 \) or \( c_2 \)). So we can write

\[
a_0 v_0 + a_1 v_1 + a_2 v_2 = 0
\]

with \( a_0, a_1, a_2 \) all positive. Now \( H_1 \cap H_2 \subset \mathbb{R}^n \Rightarrow a_0 c_0 + a_1 c_1 + a_2 c_2 < 0 \), because \( x \in H_1 \cap H_2 \subset \mathbb{R}^n \Rightarrow x \cdot v_0 > c_0 \Rightarrow x \cdot a_0 v_0 > a_0 c_0 \)

\[ \Rightarrow x \cdot (a_1 v_1 - a_2 v_2) > a_0 c_0 \Rightarrow -a_1 c_1 - a_2 c_2 > a_0 c_0. \]

We set

\[ \delta = -a_0 c_0 - a_1 c_1 - a_2 c_2 > 0. \]

Now, we pick \( \varepsilon > 0 \), and define

\[ H_\varepsilon = \{ x \in \mathbb{R}^n | x \cdot v_1 = c_1 + \varepsilon \} \]

For all \( \varepsilon \), we have \( H_\varepsilon^+ \subset H_1 \), so

\[ H_\varepsilon^+ \cap H_0^- \subset H_1^+ \cap H_0^- \]  \hspace{1cm} (*)

and for small enough \( \varepsilon \), we have

\[ H_\varepsilon^- \cap H_0^+ \subset H_2^+ \cap H_0^+ \]  \hspace{1cm} (**)}

To see this, set \( \varepsilon < \frac{\delta}{a_1} \), then \( x \in H_\varepsilon^+ \cap H_0^- \Rightarrow x \cdot v_1 > c_1 + \varepsilon \) and \( x \cdot v_0 < c_0 \). Since \( x \in H_0^- \), we need only show \( x \in H_2^+ \). But

\[
x \cdot v_2 = x \cdot \left( -\frac{a_0}{a_2} v_0 - \frac{a_1}{a_2} v_1 \right) > -\frac{a_0}{a_2} c_0 - \frac{a_1}{a_2} c_1 - \frac{a_1 \varepsilon}{a_2}
\]

\[
= - \frac{(a_0 c_0 + a_1 c_1)}{a_2} - \frac{a_1 \varepsilon}{a_2}
\]

\[
> \frac{a_2 c_2 + \delta}{a_2} - \frac{\delta}{a_2} = c_2
\]

so \( x \in H_2^+ \) as required. Since \( A_1 \subset H_0^- \), it follows from (*) and (**) that

\[ H_\varepsilon^+ \cap A_1 \subset H_1^+ \cap A_1 \]

and

\[ H_\varepsilon^- \cap A_1 \subset H_2^+ \cap A_1 \]

Thus

\[ \mu_1(H_\varepsilon^+) \leq \mu_1(H_1^+) \leq \mu_1(\mathbb{R}^n) \]

and

\[ \mu_1(H_\varepsilon^-) \leq \mu_1(H_2^+) \leq \mu_1(\mathbb{R}^n) \]

so \( H_\varepsilon \) is a median hyperplane. But then \( \mu_1 \) does not have a unique median in the direction \( v_1 \), a contradiction.

Q.E.D.

Two sets, \( B_1, B_2 \subset \mathbb{R}^n \) are separable iff there is a hyperplane \( H \subset \mathbb{R}^n \) with \( B_1 \subset H^+ \) and \( B_2 \subset H^- \). An \( n-2 \)-dimensional hyperplane is any set \( L \) of the form
$L = \{ x \in \mathbb{R}^n \mid x \cdot v_1 = c_1 \text{ and } x \cdot v_2 = c_2 \}$

where $v_1, v_2 \in S^{n-1}, c_1, c_2 \in \mathbb{R}$, and $v_1, v_2$ are linearly independent.

We now have the following definition

**Definition** A collection of sets $\{ B_i \}_{i=1}^k$ with $B_i \subseteq \mathbb{R}^n$ is

(S) **Separable** iff $\forall i, B_i$ and $B - B_i$ are separable, where

$B = \bigcup_{i=1}^n B_i$.

(N) **NonDegenerate** iff there is no $n-2$ dimensional hyperplane, $L$, with $L \cap \text{co} (B_i) \neq \emptyset$ for all $i$.

The following relation holds for any sets $B_1, \ldots, B_n \subseteq \mathbb{R}^n$

**Lemma 2** For all $n \geq 2$, $N \Rightarrow S$, for $n \leq 3$, $N \Leftrightarrow S$.

**Proof:** We prove the first assertion first. Define $B = \bigcup_{i=1}^n B_i$ and $D_i = \text{co}(B - B_i) \cap \text{co}(B_i)$. Now $D_i = \emptyset$ for all $i$, because if not, then for any $x \in D_i$, $x$ can be written as a convex combination of $n-1$ points $x_j \in \text{co}(B_j), j \neq i$. But these $n-1$ points determine a $n-2$ dimensional hyperplane which contains all $x_j$ as well as $x \in \text{co}(B_i)$, and this contradicts nondegeneracy. Hence nondegeneracy implies $D_i = \emptyset$ for all $i$. But then $\text{co}(B - B_i)$ and $\text{co}(B_i)$ are disjoint convex sets, and invoking the separating hyperplane theorem, it follows that separability is satisfied. Hence $N \Rightarrow S$, as we wished to show.

