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MULTIPLE-OBJECT, DISCRIMINATORY AUCTIONS WITH BIDDING CONSTRAINTS: A GAME-THEORETIC ANALYSIS

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Abstract

This paper examines the existence and characterization of pure strategy Nash equilibria in multiple-object auction games in which buyers face a binding constraint on exposure. There are five major results. First, symmetric Nash equilibria exist if and only if there are two or less buyers and two or less items. Second, a Nash equilibrium may not exist if the seller sets a positive reservation bid. Third, asymmetric solutions to symmetrically parameterized games typically involve "high-low" strategies: buyers submit positive bids only on some restricted subset of the items. Fourth, Nash equilibria typically generate zero "profits" to the buyers. Fifth, when asymmetric solutions exist and the buyers are identical, these solutions are never unique.

I. INTRODUCTION

This paper has two major objectives. The first is to show that the incorporation of exposure constraints into the decision problem of bidding agents may dramatically change the character of "solutions" to an auction game. A second, more fundamental objective is to demonstrate that a Nash equilibrium is of questionable value as a solution concept for auction games. It is shown that Nash equilibria exist only occasionally and, when they do, often exist non-uniquely. The non-uniqueness or non-existence problem indicates that alternative solution concepts might be more accurate predictors of behavior in many auction situations. Because auctions are frequently used allocation mechanisms, one of the more important characteristics of theoretical models of auctions is that the results be reasonably consistent with observed data. Though more empirical work remains to be done, the preliminary indications are that the model developed in this paper compares favorably with previous work in this regard.

II. MULTIPLE-OBJECT AUCTIONS

Multiple-object auctions, such as those conducted by the United States Geological Survey for outer continental shelf oil and gas leases, represent an important type of market structure for the allocation of scarce, lumpy objects. Surprisingly, the economics literature on this subject is virtually nonexistent. Before discussing briefly the work that has been done on this subject, a few distinctions should be made about specific auction institutions.

First of all, this paper examines sealed-bid auctions rather than oral auctions. Second, it examines multiple-object rather than single-object auctions. In a single-object auction, each participant submits at most one bid for whatever item is being sold. A participant either submits the highest bid and is a winner, or goes home empty-handed.

A multiple-object auction consists of at least two single-object subauctions in which a bidder may submit losing bids in several subauctions yet still win something. A multiple-object auction can be conducted either sequentially or simultaneously. In case it is sequential, participants may use information from earlier subauctions in deciding strategies for later subauctions. This information might be very useful. For example two items for sale may be highly complementary, such as a right shoe and a left shoe. The information that the bidder has at the beginning of the second auction -- specifically, knowledge of who won the first auction -- is of great value to him. Ruling out secondary markets,[1] the second shoe has no value to a bidder who has lost the first auction, but it is of considerable value to the winner of the first auction. In a simultaneous auction, all bids are submitted before the outcome on any subauction has been revealed. Bids in the auction are vectors, with each component corresponding to a subauction.

Auctions can also be distinguished by the message space of the bidders. For example, in many contractual agreements bids are submitted in which the cost, time of completion, product quality and other variables may jointly determine the winning competitive bid. This paper ignores multi-variate bidding and deals only with auctions in which a bid is a scalar, price. The message space of an auction must also specify the subsets of the set of items to be sold on which bids are to be submitted. For example, bidders may be permitted to submit sealed tenders for every subset of the set of items being auctioned. The message space considered here is one in which bids may be submitted only on singleton subsets. Thus, the auction institution examined in this paper is a simultaneous, multiple-object discriminatory auction.

The focus of much of the bidding literature in the past has been on the choice of bids in single-object auctions. Implicit in such analysis is the view that little is lost by analyzing a multiple-object auction as a series of independent single-object auctions. But single-object auction models fail to explain a number of empirical phenomena. An example is the bidding behavior in auctions for offshore and outer continental shelf drilling rights.

One of the most striking observations is that two companies with identical information often bid much differently. Capen, Clapp and Campbell (1971) document this phenomenon in the 1969 North Slope auction. They found that two joint explorers, Humble and ARCO, bid much differently on individual tracts, although neither consistently bid much greater than the other on all 55 tracts.[2] Some authors interpret this to mean that these two companies simply imputed much different value estimates to the same information. While different companies will interpret the same exploration data as indicating different amounts of oil and gas, it is hard to believe that this can

explain as much variation in bids as was observed. On one tract Humble bid 17 times as much as ARCO bid and on another the ARCO bid was 33 times as much as Humble's.

One possible alternative explanation of such divergent bids is that the two companies used randomized, or mixed, strategies. This could result in significantly different bids, even if the companies made identical value estimates. A second explanation is that the companies faced bidding constraints. For example each company might have a "target" number of tracts it wants to win, or a maximum total bonus it can afford to pay, or some mix of these two objectives. This emphasizes an important limitation in previous models of optimal bidding in auctions. When more than one item is being sold, the objective function of a firm may not be simply the cross product of the expected net values of each of the items and the probabilities of winning each. In many situations the net value of an item is linked with the total number of items that are won.

