DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

VON NEUMANN MORGENSTERN SOLUTION SOCIAL CHOICE FUNCTIONS: AN IMPOSSIBILITY THEOREM

John A. Ferejohn and Richard D. McKelvey

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ABSTRACT

Recently two game theoretic interpretations of social choice procedures have been offered. First, Wilson [1970] and Plott, [1974] suggested that, for each environment, the value of a choice function might constitute a "solution" or stable set that could arise from the play of some underlying cooperative game. In this view an important problem is to determine if and under what conditions a given solution concept (or notion of stability) can, for some game, characterize the behavior of a given social choice function.

Secondly, social choice functions have been interpreted as collections of equilibria of an underlying noncooperative game (see Gibbard [1973], Peleg [1978], Maskin [1977], and Ferejohn and Grether [1979]). In this framework, one major problem is to determine for a given equilibrium correspondence of a suitably chosen noncooperative game. A closely related problem is to determine which noncooperative games possess nonempty equilibrium correspondences of various sorts.

In this paper we pursue a cooperative game-theoretic interpretation of social choice. And, in particular, we show that if a social choice function arises as a Von Neumann Morgenstern solution in each environment then it is essentially oligarchical in exactly the same sense that "core" selecting choice functions are oligarchic. The conditions under which this conclusion is obtained are, in fact, slightly more restrictive than those for the results on core selecting choice functions but are still weak enough that our result applies to almost any commonly occurring voting rule.

VON NEUMANN MORGENSTERN SOLUTION SOCIAL CHOICE FUNCTIONS: AN IMPOSSIBILITY THEOREM

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1. INTRODUCTION

Historically, social decision rules have been given three distinct interpretations, one of which is basically normative and the others of which are positive. In the classical formulation of the problem of social choice by Bergson [1938] and Arrow [1963], a social welfare function is supposed to embody a collection of judgments about society's welfare. And, for a given configuration of individual preferences (or environment), and available technologies (or feasible set), the maximal elements of the welfare function describe the "best" courses of action open to society. On this view a principal problem of social choice theory is to determine if various collections of ethical intuitions about social welfare are consistent in the sense that allows these intuitions to be represented by a function which can then be maximized by the choice of appropriate values of the decision variables available to a hypothetical planner.

More recently two game theoretic interpretations of social choice procedures have been offered. First, Wilson [1970] and Plott, [1974] suggested that, for each environment, the value of a choice function might constitute a "solution" or stable set that could arise

from the play of some underlying cooperative game. In this view an important problem is to determine if and under what conditions a given solution concept (or notion of stability) can, for some game, characterize the behavior of a given social choice function. For example, it is known that if for any collection of preferences a nonempty-valued social choice function always chooses the core of a set of alternatives, then the underlying game must exhibit a nearly oligarchical character [see Brown, 1975]. In particular there are certain individuals who must be members of every "decisive" or "winning" coalition.

Secondly and more recently, social choice functions have been interpreted as collections of equilibria of an underlying noncooperative game (see Gibbard [1973], Peleg [1978], Maskin [1977], and Ferejohn and Grether [1979]). In this framework, one major problem is to determine for a given equilibirum concept, which social choice functions can be generated as the equilibrium correspondence of a suitably chosen noncooperative game. For example, the Gibbard-Satterthwaite theorem asserts that if for each environment the social choice function arises as a (nonempty) set of outcomes each of which can be supported by a dominant strategy equilibrium of some game, then that game must be dictatorial. A closely related problem is to determine which noncooperative games possess nonempty equilibrium correspondences of various sorts.

While the previous literature has made significant progress in characterizing those social choice functions that can be generated by noncooperative equilibrium correspondences, little attention has been given to the properties of those choice processes that can arise from cooperative play. Indeed, given the diversity of cooperative solutions to games one might suspect that such a wide variety of choice functions would be consistant with some "solution" of a cooperative game, that almost no restrictions are put on the game by the assumption that the choice process is a solution. While this question has not been addressed in earlier published literature, Plott [1974] posed it in the special case in which the choice process is also assumed to be "rational." He found that if a social choice function is rational and is a "solution" to some cooperative game, then the dominance relation of the game must be quasitransitive. And, in the context of a universal domain condition, this implies that the social choice function is oligarchical. But the restriction to rational social choice functions to be very strong since it implies the existence of a nonempty core which, by itself, forces the game to be nearly oligarchical and so, we focus instead on a much larger class of (possibly nonrational) functions.

