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DEMAND UNCERTAINTY AND THE REGULATED FIRM

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1. INTRODUCTION

Demand uncertainty is an important element of many regulated markets. Firms often must select plant size before actual demand is observed, and with some expectation of regulatory action if the actual levels of profit or rate of return do not fall within accepted ranges.

We analyze a model of a regulated firm that faces a relatively complex regime of price regulation, reflecting to at least some extent the multiple aspects suggested by Joskow (1974). The firm behaves as though it expects the current tariff to remain in effect unless, at the actual demand observed after plant size is chosen, one of two things occurs. First, if profits are negative, the firm plans to petition for and expects to receive a new tariff yielding zero economic profits. Second, if the rate of return on capital exceeds some specified maximum, the firm expects the regulator to reduce the tariff so that the firm earns only that maximum.<sup>1</sup>

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We assume that the firm chooses plant size to maximize expected profits under this regulatory regime. Certain of the results of the paper extend as well to the case of risk aversion. We do not attempt to pursue the issue of whether there is some unambiguous measure of the interests of stockholders, as others have in characterizing the behavior of the firm.<sup>2</sup>

There are important intertemporal aspects to regulation that have been discussed in some detail in the literature (for example, in Klevorick [1973], Davis [1973], Sweeney [1978], and Joskow [1974]), many of which will not be dealt with here. Instead, we employ a quite simplified, two period model of firm-regulator interactions. We have adopted this framework because it is easier to highlight the indeterminacies associated with input choices and the role of uncertainty. It is clear that these basic indeterminacies extend as well to models with longer planning horizons.

After developing the model in section two, we examine some of the effects of uncertainty on plant size and profits in section three. We define a certainty benchmark that characterizes the operation of the firm if it knows in advance of the choice of plant size that the quantity demanded will be the mean of the probability distribution on quantity, conditional on price. We show that demand uncertainty may not induce a firm to select a larger plant size than it would under certainty. Further, with demand uncertainty, the firm will not choose a plant size that guarantees that the rate of return constraint will be binding; in fact, a plant size that would assure that the

constraint is binding is too small to maximize expected profits under the regulatory regime.

In sections four and five we compare the uncertainty case to that of certainty, with the same regulatory regime. We show that the firm facing demand certainty will operate on the rate of return constraint if it is possible for at least one size of plant. Further, if the initial tariff is on an inelastic region of demand, the firm will choose a plant size that satisfies the rate of return constraint at that tariff, so that no change in the tariff results. Finally, we show that the firm may not always prefer demand certainty to uncertainty.

In section six we compare our model to an uncertainty variation of the model of Averch and Johnson. The formulation of this variant is similar to that investigated by Peles and Stein (1976, 1979) and Rau (1979), but the issues studied are different. The firm is assumed to maximize expected profit, choosing both output price and plant size before actual demand is observed. The price chosen by the firm will be changed only if the allowed rate of return is exceeded after demand is observed, and there is no profit floor. We show that the firm will not necessarily end up on the rate of return constraint with demand uncertainty, which is contrary to the well known result for the case of certainty. Finally, we show that, in contrast with our earlier model, the A-J firm will always prefer demand certainty to uncertainty.

## 2. REGULATION OF PRICE, MINIMUM PROFITS, AND MAXIMUM RATE OF RETURN

We consider a situation in which the firm produces a single nonstorable output using capital,  $K$ , and labor,  $L$ , where labor includes all inputs that are variable after capital (or plant) size is chosen. Output is produced according to the production function  $F(K,L)$ , where  $F$  is strictly quasiconcave and exhibits third order differentiability.

The firm operates under a common carrier obligation so that output must be large enough to satisfy the quantity demanded at the existing price. The quantity demanded,  $q$ , is a random variable with a known probability density function  $f(q,p)$ , where  $p$  is the price per unit of output. The common carrier requirement may be written as

$$(1) \quad F(K,L) = q$$

The level of capital must be chosen before  $q$  is observed. After  $q$  is observed, the level of the labor input,  $L$ , is chosen to satisfy the common carrier requirement (1), which implies

$$(2) \quad L = L(q,K),$$

where  $F(K,L(q,K)) = q$ .

Given a tariff,  $p$ , and a quantity demanded at that tariff,  $q$ , the profits for the firm are given by

$$(3) \quad \pi(p,q,K) = pq - wL(q,K) - rK,$$

where  $w$  represents the price of the variable factor (e.g., the wage),

and  $r$  represents the rental cost of capital. Both  $r$  and  $w$  are parametric to the firm.

The firm initially faces the regulated price  $p^0$ . Having chosen  $K$ , the firm observes  $q$  and adjusts  $L$  to meet the common carrier obligation. If the firm earns a negative profit, we assume that the firm will seek and receive a tariff increase so that the firm breaks even. We further assume demand to be inelastic, so that revenue can be increased by raising the tariff. If the firm earns positive profits, but the actual rate of return on capital is less than the allowed maximum, the prevailing tariff  $p^0$  remains in force. Finally, if the actual rate of return exceeds the maximum allowed, the regulators lower the tariff so that the firm earns only the allowed maximum.<sup>3</sup>

If we let  $s$  denote the allowed rate of return on capital,  $s > r$ , the price adjustment process can be represented as follows:

$$(4) \quad p = \begin{cases} p^0 & \text{if } 0 \leq \pi(p^0, q, K) \leq (s - r)K \\ p^1 & \text{if } \pi(p^0, q, K) < 0 \\ p^2 & \text{if } \pi(p^0, q, K) > (s - r)K, \end{cases}$$

where  $p^1$  and  $p^2$  are determined as follows. If, at  $p^0$ ,  $\pi(p^0, q, K)$  is negative, the regulator raises  $p^0$  to  $p^1$ . At  $p^1$ ,  $q(p^1)$  is observed. Then  $p^1$  is defined by the solution to  $\pi(p^1, q(p^1), K) = 0$ .<sup>4</sup>

Similary, if, at  $p^0$ ,  $\pi(p^0, q, K) > (s-r)K$ , the regulator lowers  $p^0$  to  $p^2$ . At  $p^2$ ,  $q(p^2)$  is observed. Then  $p^2$  is defined by the solution to  $\pi(p^2, q(p^2), K) = (s - r)K$ .

Given  $p^0$  and  $K$ ,  $\pi(p^0, q, K)$  is as depicted in Figure 1. In Figure 1, the values of  $q^0, q^1, q^2, q^3$ , and  $\tilde{q}$  are defined as follows:

$$(5) \quad \begin{cases} \pi(p^0, q^0, K) = \pi(p^0, q^3, K) = 0 & q^0 < q^3 \\ \pi(p^0, q^1, K) = \pi(p^0, q^2, K) = (s - r)K & q^1 < q^2 \\ \pi(p^0, \tilde{q}, K) \geq \pi(p^0, q, K) & \forall q \geq 0 \end{cases}$$

Thus for  $q < q^0$  or  $q > q^3$ , profits are negative when  $p = p^0$ ; for  $q^0 \leq q \leq q^1$  or  $q^2 \leq q \leq q^3$ , profits lie on the closed interval  $[0, (s - r)K]$ . For  $q^1 < q < q^2$ , profits exceed  $(s - r)K$  when  $p = p^0$ .<sup>5</sup>

We note for future reference that, given  $p^0$  and  $K$ ,

$$(6) \quad \pi_q = p^0 - wL_q,$$

where, from  $F(K, L(q, K)) = q$ , we have  $L_q = 1/F_L$ .

