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ASSET PRICES IN A PRODUCTION ECONOMY

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SECTION 1: INTRODUCTION

This paper develops an intertemporal general equilibrium theory of capital asset pricing. It is an attempt to put together ideas from the modern finance literature and the literature on stochastic growth models. In this way we will obtain a theory that ultimately is capable of addressing itself to general equilibrium questions such as: (1) What is the impact of an increase in the corporate income tax upon the relative prices of risky stocks? (2) What is the impact of an increase in progressivity of the personal income tax upon the relative price structure of risky assets? (3) What conditions on tastes and technology are needed for the validity of the Sharpe-Lintner certainty equivalence formula and the Ross (1976) arbitrage theory? -- and so forth.

The theory presented here derives part of its inspiration from Merton (1973). However, as pointed out by Hellwig (undated), Merton's intertemporal capital asset pricing model (ICAPM) is not a general equilibrium theory in the sense of Arrow-Debreu (i.e., the <u>technological</u> sources of uncertainty are not related to the equilibrium prices of the risky assets in Merton). We do that here and preserve the empirical tractability of Merton's formulation.

Basically what is done here is to modify the stochastic growth model of Brock-Mirman (1972) in order to put a nontrivial investment decision into the asset pricing model of Lucas (1978). This is done in such a way to preserve the empirical tractability of the Merton formulation and at the same time determine the risk prices derived by Ross (1976) in his arbitrage theory of asset pricing. Ross's price of systematic risk k at date t denoted by λ_{kt} which is induced by the source of systematic risk $\tilde{\delta}_{kt}$ is determined by the covariance of the marginal utility of consumption with δ_{kt}. In this way Ross's λ_{kt} are determined by the interaction of sources of production uncertainty and the demand for risky assets. Furthermore, our model provides a context in which conditions may be found on tastes and technology that are sufficient for equilibrium returns to be a linear function of the uncertainty in the economy. Linearity of returns is necessary for Ross's theory.

The paper proceeds as follows. Section 1 contains the Introduction. Section 2 presents an N process version of the 1 process stochastic growth model of Brock-Mirman (1972). The N process growth model will form the basis for the quantity side of the asset pricing model developed in Section 3.

In Section 2 it is indicated that optimum paths generated in the N process model are described by time independent continuous optimum policy functions a la Bellman. A functional equation is developed that determines the state valuation function using methods that are standard in the stochastic growth literature. It is also indicated that for any initial state the optimum stochastic process of investment converges in distribution to a limit distribution independent of the initial state. The detailed analysis of these questions is done in Brock (1978).

Section 3 converts the growth model of Section 2 into an asset pricing model by introducing competitive rental markets for the capital goods and introducing a market for claims to the pure rents generated by the i^{th} firm $i = 1,2,\ldots,N$. Each of the N processes is identified with one "firm." Firms pay out rentals to consumers. The residual is pure rent. Paper claims to the pure rent generated by each firm i and a market for these claims is introduced along the line of Lucas (1978).

Equilibrium is defined using the concept of rational expectations as in Lucas. That is, both sides of the economy possess subjective distributions on pure rents, capital rental rates, and share prices. Both sides draw up demand and supply schedules conditioned on their subjective distributions. Market clearing introduces an objective distribution on pure rents, capital rental rates, and share prices. A rational expectations equilibrium, abbreviated by R.E.E., is defined by the requirement that the objective distribution equals the subjective distribution at each date. I hasten to add that no problems of incomplete information will be dealt with in this paper.

In Section 3 it is shown using recent results of

Benveniste-Scheinkman (1977) that the quantity side of an R.E.E. is identical to the quantity side of the N process growth model developed in Section 2. The key idea used is the Benveniste-Scheinkman result that the standard transversality condition at infinity is <u>necessary</u> as well as sufficient for an infinite horizon concave programming problem.

The financial side of the economy is now easy to develop. A unique asset pricing function for stock i of the form $P_i(y)$ is shown to exist by use of a contraction mapping argument along the line of Lucas.

Section 4 uses a special case of the model in Section 3 to develop an intertemporal general equilibrium theory that determines the risk prices of Ross endogenously. Capital asset pricing formulae such as the Sharpe-Lintner certainty equivalence (SL) formula are derived in Section 4. It is shown there that the SL formula can be derived only if the asset pricing function is linear in the state variable.

The convergence result in Section 2 allows stationary time series methods based on the mean ergodic theorem to be used to estimate the risk prices of Ross provided that the economy is in stochastic steady state.

In Section 5 an explicit example of the N process model is solved for the optimum in closed form. The asset pricing function $P_i(y)$ turns out to be linear in output y for this case. The risk prices of Ross can also be calculated in closed form for the example.

Finally, the Appendices develop technical results that are needed but somewhat tangential to the main issue addressed in each section.

Notations

Equations are numbered consecutively within each section. Thus, for example, Equation 2 in Section 3 is written "(3.2)." Assumptions, theorems, lemmas, and remarks are numbered consecutively within each section. For example, Assumption 2 in Section 3 will be written "Assumption 3.2."

The convention is the same in the Appendix except that "A" appears to separate entities in the Appendix from those in the main text. For example, Assumption 2 in the Appendix to Section 3 will be written "Assumption A 3.2."

Finally, we should mention that after this paper was written we found the papers by Cox, Ingersoll, and Ross (1978) and by Prescott and Mehra (1977) which are similar in spirit to this paper. Other related papers are Johnsen (1978) and Richard (1978). Nevertheless, the question addressed and the methods used differ substantially in all of these papers.

SECTION 2: THE OPTIMAL GROWTH MODEL

Since the model to be given below is studied in detail in Brock (1978) we shall be brief where possible.

The model is given by

(2.1) $$\text{Maximize } E_1 \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$$

(2.2) $$\text{s.t. } c_{t+1} + x_{t+1} - x_t = \sum_{i=1}^{N} [g_i(x_{it}, r_t) - \delta_i x_{it}]$$

(2.3) $$x_t = \sum_{i=1}^{N} x_{it},\ x_{it} \geqq 0,\quad i = 1,2,\ldots,N,\quad t = 0,1,2,\ldots$$

(2.4) $$c_t \geqq 0,\quad t = 1,2,\ldots$$

(2.5) $$x_0,\ x_{i0},\ i = 1,2,\ldots,N,\ r_0 \text{ historically given}$$

where E_1, β, u, c_t, x_t, g_i, x_{it}, r_t, δ_i denote mathematical expectation conditioned at time 1, discount factor on future utility, utility function of consumption, consumption at date t, capital stock at date t, production function of process i, capital allocated to process i at date t, random shock which is common to all processes i, and depreciation rate for capital installed in process i, respectively.

The space of $\{c_t\}_{t=1}^{\infty}$, $\{x_t\}_{t=1}^{\infty}$ over which the maximum is being taken in (2.1) needs to be specified. Obviously decisions at date t should be based only upon information at date t. In order to make the choice space precise some formalism is needed. We borrow (copy) from Brock-Majumdar (1978) at this point.

The environment will be represented by a sequence $\{r_t\}_{t=1}^{\infty}$ of real vector valued random variables which will be assumed to be independently and identically distributed. The common distribution of r_t is given by a measure μ: $\mathcal{B}(R^m) \to [0,1]$

where $\mathcal{B}(R^m)$ is the Borel σ-field of R^m. In view of a well-known one-to-one correspondence (see, e.g., Loéve 1963, pp. 230-231), we can adequately represent the environment as a measure space (Ω, F, ν) where Ω is the set of all sequences of real m vectors, F is the σ-field generated by cylinder sets of the form $\prod_{t=1}^{\infty} A_t$ where

$$A_t \in \mathcal{B}(R^m),\ t = 1,2,...$$

and

$$A_t = R^m$$

for all but a finite number of values of t. Also ν (the stochastic law of the environment) is simply the product probability induced by μ (given the assumption of independence).

The random variables r_t may be viewed as the t-th coordinate function on Ω, i.e., for any $\omega = \{\omega_t\}_{t=1}^{\infty} \in \Omega$, $r_t(\omega)$ is defined by

$$r_t(\omega) = \omega_t.$$

We shall refer to ω as a possible state of the environment (or an environment sequence) and shall refer to ω_t as the environment at date t. In what follows, F_t is the σ-field guaranteed by partial histories up to period t, (i.e., the smallest σ-field generated by cylinder sets of the form $\prod_{\tau=1}^{\infty} A_\tau$ where A_x is in $\mathcal{B}(R^m)$ for all t, and $A_\tau + R^m$ for all $\tau > t$). The σ-field F_t contains all of the information about the environment which is available at date t.

In order to express precisely the fact that decisions c_t, x_t only depend upon information that is available at the time the decisions are made, we simply require that c_t, x_t be measurable with respect to F_t.

Formally the maximization in (2.1) is taken over all stochastic processes $\{c_t\}_{t=1}^{\infty}$, $\{x_t\}_{t=1}^{\infty}$ that satisfy (2.2)-(2.5) and such that for each t = 1,2,...,c_t,x_t are measurable with respect to F_t. Call such processes "admissible."

Existence of an optimum $\{c_t\}_{t=1}^{\infty}$, $\{x_t\}_{t=1}^{\infty}$ may be established by imposing an appropriate topology T on the space of admissible processes such that the objective (2.1) is continuous in this topology and the space of admissible processes is T-compact. While it is beyond the scope of this article to discuss existence presumably a proof can be constructed along the lines of Bewley (1972).

The notation almost makes the working of the model self-explanatory. There are N different processes. At date t it is decided how much to consume and how much to hold in the form of capital. It is assumed that capital goods can be costlessly transformed into consumption goods on a one-for-one basis. After it is decided how much capital to hold then it is decided how to allocate the capital across the N processes. After the allocation is decided nature reveals the value of r_t and $g_i(x_{it},r_t)$ units of new production are available from process i at the end of period t.

But $\delta_i x_{it}$ units of capital have evaporated at the end of t. Thus net new produce is $g_i(x_{it},r_t) - \delta_i x_{it}$ from process i. The total produce available to be divided into consumption and capital stock at date t+1 is given by

$$(2.6) \qquad \sum_{i=1}^{N} [g_i(x_{it},r_t) - \delta_i x_{it}] + x_t = \sum_{i=1}^{N} [g_i(x_{it},r_t) + (1-\delta_i)x_{it}]$$
$$\equiv \sum_{i=1}^{N} f_i(x_{it},r_t) \equiv y_{t+1}$$

where

$$(2.7) \qquad f_i(x_{it},r_t) \equiv g_i(x_{it},r_t) + (1-\delta_i)x_{it}$$

denotes the total amount of produce emerging from process i at the end of period t. The produce y_{t+1} is divided into consumption and capital stock at the beginning of date t+1 and so on it goes.

Note that we are assuming that it is costless to install capital into each process i and it is costless to allocate capital across processes at the beginning of each date t.

The objective of the optimizer is to maximize the expected value of the discounted sum of utilities over all consumption paths and capital allocations that satisfy (2.2)-(2.5).

In order to obtain sharp results we will place restrictive assumptions on this problem. We collect the basic working assumptions into one place:

Assumption 2.1: The functions $u(\cdot)$, $f_i(\cdot)$ are all concave, increasing, and are twice continuously differentiable.

Assumption 2.2: The stochastic process $\{r_t\}_{t=1}^{\infty}$ is independently and identically distributed. Each $r_t:(\Omega,\beta,\mu) \to R^m$ where $(\Omega,\mathcal{B},\mu)$ is a probability space. Here Ω is the space of elementary events, $\mathcal{B}$ is the σ-field of measurable sets w.r.t. μ and μ is a probability measure defined on subsets $B \subseteq \Omega$, $B \in \mathcal{B}$. Furthermore, the range of r_t, $r_t(\Omega)$, is compact.

Assumption 2.3: For each $\{x_{i1}\}_{i=1}^{N}$, r_1 the problem (1) has a unique optimal solution (unique up to a set of realizations of $\{r_t\}$ of measure zero).

Notice that Assumption 2.3 is implied by Assumption 1 and strict concavity of u, $\{f_i\}_{i=1}^{N}$. Rather than try to find the weakest possible assumptions sufficient for uniqueness of solutions to (2.1) it seemed simpler to reveal the role of uniqueness in what follows by simply assuming it. Furthermore, since we are not interested in the study of existence of optimal solutions in this article, we have simply assumed that also.

By Assumption 2.3 we see that to each output level y_t, optimum c_t, x_t, x_{it}, given y_t, may be written

$$(2.8) \qquad c_t = g(y_t),\ x_t = h(y_t),\ x_{it} = h_i(y_t)$$

The optimum policy functions $g(\cdot)$, $h(\cdot)$, $h_i(\cdot)$ do not depend upon t because the problem given by (2.1)-(2.5) is time stationary.

Another useful optimum policy function may be obtained. Given x_t, r_t Assumption 2.3 implies that the optimal allocation $\{x_{it}\}_{i=1}^{N}$ and next periods' optimal capital stock x_{t+1} is unique.

Furthermore, these may be written in the form

(2.9) $$x_{it} = a_i(x_t, r_{t-1})$$

(2.10) $$x_{t+1} = H(x_t, r_t).$$

Equations (2.9), (2.10) contain r_{t-1}, r_t respectively because the allocation decision is made after r_{t-1} is known but before r_t is revealed but the capital-consumption decision is made after y_{t+1} is revealed (i.e., after r_t is known).

Equation (2.10) looks very much like the optimal stochastic process studied by Brock-Mirman and Mirman-Zilcha. It was shown in Brock-Mirman (1972 and 1973) for the case N = 1 that the stochastic difference equation (2.10) converges in distribution to a unique limit distribution independent of initial conditions. We show in Brock (1978) that the same result may be obtained for our N process model by following the argument of Mirman-Zilcha. We collect some facts here that are established in Brock (1978).

Result 2.1: Assume Assumption 2.1. Let $U(y_1)$ denote the maximum value of the objective in (2.1) given initial resource stock y_1. Then $U(y_1)$ is concave, nondecreasing in y_1 and, for each $y_1 > 0$ the derivative $U'(y_1)$ exists and is nonincreasing in y_1.

Proof: Mirman-Zilcha (1977) prove that

(2) $$U'(y_1) = u'(g(y_1)), \text{ for } y_1 > 0$$

for the case N = 1. The same argument may be used here. The details are left to the reader.