Engelbrecht-Wiggans and Weber (1979) have constructed a "garage sale" model for the amount one should bid in each of several simultaneous auctions for identical goods, where the value of the first item won is $X > 0$ and the value of each subsequent item an agent wins is 0. Their analysis assumes an oral English auction, rather than sealed bids. The questions they ask are how many auctions to enter and how much to bid in each auction. In particular, they search for a symmetric Nash equilibrium. In one special case in which the number of bidders and the number of items are equal to n, Engelbrecht-Wiggans and Weber claim that if the number of auctions an agent can enter is limited to two, then as n goes to infinity the optimal Nash strategy to bid high on one randomly selected item and low on another.

There are a number of modifications of their model which are worth examining. These include:

(1) looking at the case where n is finite, rather than focusing on asymptotic results;

(2) examining the case where the number of agents does not equal the number of items;

(3) allowing agents to bid on as many items as they wish; and

(4) introducing an explicit constraint on the total value of the bids.

A few attempts have been made by past authors to incorporate explicit bidding constraints. Sakaguchi (1961) makes some progress characterizing Nash equilibrium pure strategy solutions when there are two items and two bidders. He offers an incomplete proof of a proposition which is presented (and correctly proved) as theorem 1 in this paper. Rothkopf (1977) formulates a decision-theoretic model of a bidder's optimal strategy in simultaneous auctions with a constraint on exposure, given a known expected payoff function, the only argument of which is an agent's own bid. The payoff function is also additively separable in the n objects at auction. Equilibrium strategies are not discussed.

Griesmer and Shubik (1963) and Cook, Kirby and Mehndiratta (1975) deal with constrained, simultaneous multi-object auctions in a slightly different context. Both papers assume that the bidders are bidding to sell (lowest bid wins), and have a resource constraint which

limits the number of auctions they can win. Cook, Kirby, and Mehndiratta (1975) use an "expected" exposure constraint rather than a certain exposure constraint. Griesmer and Shubik (1963) deal primarily with the case in which agents face the constraint that they must bid identically in all subauctions. The authors speculate that solutions often do not exist if different bids are allowed.

III. BIDDING CONSTRAINTS

The above authors have made an important contribution to the theory of competitive bidding by suggesting that solutions to an auction game can change if the total net payoff to an agent is not simply the sum of the net payoffs in each separate auction.

Nonlinear payoff functions apparently are present among agents bidding for outer continental shelf gas and oil leases. There is evidence that firms face constraints that limit the number of tracts they want to win. For example oil companies which win a substantial number of leases in a sale, sometimes resell some of them to other companies.[3] One can imagine a number of internal and external forces which might lead a firm to limit the number of tracts it bids on or the total amount of its winning bids. Because a firm does not have perfect instantaneous access to an infinite supply of capital at a constant rate of interest, the leasing division of a firm is likely to face a budget constraint. For this reason, one would expect that both the cash outlay for winning bids and the total development expenditures on all tracts won must be constrained. One might object by saying that although they face this constraint, it should not affect their bidding strategy so long as there is a secondary market in leases. Unfortunately there is a fault in that logic. If a firm submits the winning bid, it probably means that its estimate of the value of the tract was greater than all other valuations. Thus it is unlikely that the firm will receive as high a price in the secondary market as what it paid for the tract. Compounding this problem is that the attempt to resell the tract sends a signal to other firms that the tract is not worth as much to the firm as was originally believed.[4] For all the other firms know, the winning firm might have just noticed an error in its value estimate and for this reason wants to unload the tract. Hence a firm would not expect to obtain a price for the item as great as the bonus it paid in the auction.

There are several ways one might wish to formalize the budget constraint. Perhaps the most realistic is to postulate that each firm has a loss function, $L(C)$, in which C is the total amount of capital used to extract value from the items it wins. In the case of oil tracts the cost of exploration, purchase and development of each tract is

$$C = \sum_{j=1}^{J} \left[E_j + \delta_j \, (b_j + D_j) \right]$$

where

E_j = pre-auction exploration costs of tract j

b_j = bid on tract j

δ_j = 1 if b_j was the unique winning bid.

= 0 if b_j was a losing bid.

D_j = development costs of the tracts won. (Alternatively, $-D_j$ can be thought of as the capital obtained for j on a secondary market, if the item is resold instead of developed.)

One would expect that $L'(C) > 0$ and $L''(C) > 0$, to reflect costs of rapid expansion, increasing cost of capital in the lending market, and other costs which are not directly incorporated in C. If L is a smooth convex function of C then it can be loosely interpreted as a "soft" budget constraint, in the sense that the money cost of exploration purchase and development of tracts understates the true cost to the firm.

A "hard" budget constraint is an extreme case in which there exists some M such that $L(C) = \infty$ for $C > M$. Thus a second representation of the budget constraint might require that <u>in equilibrium</u> (or in the case of a mixed strategy equilibrium, <u>expected</u> in equilibrium)[5] a constraint $C \leq M$, can be satisfied.

Under the above formulation, one could model the firm's decision problem as being either static or sequential. In the sequential case, the firm first makes exploration decisions, then bidding decisions, and finally development decisions.