Hence  $\pi_q > 0$  for  $q < \tilde{q}$ ,  $\pi_q < 0$  for  $q > \tilde{q}$ . Further,

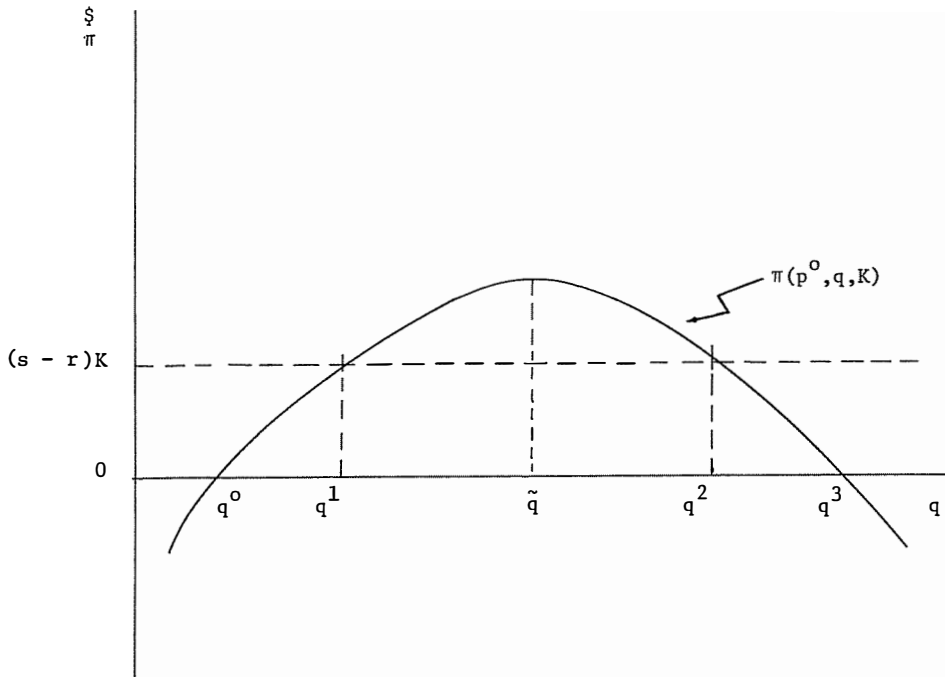
$$(7) \quad \pi_{qq} = -wL_{qq} = \frac{wF_{LL}}{F_L^3} < 0,$$

assuming  $F_{LL} < 0$ , which we shall do.

Formally, the problem of the firm is as follows:

$$(8) \quad \text{Max}_K T = \int_0^{q^0} \pi(p^1, q, K) f(q, p^0) dq + \int_{q^0}^{q^1} \pi(p^0, q, K) f(q, p^0) dq$$

FIGURE 1  
THE PROFIT FUNCTION



$$\begin{aligned}
 & + \int_{q^1}^{q^2} \pi(p^2, q, K) f(q, p^0) dq + \int_{q^2}^{q^3} \pi(p^0, q, K) f(q, p^0) dq \\
 & + \int_{q^3}^{\infty} \pi(p^1, q, K) f(q, p^0) dq
 \end{aligned}$$

By Equation 4, the first and last integrals are zero. The third integral can also be simplified since  $\pi(p^2, K, q) = (s - r)K$ . Thus, after some algebra, the first order necessary condition can be written:

$$\begin{aligned}
 (9) \quad T_K &= \int_0^{q^1} \pi_K(p^0, q, K) f(q, p^0) dq + (s - r) \int_{q^1}^{q^2} f(q, p^0) dq \\
 & + \int_{q^2}^{q^3} \pi_K(p^0, q, K) f(q, p^0) dq = 0
 \end{aligned}$$

In the work that follows, we assume that the second order condition is satisfied, i.e.,  $T_{KK} < 0$ .

### 3. UNCERTAINTY AND PLANT SIZE

We now examine the effect of uncertainty on the level of capital employed by the regulated firm. In analyzing this question, we rely on Jensen's inequality:

$$(10) \quad E g(x) \equiv \int_{-\infty}^{\infty} g(x) f(x) dx \leq g \left( \int_{-\infty}^{\infty} x f(x) dx \right) \equiv g(E(x))$$

for all probability density functions  $f(x)$  if and only if  $g$  is concave.

Consider the profit function for the regulated firm, where we incorporate into the profit function the price adjustments that occur under the regulatory process of Equation 4. Let this function be denoted by  $\pi^*(q, K)$ . Then from our discussion above,  $\pi^*$  has the following specification:

$$(11) \quad \pi^*(q, K) = \begin{cases} 0 & \text{if } q \leq q^0 \text{ or } q \geq q^3 \\ \pi(p^0, q, K) & \text{if } q^0 \leq q \leq q^1 \text{ or } q^2 \leq q \leq q^3 \\ (s - r)K & \text{if } q^1 \leq q \leq q^2 \end{cases}$$

The function  $\pi^*(q, K)$  appears as in Figure 2 for a given  $K$  and  $p^0$ .

Thus, if, given  $p^0$  and  $K$ , profits are negative ( $q < q^0$  or  $q > q^3$ ), then price is adjusted upwards to the point at which profits are zero. If profits lie between zero and  $(s - r)K$  at the price  $p^0$  (i.e., if  $q^0 \leq q \leq q^1$  or  $q^2 \leq q \leq q^3$ ), then no price adjustment occurs. Finally, if profits exceed  $(s - r)K$  at the price  $p^0$  (i.e., for  $q^1 < q < q^2$ ), then price is adjusted downward so that profits are simply  $(s - r)K$ .

Moreover,  $\pi_q^*$  and  $\pi_{qq}^*$  are given by equation 12 and equation 13:

$$(12) \quad \pi_q^*(q, K) = \begin{cases} 0 & \text{if } q < q^0 \text{ or } q > q^3 \\ p^0 - wL_q & \text{if } q^0 < q < q^1 \text{ or } q^2 < q < q^3 \\ 0 & \text{if } q^1 < q < q^2 \end{cases}$$

$$(13) \quad \pi_{qq}^*(q, K) = \begin{cases} 0 & \text{if } q < q^0 \text{ or } q > q^3 \\ -wL_{qq} = \frac{wF_{LL}}{F_L^3} < 0 & \text{if } q^0 < q < q^1 \text{ or } q^2 < q < q^3 \\ 0 & \text{if } q^1 < q < q^2 \end{cases}$$

