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COLLECTIVE CHOICE MECHANISMS FOR ACHIEVING EFFICIENT
STOCK MARKET ALLOCATIONS

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Abstract

This paper examines two collective choice mechanisms for achieving efficient stock market allocations.  The first, proposed by Helpman and Razin, is shown to have the property that an equilibrium rarely exists.  An alternative mechanism, due to Hurwicz, is examined and it is shown that the resulting equilibria under this mechanism do exist and are efficient.

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## 1.  INTRODUCTION

Much of the recent work in the theory of general equilibrium under uncertainty has been directed towards proposing and analyzing decision mechanisms for guiding the behavior of the firm when the set of markets present in the economy is not complete.  Of particular interest are mechanisms which give rise to efficient allocations.  In this context, efficiency should be understood to mean constrained Pareto optimality where the income of each individual is constrained to be a fixed share of random profits plus fixed nonrandom income.

One approach is to consider market structures under which production sets of firms are spanned by the set of existing production plans.[1]  In this case any proposed change in a firm's production plan will be unanimously approved or rejected by its stockholders.  The virture of this approach is that a firm may evaluate the change in its market value by using information which is available in the market values of all firms in the economy.

Without restricting market structure, however, any decision mechanism a firm may use must reflect the tastes of its stockholders in order for it to make efficient decisions.  Drèze [1974] has shown

that production efficiency will be attained if each firm maximizes its value according to an average of investors' marginal rates of substitution, where all investors' marginal rate of substitutions are weighted by their relative shareholdings the difficulty that arises with this mechanism is that all investors must reveal their own contingent claim prices to the firm, but, in general, they will not reveal truthfully. Much along this same line, Helpman and Razin [1978] describe a "participation equilibrium" in which stockholders make input contributions to firms and firms determine their input level as the sum of these contributions. Although this procedure provides appropriate incentives which give rise to efficient allocations, an equilibrium will, in general, fail to exist due to the particular cost sharing mechanism that is used.

The structure of the paper is as follows. In Section 2, the general model is set up and the failure of a participation equilibrium to exist is demonstrated. Making use of the recent literature on incentive compatibility and public goods, Section 3 examines an alternative cost sharing rule under which efficient equilibria exist.

## 2. THE MODEL

Consider an economy consisting of I consumers, indexed $i = 1,\ldots,I$, and J firms, indexed $j = 1,\ldots,J$. There are two periods of time labelled 0 and 1, respectively. The true state of the world

which will occur in period 1 is not known in period 0 and there is a finite number, S, mutually exclusive states which may occur in period 1. There is a single commodity which is available for consumption in period 0, $c_0$, or which may be invested in order to provide for consumption in period 1, if state of the world $\Theta$ occurs, $c_\Theta$, $\Theta = 1,\ldots,S$.

Each firm j has a market value $V_j$ and determines the level of input to use as a solution to a noncooperative I-person game among the individuals. Each individual i chooses a quantity of input $x_j^i$ given the choice made by all other individuals. The firm chooses the input level $x_j = \sum_j x_j^i$ and the firm finances these purchases by issuing debt in period 0. In period 1, each individual must pay some pre-assigned fraction $\alpha_j^i$ of firm j's debt, where $\sum_i \alpha_j^i = 1$, $j = 1,\ldots,J$. (In the Helpman-Razin formulation each individual pays the same fraction $\alpha_j^i = \frac{1}{I}$, of the cost of the quantity of input he has chosen, $x_j^i$. By requiring each individual to pay a fraction of each firm's total debt eliminates the need for lump-sum taxes to achieve a balanced budget in their model.)

The $i^{th}$ individual is endowed with $\bar{x}^i$ units of the input and a portfolio consisting of fractions $\bar{\gamma}_j^i$ of each firm j. Gross returns are distributed to shareholders according to their shareholdings, where $r_j(\Theta, x_j)$ is the gross return of firm j in state $\Theta$, using input level $x_j$. The price of the input is w, the bond price is one, and the return on one unit of bonds is r.

