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SOCIAL WELFARE FUNCTIONS FOR ECONOMIC ENVIRONMENTS WITH AND WITHOUT THE PARETO PRINCIPLE*

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ABSTRACT:

Social welfare funcitons for private goods economies with classical preferences are considered. It is shown that every social welfare function satisfying a weak nonimposition condition and the independence of irrelevant alternatives axiom is of one of the following forms. It is either null or the class of decisive coalitions is an ultrafilter or the class of anti-decisive coalitions is an ultrafilter.

Introduction

The relevance of Arrow's General Possibility Theorem [1] to economics has been questioned on various grounds. The most telling of these criticisms is that most formulations of the theorem require a social welfare function to be defined for all conceivable profiles of individual preferences. Economists usually work with a much smaller class of preference relations. The set of alternatives is usually taken to be a set of distributions of commodities, and preferences are assumed to be selfish and possess some degree of various monotonocity, smoothness and convexity properties. Arrow [1] addressed the problem of the existence of social welfare functions for these domains, but his results were not satisfactory, as noted by Blau [2]. Recently Kalai, Muller and Satterthwaite [6] and Maskin [8] have proved versions of the General Possibility Theorem for certain economic domains. Kalai-Muller-Satterthwaite prove that any social welfare function satisfying the Pareto principle (i.e., weakly Pareto superior distributions are also socially superior) and May's [9] version of the Independence of Irrelevant Alternatives (IIA), whose domain is the set of all convex, continuous, strictly monotonic preferences over distributions of at least two public goods, must be dictatorial. Maskin deals with the case where all goods are private and preferences are selfish. He requires the social welfare function to satisfy a monotonicity condition which is stronger than IIA, and proves that the social welfare function must be dictatorial.

Both Kalai-Muller-Satterthwaite and Maskin assumed that the society was a finite set of individuals. Fishburn [4] gave an example to show that there exist nondictatorial social welfare functions satisfying the Pareto principle and IIA for infinite societies. Kirman and Sondermann [7] and Hansson [5] have characterized the class of decisive coalitions for such social welfare functions with an unrestricted domain as being ultrafilters, which reduces in the case of finite societies to being dictatorial. A third line of research was pursued by Wilson [10] who dropped the Pareto principle. The reason for being interested in this case is not so much that the Pareto principle is regarded as being unreasonably strong, but that by dropping it the strength of IIA and the group rationality requirements are made more apparent. Wilson's result is basically that a social welfare function with an unrestricted domain satisfying IIA and a very weak nonimposition condition is either null (i.e., all alternatives are always socially indifferent) or dictatorial in one of two senses. The first sense is the usual one, i.e., there is some individual such that the social preference always agrees with her preference. The second sense in which a social welfare function may be dictatorial is that there is some individual such that the social preference is the reverse of hers. Such a social welfare function may more properly be called antidictatorial.

This paper unites these three strands of the theory by characterizing the decisive or antidecisive coalitions of social welfare functions which satisfy IIA and a nonimposition condition somewhat weaker than Wilson's and which have domains satisfying certain rather weak conditions which are satisfied by all the usual sets of "classical" preferences on private goods. The result is that if there are at least two private goods and the social welfare function satisfies IIA and the weak nonimposition condition, then either it is null or either the collection of decisive or antidecisive coalitions is an ultrafilter. Adding the requirement that the Pareto principle be satisfied then forces the collection of decisive coalitions to be an ultrafilter. Hence for finite societies the social welfare function must be dictatorial. Since Maskin's monotonicity condition implies IIA and the Pareto principle implies the weak nonimposition condition used here, this result yields Maskin's as a special case. It also obtains the results of Wilson, Kirman and Sondermann, and Hansson in the case where the alternatives are commodity distributions, but technically their results are not a special case of this one. The reason is that their results apply equally well to finite sets of alternatives, whereas the results here depend on the structure of the set of commodity distributions. Also, the techniques used here do not allow one to deduce the Kalai-Muller-Satterthwaite result for economies with only public goods as they involve constructing distributions in which different agents receive different consumptions. However, if one requires a social ordering to be able to rank infeasible alternatives of the sort where different agents are permitted different consumption of the public goods, then their result follows.

