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A CLASS OF DIFFERENTIAL GAMES WHERE THE CLOSED-LOOP AND OPEN-LOOP NASH EQUILIBRIA COINCIDE

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ABSTRACT

It is well known that, in general, Nash equilibria in open-loop strategies do not coincide with those in closed-loop strategies. This note identifies a class of differential games in which the Nash equilibrium in closed-loop strategies is degenerate in the sense that it depends on time (t) only. Consequently, the closed-loop equilibrium is also an equilibrium in open-loop strategies.

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1. INTRODUCTION

It is well known that, in general, Nash equilibria in open-loop strategies do not coincide with those in closed-loop strategies. Notable exceptions are reported in [1], [2]. This note identifies another class of differential games in which the Nash equilibrium in closed-loop strategies is degenerate in the sense that it depends on time (t) only. Consequently, the closed-loop equilibrium is also an equilibrium in open-loop strategies.

2. FORMULATION

Consider the following n-person nonzero-sum differential game.

2.1. n is the number of players, indexed by i.

2.2. m is the number of states, indexed by j; $m \gtrless n$.

2.3. t denotes calendar time; t ranges over $[t_0,T]$; $t_0 \in [0,T]$.

2.4. $z(t) = (z_1(t),\ldots,z_m(t))'$, a <u>column</u> vector of state variables.

2.5. $u(t,z) = (u_1(t,z),\ldots,u_n(t,z))'$, a <u>column</u> vector of (closed-loop) strategies; strategy $u_i(t,z)$ is player i's strategy.

2.6. Player i's strategy is assumed to belong to the <u>admissible set</u>

$$U_i = \{u_i(t,z): u_i(t,z) \text{ is measurable in } t \text{ for each fixed } z \text{ and continuous in } z \text{ for each fixed } t;\ |u_i(t,z)| \le k_0^i(t)(1+|z|) \text{ for } (t,z) \in [t_0,T] \times \mathbb{R}^m \text{ where } \int_{t_0}^T k_0^i(t)dt < \infty; \text{ and } |u_i(t,z) - u_i(t,\bar{z})| \le k_R^i(t)|z-\bar{z}| \text{ if } |z| \le R,\ |\bar{z}| \le R,\ t_0 < t < T\}.$$

2.7. The state variables evolve according to the kinematic equations

$$\dot{z} = A(t)u(t,z),\quad z(t_0) = z_0, \tag{1}$$

where $A(t) = [a_i^j(t)]$ is an $m \times n$ matrix; $a^j(t)$ is the j-th row vector and $a_i(t)$ is the i-th column vector. The functions $a_i^j(t)$ are assumed continuous on $[t_0,T]$.

<u>Remark 2.1</u>. We will try, in general, to use a superscript to indicate row vectors; subscripts for column vectors. Vectors with no sub- or superscripts will be column vectors.

<u>Remark 2.2</u>. Because of (2.6), there exists a unique solution $\varphi(t)$ to the system (1) on $[t_0,T]$ for any n-tuple of admissible strategies [3].

Suppose that player i's payoff functional depends on t, φ, and u in the following way:

$$2.8.\quad J^i(u;\varphi) = \int_{t_0}^T [c(t)'u + u'B(t)u]\exp(-\lambda^i\varphi)dt$$

where $\lambda^i = (\lambda_1^i,\ldots,\lambda_m^i)$ is a row vector of scalars, $c(t) = (c_1(t),\ldots,c_n(t))'$ is a column vector of continuous functions and $B(t) = [b_k^i(t)]$ is an $n \times n$ matrix of continuous functions, with $b^i(t)$ the i-th row vector and $b_k(t)$ the k-th column vector. We assume that $b_i^i(t) < 0\ \ \forall\ i$ and $[B(t) + B(t)']$ is nonsingular $\forall\ t$ with $D(t) \equiv [B(t) + B(t)']^{-1}$.

<u>Remark 2.3</u>. This problem has a linear-quadratic structure in the strategies, but is exponential in the state variables. It will be apparent that Theorem 3.1 below is true for other specifications of the payoffs and kinematic equations as well, so long as <u>exponential state dependence</u> is retained (though existence of equilibrium may be more difficult to prove since one would need to establish an instantaneous fixed point property at each t).

3. RESULTS

Definition 3.1. A strategy n-tuple $(u_1^*,\ldots,u_n^*)$ is a Nash equilibrium in pure closed-loop strategies on $[0,T] \times \mathbb{R}^m$ if, for each $(t_0,z_0) \in [0,T] \times \mathbb{R}^m$: $\forall$ i

(a) $u_i^* \in U_i$; and

(b) $J^i(u_1^*,\ldots,u_n^*;\varphi^*) \geq J^i(u_1^*,\ldots,u_{i-1}^*,u_i,u_{i+1}^*,\ldots,u_n^*;\varphi)$ $\forall\, u_i \in U_i$.

Definition 3.2. Let $\hat{u}(i) \equiv (u_1^*,\ldots,u_{i-1}^*,u_i,u_{i+1}^*,\ldots,u_n^*)$ for convenience of notation.