Thus in this paper we pursue a cooperative game-theoretic interpretation of social choice. And, in particular, we show that if a social choice function arises as a Von Neumann Morgenstern solution in each environment then it is essentially oligarchical in exactly the same sense that "core" selecting choice functions are oligarchic. Or to put things another way, a nonempty Von Neumann Morgenstern solution correspondence is well defined only for games with nearly oligarchic structures. The conditions under which this conclusion is obtained are, in fact, slightly more restrictive than

those for the results on core selecting choice functions but are still weak enough that our result applies to almost any commonly occurring voting rule. And, it should be noted that the methods of proof utilized to obtain these results are quite different from any of those found in the earlier literature.

2. DEFINITIONS AND NOTATION

We let $N = \{1,2,\ldots,n\}$ be a finite collection of individuals, and X be a set of alternatives. Let $\underline{X} = \mathcal{P}(X) - \phi$ represent the set of nonempty subsets of X, and $\underline{B} \subseteq \underline{X}$ be any subset of feasible sets. Let Θ be the set of binary relations on X, Θ^o be the set of weak orders on X, and $\underline{\Theta} = \prod_{i=1}^{n} \Theta_i$, where $\Theta_i = \Theta^o$ for all $i \in N$. A social choice function is any function $C : \underline{B} \times \underline{\Theta} \to \underline{X}$ satisfying $C(S,\underline{R}) \subseteq S$ for all $S \in \underline{B}$ and $\underline{R} \in \underline{\Theta}$.

Given a profile $\underline{R} \in \underline{\Theta}$, an n person cooperative game can be represented by an asymmetric binary relation $D(\underline{R}) \in \Theta$, which we refer to as a dominance relation. To say that $xD(\underline{R})y$, for $x,y \in X$ means that there is some coalition (set) of individuals who all strictly prefer x to y and who are also effective for x over y, in the sense that they have the power, under the rules, to implement a change from x to y, should they choose to do so. The set of dominance relations generated by a given game, as preferences vary, is then described by a function $D : \underline{\Theta} \to \Theta$ which assigns, to each profile $\underline{R} \in \underline{\Theta}$, an asymmetric binary relation $D(\underline{R}) \in \Theta$. Any such function is called a dominance structure.

Two solutions for cooperative games are well known: the core and the Von Neumann Morgenstern solution. Formally, given a set $S \in \underline{B}$, a core of the game described by $D(\underline{R})$ is any subset $K(S,\underline{R})$ of S satisfying

$$K(S,\underline{R}) = \{x \in S \mid \sim yD(\underline{R})x \text{ for all } y \in S\}.$$

A Von Neumann Morgenstern Solution is any subset $V(S,\underline{R})$ of S satisfying

(i) $\forall x,y \in S, \sim xD(\underline{R})y$

(ii) $\forall x \in S - V(S,\underline{R}), \exists y \in V(S,\underline{R})$ s.t. $yD(\underline{R})x$.

In this paper, we consider conditions under which the choice function C can be expressed as the solution of some underlying n person cooperative game. To do this, we use some definitions originally studied by Wilson [1970], which connect the social choice and game theoretic formulations:

Definition 1: A social choice function $C : \underline{B} \times \underline{\Theta} \to \underline{X}$ satisfies the core property iff there is a dominance structure $D : \underline{\Theta} \to \Theta$ such that for all $\underline{R} \in \underline{\Theta}$ and all $x \in S \in \underline{B}$, $x \in C(S,\underline{R}) \iff \sim yD(\underline{R})x$ for all $y \in S$.

Definition 2: A social choice function $C : \underline{B} \times \underline{\Theta} \to \underline{X}$ satisfies the solution property iff there is a dominance structure $D : \underline{\Theta} \to \Theta$ such that for all $\underline{R} \in \underline{\Theta}$, and all $x \in S \in \underline{B}$, $x \in C(S,\underline{R}) \iff \sim yD(\underline{R})x$ for all $y \in C(S,\underline{R})$.

It is easily verified that the solution property is equivalent to requiring that for all $S \in \underline{B}$ and $\underline{R} \in \underline{\Theta}$, $C(S,\underline{R})$ selects a von Neumann Morgenstern solution from S according to the dominance relation $D(\underline{R})$. We first note an elementary consequence of definition one:

Proposition 1 If $C : \underline{B} \times \underline{\Theta} \to \underline{X}$ is a social choice function with $\underline{B} = \underline{X} = \mathcal{P}(X) - \phi$, then if C satisfies the core property, $D(\underline{R})$ is acyclic for all $\underline{R} \in \underline{\Theta}$.