Each individual i maximizes the utility of consumption by making input proposals $x_j^i$ and by choosing a portfolio of security holdings, $\gamma_j^i$, and bond holdings, $b^i$. Thus the decision problem of each individual is[2]

to maximize $\quad U^1(c_0^i, c_1^i, \ldots, c_S^i)$
$(c^i, \gamma^i, b^i, x^i)$

subject to $\quad wc_0^i + \sum_j V_j \gamma_j^i + b^i \leq w\bar{x}^i + \sum_j V_j \bar{\gamma}_j^i$

$$c_\Theta^i \leq \sum_j r_j(\Theta, x_j)\gamma_j^i + rb^i - wr \sum_j \alpha_j^i x_j \qquad \Theta = 1, \ldots, S$$

$$\gamma_j^i \geq 0, \; x_j \geq 0 \qquad\qquad j = 1, \ldots, J$$

$$c_0^i \geq 0, \; c_\Theta^i \geq 0.$$

In this formulation a participation equilibrium is defined as a price system $(w, r, \{V\})$, a consumption allocation $(\{c_0^i, c_1^i, \ldots, c_S^i\})$, an ownership allocation $(\{\gamma_j^i\})$, a bond allocation $(\{b^i\})$, and input proposals $(\{x_j^i\})$ such that

(i) $c^i, \gamma^i, b^i$, and $x^i$ are a solution to individual i's decision problem, $i = 1, \ldots, I$,

(ii) $\sum_i c_0^i + \sum_j x_j = \sum_i \bar{x}^i$,

(iii) $\sum_i b^i = w \sum_j x_j$,

and

(iv) $\sum_i \gamma_j^i = 1, \; j = 1, \ldots, J$.

As Helpman and Razin point out, equilibrium requires that individuals own positive fractions of every firm which operates ı a nonzero level of input since any individual, say $\ell$, who owns $= 0$ of firm j will choose $x_j^\ell = -\sum_{i \neq \ell} x_j^i$. The difficulty with this ilibrium concept is even more severe due to the existence of the -assigned cost shares, $\alpha_j^i$. In general it can be shown that no ilibrium exists in this model. To see this, consider the first .er conditions to each consumer's decision problem, which are ʳen by [3,4]

$$w \sum_\Theta \rho^i(\Theta) r_j(\Theta, x_j) = V_j \qquad j = 1, \ldots, J \qquad (1)$$

$$\sum_\Theta \rho^i(\Theta) rw = 1 \qquad\qquad (2)$$

$$\sum_\Theta \rho^i(\Theta)(r_j'(\Theta, x_j)\gamma_j^i - \alpha_j^i rw) = 0 \qquad j = 1, \ldots, J \qquad (3)$$

ıere

$$\rho^i(\Theta) = \frac{\partial U^i / \partial c_\Theta^i}{\partial U^i / \partial c_0^i}, \quad \Theta = 1, \ldots, S.$$

ɪenotes individual i's imputed contingent claim price for consumption ın the state $\Theta$.

As can be seen from (3), the difficulty which arises is due to the fact that the same $x_j$ must satisfy this condition for each individual and thus the system of equations may often be over-identified. This is most easily seen by considering the case in which any proposed

change in the existing state distribution of returns lies in a subspace which is spanned by the existing return vectors. Spanning implies that there exists state-independent weights, $\beta_{jk}$, such that

$$r'_j(\theta, x_j) = \sum \beta_{jk} r_k(\theta, x_k), \theta = 1, \ldots, S. \qquad (4)$$

Using this assumption, (3) becomes

$$0 = \sum_\theta \rho^i_\theta (r'_j(\theta, x_j) \gamma^i_j - \alpha^i_j r w) = \sum_k \beta_{jk} \gamma^i_j \sum_\theta \rho^i_\theta r_k(\theta, x_k) - \alpha^i_j$$

$$= \frac{1}{w} \sum_k \beta_{jk} \gamma^i_j V_k - \alpha^i_j$$

or that

$$\gamma^i_j \sum_k \beta_{jk} V_k = \alpha^i_j w. \qquad (5)$$

Summing over i in (5) and using the fact that in equilibrium

$$\sum_i \gamma^i_j = \sum_i \alpha^i_j = 1, \text{ gives}$$

$$\sum_k \beta_{jk} V_k = w,$$

which on substituting back into (5) shows that in equilibrium that all individuals will choose a portfolio which is identical with their pre-assigned costs shares, $\gamma^i_j = \alpha^i_j$. Furthermore, if cost shares are chosen to be the initial portfolio holdings, $\bar{\gamma}^i_j$, it can be seen that a

necessary condition for an equilibrium is that no trade takes place on the securities market, i.e., $\gamma^i_j = \bar{\gamma}^i_j$. Thus, even with spanning, an equilbrium exists only in the unlikely event that the prescribed constants have been set equal to equilibrium ownership fractions.

