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Abstract

A social choice function is said to be implementable if and only if there exists a game form such that for all preference profiles an equilibrium strategy n-tuple exists and any equilibrium strategy n-tuples of the game yield outcomes in the social choice set. A social choice function is defined to be minimally democratic if and only if whenever there exists an alternative which is ranked first by n-1 voters and is no lower than second for the last voter, then the social choice must be uniquely that alternative. No constraints are placed on the social choice function for other preference profiles.

Using the usual definitions of equilibria for n-person games -- namely Nash and strong equilibria, it is shown here that over unrestricted preference domains, no minimally democratic social choice function is implementable. The same result holds in certain restricted domains of the type assumed by economists over public goods spaces. We then show that a different notion of equilibrium -- namely that of sophisticated equilibrium -- allows for implementation of democratic social choice functions also having further appealing properties. The implication is that models of democratic political processes can not be based on the standard equilibrium notions of Nash or strong equilibria.

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I. INTRODUCTION

Our purpose in this paper is to show that for <u>any</u> democratic social choice function and for the kinds of equilibria discussed by Maskin and others, mechanisms do not exist which will support acceptable" outcomes as equilibria. This result is true even if preferences are restricted to allow only preferences of the type usually assumed by economists on public goods spaces. We then give a concept of equilibrium for which intuitively appealing "implementations" of democratic social choice functions do exist.

In a recent paper, Dasgupta, Hammond, and Maskin [1978] show that if the domain of preferences is unrestricted, then no single valued nondictatorial social choice function can be implemented. More precisely, no mechanism can be constructed which for all preference profiles yields the social choice as its unique Nash or strong equilibria. This result has not drawn a great deal of attention because of its assumption of single valuedness. In fact, more recent work has seemed to indicate that even in unrestricted preference domains, the impossibility results evaporate if multiple valued social choice functions are admitted. For example, Maskin [1977] has shown that any monotonic social choice function with no veto players

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can be implemented, and Hurwicz and Schmeidler [1978] have constructed a game form which implements a social choice function whose outcomes are always Pareto optimal.

This paper shows that if the social choice function satisfies even the weakest conditions of responsiveness to individual preferences -- conditions which one would expect any democratic social choice function to satisfy -- then the impossibility result returns. The implication is that any attempt to implement "reasonable" social choice functions must rely on preference restrictions or on different equilibrium notions than the Nash and strong equilibria which are usually assumed.

For economic environments, mechanisms have been constructed which "support" as their Nash equilibria, allocations in the Core (Wilson [1978]), and the Walrasian allocations (Schmeidler [1976]). Of course, the principal reason that attractive implementations sometimes are available for social choice functions restricted to economic environments is that the collection of preference configurations on which the implementation must work is a small subset of the collection of all n-tuples of weak orders. Even in these environments, Maskin has found that the mechanisms which will implement the collection of individually rational Pareto optimal allocations in the sense of Nash equilibria requires that each agent have an extremely large strategy set. He has therefore moved in the direction of utilizing a new equilibrium concept that allows for the implementation by a more appealing mechanism. While we show that Maskin's new equilibrium concept does not allow the implementation of democratic social choice functions if the domain of admissible preference configuration is unrestricted, it might allow such implementations in certain restricted domains.

Unfortunately if one demands that a democratic social choice function at the least be able to apply to the choice among three alternatives, no natural restriction on the domain of preference configuration seems to be available. Perhaps in some applications such restrictions may turn out to be interesting and tractable with respect to the implementation of democratic procedures. For now, we remain agnostic on that issue and focus instead on introducing an alternative concept of equilibrium -- sophisticated equilibrium -- which permits the implementation of a wide class of democratic choice rules.

First we delimit the class of democratic social choice functions. Intuitively, we think of democratic social choice functions as possessing a certain minimal property. If the configuration of preferences in society is such that there exists a particular alternative that is ranked first by n-1 individuals, and no lower than second by the n^{th} individual, that alternative and only that alternative, ought to be chosen. In other words, on those rare occasions when a single alternative commands a huge majority over every other alternative then any democratic voting mechanism should choose that distinguished alternative. No further restrictions are required for a social choice function to be democratic.

It should be noted that this property has a long, and we think, honorable history in the literature on balloting systems.

Indeed, it is a generalization of a property proposed by Condorcet, and is widely held to be a desirable characteristic of a voting rule. Indeed, the failure of such rules as the Borda rule and other scoring or "positionalist" rules to satisy the <u>ordinary</u> Condorcet property is usually taken to be a strong criticism of the use of these rules. It should be noted, however, that the Borda rule as well as the other scoring rules, do satisfy our weakened condition stated above. We should also point out that our generalized Condorcet axiom has the effect of ruling out the possibility that there are oligarchies of individuals each of whom possesses the ability to veto the choice of some alternatives. This entailment seems quite unobjectionable except, possibly, in the single case of the unanimity principle. Indeed, the unanimity social choice function is the only one that is arguably democratic and which is implementable in the sense that a mechanism exists for which the outcome supported by Nash equilibria are Pareto optimal.

II. NOTATION AND DEFINITIONS

We let N stand for the set of <u>individuals</u> and X, the collection of potentially available <u>alternatives</u>. Subsets, C, of N are sometimes called <u>coalitions</u>. Each individual $i \in N$ has a complete, reflexive, transitive preference relation (or <u>weak ordering</u>) on X denoted by R_i. As usual, we let P_i and I_i represent the assymmetric and symmetric parts of R_i. The collection of all weak orders on X is denoted $\mathcal{R}$.

A <u>social choice function</u> (SCF) is a correspondence F which maps elements of R^n into subsets of X. For each $\underline{R} \varepsilon R^n$ the subset $F(\underline{R}) \subseteq X$ is thought of here, as the set of acceptable alternatives associated with the configuration $\underline{R} = (R_1, R_2, \ldots, R_n)$. The collection of subsets of a set A is written as 2^A.