Proof: Assume $D(\underline{R})$ is not acyclic for some $\underline{R} \in \underline{\Theta}$. Then there is some set $S = \{x_1, \ldots, x_k\}$ with $x_{\ell-1} D(\underline{R}) x_\ell \pmod{k}$ for all ℓ. It follows from the definition of the core property that $C(S,\underline{R}) = \phi$, a contradiction.

Q.E.D.

In this paper, we restrict our attention to games satisfying certain desirable properties. For any $\underline{R} = (R_1,\ldots,R_n) \in \underline{\Theta}$, we let $\underline{P} = (P_1,\ldots,P_n)$, where P_i is the asymmetric part of R_i for each $i \in N$. Then the first condition we impose is a monotonicity requirement:

<u>Condition 1</u> A dominance structure $D : \underline{\Theta} \to \Theta$ is <u>monotonic</u> iff $(\forall x,y \in X)(\forall \underline{R},\underline{R}^* \in \underline{\Theta})[(\forall i \in N)(xR_i y \Rightarrow xR_i^* y \text{ and } xP_i y \Rightarrow xP_i^* y) \Rightarrow (xD(\underline{R})y \Rightarrow xD(\underline{R}^*)y)]$

Thus, if we move from profile $\underline{R}$ to $\underline{R}^*$ and find that everyone's ranking of x vis a vis y is at least as good in $\underline{R}^*$ as it was in $\underline{R}$, then if x already dominated y under $\underline{R}$, it must still dominate it under $\underline{R}^*$. Next, for any ordered k-tuples $(x_1,\ldots,x_k)$ and $(y_1,\ldots,y_k)$ in XxXx...xX, for any $\underline{R},\underline{R}^* \in \underline{\Theta}$, we write $\underline{R}|_{(x_1,\ldots,x_k)} \cong \underline{R}^*|_{(y_1,\ldots,y_k)}$ iff, for all $1 \le i, j \le k$, and all $p \in N$

$$x_i R_p x_j \iff y_i R_p^* y_j$$

Now, we have the following definitions of neutrality and independence

<u>Condition 2</u> A dominance structure $D : \underline{\Theta} \to \Theta$ is <u>neutral</u> iff, for all $\underline{R} \in \underline{\Theta}$, and $x,y,z,w \in X$, $\underline{R}|_{(x,y)} \cong \underline{R}|_{(z,w)} \Rightarrow D(\underline{R})|_{(x,y)} \cong D(\underline{R})|_{(z,w)}$.

<u>Condition 3</u> A dominance structure $D : \underline{\Theta} \to \Theta$ satisfies <u>Independence of Irrelevant Alternatives</u> (IIA) iff, for all $\underline{R},\underline{R}^* \in \underline{\Theta}$, and $x,y \in \underline{X}$, $\underline{R}|_{(x,y)} \cong \underline{R}^*|_{(x,y)} \Rightarrow D(\underline{R})|_{(x,y)} \cong D(\underline{R}^*)|_{(x,y)}$.

Note that monotonicity implies independence of irrelevant alternatives.

Now if D is a dominance structure then for any given $\underline{R} \in \underline{\Theta}$, a <u>cycle</u> for $D(\underline{R})$ is any set $S = \{x_1,\ldots,x_k\} \in \underline{X}$ for which $x_{\ell-1} D(\underline{R}) x_\ell \pmod k$ for all $1 \le \ell \le k$. If $\underline{S}_{\underline{R}}$ is the set of all cycles for $D(\underline{R})$ and $\underline{S}_{\underline{R}} \ne \phi$, then set

$$k_{\underline{R}} = \min_{S \in \underline{S}_{\underline{R}}} |S|$$

Any cycle S for which $|S| = k_{\underline{R}}$ is called a <u>minimal cycle for</u> $\underline{D(R)}$. Now if $\underline{S}_{\underline{R}} \ne \phi$ for some $\underline{R} \in \underline{\Theta}$, set

$$k = \min_{\substack{\underline{R}\in\Theta \\ \underline{S}_{\underline{R}} \ne \phi}} k_{\underline{R}}$$

If $\underline{R} \in \underline{\Theta}$ and $S \in \underline{X}$ are such that S is a cycle for $D(\underline{R})$, and $|S| = k$, then S is called a <u>minimal cycle for D</u>. The integer k is called the <u>minimal cycle size</u> for D. Note that if D has no minimal cycle size (i.e., $\underline{S}_{\underline{R}} = \phi$ for all $\underline{R} \in \underline{\Theta}$), then $D(\underline{R})$ is acyclic for all $\underline{R} \in \underline{\Theta}$.