Notation and Definitions

Let $\mathcal{R}(A)$ denote the set of regular preferences[1] on the set A. When the set A is understood from the context we may write simply

$\mathcal{R}$. For any $R \in \mathcal{R}$, $\hat{R}$ denotes the asymmetric part of R and $\tilde{R}$ denotes symmetric part of R.[2] The statement xRy is interpreted to mean that subjectively, x is at least as good as y. $x\hat{R}y$ is interpreted to mean that x is strictly better than y, and $x\tilde{R}y$ means that x and y are indifferent. For $A' \subset A$, $R|_{A'}$ is that preference in $\mathcal{R}(A')$ which agrees with R. (Viewing R as a subset of $A \times A$, then $R|_{A'} = R \cap (A' \times A')$).

A T-profile of preferences R is a mapping from T to $\mathcal{R}$, i.e. $R \in \mathcal{R}^T$. (Where T is understood from the context we shall refer simply to preference profiles.) For $A' \subset A$, $R|_{A'}$ is that element of $\mathcal{R}(A')^T$ whose t^{th} coordinate is $R(t)|_{A'}$. Given a T-profile R and a subset $S \subset T$, define the partial order $\Pi^S(R)$ on A via

$$x\Pi^S(R)y \iff S \subset \{t \in T : x\hat{R}(t)y\}.$$

Π^S is then just the weak Pareto ranking for group S. When the profile is clear from the context we may write Π^S for $\Pi^S(R)$. Π^T may also be denoted simply by Π.

When the set of alternatives A has a product structure, that is $A \subset \prod_T A_t$, it is natural to introduce the notion of selfishness. A T-profile R is selfish if

$$\forall t_{\in T}\ \forall x,y,z,w_{\in A}[x(t) = w(t)\ \&\ y(t) = z(t)] \Rightarrow [xR(t)y \iff wR(t)z].$$

That is, $R(t)$ depends only on the t^{th} projection, π_t, of A. In this case $R(t)$ induces a regular preference, denoted $R^*(t)$, on $\pi_t(A)$.[3]

It should be noted that selfishness is a property of profiles, not of orders on the individual factors. A selfish profile can be constructed from orders on each of the factors as follows. For each t, let R_t be a preference order on A_t. Define $R \in R(\prod_T A_t)^T$ via

$$xR(t)y \iff x(t)R_t y(t).$$

Then R is a selfish profile on $\prod_T A_t$ and $R^*(t) = R_t$.

For a society T with the set A of social alternatives, a <u>social welfare function</u> (SWF) with domain $\mathcal{D} \subset R(A)^T$ is a mapping $\varphi:\mathcal{D} \to R(A)$. When there is no possibility of confusion $\varphi(R)$ will be denoted R. ($\varphi(R')$ will be denoted R', etc.)

An SWF φ is said to satisfy the <u>Weak Pareto Principle</u> (WP) if and only if

$$\forall\, x,y_{\in A}\ \forall R_{\in \mathcal{D}}[x\Pi(R)y \Rightarrow x\hat{\varphi}(R)y].$$

An SWF φ is said to satisfy the condition of <u>Independence of Irrelevant Alternatives</u> (IIA) if and only if

$$\forall x,y_{\in A}\ \forall R,R'_{\in \mathcal{D}}[R|_{\{x,y\}} = R'|_{\{x,y\}} \Rightarrow \varphi(R)|_{\{x,y\}} = \varphi(R')|_{\{x,y\}}].$$

An <u>ultrafilter</u> on T is a collection $\mathcal{U}$ of subsets of T such that

a) $\phi \notin \mathcal{U}$ & $T \in \mathcal{U}$.

b) $S \in \mathcal{U}$ & $S' \in \mathcal{U} \Rightarrow S \cap S' \in \mathcal{U}$.

c) $\forall S_{\subset T}\ S \in \mathcal{U}$ or $S^c \in \mathcal{U}$.

From the above follows trivially that exactly one of S and S^c belongs to $\mathcal{U}$, and that if $S' \supset S \in \mathcal{U}$, then $S' \in \mathcal{U}$. An ultrafilter $\mathcal{U}$ such that $\bigcap_{S \in \mathcal{U}} S = \phi$ is called <u>free</u>. If $\bigcap_{S \in \mathcal{U}} S = \{t\}$, then $\mathcal{U}$ is <u>fixed</u> or <u>principal</u>. Every ultrafilter on a finite set T is fixed. If T is infinte there exist free ultrafilters on T. Every ultrafilter is either free or fixed. Every free ultrafilter contains no finite set. If $\mathcal{U}$ is an ultrafilter on T and T is partitioned into a finite collection of disjoint sets, exactly one of the sets belongs to $\mathcal{U}$.