A collection of sets S_i, $i = 1, \ldots, n$ together with a function G which maps the elements of $\prod_{i=1}^{n} S_i$ into X is called a mechanism. We write $\underline{S} = \prod_{i=1}^{n} S_i$, and will denote these mechanisms by the pair $<\underline{S}, G> = \Gamma$. Elements of $\underline{S}$ are written in the form $\underline{s} = (s_1, \ldots, s_n)$, and for any $\underline{s}$, $\underline{s}^* \varepsilon \underline{S}$, and $C \subseteq N$, we write $(\underline{s}^C, \underline{s}^{*N-C})$ for the n-tuple $\underline{s}' \varepsilon \underline{S}$ such that $s'_i = s_i$ if $i \varepsilon C$ and $s'_i = s^*_i$ if $i \varepsilon N-C$. If $C = \{i\}$ for some $i \varepsilon N$, we write $(\underline{s}^C, \underline{s}^{*N-C}) = (\underline{s}_i, \underline{s}^{*N-\{i\}})$.

Given a mechanism Γ and a preference configuration $\underline{R} \varepsilon R^n$, a k-equilibrium for $<\Gamma, \underline{R}>$ is an element $\underline{s}^* \varepsilon \underline{S}$ such that for all coalitions, $C \subseteq N$ such that $|C| \leq k$, and for all $\underline{s} \varepsilon \underline{S}$, $G(\underline{s}^*)R_i$ $G(\underline{s}^C, \underline{s}^{*N-C})$ for some $i \varepsilon C$. Evidently, the Nash equilibria are the 1-equilibria and the strong equilibria are the n-equilibria. The collection of k-equilibria associated with $<\Gamma, \underline{R}>$ is conveniently written as $E^k_\Gamma(\underline{R})$.

A social choice function is said to be k-implementable if there is a mechanism, Γ, such that for all $\underline{R} \varepsilon R^n$

$$\emptyset \neq G(E^k_\Gamma(\underline{R})) \subseteq F(\underline{R}).$$

In other words, a SCF is implementable if a mechanism can be found, each of whose k-equilibrium supported outcomes are acceptable for every preference configuration.

Finally, we present a formal statement of the requirement that a SCF be democratic. For any $C \subseteq N$, $|C|$ denotes the number of elements in C. For any $S \subseteq X$, $\mu(S)$ denotes the measure of the set S. If X is finite, $\mu(S)$ is simply the number of elements of S. If X is a subset of R^n, then $\mu(S)$ is simply the Lesbeque measure of S.

Now, for any $x \varepsilon X$, and $i \varepsilon N$, define $R_i(x) = \{y \varepsilon X - \{x\} | yR_i x\}$, and $B_i(X) = \{x \varepsilon X | R_i(x) = \phi\}$. Then set $D_i(x) = R_i(x) - B_i(X)$. So $D_i(x)$ is the set of elements, other than x itself and the maximal element, which are as good as x. Now we make several definitions:

Definition: An alternative $x \varepsilon X$ is a consensus alternative if $R_i(x) = \phi$ for all $i \varepsilon N$. It is a near consensus alternative if $\exists\, C \subseteq N$ with $|C| \geq n - 1$ such that $R_i(x) = \phi$ for all $i \varepsilon C$. The alternative $x \varepsilon X$ is said to be 2+ε bounded if $\mu(D_i(x)) \leq \varepsilon$ for all $i \varepsilon N$. It is 2 bounded if it is 2+0 bounded. Finally, $x \varepsilon X$ is a Condorcet alternative iff $|\{i \varepsilon N | xP_i y\}| > \frac{n}{2}$ for all $y \varepsilon X - \{x\}$.

Definition: A social choice function $F : R^n \rightarrow 2^X$ is minimally efficient (ME) iff whenever $x \varepsilon X$ is a consensus alternative, $F(\underline{R}) = \{x\}$. F is minimally democratic (MD) iff, whenever $x \varepsilon X$ is a near consensus alternative and is 2 bounded, then $F(\underline{R}) = \{x\}$. F is ε-minimally democratic (εMD) iff whenever $x \varepsilon X$ is a near consensus alternative and is 2+ε bounded then $F(\underline{R}) = \{x\}$. F is a Condorcet extension (CE) iff whenever $x \varepsilon X$ is a condorcet alternative, then $F(\underline{R}) = \{x\}$.

A few words of interpretation are in order, especially of the notion of 2+ε bounded. An alternative is 2 bounded iff it is no lower than second in any individuals ranking. An alternative x is 2+ε bounded iff the measure of the set of alternatives ranked between x and the best alternative is no more than ε. This definition is needed for continuous alternative spaces with continuous preferences where the notion of second best may be undefined. Here, if x is 2+ε bounded, it is at most "ε away from second place" for everyone.

Now a minimally efficient social choice function need only choose consensus alternatives uniquely when they exist. I.e., in the infrequent cases when one alternative is ranked best by all voters, that must be uniquely chosen. Similarly, for the case of a minimally democratic social choice function, the social choice function need only choose near consensus alternatives which are 2-bounded when they exist. In other words, in the cases where one alternative is ranked first by n - 1 voters and no lower than second by the last voter, that alternative should be uniquely the social choice. No restrictions are placed on what the social choice should be in other configurations -- in particular it need not be single valued elsewhere.

SCF's satisfying εMD have a similar interpretation, and finally, Condorcet extensions must choose Condorcet points uniquely when they exist. Again note that no restrictions are made when Condorcet points do not exist. It should be noted that other definitions of minimal democracy trivially imply the above conditions. For example Richelson's [1978] conditions UMP and VUUMP imply both MD and εMD for all ε and any generalized Condorcet extension principle would also imply the above conditions. In addition, virtually any

specific social choice functions that have been studied, with the exception of the Pareto optimal set, satisy the conditions MD and εMD.