3. VON NEUMANN MORGENSTERN CHOICE FUNCTIONS AND DOMINANCE STRUCTURES

In this section we establish that any choice function that satisfies the solution property must be generated by an implicit dominance structure which is acyclic. More precisely we obtain the following result:

Theorem 1 Let $C : \underline{X} \times \underline{\Theta} \to \underline{X}$ be a social choice function satisfying the solution property, and let $D : \underline{\Theta} \to \Theta$ be a corresponding dominance structure. Then if Conditions 1 and 2 are satisfied, then either $D(\underline{R})$ is acyclic for all $\underline{R} \in \underline{\Theta}$, or $|X|$ is even with the minimal cycle size for D equal to $|X|$.

Proof: The proof uses a series of Lemmas which follow. Actually, only Lemmas 1 and 5 are used directly in the theorem. Lemmas 2,3, and 4 are used to prove Lemma 5. Assume the conditions of the theorem hold and that $D(\underline{R})$ is not acyclic for all $\underline{R} \in \Theta$. Then D has a minimal cycle size, say k. But if k is odd, there is some $\underline{R} \in \underline{\Theta}$ for which $D(\underline{R})$ has a minimal cycle of size k, which by Lemma 1, is impossible. So k must be even, with $k \geq 4$, since there cannot be cycles of size 2. Now, Lemmas 1-4 show that if $k < |X|$ it is possible to construct a profile $\underline{R}^*$ for which $D(\underline{R}^*)$ has minimal cycle size k + 1. The result now follows from Lemma 1, since now for $\underline{R}^*$, the solution property cannot be satisfied.

Q.E.D.

Lemma 1 Let $C : \underline{X} \times \underline{\Theta} \to \underline{X}$ be a social choice function satisfying the solution property. Let $D : \underline{\Theta} \to \Theta$ be a corresponding dominance structure, and let $\underline{R} \in \underline{\Theta}$. Then $D(\underline{R})$ cannot have minimal cycle size k where k is odd.

Proof: Let $S = \{x_1,\ldots,x_k\}$ be a minimal cycle for $D(\underline{R})$. Then $x_{\ell-1}D(\underline{R})x_\ell$ (mod k) for all ℓ, and for all other $x,y \in S$, $\sim xD(\underline{R})y$. Now let $x_i \in C(S,\underline{R})$ for some $1 \leq i \leq k$. Then $x_{i+1} \notin C(S,\underline{R})$ (mod k). But now since x_{i+1} is the only element in S which dominates x_{i+2} (mod k), it follows that $x_{i+2} \in C(S,\underline{R})$ (mod k). Thus $C(S,\underline{R})$ is equal to $\{x_1,x_3,\ldots,x_k\}$ or to $\{x_2,x_4,\ldots x_{k-1}\}$ which is a contradiction.

Q.E.D.

For the next lemmas, we introduce some additional notation. Let $\underline{R}$ be such that $D(\underline{R})$ yields a minimal cycle of size k, say $\{x_1,\ldots,x_k\}$ such that

$$x_{\ell-1}D(\underline{R})x_\ell$$

for all $1 \leq \ell \leq k$ (mod k). For each $1 \leq j \leq k$, set

$$d_j(\underline{R}) = \{i \in N | x_{\ell-1}P_i x_\ell \text{ for all } \ell \neq j (\text{mod } k)\}$$

and set

$$d_0(\underline{R}) = \{i \in N | x_\ell I_i x_p \text{ for all } 1 \leq \ell,p \leq k\}.$$

Then we have the following result.

Lemma 2 Let C be monotonic and satisfy the solution property, further assume that the dominance structure D has a minimal cycle $\{x_1,\ldots,x_k\}$. Then if $x_j P_p x_{j-1}$ for some $p \in N$ and some $1 \leq j \leq k (\text{mod } k)$, then if $\underline{R}^* \in \underline{\Theta}$ satisfies i) $R_\ell^* = R_\ell$ for $\ell \neq i$, and ii) $p \in d_j(\underline{R}^*)$, then $D(\underline{R}^*)$ also generates a minimal cycle on $\{x_1,\ldots,x_k\}$.

