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THE TRANSFER PROBLEM UNDER UNCERTAINTY: THE EXISTENCE OF PARETO-IMPROVING TRANSFERS

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ABSTRACT

This paper examines the effect of a unilateral transfer on the welfare of two countries under uncertainty. The traditional welfare effects are summarized and extended for a pure exchange economy with complete contingent claims markets. It is demonstrated that the effects of a transfer in such an economy is isomorphic to the effects in the traditional certainty case where a unilateral transfer always decreases the welfare of the transferor and increases that of the transferee. Further, in the absence of a complete set of markets, examples are exhibited in which a unilateral transfer increases the welfare of both countries.

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I. INTRODUCTION

Upon conclusion of the first World War, economists discussed the effects of a unilateral transfer -- such as German war reparations -- on the terms of trade. It was argued that any increase in unilateral payments would most likely shift the terms of trade against the paying country, and that any reduction in its unilateral payments would probably shift the terms of trade in its favor.[1]

Jones [1970] suggested that at the initial prices the real income loss represented by the transfer, the "primary" effect, might well be mitigated by the "secondary" effect of an improvement in the terms of trade in favor of the transferring country. Even in this case, however, it was recognized that though the secondary effect may indeed mitigate the income loss due to the transfer, the change in the terms of trade could not compensate for the full loss of real income.[2]

Under uncertainty, the effect of a transfer is not as clear. Fries [1979] has examined a model in which a finite number of "states of nature" may occur at the end of the period, but prior to discovering which state of nature has resulted, the two countries may trade securities at the beginning of the period. The existence of these trading opportunities permits both countries to shift their future income, at least partially, across the states of nature. When a transfer payment is made, the two countries will alter their portfolio holdings in such a way that the transferee's terms of trade may worsen in some states of nature, but improve in others. Fries has included production in his model and by concentrating solely on this "secondary" effect has left unanswered the question of whether or not the change in the terms of trade will ever dominate the initial loss in the paying country's real income due to the transfer under uncertainty.

In this paper we will concentrate on a pure exchange economy. This is because there currently does not exist a general theory of producer behavior when markets are incomplete. Further, since production in models with incomplete markets may lead to further inefficiences, we wish to examine cases in which one "rational expectations equilibrium" (before a transfer) is Pareto dominated by another "rational expectations equilibrium" (after a transfer) even when there is no production.

Employing a model of trade under uncertainty we will examine the effect of the transfer on the welfare of the two countries. We will show that given a complete contingent claims market the effect of a transfer will be to decrease the transferor's welfare and increase the transferee's welfare, regardless of the shift in the terms of trade, so long as the determinant of the Jacobian of excess supply functions is positive. In the usual two commodity model, this condition requires that the demand for the nonnumeraire commodity

is a decreasing function of the relative price.

If, however, a complete contingent claims market does not exist, we will exhibit examples in which the transferor's initial welfare loss may be more than offset by the change in terms of trade across states of nature.[3] Furthermore, it will be demonstrated that there may exist Pareto-improving transfers which increase the welfare of both countries as a result of the presence of security markets which enables a new equilibrium to be reached in which a more desirable distribution of income across states of nature can be attained.

Examples to follow will demonstrate that the welfare of the transferor may be increased in two instances. The first such example suggests that the effect of a transfer can be isomorphic to the opening of additional markets, and as such may improve the welfare of both countries by expanding trade opportunities. The second example reflects upon the fact that in economies with an incomplete set of markets, multiple equilibria are likely to exist. In these cases there may well exist equilibria which improve the welfare of the transferor relative to some of the equilibria which were attainable prior to the transfer. Further, there may also exist transfers such that the welfare of both countries may be improved.

II. A MODEL WITH COMPLETE CONTINGENT CLAIMS MARKETS

Consider a two-country model and suppose there are two goods, $x(s) = (x_1(s), x_2(s)) \varepsilon R^2_+$, and that there is one period at the end of which S distinct states of the world may occur, s = 1,...,S. We shall refer to country 1 as the home country and country 2 as the foreign country. Each country will be assumed to have preferences over consumption at the end of the period which are representable by the von Neumann-Morgenstern utility functions $V^i : R_+^2 \rightarrow R,\ i = 1,2.$ It will be assumed that each utility function is concave and twice differentiable.

Suppose further that each country receives an endowment of goods upon conclusion of the period in each state s, $w^i(s) = (w_1^i(s), w_2^i(s)) \in R_+^2$. The home country will be assumed to transfer a quantity of good 1 to the foreign country at the end of the period in state S. We shall designate this transfer as $\lambda t^1(s) = -\lambda t^2(s)$.

Now, taking good 1 in each state as the numeraire commodity, we assume that each country acts to

$$\underset{x_1^i,\ x_2^i}{\text{maximize}} \quad EV^i = \sum_S \pi^i(s)V^i(x_1^i(s),\ x_2^i(s)) \tag{1}$$

$$\text{subject to} \quad x_1^i(s) + p(s)x_2^i(s) \leq w_1^i(s) + p(s)w_2^i(s) + \lambda t^i(s)$$

$$s = 1,\ldots,S.$$

where $\pi^i(s)$ is country i's subjective probability distribution on states of the world $s = 1,\ldots,S$, and $p(s) \in R_+$ is the terms of trade expressed as the ratio of the price of good 2 to good 1 in state s. It is easily discernable that the maximization problem (1) for all states simultaneously may be decomposed into S separate problems in which each utility function, $V^i(\cdot)$, is maximized for each of the s states independently. Thus, conditional upon the occurrence of any state, s, this model is identical to the usual certainty model.