To prove the second part, we need only show for $n = 2, 3$, that $S \Rightarrow N$. This is proven by Steinhaus (1945) whose proof is sketched here. For $n = 2$, the proof is trivial. For $n = 3$, suppose the line $L$ passes through the points $x_1, x_2, x_3$, belonging to $\text{co}(B_1), \text{co}(B_2), \text{co}(B_3)$ respectively. Since $B_1, B_2, B_3$ are separable, their convex hulls have no common points, so $x_1, x_2, x_3$ are distinct.

We can suppose that $x_2$ is between $x_1$ and $x_3$. But the plane $H$ which separates $B_2$ from $B_1 \cup B_3$ puts $x_2$ on one side, and $x_1$ and $x_3$ on the other, contradicting the order of the points on the line $L$.

Q.E.D.

**Theorem 3** Let $\mu_1, \ldots, \mu_n$ be finite measures on the Borel sets of $\mathbb{R}^n$, with each $\mu_i$ having unique medians in all directions. Then if the support sets, $A_1, \ldots, A_n$ for the measures $\mu_1, \ldots, \mu_n$ satisfy nondegeneracy, the Ham Sandwich hyperplane is unique.

**Proof:** Let $H_1$ and $H_2$ be two distinct hyperplanes satisfying the conditions of Theorem 2. Then $H_1 \cap H_2$ is an $n-2$ dimensional hyperplane, and by Lemma 1, we have, for all $i$, $H_1 \cap H_2 \cap \text{co}(A_i) \neq \emptyset$. But then the sets $A_1, \ldots, A_n$ are degenerate, a contradiction.

Q.E.D.

**Corollary** If $n \leq 3$, and $\mu_1, \ldots, \mu_n$ are as in Theorem 3, then if the support sets $A_1, \ldots, A_n$ for the measures $\mu_1, \ldots, \mu_n$ are separable, the Ham Sandwich hyperplane is unique.
Proof: This follows directly from Theorem 3, along with the observation (proven in Lemma 2) that for \( n \leq 3 \), separability and nondegeneracy are equivalent.

Q.E.D.

To show that separability is not enough to guarantee uniqueness of the ham sandwich hyperplane for \( n \geq 4 \), the following example is given. Here, note \( S \not\rightarrow N \). We set

\[
\begin{align*}
    a_1 &= (0, 0, 1, 0) \\
    a_2 &= (0, 0, -1, 0) \\
    a_3 &= (0, 0, 0, 1) \\
    a_4 &= (0, 0, 0, -1)
\end{align*}
\]

and for \( 1 \leq i \leq 4 \), we define \( \mu_i \) to be the measure for which

\[
\mu_i(B) = \begin{cases} 
1 & \text{if } a_i \in B \\
0 & \text{otherwise}.
\end{cases}
\]

Clearly \( A_i = \{a_i\} \) for \( i = 1, \ldots, 4 \). Now define

\[
H_j = \{x \in \mathbb{R}^4 \mid x \cdot a_j = 1/2\}
\]

for \( j = 1, 2, 3, 4 \). The hyperplane \( H_j \) separates \( A_j \) from \( A - A_j \) for all \( j \), where

\[
A = \bigcup_{i=1}^{4} A_i.
\]

But clearly \( A \) is contained in the plane (a \( n-2 \) hyperplane of \( \mathbb{R}^4 \))

\[
L = \{x \in \mathbb{R}^4 \mid \epsilon_1 \cdot x = 0, \epsilon_2 \cdot x = 0\},
\]

where \( \epsilon_1 \) and \( \epsilon_2 \) are the first two basis vectors. So \( N \) is violated. Also, there are infinitely many three spaces (hyperplanes in \( \mathbb{R}^4 \)) which contain \( L \), each of which is a bisecting (median) hyperplane, so uniqueness of the ham sandwich plane is violated. This example can be extended straightforwardly for \( n > 4 \).
FOOTNOTES

1. The Borsuk-Ulam Theorem is as follows: If f is a continuous mapping of the n-sphere $S^n$ into $\mathbb{R}^n$ such that diametrically opposed points of $S^n$ map into points symmetric about the origin in $\mathbb{R}^n$ (i.e., $f(x) = f(-x)$), then there exists a point of $S^n$ which maps into the origin of $\mathbb{R}^n$.

REFERENCES