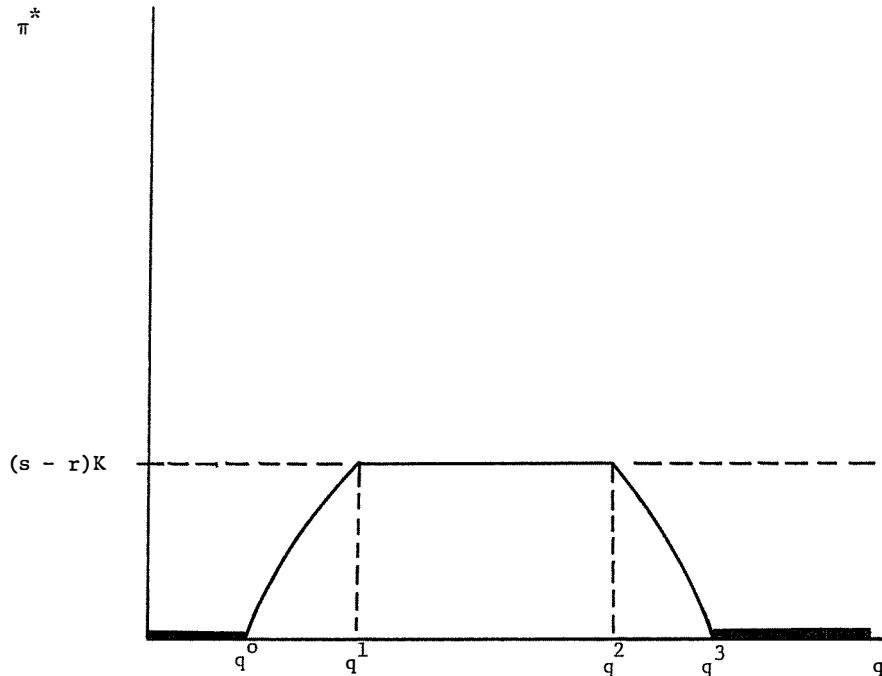
The marginal profitability of capital can be written as follows:

$$(14) \quad \pi_K^*(q, K) = \begin{cases} -wL_K - r & \text{if } q^0 < q < q^1 \text{ or } q^2 < q < q^3 \\ 0 & \text{if } q < q^0 \text{ or } q > q^3 \\ (s - r) & \text{if } q^1 < q < q^2 \end{cases}$$

where  $L_K = -F_K/F_L$

Then it follows that

$$(15) \quad \pi_{Kq}^*(q, K) = \begin{cases} -wL_{Kq} & \text{if } q^0 < q < q^1 \text{ or } q^2 < q < q^3 \\ 0 & \text{elsewhere (if defined)} \end{cases}$$

FIGURE 2. THE  $\pi^*$  FUNCTION

where,

$$L_{Kq} = -(F_{L^2 F_{KL}} - F_{K^2 F_{LL}}) / F_L^3$$

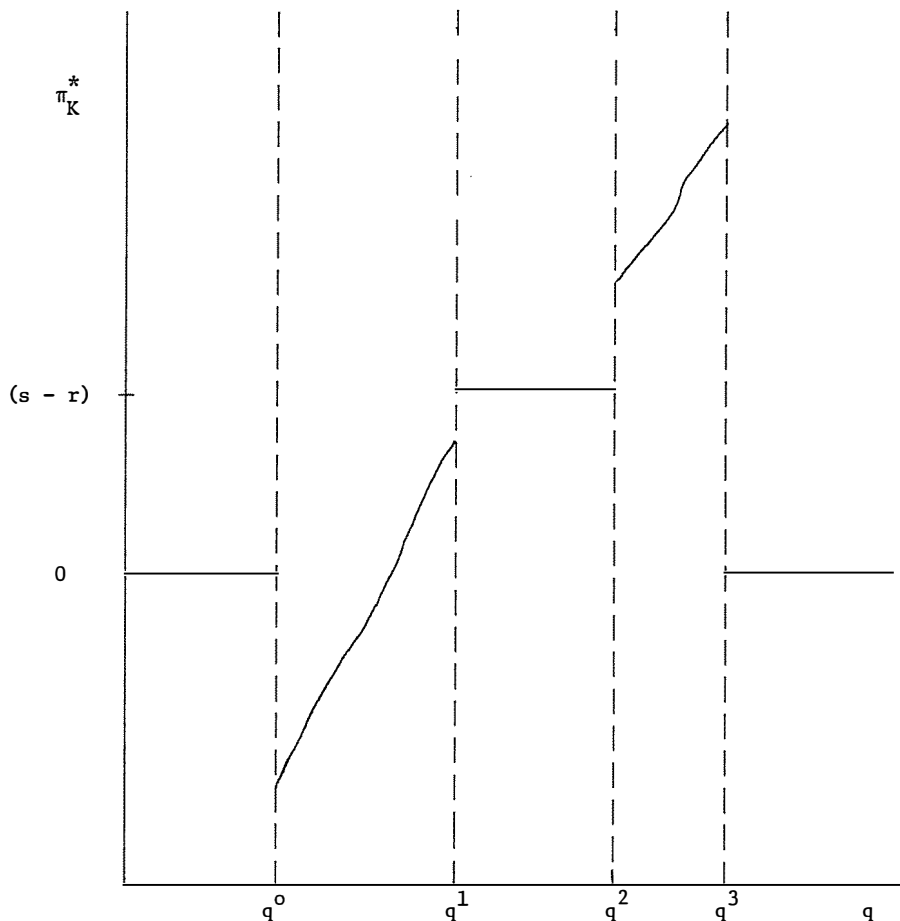
In general  $L_{Kq}$  is of indeterminate sign. However, if capital is a normal factor of production, then  $L_{Kq} < 0$ , and the graph of  $\pi_K^*$  as a function of  $q$  appears as in Figure 3. Under the assumption of normality, we know that over the ranges  $(q^0, q^1)$  and  $(q^2, q^3)$ ,  $\pi_K^*$  is increasing. In fact, over those ranges,  $\pi_{Kq}^* = \pi_{Kq}$  with  $\pi_{Kq} > 0$ , i.e.,  $\pi_K^*$  is a continuous increasing function of  $q$ . However  $\pi_K^*$  might be negative for all  $q$  (see the example at the end of this paper), given the level of  $K$  chosen as optimal by the regulated firm. In particular, there is no presumption that  $\pi_K^*$  evaluated at  $q^0$  or  $q^3$  is zero nor that  $\pi_K^*$  evaluated at  $q^1$  or  $q^2$  equals  $(s-r)$ , hence the jump discontinuities shown in Figure 3 illustrate a "typical" situation so far as the  $\pi_K^*$  function is concerned. Finally, we note that the curvature of the  $\pi_K^*$  function over the intervals  $q^0 < q < q^1$  and  $q^2 < q < q^3$  depends on the sign of

$$(16) \quad \pi_{Kqq} = w(F_{L^2 F_{KLL}} - F_{K^2 F_{LLL}} + 3F_{K^2 F_{LL}} - 3F_{L^2 F_{KL}}) / F_L^5$$

which is ambiguous.

We first contrast the choice of capital by a regulated firm under uncertainty with its choice under certainty. Under uncertainty, the firm chooses  $\hat{K}$  so that  $T_K = 0$ , that is,

FIGURE 3. THE  $\pi_K^*$  FUNCTION



$$E\pi_K^*(q, \hat{K}) = 0.$$

Under certainty, the choice of the regulated firm is  $K^*$  such that

$$\pi_K^*(E(q; K^*), K^*) = 0.$$

(Note that  $E(q)$  depends on  $K^*$ , since changes in  $K^*$  change the switching points  $q^0, q^1, q^2, q^3$ , thus change the probability distribution over the price functions  $p^1, p^2$ , and hence change the probability density function over  $q$ ).

By hypothesis,  $\pi_{KK} < 0$  and we have seen from footnote 5 that  $\pi_{KK} < 0$ , hence  $\pi_{KK}^* \leq 0$ . We have an unambiguous ordering of the amounts of capital chosen if and only if

$$0 = \pi_K^*(E(q; K^*), K^*) < E\pi_K^*(q, K^*) \quad (\iff \hat{K} > K^*)$$

$$\text{or } 0 = \pi_K^*(E(q; K^*), K^*) > E\pi_K^*(q, K^*) \quad (\iff \hat{K} < K^*).$$

Figure 4 illustrates the case where  $\hat{K} > K^*$ .