III. AN IMPOSSIBILITY THEOREM

In this section we establish the following, somewhat depressing result: if a SCF satisfies MD, then it is not k-implementable for any k. In other words if a SCF is minimally democratic there exists no mechanism with the property that all of its k-equilibria are acceptable for each $\underline{R} \in R^n$. To establish this fact we need two additional definitions and a simple lemma due to Maskin [1977].

Definition: For any $k \in \{1,2,\ldots,n\}$, $\underline{R} \in R^n$, and mechanism $\Gamma = \langle \underline{S}, G \rangle$ let $H^k(\underline{R}) = G(E^k_\Gamma(\underline{R}))$.

Definition: A function $H : R^n \to X$ is monotone if for any $x \in X$, $\underline{R}$ and $\underline{R}' \in R^n$ such that $x \in H(\underline{R})$, and $xR_i y \Rightarrow xR'_i y$ $\forall i \in N$, $\forall y \neq x$, $x \in H(\underline{R}')$.

Lemma: H^k is monotone for each k.

Proof: Assume $x \in X$ and that $\underline{R}$, and $\underline{R}'$ are such that $x \in H^k(\underline{R})$, $xR_i y \Rightarrow xR'_i y$, $\forall i \in N$, $y \neq x$. Then there is $\underline{s} \in \underline{S}$ such that $\underline{s} \in E^k_\Gamma(\underline{R})$ and $x = F(\underline{s})$. But then $\underline{s} \in E^k_\Gamma(\underline{R}')$.

Q.E.D.

<u>Theorem 1</u>: If F is k-implementable for some $k \in \{1,2,\ldots,k\}$, and $|N| \leq |X|$ then F is not minimally democratic.

<u>Proof</u>: Assume F is k-implementable for some k. Then if F is minimally democratic, set $A_i = \{i\}$ for all $i \in N$, and set $V_i = N - A_i$ for all $i \leq i \leq n$. Then pick $\{x_1,\ldots,x_n\} \subseteq X$ and construct $\underline{R} \in R^n$ to satisfy, for all $1 \leq i \leq n$, $j \in N$, and $y \in X - \{x_1,\ldots,x_n\}$,

$$x_i P_j y$$

$$x_i P_j x_{i-1} (\text{mod } n) \text{ if } j \in A_i$$

$$x_{i-1} P_j x_i (\text{mod } n) \text{ if } j \in N - A_i = V_i.$$

Now, since F is k-implementable, $x^* \in H^k(\underline{R}) = G \circ E^k_\Gamma(\underline{R})$ for some k-implementation of F, and for some $x^* \in F(\underline{R})$. There are two cases. Either $x^* \in \{x_1,\ldots,x_n\}$ or $x^* \notin \{x_1,\ldots,x_n\}$. If $x^* \in \{x_1,\ldots,x_n\}$, then $x^* = x_\ell$ for some $1 \leq \ell \leq n$, and we define $\underline{R}'$ from $\underline{R}$ by moving $x_{\ell-1}$(mod n) up as far as possible in each ordering without changing its ordering with respect to x_ℓ. Leave all else unchanged. Then $x_{\ell-1} P_j x_\ell$ for $j \in V_\ell$, and $x_{\ell-1} P_j y$ for $j \in N$ for all other $y \in X - \{x_{\ell-1}, x_\ell\}$. Hence, by MD, $\{x_{\ell-1}\} = F(\underline{R}')$. But this contradicts the monotonicity of H^k, since $(\forall j \in N)(\forall y \in X - \{x_\ell\})(x_\ell R_j y \Rightarrow x_\ell R'_j y)$ and $x_\ell \in H^k(\underline{R})$ but $x_\ell \notin H^k(\underline{R}')$. Now if $x^* \notin X - \{x_1,\ldots,x_n\}$, then construct $\underline{R}'$ from $\underline{R}$ by moving x_1 to the top of every ordering. Then by MD, $\{x_1\} = F(\underline{R}')$, which again contradicts the monotonicity of H^k. Thus, F cannot be minimally democratic.

Q.E.D.

The above result assumes an unrestricted domain of preferences. It is possible that if the set of possible preferences were to be restricted, the previous argument would not work. We present here some results addressing this question. The preference restrictions considered here are the usual preference restrictions assumed by economists over public goods spaces. Namely, it is assumed that all preferences are pseudoconcave. So let E denote the real numbers and let $X \subseteq E^m$ be convex, with a nonempty interior. Then $R \subseteq X \times X$ is said to be <u>pseudoconcave</u> iff it is representable by a differentiable function $u: E^m \to E$ satisfying $\forall x, y \in X$,

$$\nabla u(x)(y-x) \leq 0 \Rightarrow u(x) \geq u(y). \qquad (*)$$

(So R is pseudoconcave iff it is quasi concave and has a differentiable representation whose gradient is nonzero except possibly at a maximum.) We define

$$R^0 = \{R \subseteq X \times X \mid R \text{ is pseudoconcave}\},$$

and we let $\underline{R}^0 = \prod_{i=1}^{n} R^0$.