Proof: Let $\underline{R}$ and $\underline{R}^*$ be as described. Then it follows that for all $1 \leq \ell \leq k$, and all $q \in N$,

$$\left.\begin{array}{l} x_{\ell-1} P_q x_\ell \Rightarrow x_{\ell-1} P_q^* x_\ell \ (\text{mod } k) \\ x_{\ell-1} R_q x_\ell \Rightarrow x_{\ell-1} R_q^* x_\ell \ (\text{mod } k) \end{array}\right\} (*)$$

This is immediate if $q \neq p$, since here $R_q' = R_p'$. If $q = p$, then for $\ell \neq j$, (*) holds since here $x_{\ell-1} P_q^* x_\ell$. For $q = p$ and $\ell = j$, (*) holds since $x_\ell P_p x_{\ell-1}$. But now from monotonicity of C, since $x_{\ell-1} D(\underline{R}) x_\ell$ for all $\ell (\text{mod } k)$, it follows from (*) that $x_{\ell-1} D(\underline{R}^*) x_\ell$ for all $\ell (\text{mod } k)$. Thus the set $\{x_1,\ldots,x_k\}$ forms a cycle under $D(\underline{R}^*)$ of length k, which by assumption must be of minimal length.

Q.E.D.

Lemma 3 If C satisfies the solution property and its dominance structure D has a minimal cycle of size k, and if C is monotonic, then there is an $\underline{R}^* \in \underline{\Theta}$ such that $D(\underline{R}^*)$ contains a minimal cycle such that $\bigcup_{j=0}^{k} d_j(\underline{R}^*) = N$.

Proof: Let $\underline{R}$ be a profile yielding the minimal cycle $\{x_1,\ldots,x_k\}$ satisfying $x_{\ell-1} D(\underline{R}) x_\ell$ for all $1 \leq \ell < k (\text{mod } k)$. Let

$$d^*(\underline{R}) = N - \bigcup_{j=0}^{k} d_j(\underline{R})$$

Now let $K = |d^*(\underline{R})|$. Then if $K = 0$, set $\underline{R}^* = \underline{R}$, and we are done. If $K \neq 0$, then pick $p \in d^*(\underline{R})$. Since $p \notin d_o(\underline{R})$, it follows by transitivity of R_p that there is some j such that $x_j P_p x_{j-1} (\text{mod } k)$. By Lemma 2, it now follows that the profile $\underline{R}^*$ defined in Lemma 2 satisfies $d^*(\underline{R}^*) = N - \bigcup_{j=0}^{k} d_j(\underline{R}^*) = K - 1$. The result now follows immediately by induction on K.

Q.E.D.

Lemma 4 Assume C satisfies the solution property that its dominance structure D has a minimal cycle length k, and that C is monotonic and neutral. Then if $\underline{R} \in \underline{\Theta}$ and $D(\underline{R})$ contains a minimal cycle, and $\bigcup_{j=0}^{k} d_j(\underline{R}) = N$, then $d_j(\underline{R}) \neq \phi$ for all $1 \leq j \leq k$.

Proof: Let $\{x_1,\ldots,x_k\}$ be a minimal cycle for D which is attained by $\underline{R} \in \underline{\Theta}$. I.e., $x_{\ell-1}D(\underline{R})x_\ell$ for all ℓ(mod k). Now suppose $d_j(\underline{R}) = \phi$ for some $1 \leq j \leq k$. Then it follows that for all $i \in N$,

$$x_j I_i x_{j+1} \Leftrightarrow i \in d_o(\underline{R}) \Leftrightarrow x_j I_i x_{j+2} \text{ for all } j \pmod k$$

and

$$x_{j-1} P_i x_j \Leftrightarrow x_{j-2} P_i x_j \pmod k$$

But then, since $\bigcup_{j=0}^{k} d_j(\underline{R}) = N$, the above two situations exhaust the possible cases. Hence $\underline{R}|_{(x_{j-1},x_j)} \cong \underline{R}|_{(x_{j-2},x_j)}$. By neutrality, it follows that $x_{j-2}D(\underline{R})x_j$, since $x_{j-1}D(\underline{R})x_j$. But then $\{x_1,\ldots,x_k\} - \{x_j\}$ has a cycle of size k - 1, which contradicts the minimality of k. Hence $d_j(\underline{R}) \neq \phi$.