Theorem 3.1. For T sufficiently near 0, there exists a Nash equilibrium in closed-loop strategies for the game described in Section 2. Furthermore, the equilibrium depends only on t.

Proof:

Define the value functions

$$V^i(t_0,z_0) \equiv \sup_{u_i \in U_i} \int_{t_0}^{T} [c(t)'\hat{u}(i) + \hat{u}(i)'B(t)\hat{u}(i)]\exp(-\lambda^i z)dt$$

subject to $\dot{z} = A(t)\hat{u}(i)$, $z(t_0) = z_0$, where $u_k^*(t,z)$, $k \neq i$, is taken as given by player i.

This definition requires that $\forall$ i

$$V^i(T,z(T)) = 0. \qquad (2)$$

First we remark that, to be a Nash equilibrium, u must satisfy the necessary conditions below [4]: $\forall$ i

$$V_t^i + \max_{u_i \in U_i} \{V_z^i A(t)\hat{u}(i) + [c(t)'\hat{u}(i) + \hat{u}(i)'B(t)\hat{u}(i)]\exp(-\lambda^i z)\} = 0 \qquad (3)$$

at each point of differentiability of $V^i(t,z)$, where $V_z^i = (V_{z_1}^i,\ldots,V_{z_m}^i)$ is a row vector of partial derivatives of i's value function.

Performing the indicated maximization yields the candidates for a Nash equilibrium: $\forall$ i

$$V_z^i a_i(t) + \{c_i(t) + [b^i(t) + b_i(t)']u^*\}\exp(-\lambda^i z) = 0. \qquad (4)$$

Since $b_i^i(t) < 0$, these are also sufficient to guarantee a maximum in (3).

Define $y^i \equiv V_z^i a_i(t)\exp(\lambda^i z) + c_i(t)$. Then we can rewrite (4) as

$$y^i + [b^i(t) + b_i(t)']u^* = 0, \quad i = 1,\ldots,n \qquad (5)$$

or, more compactly,

$$y + [B(t) + B(t)']u^* = 0, \qquad (6)$$

where $y = (y_1,\ldots,y_n)'$ is a column vector.

The candidates for a Nash equilibrium (in feedback form) are

$$u^* = -D(t)y. \qquad (7)$$

Substituting the candidates (7) into the system (3) and recalling (2) gives the following system of partial differential equations with terminal conditions (2): $\forall\, i$

$$V_t^i - V_z^i A(t)D(t)y + [-c(t)'D(t)y + y'D(t)'B(t)D(t)y]\exp(-\lambda^i z) = 0. \qquad (8)$$

Now we note that if we have in hand a system of value functions V^* which are C^1 and solve (8)-(2), then by the verification theorem of [5], the strategies (7) form a Nash equilibrium in closed-loop pure strategies.

Claim: $V^{i*}(t,z) = h_i(t)\exp(-\lambda^i z)$, $i = 1,2,\ldots,n$ is such a system of value functions, where $h(t) = (h_1(t),\ldots,h_n(t))'$ is the unique, continuously differentiable solution of: $\forall i$

$$\dot{h}_i = -h_i\lambda^i A(t)D(t)y(t) + c(t)'D(t)y(t) - y(t)'D(t)'B(t)D(t)y(t) \qquad (9)$$

and

$$h_i(T) = 0 \qquad (10)$$

where $y_i(t) = -h_i\lambda^i a_i(t) + c_i(t)$.

To see this, substitute V_t^{*i} and V_z^{*i} into (8), note that $y_i = V_z^{*i} a_i(t)\exp(\lambda^i z) + c_i(t) = -h_i\lambda^i a_i(t) + c_i(t)$, and cancel $\exp(-\lambda^i z)$ from each term.

The solution $h(t)$ through $h(T) = 0$ exists (at least near T) and is unique because the r.h.s. of (9) is continuous in t and C^1 in h.

Since

$$u^*(t,z) = -D(t)y(t)$$

$$= D(t)\begin{pmatrix} h_1(t)\lambda^1 a_1(t) - c_1(t) \\ \vdots \\ h_n(t)\lambda^n a_n(t) - c_n(t) \end{pmatrix},$$

u^* is continuous in t (and independent of z), so $u_i^* \in U_i$ and $(u_1^*,\ldots,u_n^*)$ provides the Nash equilibrium as claimed. QED.

<u>Theorem 3.2</u>. If the functions $a_i^j(t)$, $b_k^i(t)$ and $c_i(t)$ are C^∞, then the Nash equilibrium of Theorem 3.1 is unique.

<u>Proof</u>:

If (8)-(2) has a unique solution, then the equilibrium is unique. But if all the coefficient functions are C^∞, then the system (8)-(2) is C^∞ in (t,z,V_t,V_z) and has a unique C^∞ solution $V^*(t,z)$ near $t = T$ [6].

4. EXAMPLES

The following application to research and development is reported in greater detail and generality in [7], [8].