We first prove a lemma

Lemma 1 If $R \in R^0$, and if x^*, $y^* \in X$ satisfy $y^* \in \text{Int}\,\{y \in X | yRx^*\}$, then there exists a $R' \in R^0$ satisfying

a) $\{y \in X | x^*Ry\} \subseteq \{y \in X | x^*R'y\}$

b) $yR'x^*$ for no $y \in X - \{x^*\}$

Proof: For any $x \in E^m$, and $r \in E^+$, let $B(x,r)$ denote the closed ball of radius r and center x. Let $u : E^m \to E$ be a representation of R satisfying (*). Now, by assumption, for some $\varepsilon > 0$, $B(y^*,\varepsilon) \subseteq \{y \in E^m | yR_ix^*\}$. Also, by differentiability of u, for some $\delta > 0$,

$$B\left(x^* + \delta\left(\frac{\nabla u(x^*)}{\|\nabla u(x^*)\|}\right), \delta\right) \subseteq \{y \in E^m | yR_ix^*\}.$$

take

$$D = B(y^*,\varepsilon) \cup B\left(x^* + \delta\left(\frac{\nabla u_i(x^*)}{\|\nabla u_i(x^*)\|}\right), \delta\right)$$

and set D^* to be the convex hull of D. Now for any $y \in E^m$, with $y \neq y^*$, define

$$t_y = \sup_{t>0} \{z = ty + (1-t)y^* | z \in D^*\}$$

then set

$$g(y) = \begin{cases} 0 \text{ if } y = y^* \\ -\frac{1}{t_y^2} \text{ if } y \neq y^2 \end{cases}$$

Now $g : E^m \to E$ is a differentiable function satisfying (*). So defining $R' \subseteq X \times X$ by $xR'y \Leftrightarrow g(x) \geq g(y)$ for any $x, y \in X$, it follows that $R' \in R^0$ satisfies the conditions of the Lemma.

Q.E.D.

We proceed to the theorems, and start with a result which is weaker than the results of the previous section, but which holds on small dimensional spaces.

Theorem 2: If $F : \underline{R}^0 \to 2^X$ is a Condorcet extension, where $X \subseteq E^m$, and $m \geq 2$, it is not k-implementable.

Proof: Let $\Gamma = \langle \underline{S}, G \rangle$ be a k-implementation of F, and let

$$H^k(\underline{R}) = G(E^k_\Gamma(\underline{R})) \subseteq F(\underline{R}),$$

as defined above. Then H^k is monotonic. Now let $\underline{R} \in R^0$ be any profile for which there is not a Condorcet alternative. That such profiles exist is easily seen by the following construction. Let α_i be the i^{th} basis vector, and pick $x_0 \in X$, $t_1, t_2, \in R$ such that $x_1 = x_0 + t_1\,\alpha_1$ and $x_2 = x_0 + t_2\,\alpha_2$ are both in the interior of X. Then partition N into C_0, C_1, and C_2 such that $|C_j| < \frac{n}{2}$ for all j. For i ε C_j, let preferences be based on Euclidian distance from x_j. I.e., $\forall x,y \in X$, $xR_iy \Leftrightarrow \|x - x_j\| \leq \|y - x_j\|$. Now any alternative x ε X is majority beaten by another alternative x* ε X as follows: Set $z_j = (x_j - x)$ for j = 0,1,2, and pick any two z_j, z_k which are linearly dependent. Then for small enough ε, it follows that $x^* = [x + \varepsilon \cdot (z_j + z_k)]$ is majority preferred to x. Further, for all x ε X, by convexity of X, it follows that x* ε X.

So let $\underline{R} \in \underline{R}^0$ be any profile for which there is no Condorcet alternative, and let $x \in H^k(\underline{R})$. Now, pick x' ε X with $|\{i \in N | x'P_ix\}| > \frac{n}{2}$, and let $C = \{i \in N | x'P_ix\}$. Now construct $\underline{R}' \in \underline{R}^0$ so that $\forall i \in N - C$, $R_i' = R_i$, and such that $\forall i \in N$

$$\{y \in X | xR_iy\} \subseteq \{y \in X | xR_i'y\} \qquad (*)$$

Further, for all i ε C, construct $\underline{R}'$ such that $x'P_i'y$ for all y ε X - {x'}. By Lemma 1, it follows that it is possible to construct $R_i' \in R^0$ satisfying this condition as well as (*). But, then x' is a Condorcet point for the profile $\underline{R}'$. Hence $F(\underline{R}') = \{x'\}$, since F is a Condorcet extension. So $H^k(\underline{R}') = \{x'\}$. But, by (*), for all i ε N and all y ε X, $xR_iy \Rightarrow xR_i'y$. Also $x \in H^k(\underline{R})$. So by monotonicity, we should have $x \in H^k(\underline{R}')$, a contradiction, so the result follows.

Q.E.D.

The next result shows that in large dimensional spaces, results akin to those of Theorem 1 hold. In order to prove this result, however, it is necessary to make assumptions about X, guaranteeing the smoothness of the boundary of X, if it exists. Specifically, X is said to be <u>smooth</u> if for all x ε X, $\exists z \in E^m$ with $z \neq 0$ such that for all $y \in E^m$ with $y \cdot z > 0$, $\exists \varepsilon > 0$ such that $x + \varepsilon y \in X$. This leads to the following theorem.

<u>Theorem 3</u>: Let $F : R^0 \to 2^X$ be a SCF, where $X \subseteq E^m$, $m \geq n$, and X is smooth, then for any ε > 0, if F is ε-Minimally Democratic, it is not k-implementable.

