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A HAM SANDWICH THEOREM FOR GENERAL MEASURES

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ABSTRACT

The "ham sandwich" theorem has been proven only for measures that are absolutely continuous with respect to Lesbeque measure. We prove a generalized version of the ham sandwich theorem which is applicable to arbitrary finite measures, and we give some sufficient conditions for uniqueness of the hyperplane identified by the theorem.

A HAM SANDWICH THEOREM FOR GENERAL MEASURES

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I. INTRODUCTION

The Ham Sandwich problem was first posed by Ulam [1930], and has since been examined by Borsuk [1933], Steinhaus [1945], Stone and Tukey [1942], Tucker [1945], and Dubins and Spanier [1961]. The problem derives its name from Steinhaus' picturesque formulation of the problem as that of dividing a ham, butter, and bread sandwich bv a plane into two parts each containing exactly one half of the ham, one half of the butter, and one half of the bread.

The theorem has an n-dimensional generalization which uses the following definitions: A hyperplane is any set of the form

$$H = \{x \in \mathbb{R}^n | x \cdot v > c\}$$

where $v \in S^{n-1}$, and $c \in \mathbb{R}$. Here S^{n-1} is the $n-1$ sphere of unit length vectors in $\mathbb{R}^n$. We use the notation

$$H^+ = \{x \in \mathbb{R}^n | x \cdot v > c\}$$

and

$$H^- = \{x \in \mathbb{R}^n | x \cdot v < c\}$$

to denote the positive and negative open half spaces defined by H. If μ is a finite measure on the Borel sets of $\mathbb{R}^n$, we say H <u>bisects</u> μ iff $\mu(H^+) = \mu(H^-)$. We have the following statement of the theorem:

<u>Theorem 1</u> Given any n finite measures, $\mu_1,\ldots,\mu_n$ defined on the Borel subsets of n-dimensional Euclidian Space, $\mathbb{R}^n$, if each μ_i is absolutely continuous with respect to Lebesgue measure, there exists a hyperplane which simultaneously bisects each measure.

The usual proof goes as follows:

(1) Consider the measure μ_n. We know from measure-theoretic considerations that, for each unit vector v, there exists a real number c_v such that the hyperplane $H_v = \{x \in \mathbb{R}^n \mid x \cdot v = c_v\}$ bisects μ_n.

(2) Now define a mapping f from the unit n - 1 sphere S^{n-1} to $\mathbb{R}^{n-1}$ as:

$$f_j(v) = (\mu_j(H_v^+) - \mu_j(H_v^-)), \text{ for } j = 1,\ldots,n-1.$$

Note that f is continuous and that $f(v) = -f(-v)$.

(3) Use the Borsuk-Ulam theorem[1] to infer that there exists a $v \in S^{n-1}$ which f maps into the origin in $\mathbb{R}^{n-1}$, implying $\mu_j(H_v^+) = \mu_j(H_v^-)$ for $j = 1,\ldots,n$ and proving the theorem.

The Ham sandwich theorem and its proof depend on the absolute continuity of the measures μ_i. Otherwise, the function f defined above need not be continuous. In fact, the theorem as stated is not true for general measures, as is illustrated by the example of Figure 1, where μ_1 is the atomic measure defined by setting $\mu_1(x_i) = \mu_2(y_i) = 1/3$ for $i = 1,2,3$, and $\mu_1(x) = \mu_2(y) = 0$ otherwise. Here, any bisecting line for μ_1 must pass thru one and only one x_i, with the remaining x_j's lying on either side of the line. A bisecting line for μ_2 must have similar properties. But no line passing thru one x_i and one y_i splits the remaining points in the desired fashion.

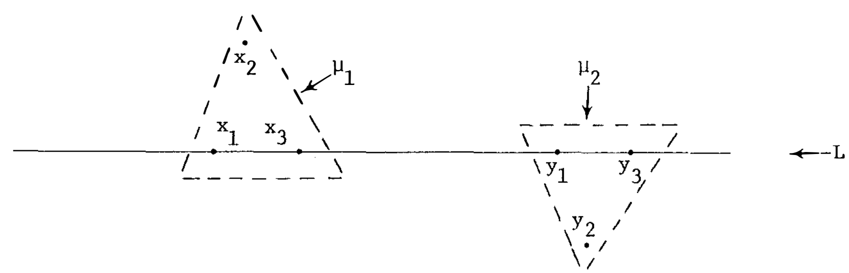

Figure 1

II. A GENERALIZED VERSION OF THE HAM SANDWICH THEOREM

Although there is no bisecting hyperplane for the example of the previous section, note that the line L is a "median" hyperplane for both measures. A median hyperplane for μ_i is defined as a hyperplane H for which $\mu_i(H^+) \leq \frac{\mu_i(\mathbb{R}^n)}{2}$ and $\mu_i(H^-) \leq \frac{\mu_i(\mathbb{R}^n)}{2}$. We prove that with this modification of the notion of bisection, the ham sandwich theorem is true whether or not the measures are absolutely continuous. Specifically, we prove the following theorem:

Theorem 2 Given any n finite measures $\mu_1,\ldots,\mu_n$ defined on the Borel sets of $\mathbb{R}^n$, there exists a hyperplane $H = \{x \in \mathbb{R}^n | x \cdot v = c\}$ with $v \in S^{n-1}$, $c \in \mathbb{R}$, such that for all $1 \leq i \leq n$,

$$\mu_i(H^+) \leq \frac{\mu_i(\mathbb{R}^n)}{2} \text{ and } \mu_i(H^-) \leq \frac{\mu_i(\mathbb{R}^n)}{2}$$

Proof: For each $\delta > 0$, and $1 \leq i \leq n$, we define the derived measure μ_i^δ by,

$$\mu_i^\delta(A) = \int \mu_i(A + x) \frac{\chi_{B_\delta}(x)}{m(B_\delta)}\, dx$$

where,

$$B_\delta = \{x \in \mathbb{R}^n | \ \|x\| \leq \delta\}$$

and $m(B_\delta)$ is the Lebesgue measure of B_δ.

Now μ_i^δ is absolutely continuous with respect to Lebesgue measure, m, because if $m(A) = 0$, then $\mu_i(A + x) = 0$ for almost every $x \Rightarrow \mu_i^\delta(A) = 0$. Hence by Theorem 1 for each δ, $\exists\, H_\delta = \{x \in \mathbb{R}^n | x \cdot v_\delta = c_\delta\}$ with $v_\delta \in S^{n-1}$, $c_\delta \in \mathbb{R}$ such that

$$\mu_i^\delta(H_\delta^+) = \mu_i^\delta(H_\delta^-) = \frac{\mu_i^\delta(\mathbb{R}^n)}{2}$$

for $i = 1,\ldots,n$.

For each δ, define $g(\delta) = c_\delta \cdot v_\delta$ and pick $r \in \mathbb{R}$ so that $\mu_i(B_{\frac{r}{2}}) > 1/2\ \mu_i(\mathbb{R}^n)$ for $i = 1,2,\ldots,n$. Then for $0 < \delta < r$, g associates with δ a vector $g(\delta)$ in the compact set B_r. Now if $\{\delta_k\}$ is a monotone decreasing infinite sequence converging to zero, then $g(\delta_k)$ is an infinite sequence in the compact set B_r. Hence, there is a subsequence $\eta(k)$ such that $g(\delta_{\eta(k)})$ converges to a point in B_r. For notational convenience, we will assume that the original sequence is such that $g(\delta_k)$ converges, and we write $\lim_{k\to\infty} g(\delta_k) = c^* \cdot v^*$, where $\|v^*\| = 1$, and $c^* \leq r$. We will show that the hyperplane

$$H = \{x \in \mathbb{R}^n | x \cdot v^* = c^*\}$$

satisfies the conditions of the theorem. Suppose, to the contrary that H* does not satisfy the conditions of the theorem. Then for some i, either

$$\mu_i(H^+) > \frac{\mu_i(\mathbb{R}^n)}{2} \text{ or } \mu_i(H^-) > \frac{\mu_i(\mathbb{R}^n)}{2}$$

We assume, without loss of generality that

$$\mu_i(H^+) > \frac{\mu_i(\mathbb{R}^n)}{2}.$$

Now, we define a sequence of sets, $A_k \subseteq \mathbb{R}^n$ as follows:

$$A_k = H^+_{\delta_k} + (\delta_k \cdot v_{\delta_k})$$

It is easily verified that

$$H^+ \subseteq \liminf_k A_k$$

Thus, from Fatou's Lemma, (see e.g., Kingman and Taylor [1966] p. 20), we have the following relation