The model used in this paper postulates a "hard" budget constraint for reasons of analytical convenience. Exploration and development costs are ignored and thus the decision problem of a firm in static rather than sequential. The constraint faced by each agent, i, is: $\sum_{j=1}^{n} b_j^i \leq M^i$. This has been referred to in the literature as a constraint on exposure. This constraint makes the problem at hand a special case of a Colonel Blotto game.[6] Blackett (1954) describes this type of game the following way:

> Two players contending N independent battlefields distribute their forces to the battlefields before knowing the opposing deployment. The payoff on the i^{th} battlefield is given by a function $P_i(x,y)$ depending only on the battlefield and the opposing forces x and y committed to the battlefield by A and B. The payoff of the game as a whole is the sum of the payoffs on the individual battlefields.[7]

In our case, armies are dollars and battlefields are items. What makes our game a rather perverse Blotto game from the military standpoint is the particular nature of the payoff function. If you win a battle, you lose all your forces -- but gain the fort. If you lose a battle, you lose no forces, but fail to gain the fort. While this may not seem realistic on a battlefield it describes an auction quite adequately.

IV. THE MODEL

In general, participants will be indexed by the superscript i, items will be indexed by the subscript j. Let:

V_j^i = the value of item j to participant i

M^i = the budget of participant i

b_j^i = the bid of participant i on item j

I = the number of participants

J = the number of items

The budget constraint imposed in this model is that

$$(1) \qquad \sum_{j=1}^{J} b_j^i \leq M^i \quad \forall i.$$

A <u>pure strategy</u> for agent i, σ^i, is a J-vector of positive numbers. Thus the strategy space for any participant is

$$R_j^+ = \{\sigma^i = (b_j^i, \ldots, b_j^i) \mid b_j^i \geq b_j^* \ \forall\ j = i, \ldots, J\}$$

where b_j^* is the seller's reservation bid. It will be demonstrated below that the presence of a strictly positive reservation bid may alter bidding strategies profoundly.

A pure strategy $\sigma^i \in \Sigma^i$ is <u>feasible</u> if $\sigma^i \cdot 1 \leq M^i$

The subset of R_J^+ which includes only and all feasible pure strategies is Σ^i.

A pure strategy is <u>full</u> if $\sigma^i \cdot 1 = M^i$. The subset of Σ^i which includes only and all full pure strategies is $\bar{\Sigma}^i$.

<u>A feasible mixed strategy</u> of participant i is a distribution function $F^i(\cdot)$ defined over Σ^i.

A mixed strategy is <u>full</u> if the domain of F is $\bar{\Sigma}^i$.

The payoff function, $\pi^i(\vec{V}^i, M^i, \sigma^1, \ldots, \sigma^I)$, for individual i, where $\sigma^1, \ldots, \sigma^I$ are strategies of <u>all</u> the players is assumed to be:

$$\pi^i = M^i + \sum_{j=1}^{J} \delta_j^i (b_j^1, \ldots, b_j^I) \left[V_j^i - b_j^i\right]$$

where

$$\delta_j^i = 1 \text{ if } b_j^i > b_j^k \quad \forall\, k \neq i$$

$$= 0 \text{ if } b_j^i < b_j^k \quad \text{some } k \neq i$$

<u>Tie-breaking Rule</u>: If several participants in the auction tie for the winning bid on an item, they evenly divide the cost and ownership of the item:

$$b_j^i \geq b_j^k \quad \forall\, k \neq i \quad \rightarrow \quad \delta_j^i = \frac{1}{K}$$

where K is the number of agents submitting identical winning bids, equal to b_j^i.

Throughout the remaining analysis, four assumptions are maintained. The first assumption is implicit in the definition of $\delta(\cdot)$.

<u>Assumption 1</u>: Values are linear in the object, in the sense that if an individual receives a share α^i of item j, the value of that share to him is $\alpha^i V_j^i$.

<u>Assumption 2</u>: Values are constant and known with certainty.

This assumption limits the comparability of the results of this paper with the results found in the standard bidding literature (e.g. Wilson [1977]). However, insight into the case where values are uncertain may well require a full understanding of the certainty case if budget constraints exist.

The assumption that values are constant precludes the possibility that winning item 1 affects the value of item 2.

This restriction is strong, for it excludes auctions for complementary items (e.g. bidding on a left shoe and a right shoe) and "duplicates" such as the extreme case in which a bidder attaches positive value only to the first item won, and resale is impossible or costly.[8]

<u>Assumption 3</u>: $I \geq 2$, $J \geq 2$, and all items $j = 1, \ldots, J$ are auctioned simultaneously.

The first part of this assumption merely rules out trivial cases. The second part of the assumption rules out sequential auctions. For interesting examples of sequential auctions and some analysis about their characteristics see Englebrecht-Wiggins (1977).[9]

<u>Assumption 4</u>: $M^i < \sum_{j=1}^{J} V_j^i$.[10]

This assumption is a necessary condition for the budget constraint to be binding.

In the analysis that follows, three types of symmetries appear. They provide a convenient classification of the cases which must be examined.

The first type of symmetry, (S1), is between values of the items to each person. Are the items the same? In this situation, for each individual, i,

$$V_j^i = V_k^i \quad \forall\, j,k \qquad \text{(S1)}$$

although it may be the case that

$$V_j^i \neq V_j^k \quad \text{for some } i,j,k.$$

A second symmetry, (S2), exists if the value of each item j, is the same for all individuals. That is

$$V_j^i = V_j^k \quad \forall \; i,k \tag{S2}$$

although it may be the case that

$$V_j^i \neq V_k^i \quad \text{for some } i,j,k.$$

The third type of symmetry, (S3), exists if all individuals have the same budget constraint. That is

$$M^i = M^k \quad \forall \; i,k. \tag{S3}$$

A second mode of classification is the scope of the market. How many items are auctioned off simultaneously? Finally, a third mode of classification is the depth of the market. How many participants are involved in the market? As will become apparent, these last two characteristics of the auction market, scope and depth, interact in very interesting ways and largely determine whether solutions to the auction game exist.