The following definitions will be useful in discussing SWFs in an economic context.

Let $\Omega_\ell = \{\xi \in \mathbb{R}^\ell : \xi_i > 0,\ i = 1,\dots,\ell\}$. The ℓ will be suppressed when it is understood from the context.

Define the partial order $>$ on Ω by $\xi > \eta \iff \xi - \eta \in \Omega$. Also, write $\eta < \xi$ for $\xi > \eta$. A preference R on Ω is <u>weakly monotonic</u> if $\xi > \eta \Rightarrow \xi \hat{R} \eta$.

For $x,y \in \Omega^T$ write $F_>(x,y)$ if

$$\forall t_{\in T} \exists i,j_{\in\{1,\dots,\ell\}}\, x(t)_i > y(t)_i \ \&\ x(t)_j < y(t)_j.$$

For real numbers α,β define $\alpha \wedge \beta = \min\{\alpha,\beta\}$ and $\alpha \vee \beta = \max\{\alpha,\beta\}$. For ℓ-vectors define $\xi \wedge \eta = (\xi_1 \wedge \eta_1,\dots,\xi_\ell \wedge \eta_\ell)$ and $\xi \vee \eta = (\xi_1 \vee \eta_1,\dots,\xi_\ell \vee \eta_\ell)$.

The set of positive commodity distributions for a society T is just Ω_ℓ^T, where ℓ is the number of commodities. It is convenient to introduce, for each $S \subset T$, the following partial order on Ω^T:

$$x \underset{S}{>} y \iff [\forall t_{\in S}\, x(t) > y(t)\ \&\ \forall t_{\in S^c}\, x(t) < y(t)].$$

Note that $x \underset{s}{>} y \iff y \underset{s^c}{>} x$, and that $\underset{s}{>}$ has no maximal or minimal elements on Ω^T.

For the theorem to be stated below we will use two conditions on the domain of the social welfare function, which guarantee that it is sufficiently rich. The conditions are not restrictive and are satisfied by any class of "classical" profiles. Indeed they are satisfied by the class of profiles whose induced preferences can be represented by utilities of the form $x \to \Sigma\alpha_i x_i + \lambda\Sigma\alpha_i \ln x_i$. Utilities of this form are strictly concave, C^∞, homothetic, strictly monotonic, and have indifference surfaces with nonvanishing Gaussian curvature which do not intersect the coordinate axes. These conditions will be satisfied by any larger class of preferences, so any reasonable definition of a "classical" set of profiles will satisfy the conditions.

A domain $\mathcal{D} \subset \mathcal{R}(\Omega^T)$ of profiles is said to be <u>complete</u> if for every pair x,y with $F_>(x,y)$ and z with $F_>(z,y)$ & $z \underset{s}{>} x$ there is a profile with z and y ranked arbitrarily and y preferred to x by S and vice versa by S^c. (See Figure 1.) Formally the condition is

$$\forall x,y,z_{\in\Omega^T}\ \forall R_{\in\mathcal{D}}\ \forall Q_{\in\mathcal{R}\{x,y\}^T}\ \forall S_{\subset T}\ [z \underset{s}{>} x \ \&\ F_>(z,y)\ \&$$
$$y\Pi^S(R)x \ \&\ x\Pi^{S^c}(R)y] \Rightarrow \exists R'_{\in\mathcal{D}}\ R'\big|\{y,z\} = Q \ \&\ R'\big|\{x,y\}$$
$$= R\big|\{x,y\}.$$

$\mathcal{D}$ is said to <u>admit nonindifference</u> if given any profile and pair of alternatives x,y there is a third alternative not indifferent to the first two and >- free with respect to each and which lies between x and y for those individuals with x preferred to y. (See Figure 2.)

Formally, the condition is

$$\forall x,y_{\in\Omega^T}\ \forall R_{\in\mathcal{D}}\exists R'_{\in\mathcal{D}}\exists x'_{\in\Omega^T} F_>(x',x)\ \&\ F_>(x',y)\ \&$$
$$\forall t_{\in T}(\neg x'\tilde{R}'(t)y\ \&\ \neg x'\tilde{R}'(t)x)\ \&\ R\big|\{x,y\} = R\big|\{x,y\}\ \&$$
$$\{t: x\hat{R}(t)y\} \supset \{t: x\hat{R}'(t)y\}.$$

<u>Theorem</u>. Let $\mathcal{D} \subset \mathcal{R}(\Omega_\ell^T)^T$, $\ell \geq 2$, and let $\varphi: \mathcal{D} \to \mathcal{R}(\Omega_\ell^T)$ satisfy IIA. Suppose that $\mathcal{D}$ satisfies the following:

(i) $\forall R_{\in\mathcal{D}}$ R is selfish.