Now assume that at the initiation of the trading period there exists pre-state contingent claims markets for both goods in all states $s = 1,\ldots,S$. Further, assume there are post-state spot markets for all states, and that no other securities exist. This regime is the familiar Arrow-Debreu system of complete contingent markets. The equivalence of such complete contingent markets with the traditional certainty structures with 2S commodities will enable us to derive results for the uncertainty case similar to those discussed in the previous section for the traditional certainty case. Let $q(s) \in \mathbf{R}_+^2$ be the prices in the pre-state market contingent on state s. Define $z^i(s) = (z_1^i(s),\ z_2^i(s)) \in R^2$ as the amount of goods 1 and 2, respectively, purchased by country i in the pre-state markets for delivery if state s occurs.

We will assume the existence of a rational expectations equilibrium which is defined formally as a set of spot prices $p(s)$, $s = 1,\ldots,S$, contingent prices, $q(s)$, $s = 1,\ldots,S$, and a set of individual consumption and contingent trades $(x^i,\ z^i)$ such that

(i) $\sum_i x_j^i(s) = \sum_i w_j^i(s)$ for each $j = 1,2,\ s = 1,\ \ldots,S;$

(ii) $\sum_i z_j^i(s) = 0$ for each $j = 1,2,\ s = 1,\ldots,S;$

(iii) For each i, $((x^i(s),\ s = 1,\ldots,S),\ (z^i))$

$$\underset{x^i,\ z^i}{\text{maximizes}} \quad EV^i = \sum_s \pi^i(s)V^i(x^i(s)) \tag{2}$$

subject to the budget constraints:

$$x_1^i(s) + p(s)x_2^i(s) \leq w_1^i(s) + p(s)w_2^i(s) + z_1^i(s) + p(s)z_2^i(s) + \lambda t^i(s)$$

$$\sum_j \sum_s q_j(s)z_j^i(s) \leq 0$$

$$x^i(s) \geq 0 \quad s = 1,\ldots,S.$$

In the Arrow-Debreu theory of general equilibrium, it is assumed that markets exist for current goods and also for contingent futures goods. Under this assumption, and assuming no transaction costs, all economic decisions may be made at one time and markets need open only once. In this case, the budget constraints in the maximization problem (2) may be combined into the single constraint

$$\textstyle\sum_s r_1(s)x_1^i(s) + \sum_s r_2(s)x_2^i(s) = \sum_s r_1(s)\{w_1^i(s) + \lambda t^i(s)\} + \sum_s r_2(s)w_2^i(s) \quad (3)$$

where $r_1(s) = q(s)$ and $r_2(s) = q(s)p(s)$ are the contingent claim prices.[4]

To find the welfare effects of the transfer, we differentiate each country's expected utility function in equilibrium with respect to λ. Employing the envelope theorem, and evaluating the derivative at $\lambda = 0$ gives

$$\partial EV^i/\partial\lambda|_{\lambda=0} = \mu^i\{\sum_s(x_1^i(s) - w_1^i(s))\partial r_1(s)/\partial\lambda + \sum_s(x_2^i(s) - w_2^i(s))\partial r_2(s)/\partial\lambda + \sum_s r_1(s)t^i(s)\} \quad (4)$$

where μ^i is the marginal utility of income of country i.

In the appendix we prove the following proposition: that if complete contingent claims markets exist and if, in equilibrium, the determinant of the Jacobian of excess supply functions is positive, then any transfer decreases the welfare of the transferor and increases the welfare of the transferee. This proposition demonstrates that a transfer under uncertainty in a traditional Arrow-Debreu system of complete contingent claims markets is similar to a transfer under certainty in that the "primary" welfare loss due to transfer indeed dominates the "secondary" effects of a change in the terms of trade.

To illustrate this proposition we provide the following example which will also form a basis for the other examples to be constructed in the remainder of the paper.

<u>Example 1</u>

Suppose there are two goods, x_1 and x_2, and that there is one period at the end of which two distinct states of the world may occur. Assume that a complete set of contingent claim markets exist at the beginning of the period. Assume further that the home and foreign countries' preferences are represented by the following von Neumann-Morgenstern utility functions;

$$V^1 = 3/4 \log x_1^1 + 3/2 \log x_2^1$$

$$V^2 = 3/2 \log x_1^2 + 3/4 \log x_2^2.$$

Each country shares the belief that state one will occur with probability 2/3 and that state two will occur with probability 1/3 so that the expected utility functions are;

$$EV^1 = 2/3(3/4 \log x_1^1(1) + 3/2 \log x_2^1(1)) + 1/3(3/4 \log x_1^1(2) + 3/2 \log x_2^2(2))$$

$$EV^2 = 2/3(3/2 \log x_1^2(1) + 3/4 \log x_2^2(1)) + 1/3(3/2 \log x_1^2(2) + 3/4 \log x_2^2(2)).$$