In what follows, an attempt is made to specify exactly when Nash equilibria exist and to characterize these Nash equilibria in terms of symmetry, profitability, and other criteria.

V. SYMMETRIC BIDDING STRATEGIES

Theorem 1:[11] Assume I = 2, J = 2 and conditions (S2), (S3) are satisfied. There will always exist a unique Nash equilibrium pure strategy at

$$\sigma^1 = \sigma^2 = (A_1, A_2)$$

where

$$A_i = \max\left\{0, \min\left[\frac{M + V_i - V_j}{2}, M\right]\right\} . \tag{*}$$

The proof of Theorem 1 requires a number of initial observations to be made.

Lemma 1: If I = 2, J ≥ 2 and (S2), (S3) are satisfied, then in equilibrium, $\sigma^1 = \sigma^2$.

Proof: Suppose $\sigma^1 \neq \sigma^2$. We can assume without loss of generality that $b_1^1 < b_1^2$. This implies that bidder 2 could be better off bidding $b_1^2 - \varepsilon$ on item 1. Hence, if $\sigma^1 \neq \sigma^2$, then (σ^1, σ^2) is not a Nash strategy pair.

□

Lemma 2: If I = 2, J ≥ 2 and (S2), (S3) are satisfied, then in equilibrium $\sum_{j=1}^{J} b_j^i = M^i$

Proof: Suppose that $\sum_{j=1}^{J} b_j^1 = \sum_{j=1}^{J} b_j^2 < M$, which we can assume from Lemma 1. The payoff for each individual is

$$\pi = M + \frac{\sum_{j=1}^{J} (V_j - b_j)}{2}$$

since they tie on each item. (Note that because bids, values and budgets are identical the superscripts can be ignored.)

By Assumption 4 we know that $b_k < V_k$ for some k. Hence, because budgets are not exahausted, player 1 can bid $b_k + \varepsilon$ on that item, and receive a new profit of

$$\pi' = M + V_k - b_k - \varepsilon + \frac{\sum_{\substack{j=1 \\ j \neq k}}^{J} (V_j - b_j)}{2} .$$

The effect on profits of raising b_k by ε is:

$$\pi' - \pi = V_k - b_k - \varepsilon - \frac{V_k - b_k}{2}$$

$$= \frac{V_k - b_k}{2} - \varepsilon .$$

Since $V_k - b_k > 0$, there exists an ε small enough so that $\pi^1 - \pi > 0$. Hence the original bid configuration was not a Nash equilibrium, and so Nash strategies must be full.

□

<u>Lemma 3</u>: Assume I = 2, J = 2 and conditions (S2), (S3) are satisfied. Let $\sigma^1 = (b_1^1, b_2^1)$ and $b_2^1 \leq b_1^1$. If (σ^1,σ^1) is an equilibrium strategy pair, then

$$(V_1^1 - b_1^1)b_2^1 = (V_2^1 - b_2^1)b_2^1.$$

<u>Proof</u>: Since, by assumption, (σ^1,σ^1) is an equilibrium, it must be the case that no unilateral bid change can make that agent better off. Since equilibrium strategies are full, and strategies are symmetric, any unilateral change of bid necessarily means that each individual wins exactly one item. Assume that the conclusion of Lemma 3 is not true, e.g., that

b_2 is not equal to zero and that $(V_1 - b_1) \neq (V_2 - b_2)$. Without loss of generality, let $V_1 - b_1 > V_2 - b_2$. The profit resulting from σ^1 is given by

$$\pi(\sigma^1) = \frac{V_1 + V_2 - M}{2} .$$

If a bidder now bids $\sigma^2 = (b_1 + \varepsilon, b_2 - \varepsilon)$, the profit becomes

$$\pi(\sigma^2) = V_1 - b_1 - \varepsilon > \frac{V_1 - b_1}{2} + \frac{V_2 - b_2}{2} = \pi(\sigma^1)$$

for small enough ε. Hence (σ^1,σ^2) is not an equilibrium strategy pair. Thus, for σ^1 to be an equilibrium, either b_2 equals zero or $(V_1^1 - b_1^1) = (V_2^1 - b_2^1)$, hence

$$(V_1 - b_1)b_2^1 = (V_2^1 - b_2^1)b_2^1 .$$

□

The proof of Theorem 1 follows immediately from Lemma 2 and Lemma 3, which produce two equations in two unknowns, b_1 and b_2, subject to the constraint that b_1, b_2 are nonnegative:

$$b_1 + b_2 = M$$

$$(V_1 - b_1)b_2 = (V_2 - b_2)b_2 .$$

This establishes that if a Nash equilibrium pure strategy exists, it must satisfy (*).