<u>Proof</u>: As in the previous proof, let $\Gamma = \langle \underline{S}, G \rangle$ be a k-implementation and define $H^k(\underline{R})$ as before. Now let α_i be the i^{th} standard basis vector in E^m, and for all i ε N, define R_i as follows: $\forall$ x,y ε X

$$xR_iy \Leftrightarrow x \cdot \alpha_i \geq y \cdot \alpha_i$$

Now since $H^k(\underline{R}) \neq \phi$, pick $x \in H^k(\underline{R})$. Now pick $z \in E^m$ such that if x is a boundary point of X, then z is the direction vector of the supporting hyperplane. If x is not a boundary point then z can be chosen arbitrarily, as long as $z \neq 0$. It follows that $x + \varepsilon \cdot z \in X$ for some ε > 0. Now pick $w \in E^m$ to satisfy $w \cdot z > 0$ and such that $w \cdot \alpha_j > 0$ for at least n - 1 j ε N. To see that this is possible, write $z = \sum_{i=1}^m a_i \alpha_i$, and $w = \sum_{i=1}^m b_i\alpha_i$. Then we have

$$w \cdot z > 0 \iff \sum_{i=1}^{m} a_i b_i > 0 \qquad (*)$$

$$w \cdot \alpha_j > 0 \iff b_j > 0$$

Now, since $z \neq 0$, it follows that $a_i \neq 0$ for at least one i. Then for all $j \neq 1$, we let $b_j = 1$. For $j = i$, pick b_j to satisfy (*). Of course we may have either $j \in N$ or $j \notin N$. Now by the assumption that X has a smooth boundary, it follows that for some ε, $x + \varepsilon \cdot w \in X$. Set $x' = x + \varepsilon \cdot w$. It follows that for all $i \in C = N-\{j\}$, that $x'P_i x$. Now, as in the proof of Theorem 2, using Lemma 1, we can construct $\underline{R}' \in \underline{R}^0$ so that $\forall i \in C$, $\{y \in X | xR_i y\} \subseteq \{y \in X | xR_i'y\}$, and such that $x'P_i'y$ for all $y \in X$. For $i \in N - C$, (if $N - C \neq \phi$) construct R_i' so that $xP_i'y$ for all $y \in X$ and so that $\mu(R_i'(x')) \leq \varepsilon$. It follows that for all $y \in X$ and all $i \in N$, that $xR_i y \Rightarrow xR_i'y$. Hence, by monotonicity of H^k, we must have $x \in H^k(\underline{R}')$. But by construction, x' is a near consensus alternative and is $2 + \varepsilon$ bounded. Hence $H^k(\underline{R}') = \{x'\}$, a contradiction. So F cannot be k-implementable.

Q.E.D.

Finally, we present a result showing that regardless of the dimensionality of the alternative space, if X is open, then not even minimally efficient social choice functions are implementable. It is easy to see that a similar result would hold if X were closed but unbounded. This theorem is rather pathological, depending as it does on the noncompactness of the alternative space, and it is presented only for completeness. It should be noted that the above

two theorems hold whether or not X is compact.

Theorem 4: If $F : R^0 \to 2^X$ is minimally efficient, and X is open, then it is not k-implementable.

Proof As in the previous proof let $\Gamma = \langle \underline{S}, G \rangle$ be a k implementation, and define $H^k(\underline{R})$ as before. Now pick any $a \in E^m$, and for all $i \in N$, define R_i as follows: $\forall\, x,y \in X$

$$xR_i y \iff x \cdot a \geq y \cdot a.$$

Now since $H^k(\underline{R}) \neq \phi$, pick $x \in H^k(\underline{R})$. Then since X is open, we can find $x' \in X$ with $x' \cdot a > x \cdot a$, i.e., such that $x'P_i x$ for all $i \in N$. Now construct $\underline{R}' \in R^0$ so that $\forall\, i \in N$, $\{y \in X | xR_i'y\} \subseteq \{y \in X | xR_i'y\}$ and further such that $x'P_i y$ for all $y \in X$. By Lemma 1, it follows that it is possible to construct $R_i' \in R^0$ satisfying these conditions. But then x' is a consensus alternative, so we must have $F(\underline{R}') = \{x'\}$. I.e., $H^k(\underline{R}') = \{x'\}$. But by monotonicity of H^k, we must have $x \in H^k(\underline{R}')$, a contradiction.

Q.E.D.

IV. AN ALTERNATIVE EQUILIBRIUM NOTION

The results of the previous section are counterintuitive. They tell us that no democratic system can be implemented: but we observe pervasive attempts in the real world to do exactly that -- namely to set up institutions which allegedly have at least minimally democratic properties. Are such institutions doomed to failure, or is there something defective in the concept of implementation that we are employing here?

The main factors which seem to be driving the impossibility results of the previous section, as well as those of Maskin, are the special properties of the equilibrium definitions being used. The definition of k-equilibrium allows us to deal simultaneously with Nash equilibria and strong equilibria. However both of these notions of equilibria have well known disadvantages. Nash equilibria generally tend to exist, but to be nonunique. A common problem with Nash equilibria in voting games is that there are frequently, in addition to the "reasonable," or "natural" Nash equilibria, a plethora of so called "bogus" equilibria, in which strategy n-tuples which are patently absurd for each individual player are nevertheless Nash equilibria, because no one player has the power to unilaterally change the outcome by a change in his own strategy. The definition of implementation used here, and elsewhere in the literature, requires all such bogus equilibria to support outcomes chosen by the social choice function. Of course, strong equilibria avoid this problem, because coalitions of players can get together and eliminate such strategies. However strong equilibria have the problem that there are generally too few of them. It is difficult to construct games where such equilibria exist for any large domain of preference profiles. Thus, by using the notion of k-equilibria, we are stuck on the horns of a dilemma. On the one hand if k is small, (so we are close to Nash equilibria), any game we design will generally have too many equilibria, and hence will

fail to implement our social choice function because it will sometimes produce bogus equilibria yielding outcomes outside of the SCF. On the other hand, if k is large (so we are close to strong equilibria), any game we design will generally have no equilibria for some profile.