(ii) $\forall R_{\in\mathcal{D}}\ \forall t_{\in T}$ $R^*(t)$ is weakly monotonic.

Suppose further that

(iii) φ is weakly non-imposed, i.e.,

$F_>(x,y) \Rightarrow [\exists R,R'_{\in\mathcal{D}}\ x\varphi(R)y\ \&\ y\varphi(R')x]$.

(iv) $\mathcal{D}$ is complete.

(v) $\mathcal{D}$ admits nonindifference.

Then either

a) $\forall x,y_{\in\Omega^T}\ \forall R_{\in\mathcal{D}}\ x\tilde{\varphi}(R)y$,

or

b) $\{S: x\Pi^S y \Rightarrow x\hat{\varphi}y\}$ is an ultrafilter,

or

c) $\{S: x\Pi^S y \Rightarrow y\hat{\varphi}x\}$ is an ultrafilter.

Remark. The condition of weak non-imposition cannot be reasonably extended to apply to all x and y for if $x \underset{s}{>} y$ for some s there is only one profile on {x,y} which is admissible.

For T finite (b) implies that there is a dictator and (c) implies that there is an antidictator.

The requirement that $\ell \geq 2$ is essential. If there is only one commodity then there is only one profile and both IIA and condition (iii) are vacuous. Any arbitrary social ordering is permissible. The fact that we take Ω^T and not $\bar{\Omega}^T$ (where $\bar{\Omega}$ is the closure in $\mathbb{R}^\ell$ of Ω) as the set of alternatives is significant. The conclusion fails to hold if Ω is replaced by $\bar{\Omega}$ as the following example (due to Blau) shows.

Let there be two individuals, $T = \{1,2\}$, and $\ell \geq 2$ commodities. Let $\mathcal{D} \subset R(\bar{\Omega}^2)^2$ be a set of selfish profiles of weakly monotonic preferences. Let $A^1 = \{x \in \bar{\Omega}^2 : x(1) = 0 \ \& \ x(2) \neq 0\}$, and $A^2 = \{x \in \bar{\Omega}^2 : x(1) \neq 0 \ \& \ x(2) = 0\}$, $A^{1,2} = \{x \in \bar{\Omega}^2 ; x(1) = 0 \ \& \ x(2) = 0\}$, and $A^\emptyset = \{x \in \bar{\Omega}^2 : x(1) \neq 0 \ \& \ x(2) \neq 0\}$.

Define the SWF φ by

$$\varphi(R)\big|_{A^\emptyset} = R(1)\big|_{A^\emptyset}$$

$$\varphi(R)\big|_{A^1} = R(2)\big|_{A^1}$$

$$\varphi(R)\big|_{A^2} = R(1)\big|_{A^1}$$

Furthermore $A^\emptyset \ \hat{\varphi} \ A^1 \ \hat{\varphi} \ A^2 \ \hat{\varphi} \ A^{1,2}$ for all profiles. No individual is a dictator for φ as the first individual always ranks A^1 below A^2 for every weakly monotonic preference, and as long as $\hat{R}(1)\big|_{A^\emptyset} \neq \hat{R}(2)\big|_{A^\emptyset}$ the second individual cannot be a dictator. On the other hand this social welfare function is easily seen to satisfy both the Pareto principle and IIA.

The failure of an analogous theorem here is due to the fact that for any selfish profile of monotonic preferences, if $x(t) = 0$ then x is $R(t)$-minimal. Since x is $R(t)$-minimal, it cannot be weakly Pareto superior to any alternative. Thus WP has little to say about such alternatives. With minor strengthening of the monotonicity condition Ω can be replaced by $\bar{\Omega} \setminus \{0\}$. See Border [3] for further details.

<u>Proof of Theorem</u>

<u>Lemma 1</u>. Suppose $x >_S y$ & $x\tilde{R}y$; then $x >_S z \Rightarrow x\tilde{R}z$ and $z >_S y \Rightarrow z\tilde{R}y$.