$$\liminf_k \mu_i(A_k) = \liminf_k \int \chi_{A_k}(x)d\mu_i(x)$$

$$\geq \int \liminf_k \chi_{A_k}(x)d\mu_i(x)$$

$$\geq \int \chi_{H^+}(x)d\mu_i(x) = \mu_i(H^+)$$

So we can pick k* such that for k > k*,

$$\mu_i(A_k) > \frac{\mu_i(\mathbb{R}^n)}{2} \qquad (*)$$

Further, by construction, for all $x \in B_{\delta_k}$, $A_k = H^+_{\delta_k} + \delta_k v_{\delta_k} \subseteq H^+_{\delta_k} + x$, so for $x \in B_{\delta_k}$,

$$\mu_i(H^+_{\delta_k} + x) \geq \mu_i(A_k) \qquad (**)$$

But now using (*) and (**) together with the definition of μ_i^δ, we have

$$\mu_i^{\delta_k}(H^+_{\delta_k}) = \int_{B_{\delta_k}} \frac{\mu_i(H^+_{\delta_k} + x)}{m(B_{\delta_k})} dx \geq \int_{B_{\delta_k}} \frac{\mu_i(A_k)}{m(B_{\delta_k})} dx$$

$$= \mu_i(A_k) \int_{B_{\delta_k}} \frac{dx}{m(B_{\delta_k})} = \mu_i(A_k) > \frac{\mu_i(\mathbb{R}^n)}{2}$$

But this implies that H_{δ_k} does not bisect $\mu_i^{\delta_k}$, a contradiction. So the hyperplane H must satisfy the conditions of the theorem.

Q.E.D.

III. UNIQUENESS

In this section we give some sufficient conditions for the uniqueness of the bisecting (or median) hyperplane identified by the ham sandwich theorem. That the bisecting hyperplane need not in general be unique be easily illustrated in Figure 2, using the two atomic measures μ_1 and μ_2 in $\mathbb{R}^2$ defined by setting $\mu_1(x_i) = \mu_2(y_i) = 1/3$

for $i = 1,2,3$. Here we have three distinct bisecting lines. Similar examples can clearly be constructed for absolutely continuous measures.

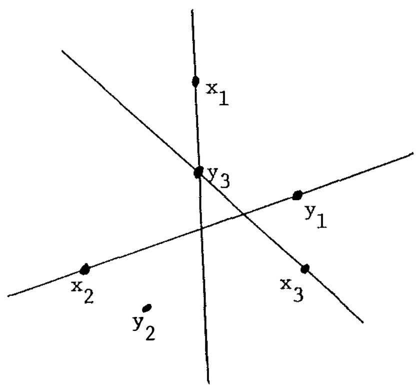

Figure 2

For this section, we let A_i be the support set for the measure μ_i, i.e., $x \in A_i \Leftrightarrow \mu_i(B_\varepsilon(x)) \neq 0$ for every open ε ball $B_\varepsilon(x)$ around x, and we let $co(A_i)$ denote the convex hull of A_i. The

measure μ_i has a <u>unique median in the direction</u> $v \in S^{n-1}$ iff there is a unique $c \in \mathbb{R}$ for which $H = \{x | x \cdot v = c\}$ is a median hyperplane for μ_i. We deal here only with measures which have a unique median in every direction. This would include any absolutely continuous measure. It would also include any measure μ_i whose support set cannot be partitioned into two seperable sets (see below for definition of separability) each containing half the measure. For example, a completely atomic measure with $\frac{1}{k}$ of the measure at each of k points has a unique median in every direction, as long as k is odd.

<u>Lemma 1</u> Let μ_i have a unique median in every direction, and let H_1 and H_2 be two distinct median hyperplanes for μ_i then $H_1 \cap H_2 \cap co(A_i) \neq \phi$.

<u>Proof</u>: Suppose that $H_1 \cap H_2 \cap co(A_i) = \phi$. Then, since $H_1 \cap H_2$ is convex, and $co(A_i)$ is convex, there is a separating hyperplane, say $H_0 = \{x \in \mathbb{R}^n | x \cdot v_0 = c_0\}$ such that

$$H_1 \cap H_2 \in H_0^+,$$

$$co(A_i) \in H_0^-.$$

Write $H_1 = \{x \in \mathbb{R}^n | x \cdot v_1 = c_1\}$, and $H_2 = \{x \in \mathbb{R}^n | x \cdot v_2 = c_2\}$, where $v_1, v_2 \in S^{n-1}$, and $c_1, c_2 \in \mathbb{R}$. Also pick v_1 and v_2 so that v_1, v_2 and v_0 positively span 0. This is possible since v_1, v_2 and v_0 are linearly dependent, any pair is linearly independent, and v_1 and v_2 can either one or both be replaced by $-v_1$, or $-v_2$ (with an appropriate change to c_1 or c_2). So we can write