Remark 2.1: Equation (a) shows that $g(y_1)$ is nondecreasing since $u''(c) < 0$ and $U'(y)$ is nonincreasing in y due to the concavity of $U(\cdot)$.

Result 2.2: Assume Assumption 2.1. Also assume that units of utility may be chosen so that $u(c) \geqq 0$, for all c. Furthermore, assume that along optima

$$E_1\beta^{t-1}U(y_t) \to 0, \text{ as } t \to \infty.$$

Then if $\{c_t\}_{t=1}^{\infty}$, $\{x_t\}_{t=1}^{\infty}$, $\{x_{it}\}_{i=1}^{N}$, t = 1,2,... is optimal then the following conditions must be satisfied:

For each i, t

(2.10a) $$u'(c_t) \geqq \beta E_t\{u'(c_{t+1})f_i'(x_{it}, r_t)\},$$

(2.10b) $$u'(c_t)x_{it} = \beta E_t\{u'(c_{t+1})f_i'(x_{it}, r_t)x_{it}\},$$

and

(2.10c) $$\lim_{t\to\infty} E_1\{\beta^{t-1}u'(c_t)x_t\} = 0$$

Proof: The proof proceeds much like the proof of Lemma 3.1 which is given in Section 3 below. For details see Brock (1978).

Lemma 2.1: Assume that $u'(c) > 0$, $u''(c) < 0$, $u'(0) = +\infty$. Furthermore, assume that $f_j(0,r) = 0$, $f_j'(x,r) > 0$, $f_j''(x,r) \leqq 0$ for all values of r. Also suppose that there is a set of r values with positive probability such that f_j is strictly concave in x. Then the function h(y) is continuous in y, increasing in y and is 0 when y = 0.

Proof: See Brock (1978)

Now by Assumption 2.3 and (2.8)-(2.10) it follows that y_{t+1} may be written

(2.11) $$y_{t+1} = F(x_t, r_t)$$

Following Mirman and Zilcha (1977) define

(2.12) $$\underline{F}(x) \equiv \min_{r \in R} F(x,r) \qquad \bar{F}(x) \equiv \max_{r \in R} F(x,r)$$

where R is the range of the random variable

$$r:(\Omega, \mathcal{B}, \mu) \rightarrow R^m$$

which is compact by Assumption 2.2. The following lemma shows that $\underline{F}$, $\bar{F}$ are well defined.

Lemma 2.2: The function F(x,r) is continuous in r.

Proof: See Brock (1978)

Let $\underline{x}$, $\bar{x}$ be any two fixed points of the functions

(2.13) $$\underline{H}(x) \equiv h(\underline{F}(x)), \quad \bar{H}(x) \equiv h(\bar{F}(x))$$

respectively. Then

Lemma 2.3: Any two fixed points of the pair of functions defined in (2.13) must satisfy

(2.14) $$\underline{x} \leqq \bar{x}.$$

Proof: See Brock (1978)

We may apply arguments similar to Brock-Mirman (1972) and prove

Theorem 2.1: There is a distribution function F(x) of the optimum aggregate capital stock x such that

$$F_t(x) \rightarrow F(x)$$

uniformly for all x. Furthermore F(x) does not depend on the initial conditions (x_1, r_1).

Proof: See Brock (1978)

(2.15) Here $F_t(x) \equiv \text{Prob}\,\{x_t \leqq x\}$.

Theorem 2.1 shows that the distribution of optimum aggregate capital stock at date t, $F_t(x)$, converges pointwise to a limit distribution F(x).

Theorem 2.1 is important because we will use the optimal growth model to construct equilibrium asset prices and risk prices. Since these prices will be time stationary functions of x_t and since x_t converges in distribution to F we will be able to use the mean ergodic theorem and stationary time series methods to make statistical inferences about these prices on the basis of time series observations.

The Price of Systematic Risk

Steve Ross (1976) produced a theory of capital asset pricing that showed that the assumption that all systematic risk free portfolios earn the risk free rate of return plus the assumption that asset returns are generated by a K factor model leads to the existence of "prices" $\lambda_o, \lambda_1, \lambda_2, \ldots, \lambda_K$ on mean returns and on each of the K factors. These prices satisfied the property that expected returns $E\tilde{Z}_i \equiv a_i$ on each asset i was a linear function of the standard deviation of the returns on asset i with respect to each factor k, i.e.,

(2.16) $$a_i = \lambda_o + \sum_{k=1}^{K} \lambda_k b_{ki}, \quad i = 1,2,\ldots,N$$

where the original model of asset returns is given by

(2.17) $$\tilde{Z}_i = a_i + \sum_{k=1}^{K} b_{ki} \tilde{\delta}_k + \tilde{\varepsilon}_i, \quad i = 1,2,\ldots,N.$$

Here $\tilde{Z}_i$ denotes random ex ante anticipated returns from holding the asset one unit of time, $\tilde{\delta}_k$ is systematic risk emanating from factor k, $\tilde{\varepsilon}_i$ is unsystematic risk specific to asset i, and a_i, b_{ki} are constants. Assume that the means of $\tilde{\delta}_k$, $\tilde{\varepsilon}_i$ are zero for each k, i, that $\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_N$ are independent and that $\tilde{\delta}_k$, $\tilde{\varepsilon}_i$ are uncorrelated random variables with finite variances for each k, i.

Ross proved that $\lambda_o, \lambda_1, \ldots, \lambda_K$ exist that satisfy (2.16) by forming portfolios $\eta \in R^N$ such that

(2.18) $$\sum_{i=1}^{N} \eta_i = 0,$$

and constructing the η_i such that the coefficients of each $\tilde{\delta}_k$ in the portfolio returns

(2.19) $$\sum_{i=1}^{N} \eta_i \tilde{Z}_i = \sum_{i=1}^{N} \eta_i [a_i + \sum_{k=1}^{K} b_{ki} \tilde{\delta}_k + \tilde{\varepsilon}_i]$$
$$= \sum_{i=1}^{N} \eta_i a_i + \sum_{k=1}^{K} (\sum_{i=1}^{N} b_{ki}\eta_i)\tilde{\delta}_k + \sum_{i=1}^{N} \eta_i \tilde{\varepsilon}_i$$

are zero, and requiring that

(2.20) $$\sum_{i=1}^{N} \eta_i a_i = 0$$

for all such systematic risk free zero wealth portfolios.

Here (2.18) corresponds to the zero wealth condition. The condition,

(2.21) $$0 = \sum_{i=1}^{N} b_{ki}\eta_i, \quad k = 1,2,\ldots,K,$$

corresponds to the systematic risk free condition. Actually Ross did not require that (2.20) hold for <u>all</u> zero wealth systematic risk free portfolios but only for those that are "well diversified" in the sense that the η_i are of comparable size so that he could use the assumption of independence of $\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_N$ to argue that the random variable

$$\sum_{i=1}^{N} \eta_i \tilde{\varepsilon}_i$$

was "small" and hence bears a small price in a world of investors who would pay a positive price only for the avoidance of risks that could <u>not</u> be diversified away.

Out of this type of argument Ross argues that the condition: for all $\eta \varepsilon R^N$

(2.22a) $$\sum_{i=1}^{N} \eta_i = 0, \quad \sum_{i=1}^{N} \eta_i b_{ki} = 0, \quad k = 1,2,\ldots,K$$

implies, that in equilibrium

(2.22b) $$\sum_{i=1}^{N} \eta_i a_i = 0,$$

should hold.

All that (2.22) says is that zero wealth, zero systematic risk portfolios should earn a zero mean rate of return. Condition (2.22) is economically compelling because in its absence rather obvious arbitrage opportunities appear to exist.

Whatever the case (2.22) implies there exists $\lambda_o, \lambda_1, \lambda_2, \ldots, \lambda_K$ such that (2.16) holds and the proof is just simple linear algebra. Notice that Ross made no assumptions about mean variance investor utility functions or normal distributions of asset returns common to the usual Sharpe-Lintner type of asset pricing theories which are standard in the finance literature.

However Ross's model, like the standard capital asset pricing models in finance, does not link the asset returns to underlying sources of uncertainty. Our growth model will be used as a module in the construction of an intertemporal general equilibrium asset pricing model where relationships of the form (2.17) are determined within the model and hence the $\lambda_o, \lambda_1, \ldots, \lambda_K$ will be determined within the model as well. Such a model of asset

price determination preserves the beauty and empirical tractability of the Ross-Sharpe-Lintner formulation, but at the same time will give us a context where we can ask general equilibrium questions such as: What is the impact of an increase of the progressivity of the income tax on the demand for and supply of risky assets and the $\lambda_o, \lambda_1, \ldots, \lambda_K$?

Let us get on with relating the growth model to (2.16). For simplicity assume all processes i are active (i.e., (2.10a) holds with equality). We record (2.10a) here for convenience.

(2.23) $$u'(c_t) = \beta E_t\{u'(c_{t+1}) f_i'(x_{it}, r_t)\}.$$

Now (2.17) is a special hypothesis about asset returns. What kind of hypothesis about "technological" uncertainty corresponds to (2.17)? Well, as an example, put for each $i = 1,2,\ldots,N$

(2.24) $$f_i(x_{it}, r_t) \equiv (A_{it}^o + A_{it}^1 \tilde{\delta}_{1t} + A_{it}^2 \tilde{\delta}_{2t} + \ldots + A_{it}^K \tilde{\delta}_{Kt}) f_i(x_{it})$$

$$\equiv r_{it} f_i(x_{it})$$

where $$A_{it}^k \equiv A_i^k$$

are constants and

$$\{\tilde{\delta}_{kt}\}_{t=1}^{\infty}$$

are independent and identically distributed random variables for each k and for each k,t the mean of $\tilde{\delta}_{kt}$ is zero, the variance is finite, and $\tilde{\delta}_{st}$ is independent of $\tilde{\delta}_{kt}$ for each s,k,t. Furthermore, assume that $f(\cdot)$ is concave, increasing, twice differentiable, $f'(0) = +\infty$, $f'(\infty) = 0$ and that there is a bound ε_o such that

$$r_{it} > \varepsilon_o > 0$$

with probability 1 for all t_1. These assumptions are stronger than necessary, but will enable us to avoid concern with technical tangentialities. Define, for all t

$$\tilde{\delta}_{ot} \equiv 1,$$

so that we may sum from k = 0 to K in (2.25) below.

Insert (2.24) into (2.23) to get for all t, k, i

$$(2.25) \qquad u'(c_t) = \beta E_t\{u'(c_{t+1})(\sum_{k=0}^{K} A_{it}^k \tilde{\delta}_{kt}) f_i'(x_{it})\}$$

$$= \sum_{k=0}^{K} ([A_{it}^k f_i'(x_{it})] E_t\{\beta u'(c_{t+1})\tilde{\delta}_{kt}\}).$$

Now set (2.25) aside for a moment and look at the marginal benefit of saving one unit of capital and assigning it to process i at the beginning of period t. At the end of period t, r_t is revealed and extra produce

$$(2.26) \qquad \tilde{Z}_{it} \equiv A_{it}^o f_i'(x_{it}) + \sum_{k=1}^{K} A_{it}^k f_i'(x_{it})\tilde{\delta}_{kt},$$

emerges.

Putting

$$(2.27) \qquad a_i \equiv A_{it}^o f_i'(x_{it}),\ b_{ki} \equiv A_{it}^k f_i'(x_{it}),\ \tilde{\delta}_{kt} = \tilde{\delta}_k,$$

equation (2.26) is identical with Ross's (2.17) with $\tilde{\varepsilon}_i \equiv 0$. We proceed now to generate the analogue to (2.16) in our model. Turn back to (2.25). Rewrite (2.25) using (2.27) thus

$$(2.28) \qquad u'(c_t) = \sum_{k=1}^{K} b_{ki} E_t\{\beta u'(c_{t+1})\tilde{\delta}_{kt}\} + a_i E_t\{\beta u'(c_{t+1})\}.$$

Hence

$$(2.29) \qquad a_i = \frac{u'(c_t)}{\beta E_t\{u'(c_{t+1})\}} - \sum_{k=1}^{K} b_{ki}(E_t\{u'(c_{t+1})\tilde{\delta}_{kt}\}/E_t\{u'(c_{t+1})\})$$

so that $\lambda_o, \lambda_1, \ldots, \lambda_K$ defined by

$$(2.30) \qquad \lambda_o \equiv u'(c_t)/\beta E_t\{u'(c_{t+1})\},$$

$$-\lambda_k \equiv E_t\{u'(c_{t+1})\tilde{\delta}_{kt}\}/E_t\{u'(c_{t+1})\}$$

$$= \lambda_o \text{ covariance } [\beta u'(c_{t+1})/u'(c_t),\ \tilde{\delta}_{kt}].$$

yields

$$(2.31) \qquad a_i = \lambda_o + \sum_{k=1}^{K} b_{ki}\lambda_k.$$

Here t subscripts are dropped to ease typing.

These results are extremely suggestive and show that the model studied in this section may be quite rich in economic content. Although the model is a normative model in the next section we shall turn it into an equilibrium asset pricing model so that the λ_k become equilibrium risk prices. Let us explore the economic meanings of (2.30) in some detail.

Suppose that K = 1 and that there is a risk free asset N in the sense that

(2.32) $$b_{N1} \equiv A^1_{Nt} f'(x_{Nt}) = 0$$

i.e.,

(2.33) $$A^1_{Nt} = 0$$

Then by (2.33)

(2.34) $$a_N = \lambda_o, \quad a_i = a_N + b_{1i}\lambda_1$$

so that for all i, j ≠ N

(2.35) $$(a_i - a_N)/b_{1i} = (a_j - a_N)/b_{1j}$$

The second part of equation (2.34) corresponds to the security market line which says that expected return and risk are linearly related in a one factor model. Equation (2.35) corresponds to the usual Sharpe-Lintner Mossin capital asset pricing model result that in equilibrium the "excess return" per unit of risk must be equated across all assets.

The economic interpretation of λ_o given in (2.30) is well known and needs no explanation here. Look at the formula for λ_k. The covariance of the marginal utility of consumption at time t+1 with the zero mean finite variance shock $\tilde{\delta}_{kt}$ appears in the numerator. Since output increases when $\tilde{\delta}_{kt}$ increases and since

$$c_{t+1} = g(y_{t+1})$$

doesn't decrease when y_{t+1} increases therefore this covariance is likely to be negative so that the sign of λ_k is positive. We will look into the determinants of the magnitudes of $\lambda_o, \lambda_1, \ldots, \lambda_K$ in more detail later. Let us show how our model may be helpful in the empirical problem in estimating the $\lambda_o, \lambda_1, \ldots, \lambda_K$ from time series data.