But by Jensen's inequality,  $\pi_K^*(E(q)) \leq E\pi_K^*(q)$  for all probability density functions  $f$  if and only if  $\pi_K^*$  is concave in  $q$ . Since  $\pi_{Kqq}^*$  is ambiguous in sign, we have the following proposition.

Proposition 1. The regulated firm operating under uncertainty many choose a level of capital greater than, less than or equal to the amount it would



select under certainty. The relative amount of capital hired depends on the specific form of the production, demand and probability density functions.

The nature of the indeterminacy highlighted in this proposition can be illustrated quite nicely by assuming that  $F$  is a Cobb-Douglas production function of the form

$$(17) \quad F = AK^\alpha L^\beta$$

where  $A$ ,  $\alpha$ , and  $\beta$  are all positive constants.

For this specific function, Equation (15) and Equation (16) reduce to:

$$(18) \quad \pi_{Kq} = \frac{\alpha w L}{K q \beta^3} > 0$$

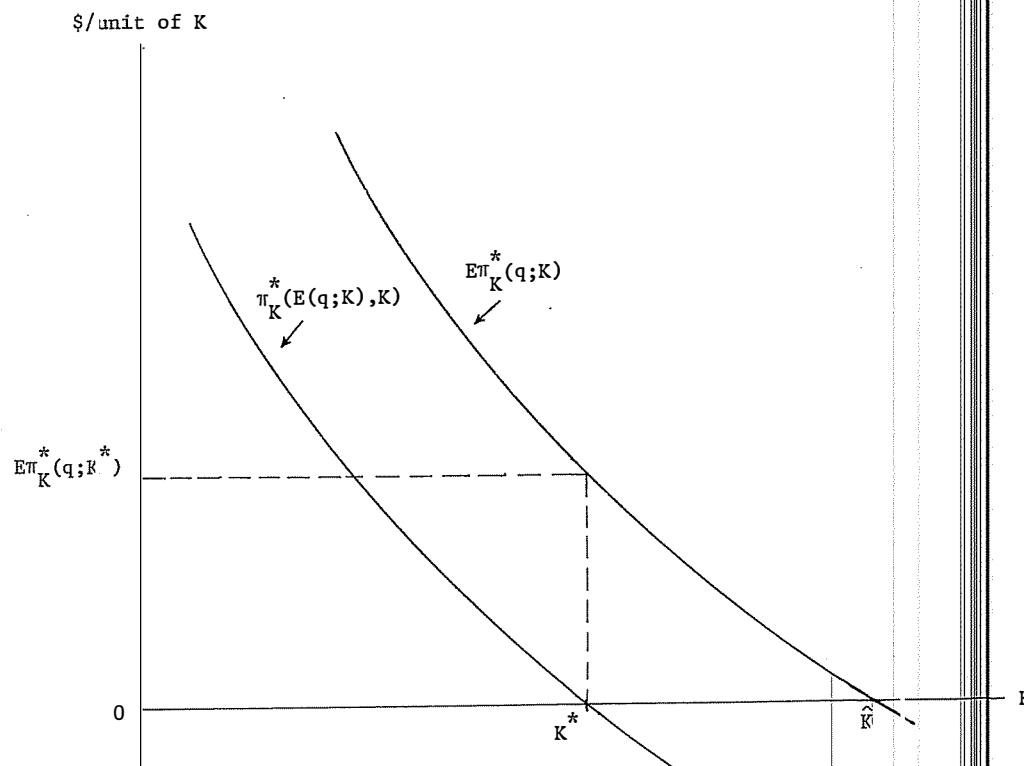
and,

$$(19) \quad \pi_{Kqq} = \frac{\alpha w L}{\beta^3 q^2 K} (1 - \beta)$$

Equation 18 verifies that  $\pi_K$  has the positive slope indicated in Figure 3. Equation 19 indicates that when capital is a normal factor of production ( $\beta < 1$ ), then  $\pi_K$  is a convex function of  $q$  since  $\pi_{Kqq}$  is positive. Consequently, if the only form of regulation were a simple price constraint ( $p = p^0$ ) with no rate of return ceiling or zero profit floor, Jensen's inequality would suggest that the firm would select a larger plant size under uncertainty than it would under certainty.<sup>6</sup>

This rather definite result becomes ambiguous in the

FIGURE 4.  $\pi_K^*(E(q;K^*), K^*) < E\pi_K^*(q, K^*)$



regulated case because both the rate of return ceiling and the profit floor constraints introduce non-convexities. A more general specification of the production function, further clouds the issue since it cannot be shown for the general case that  $\pi_K$  is convex in  $q$ .

Using the approach taken in the proof of the first proposition, one might believe that there are interesting conditions under which uncertainty has no effect on the size of plant. Suppose at  $\hat{K}$  the probability density function is positive only over regions in which  $\pi_{Kqq}$  is zero. Formally, let  $\hat{q}^0, \hat{q}^1, \hat{q}^2, \hat{q}^3$ , and  $\hat{q}$  satisfy Equation 5 at  $\hat{K}$ . Further, let  $q_L = \inf \{q | f(q, p^0) > 0\}$  and  $q_U = \sup \{q | f(q, p^0) > 0\}$ .

First, one might suppose that if  $q_U < \hat{q}^0$  or if  $q_L > \hat{q}^3$  that the plant size chosen under certainty and uncertainty would be equivalent. These conditions reflect a situation in which the firm knows with certainty that it will earn exactly zero profits for any possible  $q$  given  $p^0$ . While it is true that profits will be zero under either certainty or uncertainty, plant size is indeterminate in both cases, since  $p$  will be adjusted to yield zero profits for whatever  $K$  is chosen. Hence,  $\hat{K}$  will not generally be unique.

A second possible condition under which one might expect the plant size chosen to be invariant with uncertainty is much more interesting. Suppose the following hypothesis were true: The firm facing uncertainty may select a level of capital which guarantees that, at any possible  $q$  given  $p^0$ , it would earn at least

the allowed rate of return. Then  $\pi_{Kqq}^*(q, K)$  would be zero on the interval  $[q_L, q_U]$ , since  $\hat{q}^1 \leq q_L < q_U \leq \hat{q}^2$ , (see Figure 2). Thus, if the hypothesis were true, then by Jensen's inequality (now an equality since  $\pi_{Kqq} = 0$  on  $[q_L, q_U]$ ),  $E(\pi_K^*(q, K)) = 0 = \pi_K^*(E(q), K)$ . If this were the case then presumably the regulated firm (under either certainty or uncertainty) chooses the largest  $K$ , say  $\hat{K}$ , such that  $\pi(q, \hat{K}) \geq (s - r)\hat{K}$  for all  $q$  such that  $f(q, p^0) > 0$ .

It turns out that this argument is false since the stated hypothesis is false. Proposition 2 shows why.

Proposition 2. The firm facing demand uncertainty will not choose a plant size that guarantees that the rate of return constraint will be binding for all  $q$  where  $f(q, p^0) > 0$ . Any plant size that guarantees that the constraint will be binding for all  $q$  such that  $f(q, p^0) > 0$  is smaller than the plant size that maximizes expected profits.

Proof. Assume the contrary, i.e., that at  $\hat{K}, q^1 \leq q_L < q_U \leq q^2$ . Then by Equation (9),

$$T_K = (s - r) \int_{q^1}^{q^2} f(q, p^0) dq = (s - r) > 0$$

Thus,  $\hat{K}$  can not be an optimum. Expected profits could be increased by choosing a larger plant size.