$$a_0v_0 + a_1v_1 + a_2v_2 = 0$$

with a_0, a_1, a_2 all positive. Now $H_1 \cap H_2 \subseteq H_0^+ \Rightarrow a_0c_0 + a_1c_1 + a_2c_2 < 0$, because $x \in H_1 \cap H_2 \subseteq H_0^+ \Rightarrow x \cdot v_0 > c_0 \Rightarrow x \cdot a_0v_0 > a_0c_0$ $\Rightarrow x \cdot (-a_1v_1 - a_2v_2) > a_0c_0 \Rightarrow -a_1c_1 - a_2c_2 > a_0c_0$. We set

$$\delta = -a_0c_0 - a_1c_1 - a_2c_2 > 0.$$

Now, we pick $\varepsilon > 0$, and define

$$H_\varepsilon = \{x \in \mathbb{R}^n \mid x \cdot v_1 = c_1 + \varepsilon\}$$

For all ε, we have $H_\varepsilon^+ \subseteq H_1^+$, so

$$H_\varepsilon^+ \cap H_0^- \subseteq H_1^+ \cap H_0^- \qquad (*)$$

and for small enough ε, we have

$$H_\varepsilon^- \cap H_0^- \subseteq H_2^+ \cap H_0^-. \qquad (**)$$

To see this, set $\varepsilon < \frac{\delta}{a_1}$, then $x \in H_\varepsilon^+ \cap H_0^- \Rightarrow x \cdot v_1 > c_1 + \varepsilon$ and $x \cdot v_0 < c_0$. Since $x \in H_0^-$, we need only show $x \in H_2^+$. But

$$x \cdot v_2 = x \cdot \left(-\frac{a_0}{a_2} v_0 - \frac{a_1}{a_2} v_1\right) > -\frac{a_0}{a_2} c_0 - \frac{a_1}{a_2} c_1 - \frac{a_1\varepsilon}{a_2}$$

$$= -\frac{(a_0c_0 + a_1c_1)}{a_2} - \frac{a_1\varepsilon}{a_2}$$

$$> \frac{a_2c_2 + \delta}{a_2} - \frac{\delta}{a_2} = c_2$$

so $x \in H_2^+$ as required. Since $A_i \subseteq H_0^-$, it follows from (*) and (**) that

$$H_\varepsilon^+ \cap A_i \subseteq H_1^+ \cap A_i$$

and

$$H_\varepsilon^- \cap A_i \subseteq H_2^+ \cap A_i$$

Thus

$$\mu_i(H_\varepsilon^+) \leq \mu_i(H_1^+) \leq \frac{\mu_i(\mathbb{R}^n)}{2}$$

and

$$\mu_i(H_\varepsilon^-) \leq \mu_i(H_2^+) \leq \frac{\mu_i(\mathbb{R}^n)}{2},$$

so H_ε is a median hyperplane. But then μ_i does not have a unique median in the direction v_1, a contradiction.

Q.E.D.

Two sets, B_1, $B_2 \subseteq \mathbb{R}^n$ are <u>separable</u> iff there is a hyperplane H in $\mathbb{R}^n$ with $B_1 \subseteq H^+$ and $B_2 \subseteq H^-$. An <u>n-2 dimensional hyperplane</u> is any set L of the form

$$L = \{x \in \mathbb{R}^n \mid x \cdot v_1 = c_1 \text{ and } x \cdot v_2 = c_2\}$$

where v_1, $v_2 \in S^{n-1}$, $c_1, c_2 \in \mathbb{R}$, and v_1, v_2 are linearly independent. We now have the following definition

Definition A collection of sets $\{B_i\}_{i=1}^k$ with $B_i \subseteq \mathbb{R}^n$ is

(S) Separable iff $\forall$ i, B_i and $B - B_i$ are separable, where $B = \bigcup_{i=1}^n B_i$.

(N) NonDegenerate iff there is no n-2 dimensional hyperplane, L, with $L \cap co(B_i) \neq \phi$ for all i.

The following relation holds for any sets $B_1, \ldots, B_n \subseteq \mathbb{R}^n$

Lemma 2 For all $n \geq 2$, $N \Rightarrow S$, for $n \leq 3$, $N \Leftrightarrow S$.