To show that (*) in fact is a Nash equilibrium can be demonstrated by showing that if one agent uses (*), the other can make himself no better off by using a strategy other than (*). We need consider only the following two cases:

Case 1: Nonnegativity constraints are not binding. Fix bidder 1's strategy at (*). If bidder 2 bids something different, he wins at most one item. His profit then is:

$$\pi' = v_j - b_j \text{ where } b_j > b_j^*.$$

$$\text{Hence} \qquad \pi' = V_j - b_j < 2\,\frac{V_j - b_j^*}{2} = \pi^*.$$

Case 2: Some nonnegativity constraint is binding. Player A bids (0,M). Suppose player B bids (a,b) where $a > 0$, $b < M$. Then his profit is $[V_1 - a]$. Since the nonnegativity constraint is binding, it must be the case that

$$\frac{M + V_1 - V_2}{2} < 0$$

which implies that

$$\frac{V_1 + V_2 - M}{2} > V_1$$

If $a > 0$, then

$$\frac{V_1 + V_2 - M}{2} > V_1 - a$$

The LHS of the last inequality is player B's profit if strategy (*) is adopted, so (*) is an equilibrium.

Thus we have shown that (*) characterizes the unique Nash equilibrium strategy configuration to this bidding game.

□

Theorem 2: Assume I 2, J 2 and conditions (S2), (S3) are satisfied. If there are at least 3 items, $j = 1,2,3$ such that it is possible to have

$$V_1 - b_1 = V_2 - b_2 = V_3 - b_3$$

where b_1, b_2, b_3 are all non-negative and

$$b_1 + b_2 + b_3 = M$$

then a symmetric pure strategy Nash equilibrium cannot exist.

Proof: By Lemma 3,[12] we know that a necessary condition for a Nash equilibrium when the bid non-negativity constraints are not binding is:

$$V_j - b_j^* = V_k - b_k^*.$$

Assume that $\sigma^1 = \sigma^2 = \ldots = \sigma^I = (b_1, \ldots, b_J)$ is a Nash equilibrium strategy I-tuple. Let $\Sigma = (\sigma^1, \ldots, \sigma^I)$. Without loss of generality, assume that $b_1, \ldots, b_K$ are all positive, where $3 \leq K \leq J$, and $\sum_{j=1}^{J} b_j = M > 0$.

Let π^* be the payoff each participant receives under this strategy I-tuple:

$$\pi^* = \sum_{j=1}^{J} \frac{1}{I} (V_j - b_j) + M .$$

Let

$$\pi_K^* = \sum_{j=1}^{K} \frac{1}{I} (V_j - b_j)$$

and

$$\pi_J^* = \sum_{j=K+1}^{J} \frac{1}{I} (V_j - b_j)$$

so

$$\pi^* = \pi_K^* + \pi_J^* + M .$$

Since Σ is a Nash equilibrium I-tuple, it must be the case that no agent can unilaterally receive a larger profit by departing from Σ. In particular, an agent cannot reduce the bid on one item, redistribute it over other items and receive a greater profit. Suppose an agent bids $b_k - \varepsilon$ on item $K < J$ and increases the bids on items 1 through $K - 1$ by $\frac{\varepsilon}{K - 1}$. The new profit is:

$$\hat{\pi} = \sum_{j=1}^{K-1} \left[V_j - (b_j + \frac{\varepsilon}{K - 1})\right] + \sum_{j=K+1}^{J} \frac{1}{I} (V_j - b_j) + M .$$

This fails to improve the payoff associated with Σ if and only if

$$\sum_{j=1}^{K-1} \left[V_j - (b_j + \frac{\varepsilon}{K - 1})\right] \leq \frac{1}{I} \sum_{j=1}^{K} (V_j - b_j) .$$

This is true for all $\varepsilon > 0$ if and only if

$$\sum_{j=1}^{K-1} [V_j - b_j] \leq \frac{1}{I} \sum_{j=1}^{K} (V_j - b_j) .$$

However, in order for Σ to have been a Nash equilibrium, this condition must hold not only for item K, but for all items $k \in \{1, \ldots K\}$. Thus we have

$$\sum_{\substack{j=1 \\ j \neq k}}^{K} [V_j - b_j] \leq \frac{1}{I} \sum_{j=1}^{K} (V_j - b_j) \quad k=1, \ldots, K$$

These can be rewritten as

$$(I - 1) \sum_{\substack{j=1 \\ j \neq k}}^{K} [V_j - b_j] \leq V_k - b_k \quad k=1, \ldots, K$$

Summing these K inequalities, we obtain

$$(I - 1) \sum_{k=1}^{K} \sum_{\substack{j=1 \\ j \neq k}}^{K} [V_j - b_j] \leq \sum_{j=1}^{K} [V_j - b_j]$$

$$\Rightarrow (K - 1)(I - 1) \sum_{j=1}^{K} [V_j - b_j] \leq \sum_{j=1}^{K} [V_j - b_j]$$

$$\Rightarrow (K - 1)(I - 1) \leq 1$$

Hence, in multiple-object auctions if either $K > 2$ or $I > 2$, symmetric Nash equilibrium pure strategies cannot exist.

□

<u>Example: Bidding With a Reservation Bid Requirement</u>

In this example the concept of a reservation price is introduced. The auctioneer requires a minimum bid he will accept. In this situation, Nash equilibrium bidding strategies may not exist.

Let $I = 2$, $J = 2$

$B_R \equiv$ reservation bid

Assume S2 and S3 hold.