## 4. THE CHOICE OF PLANT SIZE UNDER CERTAINTY

For the purpose of comparison with proposition 2, we now attempt to characterize the plant size selected by a profit maximizing firm operating with demand certainty, where the firm is again subject to rate of return ceiling and profit floor constraints. Specifically we ask two questions. First, will the firm choose a level of capital that makes the rate of return constraint binding whenever that is possible? Second, will the firm ever choose a level of capital that induces the regulator to change price from  $p^0$ , where  $p^0$  is the prevailing price?

Clearly if  $\pi(p^0, K)$  is nonpositive for all  $K$ , then choice of  $K, p$  and  $q$  is not unique, since a number of  $(p^1, K)$  pairs will satisfy  $\pi(p^1, K) = 0$  as the regulator sets  $p^1$  to allow the firm to break even. Note also that it is only in the case where  $\pi(p^0, K)$  is nonpositive for all  $K$  that  $p$  will be increased, since whenever  $p$  is increased to  $p^1$  by the regulator, then  $\pi(p^1, K) = 0$ .

The more interesting cases are those for which  $\pi(p^0, K)$  is positive for some values of  $K$ . Let us restrict our attention to these cases. The firm chooses  $K$  knowing that if the rate of return constraint is violated (i.e., if  $\pi(p^0, K) > (s - r)K$ ) then price will be lowered by an amount  $\hat{p}$ . Otherwise,  $\hat{p}$  will be zero, and  $p^0$  will prevail.

Formally, the firm acts to

$$(20) \quad \max_{(\hat{p}, K)} R(p^0 - \hat{p}) - wL(q(p^0 - \hat{p}), K) - rK$$

subject to

$$\hat{p} \geq 0$$

$$wL(q(p^0 - \hat{p}), K) + sK - R(p^0 - \hat{p}) \geq 0$$

$$\hat{p}[wL(q(p^0 - \hat{p}), K) + sK - R(p^0 - \hat{p})] = 0$$

where

$R(p^0 - \hat{p})$  represents a revenue function,

$$R(p^0 - \hat{p}) = (p^0 - \hat{p})q(p^0 - \hat{p})$$

The first constraint ( $\hat{p} \geq 0$ ) indicates that  $p^0$  will be lowered by  $\hat{p}$  if the rate of return constraint is violated at  $(p^0, K)$ ; we continue to assume that demand is inelastic (but not perfectly so) at  $p^0$ . The second constraint represents the rate of return ceiling, which may not be binding. The final constraint indicates that if the rate of return constraint is not binding, then  $\hat{p}$  is zero and  $p^0$  prevails; further, it states that if  $\hat{p}$  is positive (and thus price is lowered) then the rate of return constraint must be binding.

We denote  $\lambda_1, \lambda_2$ , and  $\lambda_3$  as the Lagrange multipliers associated with these three constraints respectively, and form the Lagrangean,  $H$ :

$$(21) \quad H = R(p^0 - \hat{p}) - wL(q(p^0 - \hat{p}), K) - rK + \lambda_1 \hat{p} + (\lambda_3 \hat{p} + \lambda_2)[wL(q(p^0 - \hat{p}), K) + sK - R(p^0 - \hat{p})]$$

Among the conditions that must hold at a first order optimum are the following (we suppress the arguments of  $R$  and  $L$  to simplify notation).

$$(22) \quad H_K = -wL_K - r + (\lambda_3 \hat{p} + \lambda_2)(wL_K + s) = 0, \text{ when } K > 0.$$

$$(23) \quad H_p = R_p - wL_p + \lambda_1 + (\lambda_3 \hat{p} + \lambda_2)(wL_p - R_p) + \lambda_3(wL + sK - R) \leq 0,$$

$$\hat{p} \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0$$

We can now address the second question asked above, will the firm ever choose a level of capital that induces the regulator to change the tariff  $p^0$ ?

Proposition 3. As long as  $\pi(p^0, K) > 0$  for some  $K$ , then the regulated firm operating under certainty will never choose  $K$  such that the tariff  $p^0$  is changed.

Proof. Assume the contrary, i.e., suppose  $\hat{p} > 0$  at an optimum  $K^*$ .

Then  $\lambda_1 = 0$  and  $wL + sK - R = 0$ . By Equation (23), then  $(R_p - wL_p)(1 - \lambda_3 \hat{p} - \lambda_2) = 0$ . Since demand is inelastic,  $(R_p - wL_p) > 0$ , and thus  $(1 - \lambda_3 \hat{p} - \lambda_2) = 0$ . But this implies, by Equation (22) that  $(s - r) = 0$ , which violates the assumption that  $s > r$ . Hence  $\hat{p}$  can not be positive. Moreover, as has been noted earlier,  $\pi(p^0, K) > 0$  for some  $K$  implies that  $\hat{p} \geq 0$ . Hence  $K$  is always chosen so that  $\hat{p} = 0$ , that is,  $p^0$  is unchanged, given that  $\pi(p^0, K) > 0$  for some  $K$ .

Thus  $p^0$  will remain in force at the plant size selected by the firm, revenue is fixed at  $R(p^0)$ , and output will be  $q(p^0)$ . The only question remaining is whether the firm will minimize the cost of producing  $q(p^0)$ . If it can do so without violating the rate of return constraint, then  $\lambda_2$  is zero in Equation (22), and the firm acts as an unconstrained profit maximizer given  $p^0$ .

If the cost minimizing production violates the rate of return constraint, the firm responds by increasing  $K$  to meet the constraint rather than by incurring a change in  $p^0$ . Thus, the choice of  $K$  can be characterized as in Proposition 4:

Proposition 4. Assume that  $\pi(p^0, K) > 0$  for some  $K$ . Let  $A$  be the set  $\{K | \pi(p^0, K) = (s - r)K\}$ . Then the firm will select the largest  $K \in A$  whenever  $A$  is non-empty. If  $A$  is empty, then the firm acts as a cost minimizer in satisfying its common carrier requirement at  $p^0$ .

The comparison of plant sizes for the certain and uncertain cases is now complete. The firm facing demand uncertainty will not necessarily select a larger plant size than it would with certainty. The firm facing demand uncertainty will not choose a plant size so that the rate of return constraint will automatically be binding; with demand certainty the firm will select an operating point on the rate of return constraint whenever that is possible, but will not choose a plant size such that a change in the tariff  $p^0$  occurs, so long as  $\pi(p^0, K) > 0$  for some  $K$ .

## 5. A PREFERENCE FOR CERTAINTY?

One might ask whether a firm regulated with profit floor and rate of return constraints necessarily prefers certainty to uncertainty in demand. The notion of certainty used here is the same as for Proposition 1; the certain quantity demanded is the mean of the probability density function of quantity conditional on price.

We compare the expected profits under uncertainty with the profit level achieved with certain demand. As before, we let  $q_L = \inf \{q | f(q, p^0) > 0\}$ ,  $q_U = \sup \{q | f(q, p^0) > 0\}$ , and let  $\hat{K}$  denote the level of capital that maximizes expected profit for the firm facing uncertainty. Further, let  $K^*$  denote the profit maximizing choice of  $K$  for the firm with certain demand, and let  $\hat{q}^0$  and  $\hat{q}^3$  respectively be the levels of  $q^0$  and  $q^3$ , given  $p^0$  and  $\hat{K}$ . Then from Jensen's Inequality and Figure 2, we have

Proposition 5. Preferences of an expected profit maximizing firm between demand certainty and uncertainty depend on the subjective probability beliefs of the firm with respect to demand.