Suppose two identical firms are engaged in a race for an invention. However, there is some uncertainty regarding the feasibility of the invention. In particular, firm i's probability of success by time t depends upon $z_i(t)$, i's stock of accumulated knowledge. This probability of success is assumed to be exponential: $F_i(t) = 1 - \exp(-\lambda z_i(t))$. Player i's knowledge stock may be incremented at the rate $\dot{z}_i = u_i(t,z)$ (≥ 0) by incurring costs of $\$(1/2)u_i^2$. If player i wins the race, then i receives a patent with current value Pe^{rt} (constant present value). The game ends as soon as one player succeeds or at T, whichever occurs earlier. Discounting at rate r, player i's payoff can be written as follows:

$$J^i(u;z) = \int_0^T e^{-rt}[Pe^{rt}\lambda u_1 - (1/2)u_1^2]\exp\{-\lambda(z_1+z_2)\}dt$$

subject to $\dot{z}_j = u_j$, $z_j(0) = 0$, $u_j \geq 0$, $j = 1,2$.

That is, player i receives Pe^{rt} at t if no one has yet succeeded (this occurs with probability $(1 - F_1)(1 - F_2) = \exp\{-\lambda(z_1+z_2)\}$) and if i succeeds at t (the conditional density of success is λu_i); costs are incurred only so long as no one has succeeded.

The analogs of equations (2), (3), (7) and (8) are given below.

$$V^i_t + \max_{u_i \in U_i} \left\{ V^i_{z_i} u_i + V^i_{z_j} u_j + [P\lambda u_i - e^{-rt}(1/2)u_i^2]\exp\{-\lambda(z_1+z_2)\} \right\} = 0; \qquad (11)$$

$$u_i = V^i_{z_i} e^{rt}\exp\{\lambda(z_1+z_2)\} + P\lambda e^{rt}; \qquad (12)$$

$$V^i_t + (1/2)(V^i_{z_i})^2 e^{rt}\exp\{\lambda(z_1+z_2)\} + V^i_{z_j} V^j_{z_j} e^{rt}\exp\{\lambda(z_1+z_2)\}$$
$$+ (V^j_{z_j} + V^i_{z_i})P\lambda e^{rt} + (1/2)P^2\lambda^2 e^{rt}\exp\{-\lambda(z_1+z_2)\} = 0; \qquad (13)$$

$$V^i(T,z(T)) = 0. \qquad (14)$$

We know by Theorems 3.1-3.2 that the unique solution to (13)-(14) is of the form $V^{*i}(t,z) = h_i(t)\exp\{-\lambda(z_1+z_2)\}$. Symmetry suggests that $h_1(t) = h_2(t) \equiv h(t)$. Substituting V^*_t, $V^*_{z_1}$ and $V^*_{z_2}$ into (13)-(14) implies that h(t) must solve

$$\dot{h} = -\lambda^2 e^{rt}(1/2)\{3h^2 - 4Ph + P^2\}, \quad h(T) = 0. \qquad (15)$$

The ordinary differential equation (15) has solution

$$h(t) = \frac{P[1 - \exp\{P\lambda^2(e^{rT} - e^{rt})/r\}]}{[1 - 3\exp\{P\lambda^2(e^{rT} - e^{rt})/r\}]}.$$

Thus the Nash equilibrium strategies are: for $t \in [0,T]$,

$$u_i^* = \frac{2P\lambda e^{rt}}{3 - \exp\{P\lambda^2(e^{rt} - e^{rT})/r\}}, \quad i = 1,2.$$

Another example concerns the noncooperative exploitation of a common property resource of unknown size. That is, suppose n adjacent property-owners discover a shared pool of oil. Although i does not know the size of the reserve, S, i believes that S is distributed exponentially:

$$\Pr_i\{S \leq x\} = 1 - e^{-\lambda^i x}.$$

Let X(t) be cumulative extraction to date. Then (to i) the probability that the pool is not exhausted by t is $\Pr\{S \geq X(t)\} = e^{-\lambda^i X(t)}$. The rate of extraction (strategy) of player i is denoted by $u_i(t,X)$ and the aggregate rate of extraction is $\dot{X} = \sum_{j=1}^{n} u_j(t,X)$.

If the players sell their resource in a market with linear demand curve

$$P(t,\sum_j u_j) = a(t) - b(t)\sum_j u_j(t,X),$$

each spends $\$c_i(u_i)$ on extraction, and each discounts profits at the rate r, then player i's payoff becomes

$$J^i(u;X) = \int_0^\infty e^{-\lambda^i X(t)} e^{-rt}\{[a(t) - b(t)\sum_j u_j]u_i - c_i(u_i)\}dt$$

where $\dot{X} = \sum_j u_j(t,X)$, $X(0) = 0$.

If $c_i(u_i)$ is nonnegative and quadratic, and a(t), b(t) are positive and continuous, then by Theorem 3.1 there exists a Nash equilibrium in closed-loop strategies which is independent of X.

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