Helpman-Razin also consider a one-period formulation and maintain that existence is more likely in that case. In that formulation, however, consumption in period 0 does not enter either the utility function or the period 0 budget constraint, and the first order conditions become

$$\sum_\theta \rho^i(\theta) r_j(\theta, x_j) = V_j \qquad j = 1, \ldots, J \qquad (1')$$

and

$$\sum_\theta \rho^i(\theta)(r'_j(\theta, x_j)\gamma^i_j - \alpha^i_j w) = 0 \quad j = 1, \ldots, J \qquad (3')$$

where

$$\rho^i(\theta) = \frac{\partial U^i / \partial c^i_\theta}{r \sum_\theta \partial U^i / \partial c^i_\theta}, \qquad \theta = 1, \ldots, S.$$

Equation (5) follows as before, and the above demonstration of nonexistence continues to hold.

The following example further illustrates this difficulty.

Example

To demonstrate the nonexistence of a participation equilibrium consider an economy with two consumers, two firms, and two states of

the world where

$$r_j(\Theta, x_j) = \begin{cases} x_j & \text{if } j = \Theta \\ \\ 0 & \text{if } j \neq \Theta \end{cases}$$

and $U^i(c_0^i, c_1^i, c_2^i) = c_0^i + \beta^i \log c_1^i + (1 - \beta^i) \log c_2^i$, $0 < \beta < 1$ for $i = 1,2$.

Assume also that each individual has been assigned the same cost share for each firm ($\alpha_j^i = 1/2$ for $i,j = 1,2$). Then from the discussion above, we must have in equilibrium $\gamma_j^i = 1/2$ for $i,j = 1,2$. Using this fact in the first order conditions (3) with respect to investment proposals gives

$$\frac{\beta^i}{c_1^i} = \frac{1 - \beta^i}{c_2^i} = 1 \quad \text{for } i = 1,2.$$

thus

$$1 - \beta^i = c_2^i = x_2\gamma_2^i + r(b^i - \alpha^i w(x_1 + x_2)) \qquad (6)$$

and

$$\beta^i = c_1^i = x_1\gamma_1^i + r(b^i - \alpha^i w(x_1 + x_2)). \qquad (7)$$

Subtracting (7) from (6) gives

$$1 - 2\beta^i = x_2\gamma_2^i - x_1\gamma_1^i \qquad (8)$$

and in order for $\gamma_1^i = \gamma_2^i = 1/2$ to be an equilibrium, (8) requires that

$$\frac{1 - 2\beta^i}{x_2 - x_1} = 1/2 \quad \text{for } i = 1,2$$

which will occur only if the preferences of both individuals are identical (i.e., $\beta^1 = \beta^2$).

## 3. AN ALTERNATIVE INCENTIVE COMPATIBLE MECHANISM

As illustrated by the example above, the difficulty which arises with the Helpman-Razin mechanism is that in equilibrium, each consumer's budget constraint must have the same normal vector $(w, \alpha^i w)$ and they must be unanimous with respect to the input levels of each firm. This requires that their marginal rates of substitution at the agreed upon level of x must be proportional to the prespecified cost shares, $\alpha^i$. This problem is analogous to the one which arises in the public good literature when each consumer is confronted with a tax rule which is some prescribed share of the cost of the public good.[5]