Proof: We prove the first assertion first. Define $B = \bigcup_{i=1}^n B_i$ and $D_i = co(B - B_i) \cap co(B_i)$. Now $D_i = \phi$ for all i, because if not, then for any $x \in D_i$, x can be written as a convex combination of $n-1$ points $x_j \in co(B_j)$, $j \neq i$. But these $n-1$ points determine a $n-2$ dimensional hyperplane which contains all x_j as well as $x \in co(B_i)$, and this contradicts nondegenercy. Hence nondegenercy implies $D_i = \phi$ for all i. But then $co(B - B_i)$ and $co(B_i)$ are disjoint convex sets, and invoking the separating hyperplane theorem, it follows that separability is satisfied. Hence $N \Rightarrow S$, as we wished to show.

To prove the second part, we need only show for $n = 2,3$, that $S \Rightarrow N$. This is proven by Steinhaus (1945) whose proof is

sketched here. For $n = 2$, the proof is trivial. For $n = 3$, suppose the line L passes through the points x_1, x_2, x_3, belonging to $co(B_1)$, $co(B_2)$, $co(B_3)$ respectively. Since B_1, B_2, B_3 are separable, their convex hulls have no common points, so x_1, x_2, and x_3 are distinct. We can suppose that x_2 is between x_1 and x_3. But the plane H which separates B_2 from $B_1 \cup B_3$ puts x_2 on one side, and x_1 and x_3 on the other, contradicting the order of the points on the line L.

Q.E.D.

Theorem 3 Let $\mu_1, \ldots, \mu_n$ be finite measures on the Borel sets of $\mathbb{R}^n$, with each μ_i having unique medians in all directions. Then if the support sets, $A_1, \ldots, A_n$ for the measures $\mu_1, \ldots, \mu_n$ satisfy nondegeneracy, the Ham Sandwich hyperplane is unique.

Proof: Let H_1 and H_2 be two distinct hyperplanes satisfying the conditions of Theorem 2. Then $H_1 \cap H_2$ is a $n-2$ dimensional hyperplane, and by Lemma 1, we have, for all i, $H_1 \cap H_2 \cap co(A_i) \neq \phi$. But then the sets $A_1, \ldots, A_n$ are degenerate, a contradiction.

Q.E.D.

Corollary If $n \leq 3$, and $\mu_1, \ldots, \mu_n$ are as in Theorem 3, then if the support sets $A_1, \ldots, A_n$ for the measures $\mu_1, \ldots, \mu_n$ are separable, the Ham Sandwich hyperplane is unique.

Proof: This follows directly from Theorem 3, along with the observation (proven in Lemma 2) that for $n \leq 3$, separability and nondegeneracy are equivalent.

Q.E.D.

To show that separability is not enough to guarantee uniqueness of the ham sandwich hyperplane for $n \geq 4$, the following example is given. Here, note $S \not\Rightarrow N$. We set

$$a_1 = (0, 0, 1, 0)$$

$$a_2 = (0, 0, -1, 0)$$

$$a_3 = (0, 0, 0, 1)$$

$$a_4 = (0, 0, 0, -1)$$

and for $1 \leq i \leq 4$, we define μ_i to be the measure for which

$$\mu_i(B) = \begin{cases} 1 \text{ if } a_i \in B \\ 0 \text{ otherwise.} \end{cases}$$

Clearly $A_i = \{a_i\}$ for $i = 1,\ldots,4$. Now define

$$H_j = \{x \in \mathbb{R}^4 \mid x \cdot a_j = 1/2\} \text{ for } j = 1,2,3,4$$

The hyperplane H_j separates A_j from $A - A_j$ for all j, where $A = \bigcup_{i=1}^{4} A_i$.

But clearly A is contained in the plane (a $n-2$ hyperplane of $\mathbb{R}^4$) $L = \{x \in \mathbb{R}^4 | \varepsilon_1 \cdot x = 0,\ \varepsilon_2 \cdot x = 0\}$, where ε_1 and ε_2 are the first two basis vectors. So N is violated. Also, there are infinitely many three spaces (hyperplanes in $\mathbb{R}^4$) which contain L, each of which is a bisecting (median) hyperplane, so uniqueness of the ham sandwich plane is violated. This example can be extended straightforwardly for $n > 4$.

FOOTNOTES

1. The Borsuk-Ulam Theorem is as follows: If f is a continuous mapping of the n-sphere S^n into $\mathbb{R}^n$ such that diametrically opposed points of S^n map into points symmetric about the origin in $\mathbb{R}^n$ (i.e., $f(x) = f(-x)$), then there exists a point of S^n which maps into the origin of $\mathbb{R}^n$.

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