In this section, we investigate an alternative equilibrium notion, which has the effect of eliminating bogus equilibria without running into existence problems. In particular, we look at "sophisticated" equilibria. This approach is not at all new to this paper, and has received considerable attention in both the game theoretic literature and literature on voting theory. In fact, in a recent article, Moulin [1979] proposes exactly the same solution notion as a means of implementing efficient social choice functions which are anonymous or neutral. In the voting theory literature, sophisticated voting was defined by Farquharson [1969], and has been studied by numerous authors, including Miller [1978] and Kramer [1972]. For games arising from binary voting procedures, McKelvey and Niemi [1978] and Gretlein [1979] show that there are close connections between sophisticated equilibria and multistate equilibria, which have been studied by Shapley [1953], and others. In the game theoretic literature, ideas similar to those underlying sophisticated voting have been developed by Selten [1975] under the name "perfect equilibrium."

As in the previous section, let $\Gamma = \langle \underline{S}, G \rangle$ be a mechanism, and let $\underline{R} \in \mathcal{R}^n$. Then for any $i \in N$, and for any $s, s' \in S_i$, we say that s dominates s' if

(a) $\forall \underline{s},\underline{s}' \in \underline{S}$ with $s_i = s, s_i' = s'$ and $\underline{s}^{N-\{i\}} = \underline{s}'^{N-\{i\}}$, $G(\underline{s}) R_i G(\underline{s}')$

(b) $\exists \underline{s},\underline{s}' \in \underline{S}$ with $s_i = s, s_i' = s'$ and $\underline{s}^{N-\{i\}} = \underline{s}'^{N-\{i\}}$ such that $G(\underline{s}) P_i G(\underline{s}')$.

A strategy $s \in S_i$ is said to be primarily admissible for i in $\langle \Gamma, \underline{R} \rangle$ if it is not dominated by any other strategy in S_i. For each $i \in N$, we let $S_i^{(1)}$ be the set of primarily admissible strategies, and let $\underline{S}^{(1)} = \prod_{i \in N} S_i^{(1)}$, and $\Gamma^{(1)} = \langle \underline{S}^{(1)}, G^{(1)} \rangle$, where $G^{(1)}$ is G restricted to $\underline{S}^{(1)}$. Then, for any m > 1, and $i \in N$, we define $S_i^{(m)}$ to be the set of strategies which are primarily admissible for i in $\langle \Gamma^{(m-1)}, \underline{R} \rangle$. We set $\underline{S}^{(m)} = \prod_{i \in N} S_i^{(m)}$, and $\Gamma^{(m)} = \langle \underline{S}^{(m)}, G^{(m)} \rangle$ where $G^{(m)}$ is G restricted to $\underline{S}^{(m)}$. The sets $S_i^{(m)}$ are called the m-arily admissible strategies for i, and $\Gamma^{(m)}$ is called the m^{th} reduction of Γ. Finally, we set $S_i^u = \bigcap_{m=1}^{\infty} S_i^m$, $\underline{S}^u = \prod_{i \in N} S^{(u)}$, and $\Gamma^{(u)} = \langle \underline{S}^{(u)}, G^{(u)} \rangle$ where G^u is G restricted to $\underline{S}^u$. Then S_i^u is called the set of ultimately admissible or sophisticated strategies for i, and Γ^u is called the ultimate, or final reduction of Γ. For any $\Gamma = \langle \underline{S}, G \rangle$ and $\underline{R} \in \mathcal{R}^n$, we write

$$E_\Gamma^u(\underline{R}) = \underline{S}^u$$

to represent the set of n-tuples of sophisticated strategies.

Now, the social choice function $F: \mathcal{R}^n \to X$ is said to be implementable in sophisticated strategies if there is a mechanism, $\Gamma = \langle \underline{S}, G \rangle$ such that for all $\underline{R} \in \mathcal{R}^n$,

$$\phi \neq G(E_\Gamma^u(\underline{R})) \subseteq F(\underline{R}).$$

A few words of interpretation are in order. The sophisticated strategies in the mechanism Γ are simply the strategies which remain after successive reductions of the original mechanism, where, at each stage of the reduction each player simply eliminates presently dominated strategies. It is easily shown that if $|E_\Gamma^u(\underline{R})| = 1$, then the sophisticated strategy n-tuple is also a Nash equilibrium. In general, however there may remain strategy n-tuples $\underline{s} \in E_\Gamma^u(\underline{R})$ which are not Nash Equilibria. Note in fact that $E_\Gamma^u(\underline{R}) \neq \phi$, since S_i^m must always contain at least one strategy for all $i \in N$ and $m \geq 0$.

We now give two examples showing how sophisticated equilibria can be used to implement democratic social choice functions. The first example is a special case of the procedures considered by Farquharson [1969], namely fixed agendas using binary voting processes. The second example requires preference restrictions as well as the notion of sophisticated equilibria.

Example 1: Fixed Agenda, Binary Procedure

In a recent article, Moulin [1979] has shown that using sophisticated equilibria, the ammendment procedure implements a Pareto efficient, anonymous social choice function. Here, drawing on results from McKelvey and Niemi [1978] and Gretlein [1979], we show further that that procedure, as well as any sequential voting process based on binary procedures and a fixed agenda is a Condorcet extension. We first need a series of definitions in order to define binary voting procedures based on fixed agendas.

A voting tree over X is a pair $\Pi = (\Lambda,\Psi)$ where $\Lambda = (\Lambda,P)$ is a finite topological tree*, and $\Psi : \Lambda^{\tau} \to X$ is a function which associates with each terminal node r in Λ, an alternative $\Psi(r) \in A$. The voting tree is binary if, for each nonterminal node, $r \in \Lambda$, there are associated exactly two following nodes. In a binary voting tree, at any given node, the following branches can be indexed by 0 or 1. We can identify a node of Π in terms of the history of branches that are taken to get to it (starting from the origin) thus a node is identified by a p-tuple $r = (r_1,\ldots,r_p)$, where r_1 is the first branch that is taken, r_2 the second, etc. (note that p is the number of branches that must be tranversed to get to node r). For binary procedures it follows that for any $r \in \Lambda$, $r \in \{0,1\}^p$ for some $p \geq 0$. (Note that for the origin, $r = \Phi$, and $p = 0$). We let $\Lambda' \subseteq \Lambda$ denote the nonterminal nodes of Π and $\Lambda^{\tau} \subseteq \Lambda$ be the terminal nodes. Further, we set $K = |\Lambda'|$, and let $\phi : \Lambda' \to \{1,2,\ldots, \}$ be any one to one enumeration of these nodes. We use the notation $r^j = \phi^{-1}(j)$ to denote the j^{th} node of this enumeration.