First how is one to close Ross's model (2.17) since the $\tilde{Z}_i$ are <u>subjective</u>? The most natural way to close the model in markets as well organized as U.S. securities markets would seem to be rational expectations: The subjective distribution of $\tilde{Z}_i$ is equal to the actual or objective distribution of $\tilde{Z}_i$. We shall show that our asset pricing model under rational expectations which is developed below generates the same solution as the normative model discussed above. Hence the convergence theorem implies that $\{x_t, c_t, x_{1t}, x_{2t}, \ldots, x_{Nt}\}_{t=1}^{\infty}$ converges to a stationary stochastic process.

Thus the mean ergodic theorem which says very loosely that the <u>time</u> average of any function G of a stationary stochastic process equals the <u>average</u> of G over the stationary distribution of that process allows us to apply time series methods developed for stationary stochastic processes to estimate $\lambda_o, \lambda_1, \ldots, \lambda_K$.

Let us turn to development of the asset pricing model.

SECTION 3: AN ASSET PRICING MODEL

In this section we reinterpret the model of Section 2 and add to it a market for claims to pure rents so that it describes the evolution of equilibrium context in which to discuss the martingale property of capital asset prices, but also our model will contain a non-trivial investment decision, a non-trivial market for claims to pure rents (i.e., a stock market), as well as a market for the pricing of the physical capital stock.

We believe that there is a considerable benefit in showing how to turn optimal growth models into asset pricing models. This is so because there is a large literature on stochastic growth models which may be carried over to the asset pricing problem with little effort. Although the model presented here is somewhat artificial we believe that studying it will yield techniques that can be used to study less artificial models.

We will build an asset pricing model much like that of Lucas (1978). The model contains one representative consumer whose preferences are identical to the planner's preferences given in (2.1). The model contains N different firms who rent capital from the consumption side at rate R_{t+1} at each date so as to maximize

$$\pi_{it+1} \equiv f_i(x_{it}, r_t) - R_{i,t+1} x_{it}. \tag{3.1}$$

Notice that it is assumed that each firm i makes his decision to hire x_{it} after r_t is revealed. Here $R_{i,t+1}$ denotes the rental rate on capital prevailing in industry i at date t+1. It is to be determined within the model. These "rental markets" are rather artificial. They are introduced in order to obtain Lemma 3.2 below.

The model will introduce a stock market in such a way that the real quantity side of the model is the same as that of the growth model in equilibrium. Our model is closed under the assumption of rational expectations. The quantity side of the model is essentially an Arrow-Debreu model as is the model of Lucas. That is, we will introduce securities markets in such a way that there are enough securities such that any equilibrium is a Pareto Optimum. However, there is a separate market where claims to the rents (3.1) are competitively traded. In Arrow-Debreu the rents are redistributed in a lump sum fashion.

Market institutions may be introduced into the model of Section 2 in an alternative manner than that done here in Section 3. This alternative formulation enables us to link the theory up with the Modigliani-Miller (MM) formulation in their famous article on the invariance of firm value to dividend policy. We sketch this alternative model in the Appendix to Section 3.

The model is in the spirit of Lucas's model where each firm i has outstanding one perfectly divisible equity share. Ownership of $\alpha\%$ of the equity shares in firm i at date t entitles one to α% of profits of the firm i at date t+1. Equilibrium asset prices and equilibrium consumption, capital, and output are determined by optimization under the hypothesis of rational expectations much as in Lucas. Let us get into the model.

<u>The Model</u>

There is one representative consumer (or a "representative standin," as Lucas calls him) that is assumed to solve

$$\text{(3.2)} \qquad \text{maximize } E_1 \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$$

subject to

$$\text{(3.3)} \qquad c_t + x_t + \underset{\sim}{P}_t \cdot \underset{\sim}{Z}_t \leq \underset{\sim}{\pi}_t \cdot \underset{\sim}{Z}_{t-1} + \underset{\sim}{P}_t \cdot \underset{\sim}{Z}_{t-1} + \sum_{i=1}^{N} R_{it} x_{i,t-1} \equiv y_t$$

$$\text{(3.4)} \qquad c_t \geqq 0,\ x_t \geqq 0,\ \underset{\sim}{Z}_t \geqq 0,\ x_{it} \geqq 0, \quad i = 1,2,\ldots,N, \quad \text{all } t$$

$$\text{(3.5)} \qquad c_1 + x_1 + \underset{\sim}{P}_1 \cdot \underset{\sim}{Z}_o \leq \underset{\sim}{\pi}_1 \cdot \underset{\sim}{Z}_o + \underset{\sim}{P}_1 \cdot \underset{\sim}{Z}_o + \sum_{i=1}^{N} R_{i1} x_{io} \equiv y_1,$$

$$Z_o \equiv 1, \quad R_{i1} \equiv f_i'(x_{io}, r_o), \quad \pi_{i1} \equiv f_i(x_{io}, r_o)$$

$$- f_i'(x_{io}, r_o) x_{io}, \quad x_o, \{x_{io}\}_{i=1}^{N}$$

given, where c_t, x_t, P_{it}, Z_{it}, π_{it}, R_{it} all assumed measurable $\mathcal{F}_t$ denote consumption at date t, total capital stock owned at date t by the consumer, price of one share of firm i at date t, number of shares of firm i owned by the individual at date t,

profits of firm i at date t and rental factor (i.e. $R_{it} \equiv$ principal plus interest) obtained on a unit of capital leased to firm i. Here "•" denotes scalar product.

Firm i is assumed to hire x_{it} so as to maximize (3.1).[2] The consumer is assumed to lease capital x_{it} at date t to firm i before r_t is revealed. Hence $R_{i,t+1}$ is uncertain at date t. The consumer, in order to solve his problem at date 1 must form expectations on $\{P_{it}\}_{t=1}^{\infty}$, $\{R_{it}\}_{t=1}^{\infty}$, $\{\pi_t\}_{t=1}^{\infty}$ and maximize (3.2) subject to (3.3)-(3.5). In this way notional demands for consumption goods and equities as well as notional supplies of capital stocks and capital services to each of the N firms are drawn up by the consumer side of the economy. Similarly for the firm side. We close the model with

<u>Definition</u>: The stochastic process $\mathcal{R} \equiv \langle \{\bar{P}_{it}\}_{t=1}^{\infty}, \{\bar{R}_{it}\}_{t=1}^{\infty}, \{\bar{\pi}_{it}\}_{t=1}^{\infty}; \{\bar{x}_{it}\}_{t=1}^{\infty}, \{\bar{Z}_{it}\}_{t=1}^{\infty}, i = 1,2,\ldots,N, \{\bar{c}_t\}_{t=1}^{\infty}, \{\bar{x}_t\}_{t=1}^{\infty} \rangle$ is a <u>rational expectations equilibrium</u> (R.E.E.) if facing $\mathcal{P} \equiv \langle \{\bar{P}_{it}\}_{t=1}^{\infty}, \{\bar{R}_{it}\}_{t=1}^{\infty}, \{\bar{\pi}_{it}\}_{t=1}^{\infty} \rangle$ the consumer solves (3.2) and chooses

$$\text{(3.6)} \qquad x_t = \bar{x}_t,\ x_{it} = \bar{x}_{it},\ c_t = \bar{c}_t,\ Z_{it} = \bar{Z}_{it} \qquad \text{a.e.,}$$

and the i^{th} firm solves (3.1) and chooses

$$\text{(3.7)} \qquad x_{it} = \bar{x}_{it}$$

and furthermore

(3.8) (Asset market clears) $\bar{Z}_{it} \leq 1$, if $\bar{Z}_{it} < 1$, $\bar{P}_{it} = 0$ a.e.

(3.9) (Goods market clears) $\bar{c}_t + \bar{\bar{x}}_t = \sum_{i=1}^{N} f_i(\bar{x}_{i,t-1}, r_{t-1})$ a.e.

(3.10) (Capital market clears) $\sum_{i=1}^{N} \bar{x}_{it} = \bar{\bar{x}}_t$ a.e.

Here "a.e." means "almost everywhere." This ends the definition of R.E.E. that we will use in this paper.

It is easy to write down first order necessary conditions for an R.E.E. Let us start on the consumer side first. We drop upper bars to ease typing. At date t if the consumer buys a share of firm i the cost is P_{it} units of consumption goods. The marginal cost at date t in utils foregone is $u'(c_t)P_{it}$. At the end of period t, r_t is revealed and $P_{i,t+1}$, $\pi_{i,t+1}$ become known. Hence the consumer obtains

(3.11) $$u'(c_{t+1})(P_{i,t+1} + \pi_{i,t+1})$$

extra utils at the beginning of t+1 if he collects $\pi_{i,t+1}$ and sells the share "exdividend" at $P_{i,t+1}$. But these utils are uncertain and are received one period into the future. The expected present value of utility gained at t+1 is

(3.12) $$\beta E_t\{u'(c_{t+1})(P_{i,t+1} + \pi_{i,t+1})\}.$$

Consumer equilibrium in the market for asset i requires that the marginal opportunity cost at date t be greater than or equal to the present value of the marginal benefit of dividends and

exdividend sale price at date t+1i

(3.13a) $$P_{it}u'(c_t) \geq \beta E_t\{u'(c_{t+1})(\pi_{i,t+1} + P_{i,t+1})\}$$ a.e.

(3.13b) $$P_{it}u'(c_t)Z_{it} = \beta E_t\{u'(c_{t+1})(\pi_{i,t+1} + P_{i,t+1})\}Z_{it}$$ a.e.

Similar reasoning in the rental market yields

(3.14a) $$u'(c_t) \geq \beta E_t\{u'(c_{t+1})(R_{i,t+1})\}.$$ a.e

(3.14b) $$u'(c_t)x_{it} = \beta E_t\{u'(c_{t+1})(R_{i,t+1})\}x_{it}$$ a.e.

It would be nice if the first order necessary conditions (3.13)-(3.14) characterized consumer optima. But it is well known that a "transversality condition" at infinity is needed in addition to completely characterize optima. Recent work by Benveniste-Scheinkman (1977) allows us to prove

<u>Lemma 3.1</u>: Assume Assumption 2.1. Furthermore assume that P is such that

$$W(y_t,t) \to 0, \quad t \to \infty,$$

where $W(y_t,t)$ is defined by

(3.15) $$W(y_t,t) = \text{Maximum } E_1 \sum_{s=t}^{\infty} \beta^{s-1}u(c_s)$$

subject to (3.3)-(3.5) with "t" replaced by "s" and "1" replaced by "t." Here y_t denotes the R.H.S. of (3.3). Then given

$\{P_{it}\}_{t=1}^{\infty}$, $\{\pi_{it}\}_{t=1}^{\infty}$, $\{R_{it}\}_{t=1}^{\infty}$, $i = 1,2,\ldots N$ optimum solutions $\{Z_{it}\}_{t=1}^{\infty}$, $\{x_{it}\}_{t=1}^{\infty}$, $i = 1,2,\ldots,N$, $\{c_t\}_{t=1}^{\infty}$, $\{x_t\}_{t=1}^{\infty}$ to the consumer's problem (3.2) subject to (3.3)-(3.5) are characterized by (3.13)-(3.14) and

(3.16) $\quad \text{TVC}_\infty(\text{equity market}) \quad \lim_{t\to\infty} E_1\{\beta^{t-1}u'(c_t)\, \underset{\sim}{P}_t \cdot \underset{\sim}{Z}_t\} = 0$

(3.17) $\quad \text{TVC}_\infty(\text{capital market}) \quad \lim_{t\to\infty} E_1\{\beta^{t-1}u'(c_t)x_t\} = 0.$

Proof: Suppose $\{\bar{\underset{\sim}{Z}}_t\}$, $\{\bar{c}_t\}$, $\{\bar{x}_t\}$ satisfies (3.13)-(3.17) and let $\{\underset{\sim}{Z}_t\}$, $\{c_t\}$, $\{x_t\}$ be any stochastic process satisfying the same initial conditions and (3.3)-(3.5). Compute for each T an upper bound to the shortfall:

(3.18) $$E_1\{\sum_{t=1}^{T} \beta^{t-1}u(c_t) - \sum_{t=1}^{T} \beta^{t-1}u(\bar{c}_t)\}$$

(3.19) $$\leq E_1\{\sum_{t=1}^{T} \beta^{t-1}u'(\bar{c}_t)(c_t - \bar{c}_t)\}$$

(3.20) $$= E_1\{\sum_{t=1}^{T} \beta^{t-1}u'(\bar{c}_t)[\underset{\sim}{\pi}_t \cdot \underset{\sim}{Z}_{t-1} + \underset{\sim}{P}_t \cdot \underset{\sim}{Z}_{t-1} + \sum_{i=1}^{N} R_{it}x_{i,t-1} - \underset{\sim}{P}_t \cdot \underset{\sim}{Z}_t$$
$$- x_t - \underset{\sim}{\pi}_t \cdot \bar{\underset{\sim}{Z}}_{t-1} - \underset{\sim}{P}_t \cdot \bar{\underset{\sim}{Z}}_{t-1} - \sum_{i=1}^{N} R_{it}\bar{x}_{i,t-1} + \underset{\sim}{P}_t \cdot \bar{\underset{\sim}{Z}}_t + \bar{x}_t]\}$$

(3.21) $$= E_1\{\beta^{T-1}u'(\bar{c}_T)[\underset{\sim}{P}_T \cdot (\bar{\underset{\sim}{Z}}_T - \underset{\sim}{Z}_T) + \bar{x}_T - x_T]\}$$

(3.22) $$\leq E_1\{\beta^{T-1}u'(\bar{c}_T)[\underset{\sim}{P}_T \cdot \bar{\underset{\sim}{Z}}_T + \bar{x}_T]\} \to 0,\ T \to \infty.$$

Here equations (3.13)-(3.14) were used to telescope out the middle terms in the series of R.H.S. (3.20).

The terms corresponding to date 1 cancel each other because the initial conditions are the same. Hence only the terms of R.H.S. (3.21) remain of all the terms of R.H.S. (3.19) and (3.20). That R.H.S. (3.21) has an asymptotic upper bound of zero follows from (3.16), (3.17) and the nonnegativity of Z_T, x_T. This shows that (3.13)-(3.14), (3.16)-(3.17) imply optimality. Notice that no assumptions on $W(y_t, t)$ are needed to get this side of the proof.

