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CHANCE CONSTRAINED DYNAMIC PROGRAMMING MODEL OF WATER RESERVOIR WITH JOINT PRODUCTS

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Historically, rivers have been the focus of many international conflicts, especially in the arid and semi-arid areas of the world. However, within a particular river basin, water was relatively abundant and there was generally enough to meet the various needs of the basin's population. Water usage was limited mainly to human consumption and irrigation.

The growth in population and rising level of industrialization in many arid and semi-arid parts of the world are increasing the demands for water. However, no corresponding change in the world supply of river water occurred. It has become a scarce resource, and active planning for water utilization is under way.

An important aspect of this planning is the distribution of the benefits of the rivers over time and among uses and users. Increasingly the construction of large reservoirs is becoming the vehicle to achieve and integrate these diverse objectives. Very few reservoirs are normally dedicated to achieve a single objective. Invariably, irrigation, power generation, flood control and recreation are among the objectives listed for any dam project. That does not mean there is no hierarchy imposed on these objectives by the planner. In fact, there may exist one or two prime objectives. The absence of explicit statements on this hierarchy has become a political expedient to appease the various groups affected by the construction of the dam. Model

builders have reflected this hierarchy by directly including some variables in the objective function and others are formulated as constraints.

Some of these constraints are "soft," in the sense that they could be violated at a cost. This cost is dictated by the demand of the planner for these constraints to hold. The following analysis will focus on irrigation and power generation with soft constraints on the stock of water in the reservoir. These soft constraints reflect a trade-off between flood control and recreation purposes on the one hand and salinity control in the downstream on the other.

There are two design considerations in the process of reservoir construction: 1) the optimal reservoir size, and 2) the optimal operating rule of the reservoir. Although many scholars [12, 15] have previously pointed out that the two considerations cannot be separated, many attempt to separate the dual decisions of optimal size and optimal operating policy. The model in this paper will recognize the "jointness" of the decisions and treat them in a unified manner within the framework of dynamic programming.

An often neglected aspect in the design of impounding reservoirs in arid and semi-arid regions where evaporation losses are significant is the trade-off between two opposing considerations:

1. There are benefits from assuring a more regular flow of water and hence a "better" distribution of the river benefit over time and among users and uses.

2. There are also costs imposed by the evaporation of the impounded water in the reservoir. These costs are significant. As Quirk and Burness point out [22] for a minor river such as the Colorado with an annual mean runoff of 13.5 million acre-feet per year, evaporation losses from existing reservoirs have already reached as high as 1.5 million acre-feet per year.

To produce an outflow pattern satisfying a given economic objective, the preceding trade-off is taken into consideration in ascertaining the relationship between the hydrology of a stream and the optimal decision rule. The optimal size of the reservoir which is consistent with the chosen operating rule will be derived. Moreover, the long-run distribution of the water stock in the reservoir when the profit function from the reservoir operation has a special form will be derived.

Uncertainty will be revealed as the single most important factor affecting the optimal design and operation of a reservoir. Formally, this uncertainty may be reflected in the objective function, the constraints, or both. In the case where the uncertainty is reflected in the constraints, there is a possibility that optimal decisions will lead to violation of the constraints because of very high or very low values of inflow. This is the basic problem posed by the nature of the random constraints.

At least two different types of characterizations are available in the optimization literature to cope with the random nature of the constraints. First, there is the penalty function

approach [27] which introduces penalties for violating the random constraints. This is accomplished by adding the expected penalty costs to the objective function.

Secondly, there is the chance constrained characterization [5], [6] which puts a reliability interpretation on the constraint, such as

$$\text{prob } (b_i \geq a_i' y_i) \geq \lambda_i, \qquad 0 \leq \lambda_i \leq 1,\ i = 1,\ldots,m \tag{6}$$

Where y_i is the decision variable, λ_i is the reliability factor and b_i is a function of a random variable. The λ_i can be varied parametrically to account for the different reliability levels (alternatively, a reliability term can be added to the objective function and can be solved for optimally).