<u>Theorem 3</u>: Nash equilibrium pure strategies do not exist if $B_R < \frac{V_2}{2}$ and $M + V_2 < V_1 < M + V_2 + (V_2 - 2B_R)$.

<u>Proof</u>: The strategy of the proof is to look at the boundary solutions. By the earlier theorem we know that the unconstrained problem has a solution:

$$\sigma^1 = \sigma^2 = (M,0).$$

Under this strategy pair each individual payoff is equal to

$$\pi_1 = \pi_2 = \pi = M + \frac{V_1 + V_2 - M}{2} = \frac{M + V_1 + V_2}{2}$$

under the old rule (i.e. $B_R = 0$).

However, if $B_R > 0$, then

$$\pi_1 = \pi_2 = \pi = M + \frac{V_1 - M}{2} = \frac{M + V_1}{2} .$$

Now, by assumption

$$V_1 < M + V_2 + (V_2 - 2B_R)$$

which implies that

$$V_2 - B_R > \frac{V_1 - M}{2}$$

$$\Longrightarrow M + V_2 - B_R > \frac{M + V_1}{2} .$$

The left hand side of the inequality can be achieved unilaterally by either player (say player 1) simply by changing his bid to

$$\hat{\sigma}^1 = (A, B_R)$$

where

$$A \in [0, M - B_R].$$

Player two now has an incentive to bid less than M on item 1, in fact he will want to bid as low as A. Now, we need to check if

$$\hat{\sigma}^1 = \hat{\sigma}^2 = (M - B_R, B_R)$$

is a Nash equilibrium pure strategy.[13] This is not a Nash strategy pair because

$$V_2 - b_1 < V_1 - b_2$$

so by lemma 3, the bidder has an incentive to cut his bid on item 2 and increase his bid on item 1. In fact, this will be the case whenever $b_1 \geq 0$. Since the boundary solution ($b_1 = 0$) does not support a Nash equilibrium, the claim is demonstrated.

□

VI. ASYMMETRIC BIDDING STRATEGIES

Two questions immediately arise. When do asymmetric Nash equilibrium pure strategies exist, and what form do they take when they do exist. A first observation, that there are conditions where asymmetric Nash equilibria exist, can be made with reference to an example. This case is rather trivial, in the sense that no agent earns a profit.

Example 1: Suppose there are three agents and three items. The agents all have identical budget constraints and identical values

$$M^1 = M^2 = M^3 = 10.0$$

$$V_j^i = 4.0 \quad i = 1,2,3; \quad j = 1,2,3.$$

The following bid configuration is a Nash equilibrium.

Example 3:

Bidder \ Items	1	2	3
1	4	4	2
2	2	4	4
3	4	2	4

One can immediately see that no agent earns a profit, because all winning bids are at the value of the item. Furthermore, one can see why it is a Nash equilibrium. Clearly, for any Nash equilibrium, there have to be at least two bidders tied for the highest bid on eath item. Otherwise the winning bidder would have an incentive to cut the winning bid to just barely above the bid of the nearest competitor. This is stated more clearly below.

Lemma 4: Assume S2, $I \geq 2$.

If $\Sigma = (\sigma_1, \ldots, \sigma_I)$ is a Nash equilibrium pure strategy I-tuple, then there must be at least two bidders tied for the highest bid on each item.

Proof: Let Σ be a Nash equilibrium pure strategy K-tuple. Further suppose that for some (i,j), $b_j^i > b_j^k \; \forall\, k \neq i$. Then the payoff for item j to individual i is

$$V_j^i - b_j^i = \pi_{ij} \; .$$

Agent i can earn more profit on item j by bidding $b_j^i - \varepsilon$, for some ε small enough so that $b_j^i - \varepsilon$ is still the winning bid on item j. Agent i then receives

$$\pi'_{ij} = V^i_j - b^i_j + \varepsilon > \pi_{ij} .$$

This contradicts the assumption that Σ is an equilibrium.

□

Example 1 demonstrates the "high-low" class of strategies, in which people bid up to their value on some items and very little on the other items. In fact, bidder 1 is indifferent between bidding 2 on the third item and bidding any number between 0 and 2 on that item. Thus the example has an infinite number of pure strategy equilibria, of the form $b_1 = b_j = 4$, $b_k = a$ where $a \in [0,2]$.

One wonders if such "high-low" equilibrium strategies ever exist which support positive profits. This question can be answered in the affirmative, by giving an example.

Example 2:

$$M^1 = M^2 = M^3 = 6.0$$

$$V^i_j = 4.0$$

Consider the strategies:

Bidder \ Items	1	2	3
1	3	3	0
2	0	3	3
3	3	0	3

One can see that no player can make larger profits by departing from this

strategy. On the other hand, we can easily construct an example in which no such equilibrium exists. For instance if the auction includes a fourth player with the same parameters as the other agents, a Nash equilibrium no longer exists.

Another feature of asymmetric solutions to symmetrically parameterized auctions is that no agent can win more than 2 of the items for which both the winning bid and the net profit are strictly positive.

Lemma 5: Assume S2, S3, $I \geq 2$, $J > 3$. Let $\Sigma = (\sigma_1, \ldots, \sigma_I)$ be an equilibrium. If an agent, i, earns profits greater than 0 under Σ, then i can win at most two of the items i submitted positive bids on.