Now let $\{\bar{\underset{\sim}{Z}}_t\}$, $\{\bar{c}_t\}$, $\{\bar{x}_t\}$ be optimal given $\{\underset{\sim}{P}_t, \underset{\sim}{R}_t, \underset{\sim}{\pi}_t\}$. Since $u'(0) = +\infty$ implies that $\bar{c}_t > 0$ a.e. and W is differentiable at $\bar{y}_t$ we have by concavity of W, and $u \geq 0$ (dropping upper bars from this point on),

(3.23) $$W(y_t,t) \geq W(y_t,t) - W(y_t/2,t) \geq W'(y_t,t)(y_t/2) = \beta^{t-1}u'(c_t)y_t/2.$$

Hence

(3.24) $$E_1W(y_t,t) \to 0,\ t \to \infty \text{ implies } E_1\, \beta^{t-1}u'(c_t)y_t \to 0,\ t \to \infty.$$

But

(3.25) $$y_t \equiv \underset{\sim}{\pi}_t \cdot \underset{\sim}{Z}_{t-1} + \underset{\sim}{P}_t \cdot \underset{\sim}{Z}_{t-1} + \sum_i R_{it}x_{i,t-1}$$

so that by the first order necessary conditions

(3.26) $$E_1\, \beta^{t-1}u'(c_t)((\underset{\sim}{\pi}_t + \underset{\sim}{P}_t)\cdot \underset{\sim}{Z}_{t-1} + \sum_i R_{it}x_{i,t-1}) = E_1\, \beta^{t-2}u'(c_{t-1})\underset{\sim}{P}_{t-1}\cdot \underset{\sim}{Z}_{t-1} + E_1\, \beta^{t-2}u'(c_{t-1})x_{t-1}$$

because (in more detail) (3.13a)-(3.14b) imply

(3.27) $$x_{i,t-1}u'(c_{t-1}) = \beta E_{t-1}\{u'(c_t)R_{it}\}x_{i,t-1}$$

(3.28) $$\beta^{-1}x_{t-1}u'(c_{t-1}) = E_{t-1}\{u'(c_t)(\sum_i R_{it}x_{i,t-1})\}$$

(3.29) $$P_{i,t-1}u'(c_{t-1})Z_{i,t-1} = \beta E_{t-1}\{u'(c_t)(\pi_{it} + P_{it})Z_{i,t-1}\}$$

(3.30) $$\beta^{-1}u'(c_{t-1})\underset{\sim}{P}_{t-1}\cdot \underset{\sim}{Z}_{t-1} = E_{t-1}\{u'(c_t)(\underset{\sim}{\pi}_t\cdot \underset{\sim}{Z}_{t-1} + \underset{\sim}{P}_t\cdot \underset{\sim}{Z}_{t-1})\}.$$

Hence because $\underset{\sim}{P}_{t-1} \geqq 0$, $\underset{\sim}{Z}_{t-1} \geqq 0$, $x_{t-1} \geqq 0$ (3.24) implies

(3.31) $$E_1\beta^{t-2}u'(c_{t-1})\underset{\sim}{P}_{t-1}\cdot \underset{\sim}{Z}_{t-1} \to 0,\ t \to \infty$$

(3.32) $$E_1\beta^{t-2}u'(c_{t-1})x_{t-1} \to 0,\ t \to \infty$$

as was to be shown.

The first part of this argument follows Malinvaud (1953) and the second part is adapted from Benveniste-Scheinkman (1977). Lemma 3.1 is important because it shows that (3.13)-(3.14), (3.16), (3.17) <u>characterize</u> consumer optima.

<u>Remark 3.1</u>: The assumption that $E_1W(y_t, t) \to 0$, $t \to \infty$ restrains $\mathcal{P}$. It requires that $\mathcal{P}$ be such that along any path in $\mathcal{P}$ utils cannot grow faster than β^t on the average. A general sufficient condition on $\mathcal{P}$ for $E_1W(y_t,t) \to 0$ can be given by what should be a straightforward extension of the methods of Brock-Gale (1970) and McFadden (1973) to our setup.

An obvious sufficient condition is that the utility function be bounded, i.e., there are numbers $\underline{B} < \bar{B}$ such that for all $c \geqq 0$

$$\underline{B} \leqq u(c) \leqq \bar{B}.$$

<u>Remark 3.2</u>: The method used here of introducing a stock market into this type of model where an investment decision is present was first developed by Scheinkman (1977) in the certainty case.

A basic Lemma is

<u>Lemma 3.2(i)</u>: Let $X = \left\langle \{\bar{c}_t\}_{t=1}^{\infty},\ \{\bar{x}_{it}\}_{t=1}^{\infty},\ \{\bar{x}_t\}_{t=1}^{\infty} \right\rangle$ solve the optimal growth problem (1.1) then define

(3.33) $$\bar{R}_{it+1} \equiv f_i'(\bar{x}_{it},r_t),\ \bar{\pi}_{i,t+1} \equiv f_i(\bar{x}_{it},r_t) - f_i'(\bar{x}_{it},r_t)\bar{x}_{it}.$$

Then let $\{\bar{P}_{it}\}_{t=1}^{\infty}$, $i = 1,2,\ldots,N$ satisfy (3.27), (3.29) and (3.31).

Put

(3.34) $$\bar{Z}_{it} = 1,$$

Then $\left\langle \{\bar{P}_{it}\}_{t=1}^{\infty}, \{\bar{R}_{it}\}_{t=1}^{\infty}, \{\bar{\pi}_{it}\}_{t=1}^{\infty}, \{\bar{x}_{it}\}_{t=1}^{\infty}, \{\bar{Z}_{it}\}_{t=1}^{\infty}, i = 1,2,\ldots,N, \{\bar{c}_t\}_{t=1}^{\infty}, \{\bar{x}_t\}_{t=1}^{\infty} \right\rangle \equiv \mathcal{R}$ is an R.E.E.

Lemma 3.2(ii): Let R be an R.E.E. Then X solves the optimal growth problem (1.1).

Proof: The proof of this is straightforward and is done in the Appendix. Lemma 3.2 is central to this paper because it shows that the quantity side of any competitive equilibrium may be manufactured from solutions to the growth problem. This fact will enable us to identify the Ross prices for example. Furthermore, it will be used in the existence proof of an asset pricing function which is developed below.

Turn back to the discussion of the relationship between the growth model of Section 2 and the risk prices of Ross. This will facilitate the economic interpretation of an R.E.E. stochastic process

$$\{\bar{R}_{it}\}_{t=1}^{\infty}, \quad \{\bar{P}_{it}\}_{t=1}^{\infty}, \quad \{\bar{\pi}_{it}\}_{t=1}^{\infty}.$$

Drop upper bars off of equilibrium quantities from this point on in order to ease typing. Assume that conditions are such that all asset prices are positive with probability 1 in equilibrium. Then $\bar{Z}_{it} = 1$ with probability 1 and from (3.29) we get for each t

(3.35) $$u'(c_t) = \beta E_t\{u'(c_{t+1})Z_{it}\}, \; Z_{it} \equiv (P_{i,t+1} + \pi_{i,t+1})/P_{it}.$$

Now because profit maximization implies

(3.36) $$f_i'(x_{it},r_t) = R_{i,t+1}, \; \pi_{i,t+1} = f_i(x_{it},r_t) - f_i'(x_{it},r_t)x_{it}.$$

Turning to the rental market, suppose that all processes are used with probability 1. Then (3.36) and (3.37) give us for each i,t

(3.37) $$u'(c_t) = \beta E_t\{u'(c_{t+1})f_i'(x_{it},r_t)\}.$$

Examine the specification

(3.38) $$f_i(x_{it},r_t) = (\sum_{k=0}^{K} A_{it}^k \tilde{\delta}_{kt})f_i(x_{it}) \equiv r_{it}f_i(x_{it})$$

developed in Section 2. Now (2.28), (3.37) and (3.35) imply

(3.39) $$u'(c_t) = \sum_{k=1}^{K} b_{kit}E_t\{\beta u'(c_{t+1})\tilde{\delta}_{kt}\} + a_{it}E_t\{\beta u'(c_{t+1})\}$$

$$= \beta E_t\{u'(c_{t+1})\bar{Z}_{it}\}.$$

We are not entitled to write returns Z_{it} defined by (3.35) in the linear Ross form (2.17) unless $P_i(y_{t+1})$ is linear in y_{t+1} even for the specification (3.38) above. An example will be presented in Section 5 below where $P_i(y_{t+1})$ turns out to be linear in y_{t+1}. But first we must show that an asset pricing function exists. To that we now turn.

Existence of an Asset Pricing Function

Since in equilibrium the quantity side of our asset pricing model is the same as the N process growth model, therefore we may use the facts collected in Section 2 about the N process growth model to prove the existence of an asset pricing function

$P(y)$ in much the same way as Lucas (1978).

To begin with let us assume

Assumption 3.1: Assume for all $r \in R$,

(a) $f_i'(0,r) = +\infty, \quad i = 1,2,\ldots,N.$

(b) $\pi_i(x,r) \equiv f_i(x,r) - f_i'(x,r)x > 0$ for all $x > 0$.

Assumption 3.1(a) implies that (3.14(a)) holds with equality in equilibrium. Also Assumption 3.1(b) implies (3.13(a)) holds with equality in equilibrium. Let us search as does Lucas for a bounded continuous function $P_i(y)$ such that in equilibrium

$$(3.40) \qquad P_{it}u'(c_t) = P_i(y_t)u'(c_t) = \beta E_t\{u'(c_{t+1})(\pi_{i,t+1} + P_i(y_{t+1}))\}.$$

Convert the foregoing problem into a fixed point problem. Note first from Section 2 that

$$(3.41) \qquad u'(c_t) = U'(y_t) \quad t = 1,2\ldots$$

$$(3.42) \qquad \pi_{i,t+1} = f_i(x_{it},r_t) - f_i'(x_{it},r_t)x_{it} \equiv \pi_i(x_{it},r_t) = \pi_i(\eta_i(x_t)x_t,r_t)$$

$$= \pi_i[\eta_i(h(y_t))h(y_t),r_t] \equiv J_i(y_t,r_t)$$

$$(3.43) \qquad y_{t+1} = \sum_{j=1}^{N} f_j(\eta_j(x_t)x_t,\ r_t) = \sum_{j=1}^{N} f_j[\eta_j(h(y_t))h(y_t),\ r_t] \equiv Y(y_t,r_t)$$

Put

$$(3.44) \qquad G_i(y_t) \equiv \beta \int_{r\varepsilon R} U'[Y(y_t,r)]J_i(y_t,r)\mu(dr)$$

$$(3.45) \qquad F_i(y_t) \equiv P_i(y_t)U'(y_t)$$

$$(3.46) \qquad (T_iF_i)(y_t) \equiv G_i(y_t) + \beta \int_{r\varepsilon R} F_i[Y(y_t,r)]\mu(dr).$$

Then for each i, (3.40) may be written as

$$(3.47) \qquad F_i(y_t) = (T_iF_i)(y_t).$$

Problem (3.47) is a fixed point problem in that we search for a function F_i that remains fixed under operator T_i. In order to use the contraction mapping theorem to find a fixed point F_i we must show first that T_i sends the class of bounded continuous functions on $[0,\infty)$, call it $C[0,\infty)$, into itself. The results of Section 2 established that all of the functions listed in (3.41)-(3.46) are continuous in y_t. We need

Lemma 3.3: If $U(y)$ is bounded on $[0,\infty)$ then $G_i(y)$ is bounded.

Proof: First by concavity of U we have

$$(3.48) \qquad U(y) - U(0) \geqq U'(y)(y - 0) = U'(y)y.$$

Hence there is B such that

$$(3.49) \qquad U'(y)y \leqq B \text{ for all } y \in [0,\infty).$$

Second

$$\int_R U'[Y(y,r)]J_i(y,r)\mu(dr) = \int_R U'[Y(y,r)]Y(y,r)J_i(y,r)/Y(y,r)\mu(dr)$$

$$\leqq B \int J_i(y,r)/Y(y,r)\mu(dr)$$

$$\leqq B$$

since $f_i' \geqq 0$ implies

$$f_i - f_i'x_{it} \equiv J_i \leqq f_i,\ Y \equiv \sum_{j=1}^{N} f_j,\ J_i/Y \leq 1.$$

Thus G_i is bounded by βB. This ends the proof.

We must show that if

(3.50) $$||F_i|| \equiv \sup_{y\varepsilon[0,\infty)} |F_i(y)|$$

is chosen to be the norm on $C[0,\infty)$ then T_i is a contraction with modulus β. It is a well-known fact that $C(0,\infty)$ endowed with this norm is a Banach space.

Lemma 3.4: T_i: $C[0,\infty] \rightarrow C[0,\infty)$

is a contraction with modulus β.

Proof: We must show that for any two elements F, G in $C[0,\infty)$

(3.51) $$||T_iF - T_iG|| \leq \beta||F - G||.$$

Now for $y \varepsilon [0,\infty)$ from (i) we have

(3.52) $$|T_iF(y) - T_iG(y)| = \beta\ |\int(F[Y(y,r)] - G[Y(y,r)])\mu(dr)|$$

$$\leqq \beta \int |F[(y') - G(y')|\mu(dr)$$

$$\leqq \beta \int \sup_{y' in[0,\infty)} |F(y') - G(y')|\mu(dr)$$

$$= \beta\ ||F - G||.$$

Take the supremum of the L.H.S. of (3.52) to get

(3.53) $$||T_iF - T_iG|| \leq \beta\ ||F - G||.$$

This ends the proof.

Theorem 3.1: For each i there exists exactly one asset pricing function of the form $P_i(y)$ where $P_i \in C[0,\infty)$.

Proof: Apply the contraction mapping theorem to produce a fixed point $\bar{F}_i(y) \in C[0,\infty)$. Put

(3.54) $$P_i(t) = \bar{F}_i(y)/U'(y).$$

It is clear that $P_i(y)$ satisfies (3.40). Furthermore, by the very definition of T_i any $P_i(y)$ that satisfies (3.40) is such that $P_i(\cdot)U'(\cdot) \equiv \bar{F}_i(\cdot)$ is a fixed point of T_i. This ends the proof.