<u>Proof</u>. First note that since $x >_S y$, weak monotonicity, selfishness, and IIA imply $\varphi(R)\big|_{\{x,y\}}$ is independent of R, so it makes sense to write $x\tilde{R}y$.

Now suppose $x >_S w$ and $F_>(y,w)$. By (iii) there are R and R' such that $y\varphi(R)w$ and $w\varphi(R')y$. Thus, $x\varphi(R)w$ & $w\varphi(R')x$, but since $x >_S w, \varphi\big|_{\{x,y\}}$ is independent of R. Thus $x\tilde{R}w$.

Next we show that we can choose w so that $F_>(w,z)$ & $F_>(w,y)$ & $x >_S w$. Given such a w, applying the above argument first to y and w and then to w and z yields $x\tilde{R}z$, as desired.

To choose such a w, for each $t \in S$ choose $z(t)_1 \vee y(t)_1 < w(t)_1 < x(t)_1$ and $0 < w(t)_2 < z(t)_2 \wedge y(t)_2$. For $t \in S^c$ choose $z(t)_1 \vee y(t)_1 < w(t)_1$ and $x(t)_2 < w(t)_2 < z(t)_2 \wedge y(t)_2$. (See Figure 3).

The conclusion that $z \underset{S}{>} y \Rightarrow z\tilde{R}y$ follows from the first conclusion by replacing $\underset{S}{>}$ by $\underset{S^c}{>}$. qed

<u>Lemma 2</u>. Suppose $x \underset{S}{>} y$ and $x\tilde{R}y$. Then $\tilde{R}$ extends $\underset{S}{>}$.

<u>Proof</u>. Suppose $w \underset{S}{>} z$. We will show that below that we can choose x',y' so that $x' \underset{S}{>} y$, $x' \underset{S}{>} y'$ and $w \underset{S}{>} y'$. Then by Lemma 1 $x'\tilde{R}y$ since $x' \underset{S}{>} y$ and $x \underset{S}{>} y$ & $x\tilde{R}y$. Also by Lemma 1 $x'\tilde{R}y'$ since $x' \underset{S}{>} y'$ and $x' \underset{S}{>} y$ & $x'\tilde{R}y$. Again by Lemma 1 $w\tilde{R}y'$ since $w \underset{S}{>} y'$ and $x' \underset{S}{>} y'$ & $x'\tilde{R}y'$. Lastly by Lemma 1 $w\tilde{R}z$ since $w \underset{S}{>} z$ and $w \underset{S}{>} y'$ & $w\tilde{R}y'$. Thus we have shown that $w \underset{S}{>} z \Rightarrow w\tilde{R}z$, so $\tilde{R}$ extends $\underset{S}{>}$.

To construct x',y' as desired, for $t \in S$ choose $y'(t) < y(t) \wedge w(t)$ and for $t \in S^c$ choose $y'(t) > w(t)$. For $t \in S^c$ choose $x'(t) < y(t) \wedge w(t)$ and for $t \in S$ choose $x'(t) > y(t)$. (See Figure 4). qed

<u>Lemma 3</u>. Suppose $x \underset{S}{>} y$ and $x\tilde{R}y$. Then $\forall w,z \in \Omega^T\ \forall R \in \mathcal{D}\ w\tilde{\varphi}(R)z$, i.e., the social welfare function is null.

<u>Proof</u>. It follows from Lemma 2 that $\tilde{R}$ extends $\underset{S}{>}$. Choose u so that $u \underset{S}{>} z$ and $u \underset{S}{>} w$. (This can be done as $\underset{S}{>}$ has no maximal elements in Ω^T.) Then $u\tilde{R}z$ & $u\tilde{R}w$, thus $z\tilde{R}w$. qed

<u>Lemma 4</u>. Suppose $x \underset{S}{>} y$ and $x\hat{R}y$. Then $x \underset{S}{>} z \Rightarrow x\hat{R}z$ and $z \underset{S}{>} y \Rightarrow z\hat{R}y$.

<u>Proof</u>. Let $x \underset{S}{>} z$. Suppose first that $F_>(y,z)$. Then by (iii) there is $R \in \mathcal{D}$ such that $y\varphi(R)z$. Suppose that zRx. (Since $x \underset{S}{>} z$ this is well-defined.) Then since $x\hat{R}y$ we have $z\hat{R}y$, which contradicts $y\varphi(R)z$. Thus $x\hat{R}z$.