- (i) If  $\hat{q}^0 \leq q_L$  and  $q_U \leq \hat{q}^3$ , then the regulated firm prefers certainty to uncertainty.
- (ii) If  $q_L < \hat{q}^0$  or  $q^3 < \hat{q}^u$ , then the preferences between certainty and uncertainty depend on the specific probability distribution and production functions. No definite ranking of

the alternatives (certainty, uncertainty) is possible without such information.

Proof. For (i), we use Jensen's Inequality and the condition that the function  $\pi^*$  (see Figure 2) is concave in  $q$  on the interval  $[q_L, q_U]$  given  $\hat{K}$ . Thus

$$E(\pi^*(q, \hat{K})) < \pi^*(E(q), \hat{K}) \leq \pi^*(E(q), K^*).$$

The first inequality is strict, since Proposition 2 shows that  $q^1 \leq q_L < q_U \leq q^3$  is not possible at  $\hat{K}$ . The second inequality simply expresses the fact that  $K^*$  is the maximizer under certainty.

For (ii), we note that under the condition  $q_L < \hat{q}^0$  or  $q_U > \hat{q}^3$  the function  $\pi^*$  is neither concave nor convex in  $q$  on  $[q_L, q_U]$ ; hence no definite preference can be stated independently of information on  $f, F$ , and the demand function.

An example of the effects of uncertainty on a regulated firm operating with a rate of return ceiling and a profit floor appears in the appendix.

## 6. COMPARISON WITH AVERCH AND JOHNSON

An adaptation of our own model might be viewed as an Averch-Johnson model of the regulated firm with demand uncertainty. The regulator announces an allowed rate of return on base capital,  $s$ , and the regulated utility chooses  $K, p$ , and  $L$  to maximize expected profits. The firm selects  $K$  and  $p$  before the quantity demanded  $q(p)$  is revealed. After  $q$  is observed, the utility chooses  $L$  to meet the common carrier requirement so that  $L = L(q, K)$  where

$$F(K, L(q, K)) = q.$$

The utility chooses its decision variables to maximize expected income, subject to the rate of return constraint. If the constraint is violated after demand is observed, the regulator lowers  $p$  to  $p^2$  so that profits equal  $(s - r)K$ . This variant of the A-J model differs from our own model above in crucial respects. First, there is no break even (or profit floor) constraint. This firm could end up earning negative profits after the actual quantity demanded is observed. Second, there is no prevailing price ( $p^0$  in our model) which is in effect and will continue unless the constraint becomes binding. The firm is free to choose any  $p$  initially, although it recognizes that the regulator may change  $p$  if the constraint is violated after demand is revealed.

For any  $(p, K)$ , define  $q_1^*$  and  $q_2^*$  by

$$(24) \quad \pi(p, q_1^*, K) = pq_1^*(p, K) - wL(q_1^*, K) - rK = (s - r)K$$

$$(25) \quad \pi(p, q_2^*, K) = pq_2^*(p, K) - wL(q_2^*, K) - rK = (s - r)K$$

where  $q_1^* < q_2^*$ .

Profits for this firm are given by

$$(26) \quad \pi^A(p, q, K) = \begin{cases} pq - wL(q, K) - rK, & q \leq q_1^*(p, K) \\ (s - r)K & , \quad q_1^*(p, K) \leq q \leq q_2^*(p, K) \\ pq - wL(q, K) - rK, & q \geq q_2^*(p, K) \end{cases}$$

The function  $\pi^A(p, K, q)$  is shown in Figure 5 for given values of  $p$  and  $K$ . Thus,

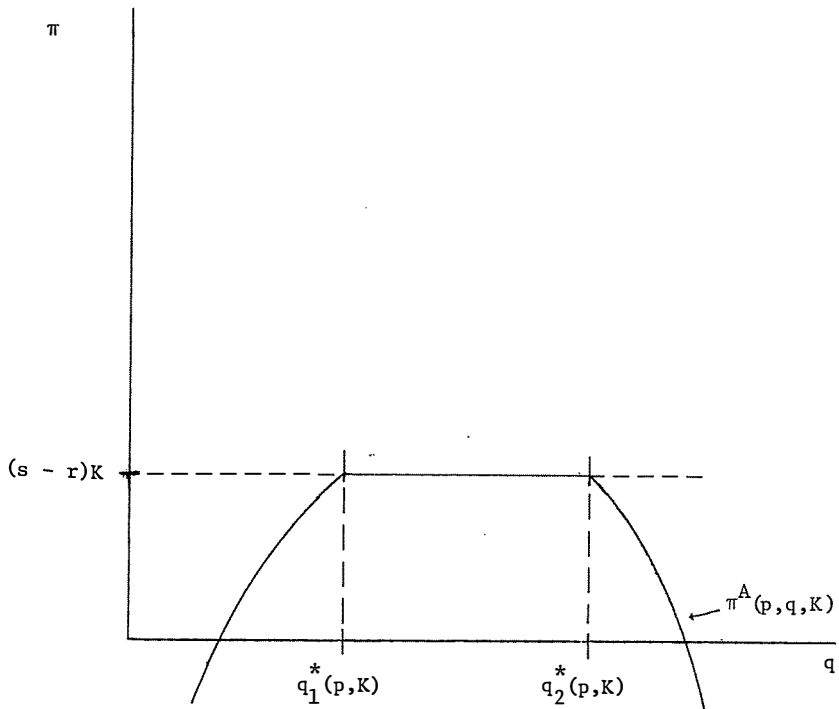
$$(27) \quad \pi_q^A(p, q, K) = \begin{cases} p - wL_q(q, K) > 0, & q < q_1^*(p, K) \\ 0 & , \quad q_1^*(p, K) < q < q_2^*(p, K) \\ p - wL_q(q, K) < 0, & q > q_2^*(p, K) \end{cases}$$

with

$$(28) \quad \pi_{qq}^A(p, q, K) = \begin{cases} -wL_{qq}(q, K) < 0, & q < q_1^*(p, K) \\ 0 & , \quad q_1^* < q < q_2^* \\ -wL_{qq}(q, K) < 0, & q > q_2^*(p, K) \end{cases}$$

Thus, the A-J firm has a profit function  $\pi^A$  that is concave in  $q$ , the variable of uncertainty, given  $p$  and  $K$ . Formally, the problem confronting the firm can be written:

$$(29) \quad \max_{(p, K)} T \equiv \int_0^{q_1^*(p, K)} \pi(p, q, K) f(q, p) dq + \int_{q_1^*(p, K)}^{q_2^*(p, K)} (s - r)K f(q, p) dq + \int_{q_2^*(p, K)}^{\infty} \pi(p, q, K) f(q, p) dq$$

FIGURE 5. THE PROFIT FUNCTION  $\pi^A(p, q, K)$ 

At an interior optimum ( $p > 0$  and  $K > 0$ ), the following two conditions must be satisfied:

$$(30) \quad T_K = \int_0^{q_1^*(p, K)} \pi_K(p, q, K) f(q, p) dq + (s - r) \int_{q_1^*(p, K)}^{q_2^*(p, K)} f(q, p) dq + \int_{q_2^*(p, K)}^{\infty} \pi_K(p, q, K) f(q, p) dq = 0$$

and,

$$(31) \quad T_p = (s - r)K \int_{q_1^*(p, K)}^{q_2^*(p, K)} f_p(q, p) dq + \int_0^{q_1^*(p, K)} [\pi(p, q, K) f_p(q, p) + \pi_p(p, q, K) f(q, p)] dq + \int_{q_2^*(p, K)}^{\infty} [\pi(p, q, K) f_p(q, p) + \pi_p(p, q, K) f(q, p)] dq = 0$$

Recall that the model of the first section of this paper, with its profit floor and its prevailing price,  $p^0$ , had no general preference for certainty over uncertainty in demand. We now contrast that result with the present version of the A-J model.