Remark 3.3: Assumption 3.1(a) is not needed for the existence theorem. Assumption 3.1(b) is needed in the theorem so that (3.40) holds with equality.

Our proof of existence, as does Lucas's, leaves begging the question of whether there exist equilibria that are not stationary, that is, equilibria that cannot be written in the form of $P_i(y)$ for some time stationary function $P_i(\cdot)$.

Indeed, the papers of Cass, Okuno, and Zilcha (1979) and Gale (1973) have brought out in a dramatic way the multitude of non time stationary equilibria that exist in overlapping generations models. If we applied the above fixed point method to overlapping generations models we would only find the time stationary equilibria. Calvo (1979) and Wilson (1978) show that the same problem may arise even in infinite horizon monetary models with only one agent type.

Fortunately for our case we may use the necessity of the transversality condition (3.16) to show that there is only one equilibrium.

Theorem 3.1': Assume the hypothesis of Theorem 3.1. For each i, t there is only one equilibrium asset price P_{it} and it can be written in the form $P_i(y_t)$.

Proof: Look at (3.13a) and develop a recursion as done in (A3.13) below. We get

$$P_{i1} = E_1 \sum_{s=2}^{T} \Pi_s \pi_{is} + E_1 \Pi_T P_{iT}.$$

We must first show that (3.16) implies

$$E_1 \Pi_T P_{iT} \to 0, \ T \to \infty.$$

In order to see this first note that $Z_{i1} = 1$ in equilibrium. Also by definition of Π_t

$$E_1 \Pi_t P_{it} = E_1\{[\beta^{t-1} u'(c_t)/u'(c_1)] P_{it}\}$$
$$= (1/u'(c_1)) E_1\{\beta^{t-1} u'(c_t) P_{it}\} \to 0, \quad t \to \infty.$$

The last statement follows directly from (3.16) since $Z_{it} = 1$ in equilibrium.

Second, we must know the values of Π_s, π_{is}. But Lemma 3.1 tells us that the quantity side of the growth model is the same as the quantity side of the "market" model in equilibrium. Hence the solution of the growth problem (2.1) determines the values of Π_s, π_{is} for all i, s.

Finally, P_{i1} is given by

$$P_{i1} = E_1 \sum_{s=2}^{\infty} \Pi_s \pi_{is}, \quad i = 1,2,\ldots,N. \tag{3.55}$$

The same argument may be used to show that

$$P_{it} = E_t \sum_{s=t+1}^{\infty} \Pi_s \pi_{is}, \quad i = 1,2,\ldots,N, \quad t = 1,2,\ldots$$

This ends the proof.

Remark 3.4: We cannot overemphasize the fact that the methods of proof used in Theorem 3.1 will not characterize all of the equilibria in general. Such methods are incapable of proving uniqueness of equilibria. In fact, one of the main contributions of our paper is to develop methods of analysis that characterize all equilibria.

Remark 3.5: It is interesting to note that (3.55) was derived by T. Johnsen (1978). He did it by iterating (3.47). Given any initial approximation the contraction mapping theorem implies that the sequence of nth iterates converges to the unique solution $P_i(y)$ as $n \to \infty$. It is important to note, as pointed out earlier, that there are examples where there are equilibria of a non time stationary form. In such cases, the approximation method will not get all of the equilibria.

SECTION 4: CERTAINTY EQUIVALENCE FORMULAE

What we shall do in this section is to use the asset pricing model of Section 3 to construct a "Sharpe-Lintner" formula for the pricing of common stocks. In equilibrium our formula must hold. Furthermore, the data used in the formula to discount future profits are observable. The closest analogue to it seems to be that of Rubenstein (1976) in that Rubenstein relates the "price of risk" to tastes and technology.

The formula will be derived from the following special case of the model of Section 3:

$$f_i(x_{it},r_t) \equiv \bar{f}_i(x_{it})(A_i^o + A_i^1\tilde{\delta}_t),\ A_i^o > 0. \tag{4.1}$$

In other words, put K = 1 in (3.38). Here $\{\tilde{\delta}_t\}_{t=1}^{\infty}$ is an independent and identically distributed sequence of random variables with zero mean and finite variance σ^2. The numbers A_i^o, A_i^1 and the random variables $\tilde{\delta}_t$ are assumed to satisfy: There is $\varepsilon_o > 0$ such that for each t

$$\text{probability } (A_i^o + A_i^1\tilde{\delta}_t \geqq \varepsilon_o) = 1. \tag{4.2}$$

Optimum profits are given by

$$\pi_i(x_{it},r_t) = \bar{f}_i(x_{it})(A_i^o + A_i^1\tilde{\delta}_t) - \bar{f}_i'(x_{it})x_{it}(A_i^o + A_i^1\tilde{\delta}_t) \tag{4.3}$$

$$\equiv \bar{\pi}_i(x_{it})(A_i^o\tilde{\delta}_t).$$

In order to shorten the notational burden in the calculations below, put

$$f_i(x_{it},r_t) = \mu_{it} + \sigma_{it}\tilde{\delta}_t \tag{4.4}$$

$$f_i'(x_{it},r_t) = \mu_{it}' + \sigma_{it}'\tilde{\delta}_t \tag{4.5}$$

$$\pi_i(x_{it},r_t) = \bar{D}_{it} + V_{it}\tilde{\delta}_t \tag{4.6}$$

where

$$\mu_{it} \equiv \bar{f}_i(x_{it})A_i^o,\ \sigma_{it} \equiv \bar{f}_i(x_{it})A_i^1,\ \mu_{it}' = \bar{f}_i'(x_{it})A_i^o,\ \sigma_{it}' = \bar{f}_i'A_i^1, \tag{4.7}$$

$$\bar{D}_{it} = \bar{\pi}_i(x_{it})A_i^o,\ V_{it} = \bar{\pi}_i(x_{it})A_i^1.$$

All quantities will be evaluated at equilibrium levels unless otherwise noted. The notation is meant to be suggestive with $\bar{D}_{it}$ standing for average dividends or profits expected at date t, V_{it} standing for the coefficient of variability of profits with respect to the process $\{\tilde{\delta}_t\}_{t=i}^{\infty}$ and so forth. For a specific parable think of the $\{\tilde{\delta}_t\}_{t=1}^{\infty}$ process as "the market." Then production and profits in all industries i = 1,2,...,N are affected by the market. High values of $\tilde{\delta}_t$ correspond to "booms" and low values correspond to "slumps." Industries i with $A_i^1 > 0$ are procyclical. Those with $A_i^1 < 0$ are countercyclical, and those with $A_i^1 = 0$ are a-cyclical.

Assumption 4.1: There is at least one industry, call it N, that is a-cyclical. The Nth industry will be called risk free. For emphasis we will sometimes say that N is systematic risk free.

In order that all industries be active in equilibrium and that output remain bounded we shall assume

Assumption 4.2(i): $\bar{f}_i'(0) = +\infty$, $i = 1,2,\ldots,N$,

Assumption 4.2(ii): $\bar{f}_i'(\infty) = 0$, $i = 1,2,\ldots,N$.

Assumption 4.2(i) guarantees that all $x_{it} > 0$ along an equilibrium. Assumption 4.2(ii) implies there is a bound B such that $x_{it} \leqq B$ with probability one for all i,t.

Although concavity of f(x) and f(0) = 0 imply optimum profits are nonnegative we shall require that profits are positive for each x > 0 i.e.,

Assumption 4.3: For all x > 0, $\pi_i(x) \equiv \bar{f}_i(x) - \bar{f}_i'(x)x > 0$.

Assumption 4.3 will be used to show that equity prices are positive in equilibrium.

By the first order necessary conditions of equilibrium (3.13)-(3.14), (4.2), and Assumption 4.2(i), Assumption 4.3, it follows that

(4.8) $$P_{it}u'(c_t) = \beta E_t\{u'(c_{t+1})(\pi_{i,t+1} + P_{i,t+1})\} \quad \text{a.e.}$$

(4.9) $$u'(c_t) = \beta E_t\{u'(c_{t+1})R_{i,t+1}\}$$
$$= \beta E_t u'(c_{t+1})\mu_{it}' + \beta E_t(u'(c_{t+1})\tilde{\delta}_t)\sigma_{it}'. \quad \text{a.e.}$$

The R.H.S. of (4.9) follows from (4.5). It is clear from Assumption 4.3 that equity prices are positive since $\pi_{i,t+1}$ is positive with probability one. Hence both (4.8), (4.9) are equalities and $Z_{it} = 1$.

The P = PDV formula will be derived from (4.8), (4.9) by recursion. Use (4.3), (4.6) to get

(4.10) $$\pi_{i,t+1} = \bar{D}_{it} + V_{it}\tilde{\delta}_t$$

In order to shorten notation put $u'(c_{t+1}) = u'_{t+1}$ for all t. From (4.8), (4.10) we get

(4.11) $$P_{it}u'(c_t) = \beta E_t u'_{t+1}\bar{D}_{it} + \beta E_t(u'_{t+1}\tilde{\delta}_t)V_{it} + \beta E_t\{u'_{t+1}P_{i,t+1}\}.$$

Notice that μ_{it}', σ_{it}', $\bar{D}_{it}$, V_{it} are (in theory at least) observable. Hence, if we recurse (4.11) forward by replacing t by t+1 in (4.8) and inserting the result into (4.11) we can use (4.9) to solve for

(4.12) $$E_t m_t \equiv \frac{\beta E_t u'_{t+1}}{u'_t}, \quad E_t n_t \equiv \frac{\beta E_t(u'_{t+1}\tilde{\delta}_t)}{u'_t}, \quad m_t \equiv \frac{\beta u'_{t+1}}{u'_t}, \quad n_t \equiv \frac{\beta u'_{t+1}\tilde{\delta}_t}{u'_t}$$

in terms of μ_{it}', σ_{it}' and build up a P = PDV formula for P_{it}.

Let us continue. From (4.11) we get

(4.13) $$P_{it} = E_t m_t \bar{D}_{it} + E_t n_t V_{it} + \beta E_t\{u'_{t+1}P_{i,t+1}\}/u'_t$$

$$= E_t m_t \bar{D}_{it} + E_t n_t V_{it} + E_t\{m_t[E_{t+1}m_{t+1}\bar{D}_{i,t+1} + E_{t+1}n_{t+1}V_{i,t+1} + \beta E_{t+1}(u'_{t+2}P_{i,t+2})/u'_{t+1}]\}$$

$$= E_t m_t \bar{D}_{it} + E_t n_t V_{it} + E_t\{m_t E_{t+1}m_{t+1}\bar{D}_{i,t+1} + m_t E_{t+1}n_{t+1}V_{i,t+1}\} + \ldots$$

$$+ E_t\{m_t E_{t+1}m_{t+1}\ldots E_{t+T}(m_{t+T}\bar{D}_{i,t+T})\}$$

$$+ E_t\{m_t E_{t+1}m_{t+1}\ldots E_{t+T-1}m_{t+T-1}E_{t+T}n_{t+T}V_{i,t+T}\}$$

$$+ E_t\{m_t E_{t+1}m_{t+1}\ldots E_{t+T}(m_{t+T}P_{i,t+T+1})\}.$$

For the next move we need

Assumption 4.4: The utility function $u(\cdot)$ is such that for all $\{P_{it}, \pi_{it}, R_{it}\}_{t=1}^{\infty}$, $i = 1,2,\ldots,N$ the TVC_∞ is necessary for a consumer's maximum. Note that, as was pointed out in Remark 3.1, boundedness of $u(\cdot)$ is sufficient for A4.4. Now the TVC_∞ implies that

(4.14) $$E_t\{m_t E_{t+1}m_{t+1}\ldots E_{t+T}(m_{t+T}P_{i,t+T+1})\} \to 0,\ T \to \infty.$$

By (4.9) we get for each t

(4.15) $$1 = E_t m_t \mu'_{it} + E_t n_t \sigma'_{it}.$$

Therefore, if $\sigma'_{Nt} \equiv 0$, (4.15) implies

(4.16) $$E_t m_t = 1/\mu'_{Nt},\ E_t n_t = [(\mu'_{Nt} - \mu'_{it})/\sigma'_{it}](1/\mu'_{Nt}) \equiv -\Delta_t/\mu'_{Nt}.$$

Note here that Δ_t is the excess marginal return over the risk free marginal return divided by marginal risk. Also μ'_{Nt} is principal plus interest gotten by employing a marginal unit in process N. It is important to observe that Δ_t is independent of i. Furthermore, the Ross risk price λ_t is determined by $\lambda_t = \Delta_t$. This follows from (2.30) and (4.16).