Upon the occurrence of state one the home country will receive the endowment represented by the vector $w^1(1) = (90, 40)$. If state two occurs the home country receives the endowment vector $w^2(2) = (90, 10)$. The foreign country's endowment vectors are given by $w^2(1) = (30, 80)$ and $w^2(2) = (30, 110)$. With complete contingent claims markets the home country maximizes EV^1 subject to

$$r_1(1)x_1^1(1) + r_2(1)x_2^1(1) + r_1(2)x_1^1(2) + r_2(2)x_2^1(2)$$

$$\leq 90r_1(1) + 40r_2(1) + 90r_1(2) + 10r_2(2)$$

$$x(1),\ x(2) \geq 0$$

and the foreign country maximizes EV^2 subject to

$$r_1(1)x_1^2(1) + r_2(1)x_2^2(1) + r_1(2)x_1^2(2) + r_2(2)x_2^2(2)$$

$$\leq 30r_1(1) + 80r_2(1) + 30r_1(2) + 110r_2(2)$$

$$x^2(1),\ x^2(2) \geq 0.$$

The equilibrium market-clearing conditions are

$$x_h^1(s) + x_h^2(s) = 120 \quad \text{for } h = 1,2 \text{ and } s = 1,2.$$

It is easy to show that the solution to this equilibrium is

$$r_1(1) = r_2(1) = 1;\ r_1(2) = r_2(2) = 1/2;\ x^1(1) = x^1(2) = (40,\ 80),$$
$$x^2(1) = x^2(2) = (80,\ 40)$$

and since the utility functions give rise to demand functions satisfying the gross substitutes assumption, this equilibrium is unique. Further, solving for $p(1)$ and $p(2)$ gives $p(1) = p(2) = (1, 1)$. In equilibrium, the value of each country's utility function is

$$EV^1 = EV^2 = 3/4 \log 40 + 3/2 \log 80 = 9.3397.$$

It may also be readily verified that, in equilibrium, the determinant of the Jacobian of the excess supply functions is positive.

Now suppose the home country transfers five units of good 1 in state 1 to the foreign country. Thus, the problem remains the same as that given above except that the home country's endowment vector in state 1 is now $w^1(1) = (85, 40)$ and the foreign country's endowment vector in state 1 becomes $w^2(1) = (35, 80)$. Setting up and solving the problem as before yields

$$x^1(1) = x^1(2) = \left(\frac{2440}{63}, \frac{2440}{31}\right), \quad x^2(1) = x^2(2) = \left(\frac{5120}{63}, \frac{1280}{31}\right)$$

and

$$r_1(1) = 1,\ r_2(1) = \frac{62}{63},\ r_1(2) = 1/2,\ r_2(2) = \frac{31}{63}.$$

Solving for the terms of trade, $p(1)$ and $p(2)$, gives $p(1) = p(2) = \left(1, \frac{62}{63}\right)$, and the value of the home and foreign countries' utility functions in equilibrium are

$$EV^1 = 9.2911 \quad \text{and} \quad EV^2 = 9.3872.$$

Thus, as was to be shown, after the transfer the welfare of the home country decreases while the welfare of the foreign country increases.

III. A MODEL WITH SECURITIES MARKETS

It has been shown thus far that the effects of a transfer in a complete contingent market is isomorphic to the effects of a transfer in the traditional certainty market with 2S commodities. However, in the real world few markets for contingent futures goods exist at any one time, and markets for such goods often reopen many times. Two main approaches reflecting the incomplete and sequential aspects of real world trading have been taken: the temporary equilibrium approach[5] and the rational expectations approach. The rational expectations approach regards expectations of future prices as variables and investigates whether there exists a set of current prices and expected prices such that all markets, both current and future, are cleared.[6]

Consider now a securities regime where at the initiation of the trading period there exists H securities over which each country may trade. Each security, h, returns an amount $a_j^h(s)$ of good j if state of the world s occurs. Let A^h represent the S x 2 matrix of returns, $a_j^h(s)$, from the securities.

As in the traditional certainty model, we wish to examine the effect of the transfer upon the terms of trade, $p(s)$ in each state $s = 1,\ldots,S$. In the model presented here, however, the terms of trade play an additional role; namely they determine the monetary returns,

$$\alpha^h(s) = a_1^h(s) + p(s)a_2^h(s)$$

from security h in state s. In turn, these monetary returns define each country's ability to shift its income across states of nature.

The assumption of rational expectations implies that both countries have "correct" self-fulfilling expectations with regard to the equilibrium distribution of prices which will prevail. Formally, a rational expectations equilibrium is defined as a set of spot prices, p(s), s = 1,...S, security prices, q_h, h = 1,...H consumption vectors, $x^i(s)$, s = 1,...,S, i = 1,2, and vectors of portfolio holdings z^i_h, i = 1,2 such that

$$\sum_i x^i(s) = \sum_i w^i(s) \quad \text{for every } s = 1,\ldots,S$$

$$\sum_i z^{ih} = 0 \qquad \text{for every } h = 1,\ldots H.$$

For each i = 1,2 $\{(x_i(s), s = 1,\ldots S)(z^{ih}, h = 1,\ldots,H)\}$ maximizes $EV^i = \sum_s \pi^i(s)U^i(x^i(s))$ subject to the budget constraints

$$x^i_2(s) + p(s)\ x^i_1(s) \leq p(s)\ w^i_1(s) + w^i_2(s) + \sum_h \alpha^h(s)z^{ih} + \lambda t^i(s)$$

$$\sum_h q_h z^{ih} \leq 0.$$

We will not consider the existence of a rational expectation equilibrium here. Hart [1975] has previously provided an example of nonexistence in this framework. Radner [1972] shows that existence can be assured in this model if security trading plans are constrained to satisfy

$$z^{ih} \leq L,\ i = 1,2,\ h = 1,\ldots,H$$

where L is a fixed positive number. The difficulty with imposing such a constraint is that the choice of L is arbitrary and the equilibrium will in general depend upon L. Following Hart, we will not impose a constraint on security plans and note that the equilibrium defined here may not exist. Thus, the results obtained here will not be a consequence of imposing an artificial constraint but instead will follow from the market sturctures we impose.