Next suppose that there is w with $x \underset{S}{>} w$ & $F_>(w,z)$ & $F_>(w,y)$. Then applying the above argument first to w and y and then to w and z, we conclude $x\hat{R}z$, as desired. Such a w is chosen as in Lemma 1. That $z \underset{S}{>} y \Rightarrow z\hat{R}y$ follows by replacing S by S^c. qed

<u>Lemma 5</u>. Suppose $x \underset{S}{>} y$ & $x\hat{R}y$. Then $\hat{R}$ extends $\underset{S}{>}$.

<u>Proof</u>. The proof of this is the same as Lemma 2 replacing $\tilde{R}$ by $\hat{R}$ and references to Lemma 1 by Lemma 4. qed

<u>Lemma 6</u>. If $\hat{R}$ extends $\underset{S}{>}$ and $\underset{S'}{>}$, then $\hat{R}$ extends $\underset{S\cap S'}{>}$ and $\underset{S\cup S'}{>}$.

<u>Proof</u>. First choose x, y, z such that

for $t \in S \setminus S'$	$y(t) > x(t) > z(t)$
for $t \in S \cap S'$	$x(t) > z(t) > y(t)$
for $t \in S' \setminus S$	$z(t) > y(t) > x(t)$
for $t \in (S \cup S')^c$	$y(t) > z(t) > x(t)$.

Then $x \underset{S}{>} z$, $z \underset{S'}{>} y$ and $x \underset{S\cap S'}{>} y$. Since $\hat{R}$ extends $\underset{S}{>}$ and $\underset{S'}{>}$, we have $x\hat{R}z$ & $z\hat{R}y$. Therefore $x\hat{R}y$. Then by Lemma 5 $\hat{R}$ extends $\underset{S\cap S'}{>}$.

Next choose x, y, z such that

for $t \in S'\setminus S$	$y(t) > x(t) > z(t)$
for $t \in (S\cup S')^c$	$x(t) > z(t) > y(t)$
for $t \in (S\setminus S')$	$z(t) > y(t) > x(t)$
for $t \in (S\cap S')$	$y(t) > z(t) > x(t)$.

Then $x >_{S^c} z$, $x >_{S'^c} y$, and $x >_{(S\cup S')^c} y$. Since $\hat{R}$ extends $\geq_S$ and $\geq_{S'}$, we have $z\hat{R}x$ and $y\hat{R}z$ so $y\hat{R}x$. But $x >_{(S\cup S')^c} y$ so $y >_{S\cup S'} x$ and since $y\hat{R}x$, by Lemma 5 we have $\hat{R}$ extends $>_{S\cup S'}$.

<u>Lemma 7</u>. Under the hypotheses of the theorem, either

a) $\forall x,y_{\in\Omega^T}\ \forall R_{\in\mathcal{D}}\ x\tilde{\varphi}(R)y$

or

b) $\{S:\hat{R}$ extends $\geq_S\}$ is an ultrafilter

or

c) $\{S:\hat{R}$ extends $\geq_{S^c}\}$ is an ultrafilter.

<u>Proof</u>. Given x,y with $x \geq_S y$ for some S, exactly one of three possibilities can occur: $x\tilde{R}y$, $x\hat{R}y$, or $y\hat{R}x$. If the first of these occurs then Lemma 3 implies that a) holds. If one of the other two possibilities occurs, then $\mathcal{S} = \{S:R$ extends $\geq_S\}$ must contain any given set or its complement. Lemma 6 says that $\mathcal{S}$ is

closed under finite unions and intersections, so that if $T \in \mathcal{S}$, then b) holds and if $\emptyset \in \mathcal{S}$ then c) holds. qed

<u>Lemma 8</u>. Suppose $\mathcal{S} = \{S:R$ extends $\geq_{S^c}\}$ is an ultrafilter. Then for $S \in \mathcal{S}$, $x\Pi^S(R)y \Rightarrow y\hat{\varphi}(R)x$, i.e., S is antidecisive.