Turn now to the K factor case. In the K factor case, put

(4.17) $$f_i(x_{it},r_t) \equiv \bar{f}_i(x_{it})(\sum_{k=o}^{K} A_i^k\tilde{\delta}_{t+1}^k),\ \tilde{\delta}_{t+1}^o \equiv 0,\ A_i^o > 0.$$

Put the same assumptions on the data as in Section 3. Then as in (4.4), (4.5), (4.6) we may write

(4.18) $$f_i(x_{it},r_t) = \mu_{it} + \sum_{k=1}^{K} \sigma_{it}^k \tilde{\delta}_{t+1}^k$$

(4.19) $$f'_i(x_{it},r_t) = \mu'_{it} + \sum_{k=1}^{K} \sigma'^k_{it}\tilde{\sigma}_{t+1}^k$$

(4.20) $$\pi_i(x_{it},r_t) = \bar{D}_{it} + \sum_{k=1}^{K} V_{it}^k\tilde{\delta}_{t+1}^k \equiv \pi_{i,t+1}$$

where the entities in (4.18)-(4.20) are defined as in (4.7). Keep the same assumptions as above. Then (4.8), (4.9) become

(4.21) $$P_{it}u'_t = \beta E_t\{u'_{t+1}(\bar{D}_{it} + \sum_{k=1}^{K} V_{it}^k\tilde{\delta}_{t+1}^k) + u'_{t+1}P_{i,t+1}\}$$

(4.22) $$u'_t = \beta E_t\{u'_{t+1}(\mu'_{it} + \sum_{k=1}^{K}\sigma'^k_{it}\tilde{\delta}_{t+1}^k)\}$$

Define

(4.23) $m_t \equiv (\beta u'_{t+1})/u'_t, \; n_t^k \equiv (\beta u'_{t+1}\tilde{\delta}^k_{t+1})/u'_t.$

Then letting N be a risk free process i.e., $\sigma'^k_{Nt} \equiv 0$, for all k, t we get

(4.24) $E_t m_t = 1/\mu'_{Nt}$

and for each i

(4.25) $1 = (E_t m_t)\mu'_{it} + \sum_{k=1}^{K} \sigma'^k_{it}(E_t n^k_t) = \mu'_{it}/\mu'_{Nt} + \sum_{k=1}^{K}\sigma'^k_{it}(E_t n^k_t)$

Hence from (4.25) it follows that

(4.26) $(\mu'_{Nt} - \mu'_{it})/\mu'_{Nt} = \sum_{k=1}^{K} \sigma'^k_{it}(E_t n^k_t), \quad i = 1,2,\ldots,N.$

It is assumed that a unique solution of (4.26) for $E_t n^k_t$ exists and is defined by

(4.27) $E_t n^k_t \equiv -\Delta^k_t/\mu'_{Nt}.$

Also the Ross price of systematic risk k, λ_{kt} satisfies $\lambda_{kt} = \Delta^k_t$. This last equality follows from (2.30) and (4.27). From (4.8) we get putting $\xi_{it} \equiv \bar{D}_{it} - V_{it}\Delta_t$

(4.28) $P_{it} = \xi_{it}/\mu'_{Nt} + E_t\{[\frac{\beta u'_{t+1}}{u'_t}\mu'_{Nt}](P_{i,t+1})\}\mu'_{Nt}$

$\equiv \{\xi_{it} + E_t\hat{P}_{i,t+1}\}/\mu'_{Nt}.$

Hence

(4.29) $E_t\hat{P}_{i,t+1} + \xi_{it} = \mu'_{Nt}P_{it}.$

Equation (4.29) says that investing P_{it} in the stock market must give the same expected return after paying for the services of risk bearing as investing it in the risk free process. It states that the stock market is a "fair game" taking into account the opportunity cost of funds and the cost of risk bearing.

Clearly restrictive assumptions on tastes and technology are necessary to get a martingale. Also, only for specific preferences is equation (4.29) testable. Its violation would signal "market inefficiency" in our model world.

A far better test would be based on (3.55). But even verification of (3.55) would not test Pareto optimality of the stock market allocation. This is so because there may exist heterogeneous consumer economics (e.g., overlapping generations models) where (3.55) holds, but the allocation is not Pareto optimal. This question remains to be investigated.

It is worth pointing out here that if the random variable

(4.30) $(\beta u'_{t+1}/u'_t)\mu'_{Nt} \equiv I_t$

is independent of $P_{i,t+1}$ at date t then (4.16) implies that (4.29) may be rewritten

(4.31) $E_t P_{i,t+1} + \xi_{it} = \mu'_{Nt}P_{it}.$

Equation (4.31) contains no subjective entities -- unlike (4.29). The problem of deriving equations like (4.31) that contain no entities that are subjective and hence are directly testable is solved abstractly by (3.55). Perhaps a formula analogous to (4.31) exists that holds in, at least, an approximate sense.

A Testable Formula

In what follows below a simple formula is developed under the hypothesis of linearity of the asset pricing functions $P_i(y)$. An example where $P_i(y)$ is linear is given in Section 5 below.

Theorem 4.1: Assume Assumptions 4.1-4.4. Furthermore, assume that there are constants K_i, L_i such that

$$(4.32)\qquad P_i(y) = K_i y + L_i, \quad i = 1,2,\ldots,N$$

Then, for each t,i.

$$(4.33)\qquad \mu'_{Nt} P_{it} = E_t P_{i,t+1} - \sum_{k=1}^{K} \Delta_t^k S_{it}^k + E_t \pi_{i,t+1} - \sum_{k=1}^{K} \Delta_t^k V_{it}^k$$

must hold. Here by (4.32) and (4.20) we may write

$$(4.34)\qquad P_{i,t+1} \equiv P_i(y_{t+1}) \equiv \bar{P}_{it} + \sum_{k=1}^{K} S_{it}^k \tilde{\delta}_{t+1}^k, \quad \bar{P}_{it} \equiv E_t P_{i,t+1}.$$

$$(4.35)\qquad \pi_{i,t+1} \equiv \bar{D}_{it} + \sum_{k=1}^{K} V_{it}^k \tilde{\delta}_{t+1}^k, \quad \bar{D}_{it} \equiv E_t \pi_{i,t+1}$$

where $E_t P_{i,t+1}$, $E_t \pi_{i,t+1}$, S_{it}^k, V_{it}^k do not depend upon y_{t+1} but depend on $(x_{1t},\ldots,x_{Nt})$ only.

Proof: In order to establish (4.33) it must be shown that (4.34), (4.35) hold. By (4.17) and the definition of y_{t+1} we have

$$(4.36)\qquad y_{t+1} \equiv \sum_{j=1}^{N} f_j(x_{j+1}, r_t) = \sum_j \bar{f}_j(x_{jt})\left(\sum_{k=0}^{K} A_j^k \tilde{\delta}_{t+1}^k\right)$$

$$\equiv L(y_t) + \sum_{k=1}^{K} M^k(y_t) \tilde{\delta}_t^k$$

Hence y_{t+1} is a linear combination of the shocks $\tilde{\delta}_{t+1}^k$ with weights that depend only upon $(x_{1t},\ldots,x_{Nt})$. So also is $P_{i,t+1}$. Thus (4.34) holds for appropriate S_{it}^k since $P_{i,t+1}$ is linear in y_{t+1}. Equation (4.20) is identical to (4.35).

Divide both sides of (4.21) by u'_t to get

$$(4.37)\qquad P_{it} = E_t\{m_t(\bar{D}_{it} + \sum_{k=1}^{K} V_{it}^k \tilde{\delta}_{t+1}^k)\} + E_t\{m_t P_{i,t+1}\}$$

Put, using (4.23),

$$(4.38)\qquad \bar{m}_t \equiv E_t m_t, \quad \bar{n}_t^k \equiv E_t n_t^k.$$

By (4.37) and (4.34) we have

$$P_{it} = \bar{m}_t\bar{D}_{it} + \sum_{k=1}^{K} V_{it}^k\bar{n}_t^k + E_t\{m_t(\bar{P}_{it} + \sum_{k=1}^{K} S_{it}^k\tilde{\delta}_{t+1}^k$$

$$= \bar{m}_t\bar{D}_{it} + \sum_{k=1}^{K} V_{it}^k\bar{n}_t^k + \bar{m}_t\bar{P}_{it} + \sum_{k=1}^{K} S_{it}^k\bar{n}_t^k.$$

But (4.24) and (4.27) imply

$$\mu'_{Nt}P_{it} = \bar{D}_{it} + \bar{P}_{it} - \sum_{k=1}^{K} \Delta_t^k(V_{it}^k + S_{it}^k).$$

This ends the proof.

It is worth pointing out that although (4.33) contains no subjective entities and, hence, is directly testable it was derived under the strong hypothesis of linearity of the asset pricing function $P_i(y_{t+1})$. The linearity hypothesis was needed to be able to write the one period returns Z_{it} to holding asset i in the linear form (2.17) of Ross. The linear form of Z_{it} was used, in turn, to derive (4.33). We suspect that strong conditions will be required on utility and technology to be able to write equilibrium asset returns in the form (2.17). Hence (4.33) is not general: it holds only as a linear approximation. Thus, it is likely to hold in continuous time relatives of our model.

The economic content of (4.33) is compelling. It is a standard "no arbitrage profits" condition. The price of risk bearing over the time interval t,t+1 sells for Δ_t^k per unit of risk of type k. At date t risk emerges from two sources: (i) $\pi_{i,t+1}$, and (ii) $P_{i,t+1}$. Profits contain V_{it}^k units of risk of type k. The price of stock i at date t+1 contains S_{it}^k units of risk of type k.

Hence the total cost of risk bearing from all sources of risk for all types of risk is

$$\sum_{k=1}^{K} \Delta_t^k(V_{it}^k + S_{it}^k)$$

Thus (4.33) just says that the risk free earnings from an investment of P_{it} must equal the sum of risk adjusted sale value of stock i at date t+1 and risk adjusted profits.

Remark: The formula (4.33) is exactly the Sharpe-Lintner formula of finance. While the formula itself is textbook knowledge, the advantage of deriving it from a general equilibrium model is that we can study exactly what conditions on tastes and technology are required for its validity. Vis-à-vis, tastes and technology must be such that the asset pricing function is linear in y.

A set of approximate formulae of "accuracy" a may be derived from (4.37) by expanding

$$(4.39) \qquad P_{i,t+1} \equiv P_i(y_{t+1}) = P_i[L(y_t) + \sum_{k=1}^{K} M^k(y_t)\tilde{\delta}_t^k]$$

in a Taylor series about $L(y_t)$ and discarding terms of order higher than a. The Sharpe-Lintner formula (4.33) corresponds to a = 1. In order to see how this type of development goes, we calculate the case a = 2, K = 1 and discard terms of order higher than 2. Doing this we get, putting $M^1(y) = M(y)$

(4.40) $$P_i[L(y_t) + M(y_t)\tilde{\delta}_t] = P_i[L(y_t)] + P_i'[L(y_t)]M(y_t)\tilde{\delta}_t + \frac{1}{2} P_i''[L(y_t)]M^2(y_t)\tilde{\delta}_t^2$$

Inserting (4.40) into (4.37) we get for i = i,2,...,N

(4.41) $$P_i(y_t) \equiv P_{it} = \bar{m}_t\bar{D}_{it} + \bar{n}_tV_{it} + \bar{m}_tP_i[L(y_t)] + \bar{n}_tP_i'[L(y_t)]M(y_t) + \bar{o}_t \frac{1}{2} P_i''[L(y_t)]M^2(y_t).$$

where

(4.42) $$\bar{o}_t \equiv E_t(m_t\tilde{\delta}_t^2).$$

Since (4.41) holds for all i the subjective entities $\bar{m}_t$, $\bar{n}_t$, $\bar{o}_t$ may be expressed in terms of observables as before in the a = 1 case.

Space limitations prevent us from pursuing the development of asset pricing formulae further.

SECTION 5: EXAMPLE

In this section we present a solved example where equilibrium returns are linear in the stocks. Let the data be given by

(5.1) $$u(c) = \log c$$

(5.2) $$f_i(x_i,r) = A_i(r)x_i^\alpha, \quad i = 1,2,\ldots,N, \quad 0 < \alpha < 1.$$

We shall assume that for all i,

$$A_i(r) > 0 \quad \text{for all } r \in R,$$

and $A_i(r)$ is continuous in r. Since R is compact each $A_i(r)$ has a positive lower bound $\underline{A}_i > 0$.

First order necessary conditions (1.10a), (1.10b) become for all t,

(5.3a) $$\frac{1}{c_t} \geqq \beta\alpha E_t\{\frac{1}{c_{t+1}} A_i(r_t)x_{it}^{\alpha-1}\}$$

(5.3b) $$\frac{1}{c_t} x_{it} = \beta\alpha E_t\{\frac{1}{c_{t+1}} A_i(r_t)x_{it}^{\alpha-1}\}x_{it}, \quad i = 1,2,\ldots,N$$

(5.3c) $$\lim_{t\to\infty} E_1\{\frac{\beta^{t-1}}{c_t} x_t\} = 0.$$

Conjecture an optimum solution of the form

(5.4) $$c_t = (1 - \lambda)y_t, \quad x_t = \lambda y_t, \quad x_{it} = \eta_i x_t, \quad \sum_{i=1}^N \eta_i = 1,$$

where

(5.5) $$\lambda > 0, \quad \eta_i \geqq 0, \quad i = 1,2,\ldots,N.$$

Inset (5.4) into (5.3a), (5.3b) solve for λ, $\{\eta_i\}_{i=1}^N$ and check that (5.3c) is satisfied. Doing this we get

(5.6) $$\frac{1}{(1 - \lambda)y_t} \geqq \beta\alpha E_t\left\{\frac{1}{(1 - \lambda)y_{t+1}} A_i(r_t)x_{it}^{\alpha-1}\right\}$$

$$\text{iff} \qquad \frac{1}{y_t} \geqq \beta\alpha E_t \left\{ \frac{A_i(r_t)}{\sum_{j=1}^{N} A_j(r_t) x_{jt}^{\alpha}} x_{it}^{\alpha-1} \right\}$$

$$\text{iff} \qquad \frac{1}{y_t} \geqq \beta\alpha E_t \left\{ \frac{\eta_i^{\alpha-1} x_t^{\alpha-1} A_i(r_t)}{\sum_{j=1}^{N} A_j(r_t) \eta_j^{\alpha} x_t^{\alpha}} \right\}$$

$$\text{iff} \qquad \frac{1}{y_t} \geqq \beta\alpha E_t \left\{ \frac{A_i(r_t)}{\sum_{j=1}^{N} A_j(r_t) \eta_j^{\alpha}} \right\} \frac{\eta_i^{\alpha-1}}{x_t} \equiv \beta\alpha \eta_i^{\alpha-1} \Gamma_i / x_t$$

iff

$$x_t \geqq \beta\alpha \eta_i^{\alpha-1} \Gamma_i y_t. \tag{5.7}$$

Set (5.7) aside for the moment. From (5.3b) following the same steps that we used to get (5.7) we are led to

$$\frac{x_{it}}{y_t} = \beta\alpha \eta_i^{\alpha} \Gamma_i. \tag{5.8}$$

iff

$$x_{it} = \beta\alpha \eta_i^{\alpha} \Gamma_i y_t = \eta_i x_t. \tag{5.9}$$

Hence (5.7) holds with equality for all t,i. Since it is well known and is easy to see that for N = 1,

$$\lambda = \beta\alpha$$

it is natural to conjecture for $N \geqq 1$ that

$$\lambda = \beta\alpha, \quad \eta_i^{\alpha-1} \Gamma_i = 1, \quad i = 1,2,\ldots,N \tag{5.10}$$

and test (5.3c). If (5.10) satisfies (5.3c) then we have found an optimum solution and hence the unique optimum solution.

Continuing we have

$$\Gamma_i \equiv E \left| \frac{A_i(r)}{\sum_{j=1}^{N} A_j(r) \eta_j^{\alpha}} \right| = \eta_i^{1-\alpha}, \quad \sum_{j=1}^{N} \eta_j = 1. \tag{5.11}$$

It is shown in the Appendix that (5.11) has a unique solution $\{\bar{\eta}_i\}_{i=1}^{N}$.