There are two types of market structures which will lead to "unorthodox effects" as a result of a transfer. Both result from the effect of the terms of trade on the vectors of monetary returns present in the economy. In the first type of structure we wish to examine, the effect of the transfer is that it shifts the terms of trade in such a way as to change the number of linearly independent vectors of monetary returns. By doing so, both countries ability to trade income across states of nature are altered and as such it is possible that the total effect of a transfer may either increase or decrease the welfare of both countries. We now demonstrate this by means of the following example.

Example 2

The following example is identical to Example 1, except that we replace the contingent claims market with a securities market. Assume that there are two securities available at the beginning of the period with return structures represented by

$$A^1 = \begin{bmatrix} 20 & 20 \\ 20 & 20 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} \frac{290}{7} & 18 \\ \frac{164}{7} & 36 \end{bmatrix}$$

and possessing monetary returns

$$\alpha^2 = \begin{bmatrix} 20(1 + p_2(1)) \\ 20(1 + p_2(2)) \end{bmatrix} \quad \text{and} \quad \alpha^2 = \begin{bmatrix} \frac{290}{7} + 18p_2(1) \\ \frac{164}{7} + 36p_2(2) \end{bmatrix}.$$

If the vectors of monetary returns are linearly independent then the market structure is equivalent to one with a complete contingent claims market. From Example 1, however, we know that in this case $p(1) = p(2) = 1$ and thus

$$\alpha^1 = \begin{bmatrix} 40 \\ 40 \end{bmatrix} \quad \text{and} \quad \alpha^2 = \begin{bmatrix} \frac{416}{7} \\ \frac{416}{7} \end{bmatrix}.$$

Since these are linearly dependent, a complete market equilibrium does not exist.

Alternatively, if the vectors of monetary returns are linearly dependent, then the market structure is equivalent to one with no contingent claims market. Thus there is no way of trading wealth in state 1 for wealth in state 2 and we may assume without loss of generality that no trading takes place at the beginning of the period.

If state 1 occurs at the end of the period, the home country maximizes $3/4 \log x_1^1(1) + 3/2 \log x_2^1(1)$ subject to

$$p(1) \cdot x^1(1) \geq 90p_1(1) + 40p_2(1),\ x^1(1) \geq 0,$$

and the foreign country maximizes $3/2 \log x_1^2(1) + 3/4 \log x_2^2(1)$ subject to

$$p(1) \cdot x^2(1) \leq 30p_1(1) + 80p_2(1),\ x^2(1) \geq 0.$$

The equilibrium condition is $x_h^1(1) + x_h^2(1) = 120$ for $h = 1,2$ and the unique equilibrium is given by $x^1(1) = \left(44, \frac{1760}{21}\right)$, $x^2(1) = \left(76, \frac{760}{21}\right)$ and $p(1) = \left(1, \frac{21}{20}\right)$.

If state 2 occurs, the home country maximizes $3/4 \log x_1^1(2) + 3/2 \log x_2^1(2)$ subject to

$$p(2) \cdot x^1(2) \leq 90p_1(2) + 10p_2(2), x^1(2) \geq 0$$

and the foreign country maximizes $3/2 \log x_1^2(2) + 3/4 \log x_2^2(2)$ subject to

$$p(2) \cdot x^2(2) \leq 30p_1(2) + 110p_2(2),\ x^2(2) \geq 0.$$

The equilibrium condition is $x_h^1(2) + x_h^2(2) = 120$ for $h = 1,2$ and the unique equilibrium is given by $x^1(2) = \left(\frac{760}{23}, \frac{1520}{21}\right)$, $x^2(2) = \left(\frac{2000}{23}, \frac{1000}{21}\right)$ and $p(2) = \left(1, \frac{21}{23}\right)$.

It follows that the monetary returns are,

$$\alpha^1 = \begin{bmatrix} 41 \\ \\ \frac{880}{23} \end{bmatrix} \quad \text{and} \quad \alpha^2 = \begin{bmatrix} \frac{4223}{70} \\ \\ \frac{9064}{161} \end{bmatrix}$$

and since $\alpha^1 = \frac{70}{103}\alpha^2$ the monetary return vectors have remained linearly dependent. Thus, this is the only equilibrium before the transfer, and in equilibrium, the values of the utility functions of the two countries are given by

$$EV^1 = 1/2 \log 44 + \log \frac{1760}{21} + 1/4 \log \frac{760}{23} + 1/2 \log \frac{560}{13} = 9.0766$$

and (5)

$$EV^2 = \log 76 + 1/2 \log \frac{760}{21} + 1/2 \log \frac{2000}{23} + 1/4 \log \frac{1000}{21} = 9.3236.$$

As in Example 1, suppose the home country transfers 5 units of good 1 in state 1 to the foreign country. We wish to show that the welfare of both countries is increased. The problem remains the same as that above except that the home country's endowment vector in state 1 is now $w(1) = (85,40)$ and the foreign country's endowment vector in state 1 becomes $w^2(1) = (35,80)$.