Proof: Suppose that agent i is tied with other bidders on three items, 1, 2, and 3, earning profits at most:

$$\pi_i = \sum_{j=1}^{3} (V^i_j - b^i_j) \frac{1}{2} .$$

Suppose, without loss of generality, that $V^i_1 - b^i_1 \leq V^i_2 - b^i_2 \leq V^i_3 - b^i_3$. If i bids $b^i_1 - \varepsilon$ on the first item and $b^i_2 + \frac{\varepsilon}{2}$, $b^i_3 + \frac{\varepsilon}{2}$ on the second and third items, then profits become:

$$\pi_i = \sum_{j=2}^{3} (V^j_i - b^j_i) - \varepsilon > \sum_{j=1}^{3} (V^j_i - b^j_i) \frac{1}{2}$$

for some $\varepsilon > 0$.

□

An additional proposition is demonstrated below:

Lemma 6: Assume S2, S3, $I \geq 2$, $J \geq 2$. Let $\Sigma = (\sigma_1, \ldots, \sigma_I)$ be an equilibrium. If i earns positive profits under Σ, then if $b_j^i < b_j^k$ for some $k \neq i$, then $b_j^i = 0$.

Proof: If $b_j^i > 0$, then agent i can bid $b_j^i - \varepsilon$ on j and $b_k^i + \varepsilon$ on some item k for which b_k^i is tied for the winning bid. Agent i will then be the sole highest bidder on item k thereby capturing all of its value.

□

In addition, from Lemma 3 an agent i must be earning equal profits on the items i wins with a positive bid. That is

$$V_j^i - b_j^i = V_k^i - b_k^i$$

whenever i is a winner in the $j^{\underline{th}}$ and $k^{\underline{th}}$ items. These restrictions are really quite strong, and seem to limit to only a few special cases the situations in which asymmetric Nash equilibria exist that generate positive profits when 3 or more people bid for 3 or more items.

As stated above, Nash solutions generally are not unique. In particular, example 1 has six permutations of the given individual strategies, all of which are Nash equilibria. Given that this is the case, how would an individual decide which strategy to use? In this situation one can hardly expect a Nash equilibrium to be achieved by non-cooperative behavior, because the agents must, in a sense, agree beforehand which equilibrium strategy I-tuple to play.

The preceeding discussion assumes symmetry in agents and

values of items. The following example shows that if budget constraints are "not too binding," it is the symmetry between individuals, rather than symmetry between items, which leads to the existence of non-unique Nash equilibria. Suppose that the individuals are identical and V_1, V_2 have the greatest value. Furthermore, suppose that there are fewer items then there are bidders. Also, assume that $M \geq V_1 + V_2$. Then a Nash equilibrium exists. But if there are "too many" more items than bidders, so that the budget constraint makes it impossible for at least 2 agents to bid their value on each item, then Nash equilibria may not exist. The following sequence of examples illustrates these points.

Example 3a: I = 3 J = 3

$V_1^i = 5 \quad \forall i$

$V_2^i = 3 \quad \forall i$

$V_3^i = 2 \quad \forall i$

$M_i = 8 \quad \forall i$

One permutation of Nash strategy triples is shown below:

Bidder \ Items	1	2	3
1	5	1	2
2	3	3	2
3	5	3	0

Example 3b: Same parameters as example 3a, except that

M = 7.

By reducing the budgets no Nash equilibrium exists. The reason is that it is no longer possible for any participant to bid the full value of items one and two simultaneously.

On the other hand, if there are four bidders with these parameters, Nash equilibria do exist. One such bid configuration is shown in the following table.

Example 3c:

Bidder \ Items	1	2	3
1	5	0	2
2	5	0	2
3	2	3	2
4	2	3	2

In both 3a and 3c the Nash equilibria all generated zero profits. This will always be the case when values are different across items (but identical across individuals) budgets are identical, and both the number of items and the number of bidders is greater than two. This is stated more precisely in the next theorem.

Theorem 4: Let $I \geq 3$, $J \geq 3$. Assume S2, S3

If

(1) $\exists\, j_1, j_2$ s.t. $V_{j_1} \neq V_{j_2}$;

(2) everyone submits positive bids on at least 2 items;

(3) all items are bid on,

then at any Nash equilibrium pure strategy I-tuple, $\Sigma = (\sigma^1, \ldots, \sigma^I)$,

$$\sum_{j \in A^i} V_j^i - b_j^i = 0 \quad i = 1, \ldots, I$$

where

$$A^i = \{j \mid b_j^i \geq b_j^k \quad k=1, \ldots, I\}$$

Proof: Suppose that some agent, i, makes a positive profit. First we show that in equilibrium this agent must make an equal profit on all items for which he submitted a positive winning bid.

Suppose that agent i has submitted winning bids on K items, and does not earn an equal profit on all K items. From Lemma 4, in equilibrium at least two agents must have submitted winning bids on each of these items. From lemma 5 and lemma 6 we know that agent i submitted strictly positive winning bids on at least two items, say j and k. Suppose, without loss of generality,

$$V_j^i - b_j^i > V_k^i - b_k^i .$$

Then i can earn greater profits by bidding slightly more on j and slightly less on k, since $V_j^i - b_j^i > \frac{1}{2}(V_j^i - b_j^i + V_k^i - b_k^i)$.