Proposition 6. The A-J firm prefers certainty to uncertainty in demand.

Proof. Let  $(\hat{K}, \hat{p})$  satisfy Equation 30 and Equation 31, and let  $(K^*, p^*)$  maximize profit subject to a rate of return constraint in the standard A-J model in which demand is certain. Note that  $\pi^A$  is concave in the random variable  $q$ , given  $(p, K)$ , by Equation 28. Then  $E\pi^A(\hat{p}, \hat{K}) < \pi^A(E(q|\hat{p}), \hat{K}) \leq \pi^A(E(q|p^*), K^*)$ .

As before, demand certainty corresponds to a knowledge that the quantity demanded will equal the mean of the probability distribution of quantity given price.

In another respect the A-J model and our earlier model yield similar results. Proposition 2 showed that the firm facing a profit floor constraint, a rate of return ceiling, and a prevailing price would not choose a plant size that would guarantee that the rate of return constraint would be binding. For the A-J firm we have:

Proposition 7. The A-J firm facing demand uncertainty will not choose a plant size and price level that guarantees that the rate of return constraint will be binding.

Proof. Let  $q_L(\hat{p}) = \inf \{q | f(q, \hat{p}) > 0\}$  and  $q_U(\hat{p}) = \sup \{q | f(q, \hat{p}) > 0\}$ . Assume the contrary to Proposition 7, that is, assume  $q_1^*(\hat{p}, K) < q_L(\hat{p}) < q_U(\hat{p}) \leq q_2^*(\hat{p}, K)$ . Then, by equation (30),

$$T_K = (s - r) \int_{q_1^*(\hat{p}, \hat{K})}^{q_2^*(\hat{p}, K)} f(q, \hat{p}) dq = (s - r) > 0.$$

Thus  $\hat{K}$  can not be an optimum at  $\hat{p}$ , since expected profits would increase at  $\hat{p}$  if  $K$  were larger.

We note here that it would be possible to set forth a proposition similar to Proposition 1, except directed toward the A-J firm. For a given  $(p, K)$ , we could construct a curve for  $\pi_{Kqq}$ , and show that it is generally neither convex nor concave. Thus, Jensen's Inequality would not yield determinate results about the relative plant size of the A-J firm under certainty and uncertainty. More knowledge about the functional forms would be required to state determinate results.

## 7. CONCLUSION

We have constructed and analyzed a model of regulation that combines many of the features often observed in regulated markets. These features include demand uncertainty, the need to choose plant size before actual demand is observed, a common carrier obligation, and a form of price regulation. The latter specifies that a prevailing price will continue unless a rate of return ceiling is violated or a specified minimum level of profit is not achieved after demand is observed.

We have shown that demand uncertainty will not necessarily induce a firm to select a larger plant size that it would under



certainty. If the firm facing demand certainty is able to select any plant size that places it on the rate of return constraint, then the rate of return constraint will be binding at an equilibrium and the prevailing price will continue. In contrast, the firm confronted by uncertain demand will not necessarily operate where the actual rate of return is the maximum allowed; in fact, any plant size that would guarantee that the rate of return constraint would be binding is too small to maximize expected profits under the specified regulatory regime. In addition, we have shown that with a profit floor in effect, the firm may not even prefer certainty to uncertainty in demand.

We have also compared our model with an uncertainty version of the Averch-Johnson model, in which the firm chooses both price and plant size before actual demand is observed. The price chosen by the firm is changed only if the allowed rate of return is exceeded, and no profit floor is specified. When demand is certain, it is well known that such a firm will operate on the rate of return constraint if it is possible for at least one combination of price and plant size. However, if the firm faces demand uncertainty, then the firm will not necessarily end up on the rate of return constraint. Finally, we note that, in contrast with the two-constraint model of the earlier sections of the paper, the A-J firm will always prefer demand certainty to uncertainty.

## APPENDIX

Consider the case of a firm that produces power with fixed proportions of capital and fuel. Assume that power is produced according to the production function  $K^\alpha$ , where reference to fuel is suppressed because factor proportions are fixed.

The firm is able to meet its output under the common carrier obligation ( $q$ ) by producing electricity itself, or by buying power from a pool at a fixed price  $w$  per unit of power. Then  $q = K^\alpha + L$  where  $L$  represents the number of units of power purchased from the pool. If  $q > K^\alpha$ , then  $L > 0$ . If  $q < K^\alpha$ , then  $L < 0$ , that is, the firm can also sell units of power into the pool and receive a price of  $w$ .

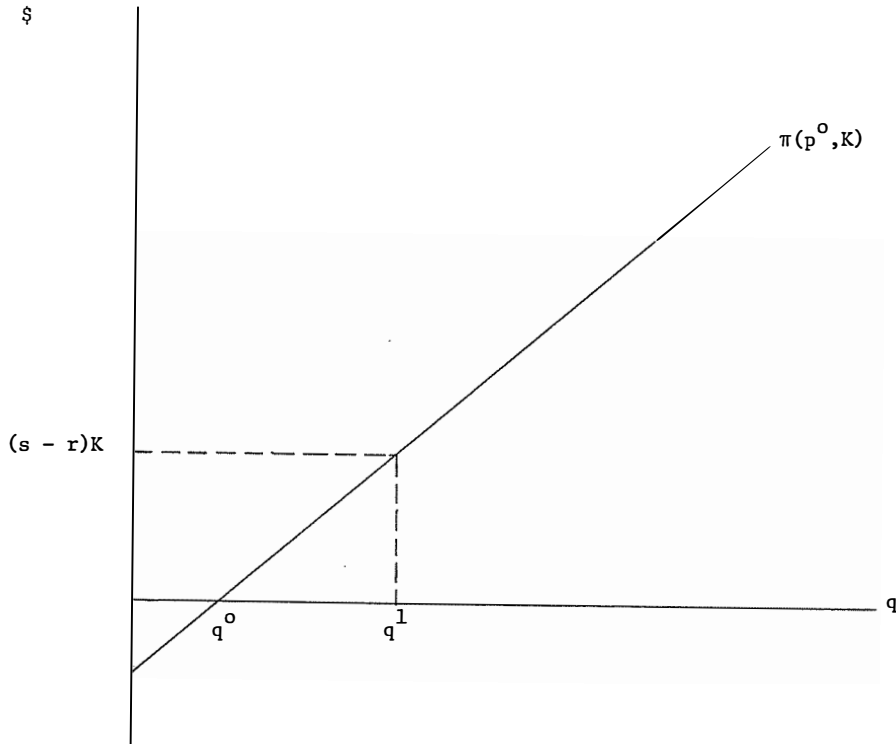
The firm operates under a rate of return ceiling, a zero profit floor, and demand uncertainty. It must choose  $K$  before actual demand is revealed at the prevailing price  $p^0$ , that regulators have set for sales to customers outside the power pool. The regulator has set  $p^0 > w$ .