<u>Proof</u>. Suppose that for some $S \in \mathcal{S}$ we have $x\Pi^S(R)y$ and that $x\varphi(R)y$. Consider first the case where $F_>(x,y)$ and $\forall t_{\in T}\ \neg x\tilde{R}(t)y$. Put $S' = \{t:y\hat{R}(t)x\}$. Since $x\Pi^S y$ we have $S \subset S'^c$, so $S'^c \in \mathcal{S}$, hence $\hat{R}$ extends $\geq_{S'}$. Choose z such that $z >_{S'} x$ & $F_>(z,y)$. (This can be done since $F_>(x,y)$. See Figure 5). Since φ is weakly nonimposed there exists $R' \in \mathcal{D}$ such that $y\varphi(R')z$. Since $\mathcal{D}$ is complete there is a profile $R'' \in \mathcal{D}$ such that

$$R''\big|_{\{y,z\}} = R'\big|_{\{y,z\}} \ \&\ R''\big|_{\{x,y\}} = R\big|_{\{x,y\}}.$$

Thus by IIA we have

$$x\varphi(R'')y\varphi(R'')z.$$

But $z >_{S'} x$, so this contradicts the fact that $\hat{R}$ extends $\geq_{S'}$. Thus $y\hat{\varphi}(R)x$.

The general case, where it is not necessarily true that $F_>(x,y)$ or $\forall t_{\in T}\ \neg x\hat{\varphi}(t)y$, can be reduced to the previous case. Since

$\mathcal{D}$ admits nonindifference there exists x' s.t. $F_>(x,x')$ & $F_>(y,x')$ & $\forall t_{\in T}(\neg x'\tilde{R}(t)x \;\&\; \neg x'\tilde{R}(t)y)$ & $x\Pi^S(R)x'\Pi^S(R)y$. Then from the previous case $y\hat{\varphi}(R)x'\hat{\varphi}(R)x$. qed

Lemma 9. Suppose $\mathcal{S} = \{S : R \text{ extends } >_S\}$ is an ultrafilter. Then for $S \in \mathcal{S}$

$$x\Pi^S(R)y \Rightarrow x\hat{\varphi}(R)y,$$

i.e., S is decisive.

Proof. The proof is virtually identical to that of Lemma 8. qed

The Theorem follows immediately from Lemmas 7, 8, and 9.

QED

The Pareto Principle

By requiring in addition to the hypotheses of the theorem that the social welfare function also satisfy the weak Pareto principle, we rule out possibilities (a) and (c) of the theorem. (Just consider $x >_T y$. Then monotonicity and the Pareto principle imply $x\hat{R}y$. This means that the social welfare function is not null and that $\{S : \hat{R} \text{ extends } >_{S^c}\}$ is not an ultrafilter since it does not contain T.) Thus for finite societies the social welfare function must be dictatorial, even for classical economic domains.

FOOTNOTES

1. A regular preference on A is a binary relation R on A which is total ($\forall w,y_{\in A}\, x \neq y \Rightarrow [xRy \text{ or } yRx]$), reflexive ($\forall x_{\in A}\, xRx$), and transitive ($\forall x,y,z_{\in A}[xRy \;\&\; yRz] \Rightarrow xRz$). The terms preference ordering or preference ranking or preference relation or simply ordering or ranking will sometimes be used.

2. $\hat{R}$ is defined by $x\hat{R}y \Leftrightarrow xRy \;\&\; \neg\, yRx$. $\tilde{R}$ is defined by $x\tilde{R}y \Leftrightarrow xRy \;\&\; yRx$. If R is regular then $\hat{R}$ is irreflexive ($\forall x_{x\in A}\; \neg\, xRx$), asymmetric ($\forall x,y_{\in A}\, x\hat{R}y \Rightarrow \neg\, y\hat{R}x$), and transitive. $\tilde{R}$ is symmetric ($\forall x,y_{\in A}\, x\tilde{R}y \Rightarrow y\tilde{R}x$), reflexive, and transitive.

3. $R^*(t)$ is defined by

$$\xi R^*(t)\eta \Leftrightarrow \exists x,y_{\in A}\, \pi_t(x) = \xi \;\&\; \pi_t(y) = \eta \;\&\; xR(t)y.$$

If R is selfish then $R^*(t)$ is well-defined and regular on $\pi_t(A)$.

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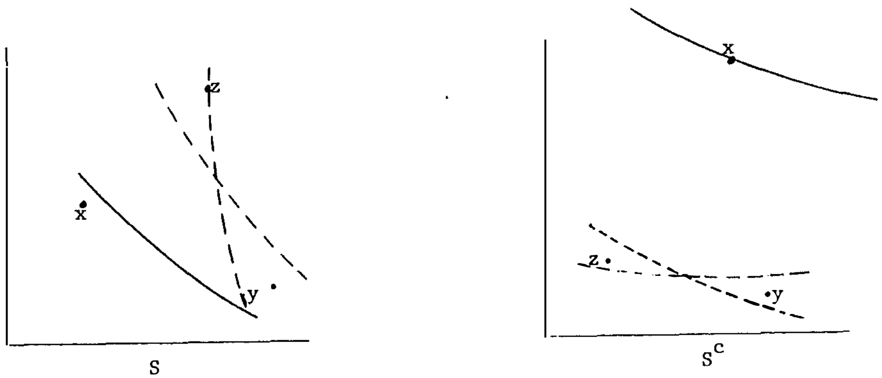

Figure 1.