Now for any vector $d = (d_1,\ldots,d_K) \in \{0,1\}^K$, define a_d^* as follows: Set

$$r_1^* = d_{\phi(\Phi)}$$

*See McKelvey and Niemi, p. 9 for a formal definition.

and for $j > 1$,

$$r_j^* = d_{\phi(r_1^*,\ldots,r_{j-1}^*)} \text{ as long as } (r_1^*,\ldots,r_{j-1}^*) \in \Lambda'.$$

Then set

$$a_d^* = \psi(r_d^*) = \psi(r_1^*,\ldots,r_j^*)$$

where r_d^* is the terminal node in the above process.

Next, let $v : \{0,1\}^n \to \{0,1\}$ be the binary majority rule function, i.e., for any $s = (s_1,\ldots,s_n) \in \{0,1\}^n$

$$v(s) = \begin{cases} 1 \text{ if } \sum s_i > \frac{n}{2} \\ 0 \text{ if otherwise} \end{cases}$$

and define $V : (\{0,1\}^K)^n$ as follows: For any $\underline{s} = (s_1,\ldots,s_n) \in (\{0,1\}^K)^n$, and all $0 \leq k \leq K$ set $s^k = (s_{1k},s_{2k},\ldots,s_{nk})$, and then

$$V(s) = (v(s^1),v(s^2),\ldots,v(s^K))$$

Then the pair (Π,V) is called a binary voting procedure based on majority rule.

Now we define a mechanism, $\Gamma = \langle \underline{S},G \rangle$ on the basis of the above definition. For each $i \in N$, set $S_i = \{0,1\}^K$. Now $G : \underline{S} = \prod_{i \in N} S_i \to X$ is defined by, for any $\underline{s} \in \underline{S}$,

$$G(\underline{s}) = a^*_{V(\underline{s})}$$

It now follows from theorem 1 and corollary 1 of McKelvey and Niemi [1978] together with theorem 2 of Gretlein [1979] that $G(E^u_\Gamma(\underline{R}))$ is single valued for all $\underline{R} \in R^n$, and that it is also a Condorcet extension. Hence

$$F(\underline{R}) = G(E^u_\Gamma(\underline{R}))$$

is a social choice function, which, by construction, is implementable in sophisticated strategies by Γ.

Example 2: Two Candidate Competition

In this example, a type of implementation is arrived at by introducing two political entrepeneurs (candidates) who's sole function is to adopt policy positions in an attempt to win an election. Their preferences are thus restricted, and in the resulting game they try and provide the voters with "socially desirable" outcomes. The mechanism is constructed so that it is a dominant strategy for voters to provide correct information about their preferences. From the point of view of the voters, the resulting mechanism implements a Condorcet extension which is neutral, anonymous, and Pareto efficient.

We set

$$N' = \{1,2,\ldots,n\} \cup \{n+1,n+2\} = N \cup J$$
$$X' = J \times X.$$

We can think, then of N as the set of voters, and J as the set of candidates. The set X' is the set of final outcomes, which consist of a policy $x \in X$, together with a candidate $j \in J$ to carry out the policy. Then define

$$S_i = X \times X \rightarrow J \quad \text{for } i \in N$$

$$S_i = X \quad \text{for } i \in J$$

So candidates choose policy positions, while voters choose candidates, based on their policy positions. Set

$$S^N = \prod_{i \in N} S_i, \quad S^J = \prod_{i \in J} S_i, \quad \underline{S} = S^N \times S^J.$$

For any $\underline{s} = (s^N, s^J) = (s_1, \ldots, s_n, s_{n+1}, s_{n+2}) \in \underline{S}$,

define

$j^*(\underline{s}) =$ smallest $j^* \in J$ such that

$$|\{i \in N | s_i(s^J) = j^*\}| = \max_{j \in J} |\{i \in N | s_i(s^J) = j\}|$$

and set

$$x^*(\underline{s}) = s_{j^*(\underline{s})}.$$

Thus, $j^*(\underline{s})$ is the candidate with the largest number of votes, with ties broken arbitrarily (i.e., the first candidate wins in a case of a tie), and $x^*(\underline{s})$ is the policy adopted by the winning candidate. Then set

$$G(\underline{s}) = (j^*(\underline{s}), x^*(s))$$

Now it is assumed that for each $i \in N$, there is a weak order R_i on X, representing i's preferences over the basic alternatives, and a weak order $\hat{R}_i$ on J representing i's preferences over the candidates. It is assumed that for $j,k \in J$, if $j \neq k$, then $j\hat{P}_j k$. These preferences are extended to preferences R_i' over X' as follows: For any $(j,x),(k,y) \in X'$,

a) If $i \in N, (j,x)R_i'(k,y) \Leftrightarrow xR_i y$

b) If $i \in J, (j,x)R_i'(k,y) \Leftrightarrow j\hat{R}_i k$

Thus, voters care only about what policy is adopted, having no preference for which candidate is elected, and candidates care only about being elected, having no policy preferences.*