Assuming that the vectors of monetary returns are linearly dependent we may set up and solve the problems as before. This yields the solution

$$x^1(1) = \left(42, \frac{3360}{41}\right),\ x^2(1) = \left(78, \frac{1560}{41}\right),\ p(1) = \left(1, \frac{41}{40}\right),$$

$$x^1(2) = \left(\frac{760}{23}, \frac{1520}{21}\right),\ x^2(2) = \left(\frac{2000}{23}, \frac{1000}{21}\right),\ p(2) = \left(1, \frac{21}{23}\right),$$

so that the vector of monetary returns are

$$\alpha^1 = \begin{bmatrix} \frac{81}{2} \\ \\ \frac{880}{23} \end{bmatrix} \quad \text{and} \quad \alpha^2 = \begin{bmatrix} \frac{8383}{140} \\ \\ \frac{9064}{161} \end{bmatrix}$$

and it may be verified that these two vectors are linearly independent so that this equilibrium will not exist.

Alternatively, if the vectors of monetary returns are linearly independent then the market structure is equivalent to one with a complete contingent claims market. From Example 1, we know that $p(1) = p(2) = \left(1, \frac{62}{63}\right)$ so that

$$\alpha^1 = \begin{bmatrix} \frac{2500}{63} \\ \\ \frac{2500}{63} \end{bmatrix} \quad \text{and} \quad \alpha^2 = \begin{bmatrix} \frac{4726}{63} \\ \\ \frac{3708}{63} \end{bmatrix}$$

and it may be verified that these two vectors are linearly independent. Thus, from Example 1,

$$EV^1 = 9.2911 \text{ and } EV^2 = 9.3872. \qquad (6)$$

Comparing the value of the utility functions in equation (6) in equilibrium after the transfer, with the value of the utility functions in equation (5) in equilibrium prior to the transfer demonstrates the increase in both countries welfare.

This example serves to illustrate that trading opportunities depend upon the expected prices. Before the transfer, α^1 and α^2 are linearly dependent and the trading opportunities are the same as they would be if no securities market existed. After the transfer, α^1 and α^2 are linearly independent and the trading opportunities are the same as they would be if a complete set of contingent claim markets existed at the beginning of the period. Thus, the transfer has the effect of opening a new market which completes the set of markets and the gains realized from trading wealth in state 1 for wealth in state 2 exceeds the losses due to the transfer. Prior to the transfer the home country would have liked to have traded some of its state 1 income for state 2 income but is unable to do so since a complete market equilibrium fails to exist. After the transfer, a complete market equilibrium does exist and the home country may now trade income in state 1 for income in state 2 (although to a lesser extent than it would have previously). The gains from doing so then result in more than offsetting the initial effect of the transfer. It should be

realized that this result is sensitive to the size of the transfer since it may be verified that if the home country transfers ten units of good one in state one then in the new equilibrium it will be worse off than it was prior to the transfer, while the welfare of the foreign country improves.

This example does not appear in any way pathological and it is generally true that given the preferences and endowments of two countries, if the terms of trade shift as a result of a transfer, and if there exists a security structure such that before the transfer the securities are linearly dependent and after the transfer they become linearly independent, then there will exist some transfer such that the welfare of both countries is increased.

Indeed, even if there exists as many securities as states, the existence of a complete market representation depends upon the prices which result. In fact, if the security structures reflect the terms of trade prior to the transfer, the vectors of monetary returns are always linearly dependent in the intial equilibrium. For example, if we had return structures represented by

$$A^1 = \begin{bmatrix} -p(1) & 0 \\ -p(2) & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

we would define the monetary returns

$$\alpha^1 = \begin{bmatrix} -p(1) \\ -p(2) \end{bmatrix} \qquad \alpha^2 = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}.$$

After the transfer if the terms of trade shift to $p'(s)$, $s = 1,2$, the monetary returns become

$$\alpha^1 = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix} \quad \text{and} \quad \alpha^2 = \begin{bmatrix} p'(1) \\ p'(2) \end{bmatrix}.$$

Thus, if the transfer does not shift all prices proportionality, it will increase the number of linearly independent monetary returns, and thus is isomorphic to the opening of a new market. One caveat that must be noted here however is that unless the effect of the transfer is to open all markets, then we cannot be in general assured that it will be a Pareto improvement. In fact, if the markets remain incomplete, then both countries welfare may decrease (for an example of this see Hart [1975] and Bhattacharya [1979]).

It should also be noted that Example 2 may also be used to illustrate a transfer which results in a Pareto inferior allocation. Namely, if we begin at the final equilibrium and transfer 5 units of good 1 from the foreign country to the home country, we will move to the initial equilibrium in that example. In this instance, the welfare of both countries is decreased.

The second type of market structure we wish to examine is one in which the set of linearly independent securities remain the

same both before and after the transfer but there are fewer securities than states of nature. The following example examines the welfare effects in an economy with an incomplete set of security markets. It is well known that such security markets have multiple equilibria each possessing a distinct contingent claims representation. In this instance there may exist an equilibrium after the transfer that Pareto dominates the earlier (pre-transfer) equilibrium, as the following example demonstrates.