Q.E.D.

Lemma 5 Let C satisfy the solution property and assume that its dominance structure has minimal cycle length $k \geq 4$ where if X is finite $|X| > k$. Assume C is monotonic and neutral. Then there is an $\underline{R}^* \in \underline{\Theta}$ such that $D(\underline{R}^*)$ contains a minimal cycle of length k + 1.

Proof: By lemmas 3 and 4, it follows that there is a profile $\underline{R} \in \underline{\Theta}$ such that $D(\underline{R})$ contains a minimal cycle of length k, say $\{x_1,\ldots,x_k\} = B$, and such that $\underline{R}$ satisfies.

i) $\bigcup_{j=0}^{k} d_j(\underline{R}) = N$

ii) $d_j(\underline{R}) \neq \phi$ for $1 \leq j \leq k$

Now we pick $x^* \in X - \{x_1,\ldots,x_k\}$, and we define the new profile $\underline{R}^* \in \underline{\Theta}$ as follows for any $x,y \in X$, and $p \in N$, then

xI_p^*y if $x,y \in X - \{x_1,\ldots,x_k,x^*\}$.

xP_p^*y if $x \in \{x_1,\ldots,x_k,x^*\}$ and $y \in X - \{x_1,\ldots,x_k,x^*\}$.

$xR_p^*y \Leftrightarrow xR_py$ if $[x,y \in \{x_1,\ldots,x_k\}$ and $p \notin d_{k-1}(\underline{R})]$ or $[\{x,y\} \neq \{x_1,x_k\}]$

$xP_p^*y \Leftrightarrow yP_px$ if $\{x,y\} = \{x_1,x_k\}$ and $p \in d_{k-1}(\underline{R})$

$x_k P_p^* x^*$ if $p \in d_1(\underline{R})$

$x^* P_p^* x_{k-1}$ if $p \in d_{k-1}(\underline{R})$

$x^* I_p^* x_k$ if $p \in d_o(\underline{R})$

and

$$\left.\begin{array}{l} x_k P_p^* x^* \\ x^* P_p^* x_1 \end{array}\right\} \text{ otherwise}$$

Now on $\{x_1,\ldots,x_k\}$, we have, for $\{x,y\} \neq \{x_k,x_1\}$,

$$\underline{R}^*\big|_{(x,y)} \cong \underline{R}\big|_{(x,y)}$$

Hence for $\ell \neq 1\,(\text{mod } k)$,

$$x_{\ell-1} D(\underline{R}^*) x_\ell$$

Further, it follows that

$$\sim x_k D(\underline{R}^*) x_1$$

To see this, note that if $x_k D(\underline{R}^*) x_1$, then $D(\underline{R}^*)$ generates a minimal cycle on $\{x_1,\ldots,x_k\}$. But $d_{k-1}(\underline{R}^*) = \phi$, and for all $p \in d^*(\underline{R}^*)$ $= N - \bigcup_{j=0}^{k} d_j(\underline{R}^*)$, we have, by construction, that $x_1 P_p x_k$. Hence, applying Lemma 2, we can construct a third profile, $R^{**} \in \underline{\Theta}$ for which $D(\underline{R}^{**})$ also has a minimal cycle in $\{x_1,\ldots,x_k\}$ such that $\bigcup_{j=0}^{k} d_j(\underline{R}^{**}) = N$, but where $d_{k-1}(\underline{R}^{**}) = \phi$. This contradicts Lemma 4, so we must have $\sim x_k D(\underline{R}^*) x_1$, as claimed. Now, by construction it follows that.

1) $\underline{R}^*\big|_{(x^*,x_1)} \cong \underline{R}\big|_{(x_k,x_1)}$

2) $\underline{R}^*\big|_{(x_k,x^*)} \cong \underline{R}\big|_{(x_{k-2},x_{k-1})}$

3) $\underline{R}^*\big|_{(x^*,x_{k-1})} \cong \underline{R}\big|_{(x_k,x_{k-2})}$

4) $\underline{R}^*\big|_{(x^*,x_2)} \cong \underline{R}\big|_{(x_k,x_2)}$

Thus, by neutrality, we get

$$x^* D(\underline{R}^*) x_1$$

$$x_k D(\underline{R}^*) x^*$$

$$\sim x^* D(\underline{R}^*) x_{k-1} \text{ and } \sim x_{k-1} D(\underline{R}^*) x^*$$

$$\sim x^* D(\underline{R}^*) x_2 \text{ and } \sim x_k D(\underline{R}^*) x_2$$

Thus $\{x_1, \ldots x_k, x^*\}$ form a cycle of size k + 1. Further this is a minimal cycle in $D(\underline{R}^*)$ by construction.