Profits for the firm are given by

$$\pi = pq - wL - rK = (p - w)q + wK^\alpha - rK, \text{ so that}$$

$\pi_q = p - w > 0$  and  $\pi_K = w\alpha K^{\alpha-1} - r$ . Profits as a function of  $q$  are as in Figure 6.

$$\text{Then } q^0 = \frac{rK - wK^\alpha}{(p^0 - w)}, \quad q^1 = \frac{sK - wK^\alpha}{(p^0 - w)}, \text{ with}$$

FIGURE 6.  $\pi(p^0, K)$ 

$$T = \int_0^1 [(p^0 - w)q + wK^\alpha - rK]f(q, p^0) dq + (s - r)K[1 - f^c(q^1, p^0)],$$

where  $f^c$  denotes the cumulative density function corresponding to  $f$ . The optimal value of  $K, \hat{K}$ , is obtained by setting  $T_K = 0$ , giving

$$\hat{K}^\alpha = \frac{\hat{K}}{[f^c(q^1, p^0) - f^c(q^0, p^0)]\alpha w} \{r[f^c(q^1, p^0) - f^c(q^0, p^0)] - (s - r)[1 - f^c(q^1, p^0)]\}$$

In particular, assume that  $f(q, p)$  is uniform with  $f(q, p) = \frac{1}{q}, 0 \leq q \leq q^1$ .

$$\text{Then } \hat{K}^\alpha = \frac{\hat{K}}{\alpha w(q^1 - q^0)} \{s(q^1 - \hat{q}) + r(\hat{q} - q^0)\}.$$

By tedious calculations, it can be shown that

$$\hat{K}^\alpha = \frac{(s + r)\hat{K} - \hat{q}(p^0 - w)}{(\alpha + 1)w}$$

For the special case where  $\alpha = 1/2$ , a closed form solution to this equation can be obtained. (The fact that  $\alpha = 1/2$  implies decreasing returns to scale is not important in this example.) In particular, take  $p^0 = 1, w = .5, s = .1, r = .0667, \hat{q} = 100$ . Then it can be verified that

$$\hat{K} = 388.72, q^0 = 32.14, q^1 = 58.02.$$

Note that  $\pi_K = w\alpha K^{\alpha-1} - r = -.054 < 0$  and constant for all  $q$ . On the other hand

$$\pi_K^* = \begin{cases} 0 & \text{for } q < 32.14 \\ \pi_K \equiv -.054 & \text{for } 32.14 < q < 58.02 \\ s - r = .0333 & \text{for } 58.02 < q < 100 \end{cases}$$

Intuitively, what is going on is that the firm has chosen a larger stock of capital than if the firm operated under certainty with a fixed price  $p^0$  with no rate of return constraint. In such a case it chooses  $\pi_K = 0$ , with  $K^* = 14.05$  and  $L = 46.25$  ( $q = 50$ ). The overinvestment in capital in the regulated case is to take advantage of the allowed rate of return once demand is above  $q^1 \equiv 58.02$  units.

Figure 7, 8 and 9 show  $\pi^*$ ,  $\pi_q^*$  and  $\pi_K^*$  as functions of  $q$ .

FIGURE 7. THE  $\pi^*$  FUNCTION

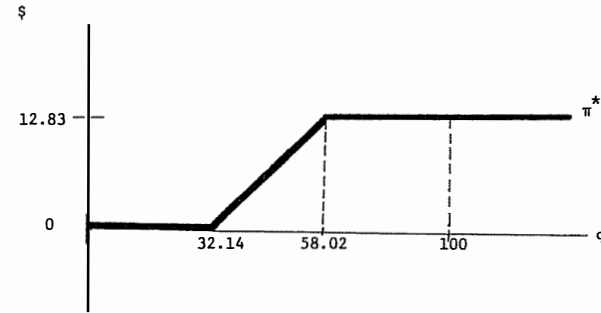


FIGURE 8. THE  $\pi_q^*$  FUNCTION

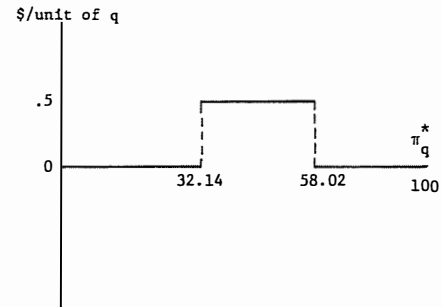
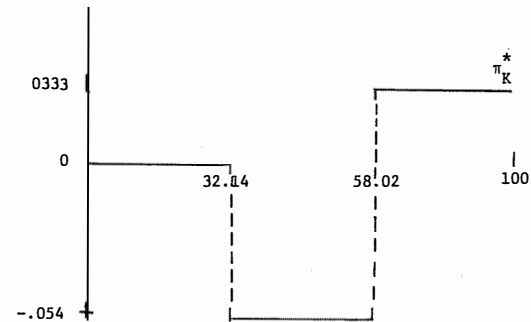


FIGURE 9. THE  $\pi_K^*$  FUNCTION



## FOOTNOTES

1. Burness, Montgomery and Quirk (forthcoming) have analyzed a regime of price regulation with such a bankruptcy constraint. However, the present paper differs from that analysis in two major respects. Their paper does not analyze the case of a simultaneous zero profit floor and allowed rate of return ceiling. Also, it addresses the effects of uncertainty about the price of capital rather than the level of demand.
2. See, for example, Ekern and Wilson (1974) and Leland (1972). Baran and Taggart (1977) have used the notion of stockholder unanimity to examine how demand uncertainty affects a regulated firm that is required to choose the level of capital before demand is revealed. Their work also differs from ours in that, while the firm faces a rate of return ceiling, it operates with no specific expectation that regulators will act to eliminate negative profits.

3. The telescoping of the rate adjustment mechanism in this fashion results in a bypassing of the process by which the firm adjusts inputs in early periods to increase profits in late periods through "test period" observations over time. See, for example, Sweeney (1977) on this point. While this is an interesting and important phenomenon in certain aspects of the theory of regulation, it is an unnecessary complication for our purposes, hence the simplifying assumption of instantaneous adjustments.

4. For example, suppose that  $f(q,p) = E(q|p) + g(\varepsilon)$  where  $\varepsilon = q - E(q|p)$ ,  $-\infty < \varepsilon < +\infty$ . Then  $p$  determines only the expected value of  $q$ , and hence  $q(p^1) = E(q|p^1) + q(p^0) - E(q|p^0) = E(q|p^1) + \varepsilon$ .

5. Of course it might be the case that for no values of  $q$  is it true that  $\pi = 0$  or  $\pi = (s - r)K$ . Note that

$$\pi_K = -wL_K - r \equiv w \frac{F_K}{F_L} - r. \quad \text{For a given } q, \text{ define } K(q)$$

as that value of  $K$  such that  $w \frac{F_K(K(q))}{F_L(K(q))} = r$ . Moreover,

$$\pi_{KK} = \frac{w}{F_L} \left\{ F_{LK}^2 + F_{KL}^2 - 2F_{LKL} \right\} < 0 \text{ since } F \text{ is}$$

strictly quasi-concave.  $\pi_K > 0$  for  $K < K(q)$ ,  $\pi_K < 0$  for  $K > K(q)$ ,

with  $\lim_{K \rightarrow \infty} \pi = -\infty$ . Thus for  $K$  sufficiently large,  $\pi < 0$  for

all  $q$ , hence  $q^0, q^1, q^2, q^3$ , do not exist.

6. This result is quite in line with the expectation that a firm might react to uncertainty in demand by adding additional capital to make the production scale of the plant more flexible. Note, though, that this principle need not hold for a more general production function, for which  $\pi_{kqq}$  might not be positive everywhere.

7. In general,  $\hat{q} = \hat{q}(p)$ ; here  $\hat{q} \equiv \hat{q}(p^0)$ .

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