For overcoming this difficulty, the literature on public goods provides some alternative mechanisms which may be adopted for this stock market model. As an illustration, consider the Shared Cost Mechanism presented by Hurwicz [1976]. This mechanism is in the spirit of the Helpman-Razin mechanism except that cost shares are no longer exogenous but rather are choice variables of each

consumer. Once again each individual is asked to make input suggestions, $x_j^i$, and the firm ocntinues to operate at $x_j = \sum_i x_j^i$ through debt financing. In addition, each individual is asked for the share of the cost of the debt financing which he is willing to transfer to others, $a_j^i$. The amount which individual i must pay of firm j's debt is given by

$$t_j^i = \left(1 - \sum_{\ell \neq i} a_j^\ell\right) rwx_j + M\left(1 - \sum_\ell a_j^\ell\right)^2 \qquad (9)$$

where $M > 0$. Each individual's set of period 1 budget constraints become

$$c_\Theta^i \leqq \sum_j r_j(\theta, x_j)\gamma_j^i + rb^i - \sum_j t_j^i. \qquad (10)$$

By allowing cost shares to be decision variables, this mechanism circumvents the existence problem which arose in the example of the previous section. Further, at a Nash equilibrium $\sum_\ell a_j^\ell = 1$, $j = 1,\ldots,J$. To see this, note that $a_j^i$ affects consumer i only through the term $M\left(1 - \sum_\ell a_j^\ell\right)^2$ and thus $t_j^i$ can be decreased (and hence each $c_\Theta^i$ increased) simply by changing $a_j^i$ if $\sum_\ell a_j^\ell \neq 1$. In equilibrium, $a_j^i = 1 - \sum_{\ell \neq i} a_j^\ell$ becomes the share of cost paid by consumer i. In equilibrium, with these endogenous cost shares, the budget set of each consumer coincides with the set each consumer faces in the Helpman-Razin formulation and their proof can be used to show that this mechanism achieves a constrained Pareto Optimum.

For the example in the previous section, it is easy to see that equilibrium cost shares will be chosen as equal to equilibrium ownership fractions. Since production decision are unanimously supported in this case there is no reason for cost shares to differ from ownership fractions. In general, however, this need not be the case.

## 4. CONCLUSION

The set of incentive compatible mechanisms, including the Helpman-Razin mechanism, are subject to the criticism that stockholders must have costs shares which are not, in general, the same as the ownership shares of each firm they hold. This will always occur when shareholdings are determined in some other manner. Under any mechanism which provides costs shares which are equal to ownership shares (i.e., $\gamma_j^i = \alpha_j^i$), then, as may be seen from (3), it must always be the case that the production decisions of the firm are unanimously supported. As shown by Leland [1973] unanimity will occur only if the spanning condition is satisfied.

Thus, in order to construct a more satisfactory theory of the firm under general market structures, other nonmarket mechanisms must be explored. Hart [1977] has taken this approach in the study of take-over bid equilibrium and Benninga and Muller [1977] have studied the behavior of the firm under a majority rule mechanism. Unfortunately, neither of these approaches can insure equilibrium which are efficient.

FOOTNOTES

6.  An overview of these mechanisms is given in Groves [1979].

1.  See Ekern and Wilson [1974], Leland [1973, 1974], and Radner [1974].

2.  In the Helpman-Razin formulation, the budget constraint for consumption in state of the world $\Theta$ is given by

$$c_\Theta^i \leqq \sum_j r_j(\Theta, x_j) \, \gamma_j^i + rb^i - \frac{1}{I} \, wr \sum_j x_j^i - T^i, \quad \Theta = 1, \ldots, S$$

where $T^i$ is a lump-sum tax on individual i, and $\sum_i T^i = \frac{(I-1)}{I} \, wr \sum x_j$

is required in order to achieve a balance budget.

3.  Helpman and Razin assume that consumers take the value of each firm as independent of their actions. Since each consumer provides inputs, this "competitive" behavior seems to require a large number assumption. Alternatively one could make the competitivity assumption found in the unanimity literature (see Baron [1979]). This assumption requires that all consumer forecast valuation changes taking their own contingent claims prices as given. In this case, $\bar{\gamma}_j^i$ would replace $\gamma_j^i$ in (3).

4.  If the Helpman-Razin formulation is used (see footnote 3), it may be verified that the first order conditions remain the same with $\alpha_j^i$ replaced by $\frac{1}{I}$ in (3).

5.  In a pure public goods context, this problem has previously been pointed out by Groves and Ledyard [1977].

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