<u>Example 3</u>

Suppose now that there are three states of the world which may occur at the end of the period. The home country's preferences are represented by the von Neumann-Morgenstern utility function $1/2 \log x_1^1 + \log x_2^1$ and the foreign country's preferences are represented by the von Neumann-Morgenstern utility function $\log x_1^2 + 1/2 \log x_2^2$. Each country has the probability beliefs, 1/3, 1/6, and 1/2 that states 1, 2 and 3 occur respectively, so that

$$EV^1 = 1/6 \log x_1^1(1) + 1/3 \log x_2^1(1) + 1/12 \log x_1^1(2) + 1/6 \log x_2^1(2) + 1/4 \log x_1^1(3) + 1/2 \log x_2^1(3)$$

and

$$EV^2 = 1/3 \log x_1^2(1) + 1/6 \log x_2^2(1) + 1/6 \log x_1^2(2) + 1/12 \log x_2^2(2) + 1/2 \log x_1^2(3) + 1/4 \log x_2^2(3).$$

There are two securities available at the beginning of the period with return structures given by

$$A^1 = \begin{bmatrix} 1 & 0 \\ 10 & 0 \\ 6 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} \frac{19}{5} & 0 \\ 26 & \frac{92}{5} \\ 26 & 0 \end{bmatrix}$$

The home country's endowment vectors are $w^1(1) = (90, 40)$ $w^1(2) = (90, 10)$ and $w^1(3) = (40, 20)$ and the foreign country's are $w^2(1) = (30, 80)$ $w^2(2) = (30, 110)$ and $w^2(3) = (80, 100)$. We will show that there is an equilibrium such that both countries can only trade income in state 1 for income in state 2. If this is the case then this is the equivalent of the existence of contingent claims markets for states 1 and 2 only. Thus, both countries maximize the expected utility of states 1 and 2 subject to a single budget constraint. That is, the home country

maximizes $1/6 \log x_1^1(1) + 1/3 \log x_2^1(1) + 1/12 \log x_1^1(2) + 1/6 \log x_2^1(2)$

subject to $r_1(1)x_1^1(1) + r_2(1)x_2^1(1) + r_1(2)x_1^1(2) + r_2(2)x_2^1(2)$

$\leq 90r_1(1) + 40r_2(1) + 90r_1(2) + 10r_2(2)$

$x^1(1),\ x^1(2) \geq 0$

and the foreign country

maximizes $1/3 \log x_1^2(1) + 1/6 \log x_2^2(1) + 1/6 \log x_1^2(2) + 1/12 \log x_2^2(2)$

subject to $r_1(1)x_1^2(1) + r_2(1)x_2^2(1) + r_1(2)x_1^2(2) + r_2(2)x_2^2(2)$

$\leq 30r_1(1) + 80r_2(1) + 30r_1(2) + 110r_2(1)$

$x^2(1),\ x^2(2) \geq 0.$

The equilibrium conditions are thus

$$x_h^1(s) + x_h^2(s) = 120 \quad \text{for } h = 1,2 \text{ and } s = 1,2.$$

Since the utility functions of both countries are 1/3 of what they were in the first example, it is easily seen that this is the same problem as the before transfer case in that example. Thus,

$$x^1(1) = x^1(2) = (40, 80),\ x^2(1) = x^2(2) = (80, 40), \text{ and}$$

$$p(1) = p(2) = (1,1).$$

For state 3, the home country

maximizes $1/4 \log x_1^1(3) + 1/2 \log x_2^1(3)$

subject to $p_1(3)x_1^1(3) + p_2(3)x_2^1(3) \leq 40p_1(3) + 20p_2(3),\ x^1(3) \geq 0,$

and the foreign country

maximizes $1/2 \log x_1^2(3) + 1/4 \log x_2^2(3)$

subject to $p_1(3)x_1^2(3) + p_2(3)x_2^2(3) \leq 80p_1(3) + 100p_2(3),\ x^2(3) \geq 0$

where $x_h^1(3) + x_h^2(3) = 120$ for $h = 1,2$, in equilibrium. It may be verified that the solution to the problem is given by

$$x_1^1(3) = \frac{200}{11},\ x_2^1(3) = 50,\ x_1^2(3) = \frac{1120}{11},\ x_2^2(3) = 70$$

$$p(3) = (1, \frac{8}{11}).$$

Furthermore, these prices and consumption allocations may be shown to be an equilibrium to the economy with the two securities given above, where the equilibrium security holdings are

$$z_1^1 = -\frac{325}{4},\ z_2^1 = \frac{75}{4},\ z_1^2 = \frac{325}{4},\ z_2^2 = -\frac{75}{4}$$

and the security prices are $q_1 = 6$ and $q_2 = 26$. Thus, in the equilibrium the utility levels of both countries are

$$EV^1 = 1/2\left(1/2 \log 40 + \log 80\right) + 1/2\left(1/2 \log \frac{200}{11} + \log 50\right) = 5.7943$$

and

$$EV^2 = 1/2\left(\log 80 + 1/2 \log 40\right) + 1/2\left(\log \frac{1120}{11} + 1/2 \log 70\right) = 6.4869.$$