We now consider properties of $G(E_\Gamma^u(\underline{R}'))$. We are concerned only with the voters $i \in N$, and so since voters have no preferences over which candidate is elected, we can equivalently look at $x^*(E_\Gamma^u(\underline{R}'))$, and look at this as a function of the preferences $\underline{R} = (R_1,\ldots,R_n)$ of voters in N. Thus let

$$F(\underline{R}) = x^*(E_\Gamma^u(\underline{R}')).$$

*The same results follow if the following lexicographic assumption is made about preferences

a) If $i \in N, (j,x)R_i'(k,y) \Leftrightarrow xP_i y$ or ($xI_i y$ and $j\hat{R}_i k$)

b) If $i \in J, (j,x)R_i'(k,y) \Leftrightarrow j\hat{P}_i k$ or ($j\hat{I}_i k$ and $xR_i y$)

Under the above assumptions, it follows that for $i \in N$, s_i is an admissible strategy iff $\forall s_i' \in S_i$ and $\forall s^J \in S^J$,

$$s_{s_i}(s^J) R_i s_{s_i'}(s^J)$$

In other words, setting $J = \{j,k\}$, s_i is admissible iff $\forall s^J = (s_j, s_k) \in S^J$

$$s_i(s^J) = \begin{cases} j & \text{if } s_j P_i s_k \\ k & \text{if } s_k P_i s_j \end{cases}$$

note that $s_i(s^J)$ is unrestricted if $s_j I_i s_k$. However, in general, <u>any</u> strategy is primarily admissible for $i \in J$. Thus, after the first reduction, we have, for all $\underline{s} \in \underline{S}^{(1)}$,

$$x^*(\underline{s}) = \begin{cases} s_j & \text{if } |\{i \in N | s_j P_i s_k\}| \geq |\{i \in N | s_k R_i s_j\}| \\ s_k & \text{if } |\{i \in N | s_k P_i s_j\}| \geq |\{i \in N | s_j R_i s_k\}| \end{cases}$$

It follows, that if there is a Condorcet point, $x^* \in X$, then for all $y \in X$

$$|\{i \in N | x^* P_i y\}| > \frac{n}{2}$$

Now, for $j \in J$, $s_j = x^*$ dominates any other strategy $x \in S_j^{(1)}$, because, picking $k \in J-\{j\}$, for any $\underline{s}, \underline{s}^* \in \underline{S}^{(1)}$ with $s_j = x, s_j^* = x^*$, and $s^{N'-\{j\}} = s^{*N'-\{j\}}$,

$$j^*(s^*) = \begin{cases} j & \text{if } s_k \neq x^* \\ j \text{ or } k & \text{if } s_k = x^* \end{cases}$$

$$j^*(\underline{s}) = \begin{cases} j \text{ or } k & \text{if } s_k \neq x^* \\ k & \text{if } s_k = x^* \end{cases}$$

Hence, it follows that if there is a Condorcet point x^*, then the second reduction has $\underline{S}^{(2)}$ satisfying

For $i \in N$, $S_i^{(2)}$ is the set of strategies satisfying (*)

For $i \in J$, $S_i^{(2)} = \{x^*\}$.

It follows, for all $\underline{s} \in S^{(2)}$, that $G(\underline{s}) = (j, x^*)$ for some $j \in J$. Note that the second reduction is also the final reduction. Thus, any sophisticated strategy in this mechanism picks out a Condorcet point if there is one.

Further, it follows from results of McKelvey and Ordeshook [1976], that regardless of whether or not there is a Condorcet point, that for all $\underline{s} \in \underline{S}^{(2)}$, and hence for all $\underline{s} \in \underline{S}^u$, that $x^*(\underline{s})$ is pareto optimal. Finally, it is easily shown that

$F(\underline{R}) = x^*(E_\Gamma^u(\underline{R}'))$ is neutral and anonymous. This example shows then, how a combination of preference restrictions and a revised equilibrium concept can lead to implementability of a democratic social choice function.

REFERENCES

Dasgupta, P., P. Hammond, and E. Maskin, "The Implementation of Social Choice Rules," Review of Economic Studies, 46 (April 1979), pp. 185-215.

Farquharson, R., Theory of Voting, (Yale University Press, New Haven, 1969).

Gretlein, R., "On the Relation Between Farguharson's Method of Sophisticated Voting and other Methods of Successive Elimination of Dominated Strategies in Finite Alternative Games," Carnegie Mellon University, mimeo, 1979.

Hurwicz, L., and D. Schmeidler, "Construction of Outcome Functions Guaranteeing Existence and Pareto Optimality of Nash Equilibria," Econometrica, 46 (1978), 1447-1474.

Kramer, G., "Sophisticated Voting over Multidimensional Choice Spaces," Journal of Mathematical Sociology, 2 (1972)

Maskin, E., "Nash Equilibrium and Welfare Optimality," Mathematics of Operations Research, (forthcoming).

McKelvey, R., and R. Niemi, "A Multi-Stage Game Representation of Sophisticated Voting for Binary Procedures," Journal of Economic Theory, 18. (June 1978), pp. 1-22.

McKelvey, R., and P. Ordeshook, "Symmetric Spatial Games without Majority Rule Equilibria," American Political Science Review, 70 (1976) 1172-1184.

Miller, N., "Beyond Farquharson: Game Theoretical Applications to Voting Bodies," University of Maryland, mimeo, 1978.

Moulin, H., "Dominance Soluable Voting Schemes," Econometrica, 47 (November, 1979), 1337-1351.

Richelson, J., "A Comparative Analysis of Social Choice Functions, II, Behavioral Science,

Schmeidler, D., "A Remark on Microeconomic Models of an Economy and on a Game Theoretic Interpretation of Walras Equilibria," mimeo, University of Minnesota, 1976.

Selten, R., "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," International Journal of Game Theory, 4 (1975) 25-55.

Shapley, L. S., "Stochastic Games," Proceedings of the National Academy of Science, 39 (1953) 327-332.

Wilson, R., "Competitive Exchange," Econometrica, 46 (May 1978) pp. 577-585.