Q.E.D.

4. DISCUSSION

That a social choice function arises either as a solution or a core to a neutral and monotone cooperative game evidently requires that game to possess a rather well-behaved dominance structure in the following sense. First, we know that the dominance structure must satisfy the independence of irrelevant alternatives and be acyclic valued. Further, if for some $S \varepsilon \underline{X}$ and $\underline{R} \varepsilon \underline{\Theta}$, $C(S,\underline{R}) \neq S$, then Arrow's Pareto Principle (PP) must be satisfied too.

<u>Proposition 2</u>: If C satisfies the solution property (or the core property) relative to D, and D satisfies conditions 1 and 2 and if there is an $S \varepsilon \underline{X}$ and $\underline{R}^* \varepsilon \underline{\Theta}$ with $C(S,\underline{R}^*) \neq S$ then

$$(PP): \forall \underline{R} \varepsilon \underline{\Theta} \text{ such that } xP_iy \ \forall i \varepsilon N, \ xD(\underline{R})y.$$

<u>Proof</u> $z \varepsilon S - C(S,R^*) \Rightarrow [\exists w \varepsilon C(S,\underline{R}^*)$ if the solution property is satisfied or $\exists w \varepsilon S$ if the core property is satisfied] such that $wD(\underline{R}^*)z$. Now if $\underline{R}$ is such that $xP_iy \ \forall i \varepsilon N$; define $\underline{R}^{**}$ so that $wP_i^{**}z$ and $xP_i^{**}y \ \forall i \varepsilon N$. Then $wD(\underline{R}^{**})z$ by monotonicity and $xD(\underline{R}^{**})y$ by neutrality. Finally $xD(\underline{R})y$ by monotonicity.

Q.E.D.

A theorem of Brown's [1975] indicates that games with dominance structures which satisfy the above axioms must actually be oligarchical in a certain sense. A coalition, V, is called <u>decisive</u> just in case for any $\underline{R} \varepsilon \underline{\Theta}$ and $x,y \varepsilon X$ such that xP_iy for all $i \varepsilon V$, $xD(\underline{R})y$. Brown showed that if $D : \underline{\Theta} \to \Theta$ is acyclic valued, satisfies independence of irrelevant alternatives, and the Binary Pareto Principle then there is some individual who is in every decisive set.

A straightforward application of Brown's theorem and our theorem 1 establishes the following result.

<u>Theorem 2</u> If $C : \underline{B} \otimes \underline{\Theta} \to \underline{X}$ satisfies conditions 1 and 2 and the solution property relative to the Dominance structure, D, and if $\underline{B}$ contains the finite subsets of X, then either

i) there is some individual who is in every decisive set,

or

ii) $|X|$ is even with minimal cycle size for D equal to $|X|$

This result constitutes an answer to the problem of "revealed" institutions posed by Plott [1974]: if a social choice function which is neutral and monotone is known to be generated as the solution of some underlying cooperative game, what can be said about the structure of this cooperative game? Theorem 2 indicates that we can conclude in this case that if the set of environments over which the social choice function operates is sufficiently diverse then the game structure must be oligarchic in a certain sense. And, in fact, we can identify exactly those individuals who are in every decisive coalition.

As a final example note that among the neutral and monotone social decision procedures are the special majority rules discussed in Ferejohn and Grether [1974]. A social choice function might be called an S-extended special majority rule if on the two element sets it is a special majority rule, while on the larger sets it chooses solutions with respect to some dominance structure. It's easy to see that in this case the dominance structure must agree with the special majority rule on the two element sets. And, if $|X| = m$ is even, we can employ Theorem 1 of Ferejohn and Grether along with Theorem 2 of this paper to conclude that the special majority must contain at least $n \frac{m-1}{m}$ individuals. If $|X|$ is odd or infinite, then the only special majority rule that satisfies the condition of Theorem 2 is the unanimity rule (which is of course oligarchic by our definition).

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