It is straightforward to check that (5.4) with $\bar{\lambda} \equiv \alpha\beta$, $\eta_i \equiv \bar{\eta}_i$, $i = 1,2,\ldots,N$ generates a solution that not only satisfies (5.3a), (5.3b) by construction but also satisfies (5.3c). We leave this to the reader.

Let us use the solution to calculate an example of an equilibrium asset price function from the work of Section 3. From (3.13a) and (3.37) we get

$$\begin{aligned} E_t[u'(c_{t+1})(P_{i,t+1} + \bar{\pi}_{i,t+1})/\bar{P}_{it}] &= E_t[\alpha A_i(r_t) \bar{x}_{it}^{\alpha-1} u'(c_{t+1})] \\ &= E_t[\alpha A_i(r_t) \bar{\eta}_i^{\alpha-1} \bar{x}_t^{\alpha-1} u'(c_{t+1})] \end{aligned} \tag{5.12}$$

$$\begin{aligned} \bar{\pi}_{i,t+1} &= A_i(r_t) \bar{x}_{it}^{\alpha} - \alpha A_i(r_t) \bar{x}_{it}^{\alpha-1} \bar{x}_{it} \\ &= (1-\alpha) A_i(r_t) \bar{x}_{it}^{\alpha} = (1-\alpha) A_i(r_t) \bar{\eta}_i^{\alpha} \bar{x}_t^{\alpha}. \end{aligned} \tag{5.13}$$

Hence, the first order necessary condition for an asset pricing function of the form $P_{it} = P_i(y_t)$ becomes for $u(c) = \log c$, using $c_t = (1 - \bar{\lambda})y_t$:

$$(5.14) \qquad P_i(y_t)/y_t = \beta E_t\{(P_i(y_{t+1}) + \pi_{i,t+1})/y_{t+1}\}$$

Equations (5.13) and (5.14) give us

$$(5.15) \qquad P_i(y_t)/y_t = \beta E_t\{(1 - \alpha)A_i(r_t)\bar{\eta}_i^{-\alpha}x_t^{-\alpha}/[\sum_{j=1}^{N} A_i(r_t)\bar{\eta}_j^{-\alpha}x_t^{-\alpha}] + P_i(y_{t+1})/y_{t+1}\}$$

$$\equiv \beta(1 - \alpha)\bar{\eta}_i + \beta E_t\{P_i(y_{t+1})/y_{t+1}\}, \quad i = 1,2,\ldots,N.$$

Here by (5.11)

$$(5.16) \qquad \bar{\eta}_i = E_t\{A_i(r_t)\bar{\eta}_i^{\alpha}/(\sum_{j=1}^{N} A_j(r_t)\bar{\eta}_j^{\alpha})\}.$$

The system of equations (5.15) is in particularly suitable form for the application of the contraction mapping theorem to produce a unique fixed point $\bar{P}(y) \equiv (\bar{P}_1(y),\ldots,\bar{P}_N(y))$ that solves (5.15). Rather than do this we just conjecture a solution of the form

$$(5.17) \qquad \bar{P}_i(y) = \bar{K}_i y, \quad i = 1,2,\ldots,N.$$

and find $\bar{K}_i$ from (5.15) by equating coefficients. Obviously from (5.15) $\bar{K}_i$ satisfies

$$(5.18) \qquad \bar{K}_i = \beta(1 - \alpha)\bar{\eta}_i + \beta\bar{K}_i, \quad i = 1.2,\ldots,N,$$

so that

$$(5.19) \qquad \bar{K}_i = (1 - \beta)^{-1}\beta(1 - \alpha)\bar{\eta}_i, \quad i = 1,2,\ldots,N.$$

Since R.H.S. (5.15) is a contraction of modulus β on the space of bounded continuous functions on $[0,\infty)$ with values in R^N the solution (5.17) is the only solution such that each $P_i(y)/y$ is bounded and continuous on $[0,\infty)$.

We now have a solved example. It is interesting to examine the dependence of $P_i(y)$ on the problem data from (5.17), (5.19).

First, in the one asset case we find $\eta_N = 1$ from (5.16) so that

$$(5.20) \qquad P(y) = \frac{\beta}{1 - \beta}(1 - \alpha)y$$

Hence (i) the asset price decreases as the elasticity of output with respect to capital input increases. (ii) The variance of output has no effect on the asset price function. (iii) Asset price increases, when β increases.

Result (i) follows because profit's share of national output is inversely related to α. One would expect (ii) from the log utility function. One would expect (iii) because as β increases the future is worth more relative to the present hence savings should increase forcing asset prices to rise.

Furthermore, (5.20) says that asset price increases as y increases.

Secondly, in the multi asset deterministic case we have

$$P_i(y) = \frac{\beta}{1-\beta}(1-\alpha)\bar{\eta}_i y \quad i = 1,2,\ldots,N \tag{5.21}$$

where $\bar{\eta}_i$ is given by equation (A5.4) in the Appendix. We can see that if the coefficient A_i measures the productivity of firm i using the common technology x^α so that output of i is $A_i x^\alpha$ then firms that are relatively more productive bear higher relative prices for their stock. Absolute productivity does not effect relative prices. This is so because $\bar{\eta}_i$ is homogenous of degree zero in $(A_1,\ldots,A_N)$.

This is again one of those results that looks intuitively clear after hindsight has been applied. The consumers in this economy have no other alternative but to lease capital or to invest in stock in the N firms. Hence if the productivity of all of them is halved the constellation of asset price relatives will not change although output will drop. This type of result is specific to the log utility and Cobb Douglas production technologies.

The technique of Mirman-Zilcha (1975) may be applied to find the closed form solution for the limit distribution F mentioned in Section 2. Once F is known the limit distribution of asset prices may be found from (3.40) and the limit distribution of Ross's risk prices may be calculated, from (2.30). We leave that to the reader.

SECTION 6: SUMMARY, CONCLUSIONS, COMMENTS, AND SUGGESTIONS FOR FURTHER RESEARCH

Most of the results of this paper are summarized in the Introduction. Therefore, we will first comment on what we think has been done here. What has been done is to turn normative

stochastic growth theory into positive theory by introducing market institutions into received stochastic growth theory.

Furthermore, we have specialized the model so that received stochastic growth theory may be modified to generate the recursive structure that is so useful for preserving the empirical tractability of Merton's (1973) ICAPM. This has been done in such a way to link our theory up with the K factor arbitrage theory of Ross (1976).

The reader may ask why not decentralize the N process growth model along the lines of Arrow-Debreu where the pure rents are redistributed lump sum, assume constant returns to scale so that pure rents are zero, and price the capital stock along Arrow-Debreu lines. The reason we did not do this is because it has already been done in the stochastic growth literature for the general N process multisector case. However, implications of this type of model for finance have not been explored in any great detail yet. But what we have done here may easily be modified to include this case.

This literature has been surveyed by Roy Radner (1974). It was pointed out in my comment (August 1974) on Radner that simple stochastic growth models could be turned into "rational expectations models" by introducing a representative firm and consumer and finding decentralizing prices for them along standard Malinvaud (1953) lines provided that the initial Malinvaud price is chosen so that the consumer's transversality condition at infinity is satisfied. For Malinvaud prices, see, for example, the papers in Los, J. and

Los, M (1974) on "stimulating prices" for the Russian literature and I. Zilcha (1976) and his references for the Western literature.

By our modification of the "Malinvaud" price technique mentioned above all stochastic growth models may be turned into rational expectations models by introducing a representative consumer that has the same preferences as the planner in the growth model and using the resulting "decentralizing prices" as the rational expectations prices. After choosing the initial Malinvaud price so that the TVC_{∞} holds for the representative consumer, growth models become "asset pricing models" by this device.

More advances should be expected along the lines of introducing imperfect information and inquiring into what rules firm managers should follow in order to maximize equilibrium welfare of the representative consumer when some contingency markets are absent.

Existing results on stochastic stability in the multi-sectoral growth literature could be used to extend the stochastic stability theorem that was presented here to the multisector case.

It should be straightforward to extend the pricing results themselves to the multisector case.

More difficult and more interesting would be to introduce heterogeneous consumers so that borrowing on future income may be introduced and investigate the impact of this new institution on the price of risk. For example, in a finite horizon model where the individual is constrained to plans that require only that the

expected wealth at horizon T conditioned at date 1 is nonnegative then one suspects that the price of risk may be small and the security market line may be quite flat. But care must be taken since "for each lender there must a borrower be." Thus, the institutional requirements on wealth at date T and the penalties for insolvency should have an impact on the price of risk. Furthermore, following the same line of reasoning, the work of Truman Bewley (1977) on the self insurance behavior embedded in the permanent income hypothesis of Milton Friedman via borrowing and lending leads to the belief that the security market line SML generated by such a modification of our model will be flatter than the SML predicted by the standard CAPM. This observation may provide an additional clue to why the observed SML is flatter than the SML predicted by the CAPM. See Merton (1973) and Fama (1976) for a discussion of this issue. What we have said here about the issue is highly speculative at best.

We close this paper with the hope that the methods developed here should be of some use to economics and finance.

APPENDIXES

APPENDIX TO SECTION 3

Let us prove Lemma 3.2 first. Let X solve the optimal growth problem (1.1). It is obvious that $\mathcal{R}$ satisfies the first order necessary conditions for an R.E.E. by its very definition. What is at issue is the TVC_∞ (3.16), (3.17).

Put

(A 3.1) $$V(x_{t-1}, t-1) \equiv \text{Maximum } E_1 \sum_{s=t}^{\infty} \beta^{s-1} u(c_s)$$

$$\text{s.t. } c_s + x_s = \sum_{j=1}^{N} f_j(x_{j,s-1}, r_{s-1})$$

(A 3.2) $$\sum_{j=1}^{N} x_{js} \equiv x_s, \quad x_{js} \geqq 0, \quad j = 1,2,\ldots,N. \quad c_s \geqq 0, \; x_s \geqq 0,$$

$$s = t, t+1, \ldots, x_{t-1} \text{ given.}$$

Then, following a similar argument as that in (3.23)-(3.32), we have, since u is bounded, that for any $x_t \geqq 0$, $V(x_t,t) \to 0$, $t \to \infty$ and

(A. 3.3) $$V(x_t,t) \geqq V(x_t,t) - V(x_t/2,t) \geqq V'(x_t,t)x_t/2.$$

$$= E_1\{\beta^t u'(c_{t+1}) f_i'(x_{it}, r_t) x_t/2\}$$

$$= E_1\{\beta^{t-1} u'(c_t) x_t/2\} \geqq 0$$

Since the L.H.S. of (A 3.3) must go to zero, the R.H.S. must also. Hence

(A 3.4) $$E_1\{\beta^{t-1} u'(c_t) x_t\} \to 0, \quad t \to \infty$$

along any optimum program. This establishes (3.17).

What about (3.16)? Here the stochastic process $\{\bar{P}_{it}\}_{t=1}^{\infty}$ was assumed to have been constructed from the quantity side of the model by use of (3.29) so that the TVC_∞ (3.31) was satisfied. Hence TVC_∞ (3.16) is satisfied by the very construction of $\{\bar{P}_{it}\}_{t=1}^{\infty}$. This establishes the implication: (i) implies (ii).

In showing that (ii) implies (i) it is clear that the first order necessary conditions for the quantity side of an R.E.E. boil down to the first order conditions for the optimal growth problem. What must be established is the TVC_∞ (A 3.4). But this follows from (3.17) of Lemma 3.1. This ends the proof of Lemma 3.2.

<u>Remark A 3.1</u>: Lemma 3.2 is not really useful as it stands because given the quantity side of the growth model it was assumed that $\{\bar{P}_{it}\}$ was constructed from use of (3.29) so that (3.31) held. How can we be sure that such a solution to the stochastic difference equation (3.29) exists even though $Z_{it} = 1$ for all i, t and π_t is given by (3.33) from the quantity side of the growth problem? Even though we have assumed that pure rents are positive so that equity prices must be positive in <u>any</u> equilibrium so that $Z_{it} = 1$ for all i,t there is still a problem to show that a solution of (3.29) exists such that (3.31) holds.

Theorems 3.1 and 3.1' take care of this problem. They establish the existence of a solution of (3.29) that satisfies (3.31) under mild restrictions on the quantity side of the growth problem. The reason Theorem 3.1 can be used is that standard arguments (cf. Brock-Mirman (1972); Mirman-Zilcha (1975, 1976, and 1977) using dynamic programming establish that the quantity side of the growth model is recursive. Hence the quantity side of any R.E.E. must be recursive too.

APPENDIX TO SECTION 5

It is straightforward to show by direct calculation that for the example of Section 5 the solution to the Bellman equation

$$\text{(A 5.1)} \qquad U(y_1) = \text{maximum } \{u(y_1 - x_1) + \beta EU[\sum_j f_j(x_{j1}, r_1]\}$$

is of the form

$$\text{(A 5.2)} \qquad U(y_1) = K_1 + [1/(1 - \alpha\beta)] \log y_1$$

for some constant K_1.

Hence for any given x_t the allocation functions $x_{it} \equiv \eta_i(x_t)$ are given by solving the problem

$$\text{(A 5.3)} \qquad \text{maximize } \int \log [\sum_{i=1}^{N} A_i(r_t)\eta_i^{\alpha} x_t]\mu(dr_t)$$

$$\text{s.t. } \eta_i \geqq 0, \quad i = 1,2,\ldots,N, \quad \sum_{i=1}^{N} \eta_i = 1.$$

But the solution η to problem (A 5.3) is the same as the solution η to the problem

$$\text{(A 5.4)} \qquad \text{maximize } \int \log [\sum_{i=1}^{N} A_i(r_t)\eta_i^{\alpha}]\mu(dr_t)$$

because the log function is multiplicatively additive.

By strict concavity and monotonicity of the logarithm there is just one solution η to (A 5.4). It may be readily studied by use of (A 5.4) and we leave this to the reader.

<u>Alternative Setup Where Firms Carry Capital and Maximize Value</u>

Let equity i now represent a claim on the dividends of firm i. Also let Z_{it}, d_{it}, and x_{it} be chosen by the firm. The budget constraint of firm i is

$$\text{(A 3.5)} \qquad P_{it}(Z_{it} - Z_{i,t-1}) + g_i(x_{i,t-1}, r_{t-1})$$

$$= x_{it} - x_{i,t-1} + \delta_i x_{i,t-1} + d_{i,t-1} Z_{i,t-1}.$$

Here the new symbol d_{it} which denotes dividends per share paid at the end of period t. We will <u>derive</u> an expression for the value of the firm from the consumer side of the model.