As before, suppose the home country transfers 5 units of good one in state 1 to the foreign country. We now wish to show that there is an equilibrium such that both countries can only trade income in state 1 for income in state 3. Now both countries maximize the expected utility of states 1 and 3 subject to a single budget constraint. The home country

maximizes $\quad 1/6 \log x_1^1(1) + 1/3 \log x_2^1(1) + 1/4 \log x_1^1(3) + 1/2 \log x_2^1(3)$

subject to $\quad r(1)x^1(1) + r(3)x^1(3) \leq 85r_1(1) + 40r_2(1) + 40r_1(3) + 20r_2(3)$

$$x^1(1),\ x^1(3) \geq 0$$

Similarly the foreign country

maximizes $\quad 1/3 \log x_1^2(1) + 1/6 \log x_2^2(1) + 1/2 \log x_1^2(3) + 1/4 \log x_2^2(3)$

subject to $\quad r(1)x^2(1) + r(3)x^2(3) \leq 35r_1(1) + 80r_2(1) + 80r_1(3) + 100r_2(3)$

$$x^2(1),\ x^2(3) \geq 0$$

where $\quad x_h^1(s) + x_h^2(s) = 120$, for $h = 1, 2$, $s = 1,3$ in equilibrium.

This equilibrium is given by

$$x_1^1(1) = x_1^1(3) = \frac{1440}{53},\ x_2^1(1) = x_2^1(3) = \frac{5760}{89}$$

$$x_1^2(1) = x_1^2(3) = \frac{4920}{53},\ x_2^2(1) = x_2^2(3) = \frac{4920}{89},\ \text{and}$$

$$p(1) = p(3) = (1, \frac{89}{106}).$$

For state 2, the home country

maximizes $\quad 1/12 \log x_1^1(2) + 1/6 \log x_2^1(2)$

subject to $\quad p_1(2)x_1^1(2) + p_2(2)x_2^1(2) \leqq 90p_1(2) + 10p_2(2),\ x^1(2) \geq 0,$

and the foreign country

maximizes $\quad 1/6 \log x_1^2(2) + 1/12 \log x_2^2(2)$

subject to $\quad p_1(2)x_1^2(2) + p_2(2)x_2^2(2) \leqq 30p_1(2) + 110p_2(2),\ x^2(2) \geq 0.$

Again, $x_h^1(2) + x_h^2(2) = 120$, for h = 1,2 in equilibrium. The equilibrium is given by

$$x_1^1(2) = \frac{760}{23},\ x_2^1(2) = \frac{150}{21},\ x_1^2(2) = \frac{2000}{23},\ x_2^2(2) = \frac{1000}{21},\ p(2) = (1, \frac{21}{23}).$$

These prices and consumption allocations may also be shown to be an equilibrium to the economy with the two securities given above, where the equilibrium security holdings are

$$z_1^1 = -\frac{5(14017)}{4(53)},\ z_2^1 = \frac{5(3275)}{4(53)},\ z_1^2 = \frac{5(14017)}{4(53)},\ z_2^2 = -\frac{5(3275)}{4(53)}$$

and the security prices are $q_1 = 10$ and $q_2 = \frac{214}{5}$. In this equilibrium the utility levels of both countries are

$$EV^1 = 5/6\left[1/2 \log \frac{1440}{53} + \log \frac{5760}{89}\right] + 1/6\left[1/2 \log \frac{760}{23} + \log \frac{1520}{21}\right]$$
$$= 5.8561.$$

and

$$EV^2 = 5/6\left[\log \frac{4920}{53} + 1/2 \log \frac{4920}{89}\right] + 1/6\left[\log \frac{2000}{23} + 1/2 \log \frac{1000}{21}\right]$$
$$= 6.5135,$$

which Pareto dominates the equilibrium which we characterized before this transfer was made. Thus, the transfer acts as a vehicle through which, in an economy with an incomplete market structure, the welfare of both countries is improved.

Again a caveat is necessary. It has been demonstrated that there exists an equilibrium after a transfer which Pareto dominates an equilibrium attainable prior to the transfer. It should be noted here that such equilibria after transfer may indeed be Pareto inferior to some equilibria prior to the transfer.

VII. CONCLUSION

We have shown that the effect of a transfer when there are incomplete opportunities for risk sharing differs markedly from both the usual certainty case and the complete markets case. In particular, the transfer may increase or decrease the welfare of either country depending upon its effect on each country's opportunity set. In a sequel, we will examine opportunities for Pareto-improving transfer when production is included in this model. This will have the further effect that the returns on securities will become endogenous since by introducing a market for shares in firms will tie the return structure to each firm's profit structure.

Appendix

We wish to evaluate the sign of the derivative in (4) to determine the effect of a transfer on the welfare of each country when the set of markets are complete. In this case, there are 2S goods, which in this section we will index by j = 1,...,2S. We will also need to examine the Jacobian of the market excess supply function in equilibrium which is defined by

$$E(r) = \left[e_{jk}(r)\right] = \left[\frac{\partial E_j(r)}{\partial r_k}\right]$$

where $E_j(r)$ is the market excess supply function for good j. Thus, we have in equilibrium

$$E_j(r;M^1, M^2) = \sum_i (w_j^i - x_j^i) = 0, \quad j = 1,\ldots,2S - 1. \tag{A.1}$$

where, using Walras' law, we have omitted $E^{2S}(r)$. From Slutsky's equation we know that

$$E(r) = -\sum_{i=1}^{2} S^i(r) + \sum_{i=1}^{2} I^i(r) \cdot [x^i - w^i] \tag{A.2}$$

where the first term is the sum of positive definite matrices of substitution effects and the second term is the sum of income effects. Since, in equilibrium, $x^1 - w^1 = w^2 - x^2$, (A.2) may be written as

$$E(r) = S(r) + I(r) \cdot [x^1 - w^1]',$$

where

$$S(r) = \sum_{i=1}^{2} S^i(r) \quad \text{and} \quad I(r) = I^1(r) - I^2(r).$$

For future use, we state the following lemma:

Lemma (Graybill [1969], Theorem 8.9.3) Since S(r) is nonsingular, then

$$\det E(r) = |E(r)| = |S(r)|(1 + [x^1 - w^1]' \cdot [S(r)]^{-1} \cdot I(r)) \tag{A.3}$$

and

$$[E(r)]^{-1} = [S(r)]^{-1} - \frac{[S(r)]^{-1} \cdot I(r) \cdot [x^1 - w^1]' \cdot [S(r)]^{-1}}{(1 + [x^1 - w^1]' \cdot [S(r)]^{-1} \cdot I(r))} \tag{A.4}$$

We may now proceed to demonstrate our proposition, namely

Proposition: Consider an economy with a complete set of markets. If E(r) has a positive determinant in equilibrium then a transfer decreases the welfare of the transferor and increases the welfare of the transferee.

Proof: From (4), we need to evaluate the sign of

$$\partial EV^i/\partial\lambda|_{\lambda=0} = \mu^i\{[w^i - x^i] \cdot [\partial r/\partial\lambda] + \sum_s r_i(s)t^i(s)\} \tag{A.5}$$

The change in the contingent claim prices as a result of the transfer,

$[\partial r/\partial\lambda]$, is found by differentiating the system of excess supply functions, (A.1), to obtain

$$[\partial E_j/\partial r]\cdot[\partial r/\partial\lambda] + \sum_i \left(\partial E_j^i/\partial M^i\right)\left(\partial M^i/\partial\lambda\right) = 0, \quad j = 1,\ldots,2S-1. \tag{A.6}$$

where, $M^i = \sum_s \lambda r_i(s)t^i(s)$, is the amount of the transfer and

$$\partial M^1/\partial\lambda = \sum_s r_1(s)t^1(s) \text{ and } \partial M^2/\partial\lambda^2 = \sum_s r_1(s)t^2(s) = -\sum_s r_1(s)t^1(s)$$

since $t^1(s) = -t^2(s) < 0$. The system of equations, (A.6), thus may written in a matrix notation as

$$\begin{aligned} E(r)\cdot[\partial r/\partial\lambda] &= \{[\partial E^1(r)/\partial M^1] - [\partial E^2(r)/\partial M^2]\}\sum_s r_1(s)t^1(s) \\ &= \{I^1(r) - I^2(r)\}\sum_s r_1(s)t^1(s) \qquad \text{(A.7)} \\ &= I(r)\sum_s r_1(s)t^1(s) \end{aligned}$$

since $\partial E^i(r)/\partial M^i$ may be seen to be the change in country i's demands as its income, μ^i, varies. Solving (A.7) for $[\partial r/\partial\lambda]$ and substituting into (A.5) yields

$$\partial EV^i/\partial\lambda|_{\lambda=0} = \mu^i\sum_s r_1(s)t^1(s)\ \{[w^i - x^i]'\cdot[E(r)]^{-1}\cdot I(r) + 1\}$$

Since μ^i, the marginal utility of income, is positive and the transfer, $t^i(s)$,

is negative in each state, the qualitative effect of the transfer on country i's welfare will be of opposite sign of the expression within the braces. To evaluate this expression we use (A.4) from the above lemma to show that for the home country

$$[w^1 - x^1]'\cdot[E(r)]^{-1}\cdot I(r) + 1 = \{1 + (x^1 - w^1)'\cdot[S(r)]^{-1}\cdot I(r)\}^{-1}$$

which, by (A.3) of the lemma becomes

$$[w^1 - x^1]'\cdot[E(r)]^{-1}\cdot I(r) + 1 = |S(r)|/|E(r)|. \tag{A.8}$$

The reader may verify that for the foreign country, this expression is of opposite sign. Thus, since $S(r)$ is positive definite, we know that $|S(r)| > 0$ and since by assumption $|E(r)| > 0$, then the welfare of the transferor has decreased and the welfare of the transferee has increased.

In conclusion, we should also note that this is exactly the condition used by Varian [1975] to show the uniqueness of an equilibrium in a pure exchange economy.

FOOTNOTES

1. Keynes [1929] makes this presumption. On the other hand, Samuelson [1952] concludes that there should be presumption on how the terms of trade should change. However, in a second paper, Samuelson [1954] shows how Keynes' presumption could be restored when certain trade impediments are introduced.

2. See Caves and Jones [1977], pages 456-60.

3. Hart [1975, p. 442] gives an example of a Pareto-improving transfer. However, in his model there are three trading periods with uncertainty entering only in the final period. Further, he requires that one of the goods does not enter one of the country's utility functions.

4. When the set of markets are complete, it follows that, for each $s = 1,\ldots,S$, there exists a set of scalars $q(s)$ such that $r_1(s) = q(s)p_1(s)$ and $r_2(s) = q(s)p_2(s)$. (For a proof of a more general result see Radner [1972].)

5. For a survey of this approach, see Grandmont [1977].

6. This is the approach adopted by Bhattacharya [1979], Fries [1979] and Hart [1975], which we will continue here.

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