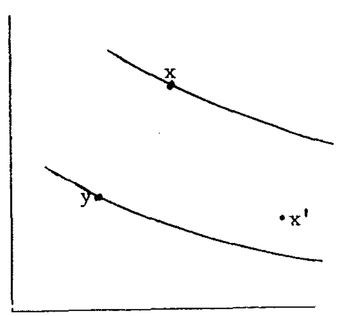

Figure 2.

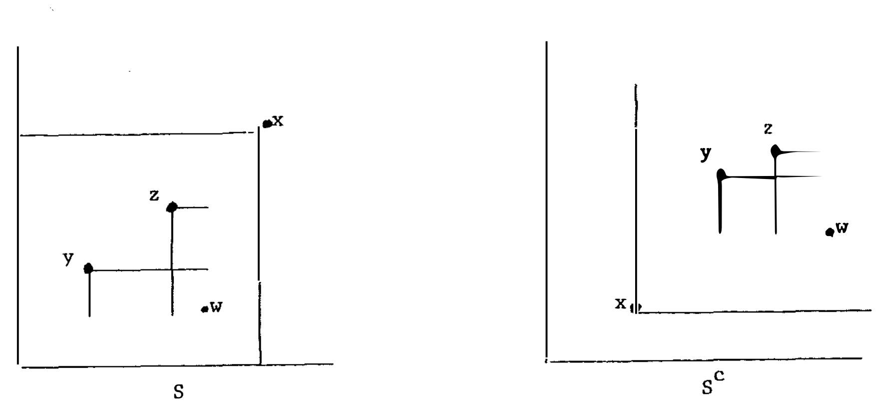

Figure 3.

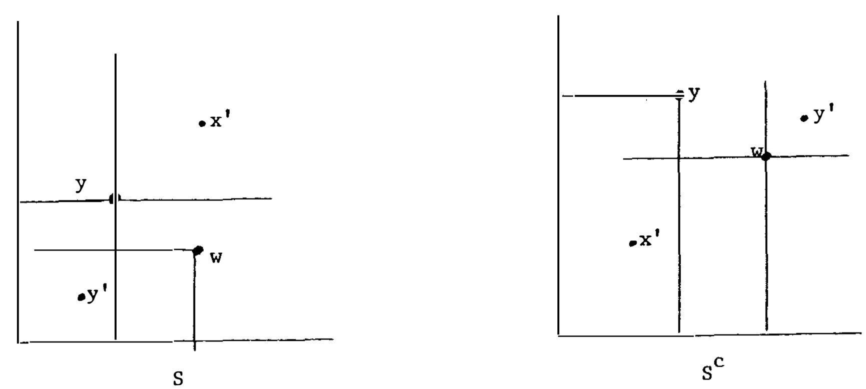

Figure 4.

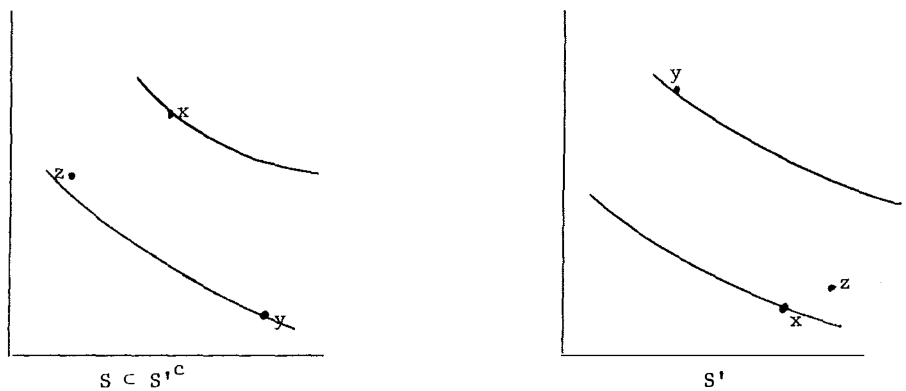

Figure 5.