The budget equation for the consumer is from (3.3)

$$\text{(A 3.6)} \qquad c_t + \underset{\sim}{P}_t \cdot (\underset{\sim}{Z}_t - \underset{\sim}{Z}_{t-1}) \leqq \underset{\sim}{d}_{t-1} \cdot \underset{\sim}{Z}_{t-1} \equiv y_t.$$

The consumer faces $\{\underset{\sim}{P}_t\}$, $\{d_{it}\}$ parametrically and maximizes (3.2) subject to (A 3.6) and $\underset{\sim}{Z}_t \geqq 0$, $t = 1,2\ldots$ Note that we do not allow short selling. There is not enough space to treat short selling.

Arguments analogous to those of Section 3 allow us to show that the necessary and sufficient conditions for a solution to the consumer's problem are

(A. 3.7) $$P_{it} \geqq E_t\{\Gamma_{t+1}(d_{it} + P_{i,t+1})\}$$

(A 3.8) $$P_{it}Z_{it} = E_t\{\Gamma_{t+1}(d_{it} + P_{i,t+1})\}Z_{it}$$

(A. 3.9) $$\lim_{t\to\infty} E_1\{\beta^{t-1}u'(c_t)\underset{\sim}{P}_t \cdot \underset{\sim}{Z}_t\} = 0.$$

Here Γ_{t+1} is defined by

(A 3.10) $$\Gamma_{t+1} = \beta u'(c_{t+1})/u'(c_t).$$

Equation (A 3.8) may be rewritten to derive a recursion for the value of the firm

(A 3.11) $$V_{it} \equiv P_{it}Z_{it}.$$

We have from (A 3.5), (A 3.8)

(A 3.12) $$V_{it} \equiv P_{it}Z_{it} = E_t\{\Gamma_{t+1}(d_{it}Z_{it} + P_{i,t+1}Z_{it+1} + P_{i,t+1}Z_{it}$$

$$- P_{i,t+1}Z_{i,t+1})\} = E_t\{\Gamma_{t+1}(V_{i,t+1} + d_{it}Z_{it} - P_{i,t+1}(Z_{i,t+1} - Z_{it}))\}$$

$$= E_t\{\Gamma_{t+1}(V_{i,t+1} + g_i(x_{it},r_t) - (x_{i,t+1} - x_{it} + \delta_i x_{it}))\}$$

$$\equiv E_t\{\Gamma_{t+1}(V_{i,t+1} + N_{i,t+1})\}.$$

The L.H.S. and extreme R.H.S. of (A 3.12) corresponds to Modigliani-Miller's (MM) equation (5) (Miller-Modigliani 1961, p. 414). They use this equation to demonstrate that the firm's value is invariant to dividend policy. The same conclusion obtains in our general equilibrium model. In order to see it develop the recursion

(A 3.13) $$V_{i1} = E_1\{\Gamma_2(V_{i2} + N_{i2})\} = E_1(\Gamma_2 N_{i2}) + E_1(\Gamma_2\Gamma_3 N_{i2}) + E_1(\Gamma_2\Gamma_3 V_{i3})$$

$$= \ldots = E_1(\Gamma_2 N_{i2}) + \ldots + E_1(\prod_{s=2}^{s=T}\Gamma_s N_{iT}) + E_1\{(\prod_{s=2}^{s=T}\Gamma_s)V_{iT}\}$$

The $\{\Gamma_t\}$ sequence is a sequence of random discount factors. They were exogenously given in MM's model and were not endogenously determined by tastes and technology as in our setup. Hence MM had to <u>assume</u> that dividend policy did not effect them in order to get the invariance result.

More fundamentally, however, MM had to <u>assume</u> that

(A 3.14) $$\lim_{T\to\infty} E_1\{(\prod_{s=2}^{s=T}\Gamma_s)V_{iT}\}$$

was not effected by dividend policy in order to get their invariance result.

We can demonstrate that the consumer's TVC_∞ implies the limit (A 3.14) is zero in equilibrium as soon as we define equilibrium.

<u>Definition:</u>[3] A rational expectations equilibrium (R.E.E.) is a

stochastic process $\mathcal{R} \equiv < \{\bar{P}_{it}\}_{t=1}^{\infty}, \{\bar{x}_{it}\}_{t=1}^{\infty}, \{\bar{Z}_{it}\}_{t=1}^{\infty}, \{\bar{d}_{it}\}_{t=1}^{\infty}$, $i = 1,2,\ldots,N$; $\{\bar{c}_t\}_{t=1}^{\infty}, \{\bar{\Gamma}_t\}_{t=2}^{\infty} >$ if facing $\mathcal{P} \equiv < \{\bar{P}_{it}\}_{t=1}^{\infty}, \{\bar{d}_{it}\}_{t=1}^{\infty} >$ the consumer chooses

(A 3.15) $$c_t = \bar{c}_t, Z_{it} = \bar{Z}_{it},\ t = 1,2,\ldots;\ \ i = 1,2,\ldots,N$$

and if facing $\{\bar{\Gamma}_t\}_{t=2}^{\infty}$ the ith firm chooses

(A 3.16) $$x_{it} = \bar{x}_{it},\ t = 1,2,\ldots$$

and the ith firm accommodates the optimum investment plan (A 3.16) by setting

(A 3.17) $$Z_{it} = \bar{Z}_{it}, d_{it} = \bar{d}_{it},\ t = 1,2,\ldots$$

Firms are assumed to solve

(A 3.18) $$\underset{\{x_{it}\}_{t=1}^{\infty}}{\text{maximize}}\ \bar{V}_{i1} - x_{i1}$$

$$\text{s.t.}\ \ x_{it} \geqq 0,\ \ t = 1,2,\ldots$$

where

(A 3.19) $$\bar{V}_{i1} \equiv E_1\{\sum_{T=2}^{\infty} (\prod_{s=2}^{s=T} \bar{\Gamma}_s N_{iT})\}.$$

Furthermore, firm's expectations on the sequence of random discounts must be "rational" in the sense that

(A 3.20) $$\bar{\Gamma}_s = \beta u'(\bar{c}_s)/u'(\bar{c}_{s-1}),\ \ s = 2,3,\ldots$$

Finally, material balance must obtain

(A 3.21) $$\bar{c}_t + \bar{x}_t = \sum_{i=1}^{N} f_i(\bar{x}_{i,t-1}, r_{t-1}),\ \sum_{i=1}^{N} \bar{x}_t,\ \ t = 1,2,\ldots$$

This ends the definition of R.E.E.

It is fairly straightforward to use the same argument as used in Section 3 to demonstrate that necessity of the TVC_{∞} from the consumer's side implies that the limit in (A 3.14) is zero in equilibrium. It is also fairly straightforward to show that $< \bar{c}_t, \bar{x}_t, \{\bar{x}_{it}\}_{i=1}^{N} >_{t=1}^{\infty}$ is equilibrium if it solves the problem (2.1). Furthermore, the fixed point argument that was applied to (3.40) to produce the asset pricing function of Section 3 may be adapted to produce a value function $V_i(y_t)$ from the recursion (A 3.12).

Hence value is independent of dividend policy. Not only that the function $V_i(y_t)$ may be used in conjunction with the "policy function form" of (A 3.12)

(A 3.22) $$V_i(y_t) = E_t\{\Gamma_{t+1}(V_i(y_{t+1}) + N_i(y_{t+1}))\}$$

to develop valuation formulae for the firm as we did in Sections 3 and 4 above.

For example, at date t, y_{t+1} is a random variable. Suppose following the development in Section 4 that y_{t+1} may be written

(A 3.23) $$y_{t+1} = \bar{y}_t + \sum_{k=1}^{K} \psi_t^k \tilde{\delta}_t^k.$$

Follow the development in (4.39)-(4.32), expanding V_i, N_i in Taylor series about $\bar{y}_t$, keeping only first order terms, we get

(A 3.24) $$V_i(y_t) = E_t\{\Gamma_{t+1}(V_i(\bar{y}_t)) + (\sum_{k=1}^{K} \psi_t^k \tilde{\delta}_t^k)V_i'(\bar{y}_t) + N_i(\bar{y}_t) + (\sum_{k=1}^{k} \psi_t^k \tilde{\delta}_t^k)N'(\bar{y}_t))\}$$

$$= \bar{\Gamma}_{t+1}(V_i(\bar{y}_t) + N_i(\bar{y}_t)) + (\sum_{k=1}^{K} \bar{\Theta}_{t+1}^k \psi_t^k)(V_i'(\bar{y}_t) + N_i'(\bar{y}_t)).$$

where

(A 3.25) $$\bar{\Gamma}_{t+1} \equiv E_t\Gamma_{t+1}, \quad \bar{\Theta}_{t+1}^k \equiv E_t(\Gamma_{t+1}\tilde{\delta}_t^k).$$

Formula (A 3.24) is the Sharpe Lintner formula for firm value.

Banz-Miller (1978), Breedon-Litzenberger (1978) (BMBL) propose a procedure that can be used to estimate $\{\Gamma_t\}$ from market data. Using their methods (A 3.22) may be implemented empirically. We mention their methods here not only to implement (A 3.22) empirically, but also to counter the objection that firms have no way of inferring $\{\Gamma_t\}$ from consumer behavior and, hence, there is no operational way that firms can solve (A 3.18).

The BMBL idea is to use option pricing theory to price Arrow Debreu elementary securities. Prices of these securities at date t reveal the marginal rate of substitution between goods

at date t and date event pairs at t+1. Since Γ_{t+1} is this marginal rate of substitution therefore it is revealed. Furthermore in recursive systems like ours which can be written as functions of a state variable the number of Arrow-Debreu securities that are needed to reveal $\{\Gamma_t\}$ can be greatly economized upon.

The prices of the Arrow-Debreu securities that are needed to reveal $\{\Gamma_t\}$ may be found to any degree of accuracy desired by writing options that pay off on certain intervals of values of the state variable and using Black-Scholes theory to price such options. This is the heart of the Banz-Miller, Breedon-Litzenberger theory. We do not have space to discuss it any more here. At any rate using it firms can, in principle, at least, get enough information from market data to solve (A 3.18) to some degree of accuracy.

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2. A parable may be helpful. There is one good. Call it "shmoos." Imagine that there are N "cottage" industries that consumers operate. Industry i costlessly turns one shmoo into capital of type i with a one period lag. The consumer, at date t, must commit x_{it} shmoos to cottage technology i before r_t is revealed.

After r_t is revealed by nature the one period lag production process of type i emits x_{it} units of capital of type i. Hence after r_t is revealed this precommitted capital is inelastically supplied. It cannot be changed until period t+1.

Now imagine that there are a large number of firms of each type and a large number of consumers of each type so that the price taking assumption makes sense. The demand for capital services of type i at date t is determined by the marginal physical product of capital. The intersection of demand for capital services of type i with the perfectly inelastic supply x_{it} determines $R_{i,t+1}$. At the beginning of t+1 capital becomes "unfrozen." It is reallocated by the consumption side to supply $x_{i,t+1}$ before r_{t+1} is revealed and so on it goes.

Notice that the fact that capital is frozen into capital of type i is what causes risk to be borne. If capital can be instantly adjusted when r_t is revealed then there is no risk to be borne. Adjustment costs give rise to risk in our model.

It may be helpful for the reader to think of Z_{it} as units of perfectly divisible "land" and $\pi_{i,t+1}$, given by (3.1), to be the landowner's period earnings. The supply of land of type i is perfectly inelastic at unity. The price P_{it} is just the price of a unit of land of type i at date t.

3. It may be easier for the reader to follow this discussion if we operate in a slightly different space.

Suppose that consumers have read accounting textbooks so that they know (A 3.5) in forming their expectations. Let

$$s_{it} = Z^d_{it}/Z^s_{it},$$

denote the percentage of firm i's shares demanded by the consumer. Upper d,s denote demand and supply, respectively. Using (A 3.5) rewrite the consumer's budget constraint (A 3.6) thus

$$c_t + \sum_{i=1}^{N} P_{it} Z^d_{it} = c_t + \sum_{i=1}^{N} s_{it} V_{it}$$

$$= \sum_{i=1}^{N} (d_{i,t-1} + P_{it}) Z^d_{i,t-1}$$

$$= \sum_{i=1}^{N} s_{i,t-1}(V_{it} + N_{it}).$$

The last equality follows from (A 3.5). Hence view the consumer as choosing $\{c_t, s_{it}\}$ to solve

maximize $\quad E_1\{\sum_{t=1}^{\infty} \beta^{t-1} u(c_t)\}$

s.t. $\quad c_t + \sum_{i=1}^{N} s_{it}V_{it} \leqq \sum_{i=1}^{N} s_{i,t-1}(V_{it} + N_{it}),\ t = 1,2,\ldots$

$$s_{it} \geqq 0,\quad i = 1,2,\ldots N,\quad t = 1,2,\ldots$$

$$s_{io} = 1,\quad i = 1,2,\ldots,N.$$

Here the consumer faces $\{N_{it}, V_{it}\}$ parametrically. Notice that MM value invariance is imbedded in the consumer's expectation that the value at t plus net cash flow at t (i.e., $V_{it} + N_{it}$) must equal $(d_{i,t-1} + P_{it})Z^s_{i,t-1}$ via the firm's accounting constraint (A 3.5).

We may now define R.E.E. as above. The only difference is that the consumer faces $\{\bar{V}_{it}, \bar{N}_{it}\}$ and chooses $\{s_{it}\}$ instead of choosing $\{Z_{it}\}$. In equilibrium we require the optimal choice of the consumer to satisfy

$$\bar{s}_{it} = 1,\quad i = 1,2,\ldots,N,\quad t = 1.2,\ldots$$

It is easy to follow the argument of Section 3 (i.e., Lemma 3.2) and use the necessity of the transversality condition at infinity from the consumer's side to prove

$$V_{it} = E_t \sum_{s=t+1}^{\infty} \Pi_s N_s,$$

where $\{\Pi_s, N_s\}$ are evaluated from the planner's problem (2.1).

Notice that only V_{it} is unique in equilibrium. Any P_{it}, Z_{it} such that

$$P_{it} Z_{it} = V_{it}$$

is equilibrium.

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