In fact, in equilibrium, if the budget constraint is binding, all strictly positive bids must be winning bids, by lemma 6. Since all agents have the same budget and the same values, it must be the case that

$$V_j - b_j^* = V_k - b_k^* \quad \forall\, j,k$$

where b_j^* and b_k^* are the winning bids of j and k, respectively.

Therefore, if A and B are the two least valuable items, then

$$b_A^* + b_B^* < M$$

where b_A^* and b_B^* are the winning bids on items A and B. In this case any agent can obtain greater profits by bidding $b_A^* + \varepsilon$ on A, $b_B^* + \varepsilon$ on B, and 0 on all other items. This contradicts the Nash assumption, so the theorem is proven.

□

Assumption (2) in the statement of the theorem is actually stated just to rule out two special cases. One such case occurs if one item is so much less valuable than the other items that nobody bids on it. Referring back to example 4, if there were a fourth item valued at $\frac{1}{8}$, then the following bid configuration is a Nash equilibrium.

Example 4:

Bidder \ Items	1	2	3	4
1	3	3	0	0
2	0	3	3	0
3	3	0	3	0

Each agent is bidding on at least two items. However, all items receiving positive bids have the same value, so we are essentially back in the "identical value" type of auction. An alternative assumption to avoid this special case is that

$$V^{\alpha} - V^{\beta} < M$$

where V^{α} os the highest value of all items and V^{β} is the second highest value ($V^{\alpha} \neq V^{\beta}$). A third possibility is to require that

$$i \neq j \Rightarrow V_k \neq V_j \quad \forall i, j.$$

The second special case to rule out is when some players bid on only one very valuable item. One way this can occur is if every item is more valuable than the budget of each agent. In this case a Nash equilibrium may exist in which each agent's entire budget is bid on one item, as in the following example.

Example 5:

$$M = 6 \quad V_1 = 18 \quad V_2 = 16 \quad V_3 = 15$$

A Nash equilibrium strategy 6-tuple is:

Bidder \ Items	1	2	3
1	6	0	0
2	6	0	0
3	0	6	0
4	0	0	6
5	0	6	0
6	0	0	6

This special case is ruled out by considering only cases in which everyone bids on at least two items. It is actually only necessary to require that at least one individual bid on more than one item.

The point of this discussion has been to demonstrate that assumption (2) in the theorem is not as strong as it may at first appear. There exist fairly weak sufficient conditions for (2) to hold. Furthermore, it rules out cases which are, for the most part uninteresting.

VII. CONCLUSION

This paper has demonstrated several properties of Nash equilibria in multiple-object simultaneous sealed-bid auctions in which the participants face a constraint on exposure. First of all it has been shown that if there are more than two bidders and more than two objects, symmetric pure strategy Nash equilibria do not exist. Second, the presence of a reservation bid requirement can also result in the nonexistence of Nash equilibria, even if there are only two bidders and two objects. Third, when there are more than two bidders and two objects, sufficient conditions were derived for Nash equilibria to result in zero profits to the buyers. The conditions were fairly weak, indicating that when a Nash equilibrium exists, profits will often be zero.

The lack of symmetric pure strategy Nash equilibria is particularly interesting. The implication is that Nash equilibria, when they exist, can be realistically achieved only if the bidders cooperate with each other. Referring back to example 3, bidder 1 will submit (4,4,2) only if he knows bidders 2 and 3 will submit (4.2.4) and 2.4.4). Otherwise, the first agent's optimal response will be something else. Collusion is required for the buyers to coordinate their bids. Such collusion, unlike prisoner's dilemma situations which characterize many collusive arrangements such as cartels, is stable, for the point of collusion is a Nash equilibrium.

FOOTNOTES

1. In the analysis that follows, it is assumed that secondary markets do not exist.

2. Capen, Clapp, and Campbell (1971), pp. 642-643.

3. Engelbrecht-Wiggans and Weber (1979) cite a congressional study (1976), "An Analysis of the Economic Inpact of the Current OCS Bidding System," prepared for Representative Hughes (D. N.J.).

4. This is perhaps analogous to the "market for lemons" problem in which the bad drives out the good. In this case, one would expect resale value for even very good tracts to be far below the cash bonus originally paid for them.

5. See Engelbrecht-Wiggans (1979), p. 37; and Cook, Kirby, and Mehndiratta (1975), p. 729 ff.

6. This connection was first noticed by Sakaguchi (1962).

7. Blackett (1954), p. 55.

8. Engelbrecht-Wiggans and Weber (1979) deal with auctions in which the items were duplicates.

9. Of particular interest is his brief discussion of horse auction. See also Schotter (1974), Engelbrecht-Wiggans (1977), and Brams and Straffin (1979).

10. This merely rules out trivial cases.

11. This theorem can be found in Sakaguchi (1962), where it was first stated.

12. Although lemma 3 was proved for the two-bidder, two-item case, one can easily see that the "equal profits condition" is necessary for an equilibrium in the n-bidder, m-item case as long as non-negativity constraints on bids are not binding. If profits are not equal between two items which an agent submits positive bids on, then the agent has an incentive to bid slightly lower on the item with less profit and slightly higher on the item which has a greater profit.

13. We need not examine (A, B_R) where

$$A < M - B_R$$

because either agent could unilaterally bid $A + \varepsilon$ and be better off.

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