In this characterization, the chance constraint is reduced to an equivalent deterministic constraint [6] by the use of the marginal distribution function of b_i: $\phi(b_i)$. The existence of a fractile $\overline{b}_i$ such that

$$P(b_i \geq a_i' y) \geq \lambda_i \iff b_i(1 - \lambda_i) \geq a_i' y \tag{10}$$

makes this reduction possible. To facilitate this transformation in the reservior models, the optimal decision rule is restricted to the class of linear functions [16, 17, 19]. Additionally, it is sometimes assumed that the random variable is distributed normally or truncated normal at zero [7, 28]. This technique suffers from a number of shortcommings:

1. The continuity equation, used to develop the deterministic equivalent for the chance constraints and the steady state distribution

of the stocks, ignores the overspill. The overspill occurs because the constraints may be violated in these models.

2. The net return function in these models does not reflect the probability that the constraints could be violated. Violation could occur as a result of the optimizing program, yet the net return is not affected. This condition raises a question regarding the incentive structure in these kinds of models.

3. The ad hoc specification of the reliability levels (λ_i's) raises objections from many planners and politicians. No decision-maker would risk making an explicit statement on reliability. The problem of the choice of the weight w_i given to the reliability term in the composite objective function persists, even if the choice of λ_i's is included within the optimizing framework, as in the model shown in (7-9).

The model in this paper uses a dynamic programming approach in conjunction with a penalty function. The penalty is a convex increasing function of the magnitude of the violations. These penalty costs differ from the fixed accounting costs employed by Askew [2, 3]. Accounting costs are never actually intended to be paid, but are merely devices to ensure optimal behavior by the management. Penalty costs in this model, however, are actually economic costs imposed on the manager to correct for stock deficiency or surplus which results from his decisions and the random flow of the river.

A Dynamic Programming Model with Penalty Function

In this model the penalty function approach is utilized to account for the possibility of violating the constraints within a

dynamic programming framework. These penalty costs are more than "accounting" costs used to insure that the dam manager takes the imposed "soft" constraints into consideration in arriving at his decision rule [2], [3]. They are costs actually paid by the dam manager for importing or exporting water to compensate for violating the constraints.

Although no a priori form for the optimal decision rule is imposed, it will become evident that this formulation implies a simple "one part" decision rule with "predictable" characteristics. Further, the linear decision rule, which implies constant optimal stock policy, will be shown as a consequence of certain restrictions on the form of the profit function.

It will also be shown that the long-run distribution for the stock of water in the reservoir exists and can be derived when the linear decision rule applies. This formulation will not suffer from the shortcomings of the chance constraints-deterministic equivalent approach. Moreover, the analysis will be expanded to include profits from the generation of electricity directly in the profit function and will be shown to affect the optimal policy and the optimal reservoir size.

The Objective Function

The manager of the reservoir is maximizing at every period, a concave objective function of the form $\pi(y,x-p)$ where $\pi_{12} \geq 0$, y is the release at the start of the period and x is the stock at the start of the period. The first argument of the objective function, y, reflects the payoff to agricultural downstream users from releases and the second argument reflects the payoff from power generated by the electrostatic head provided by the stock of water after releasing y.

The objective function, π, may be interpreted in various ways:

1. In a socialist economy, π might be the criterion function provided by the central planners. π is, then, the expected net social revenue which equals the expected revenue minus expected cost, all inputs and outputs being evaluated at prices set by the planners. The manager carries on the maximization procedure treating these prices as parameters. Under ideal conditions, the prices for inputs and outputs set by the central planners would be prices consistent with a Pareto optimum. In this special case, the optimality rule derived from the maximization procedure is also optimal from the point of view of welfare maximization. Under more realistic conditions, the criterion function simply reflects the central planners' evaluation of all the alternatives in the economy.
2. In a private economy operating under the appropriative doctrine, property rights to water are held by users of water. A possible situation is one in which the reservoir manager is instructed to operate so as to maximize aggregate expected profits, $E\pi$, of down stream users where $\pi = \{\sum_i \pi_i + \pi_E\}$ and π_i is the profit of downstream user i and π_E is the profit from power generation. Such a scenario is approximately the situation for the Colorado river where downstream users hold appropriative rights to the water in the reservoir and the Bureau of Reclamation operate the reservoir system for them under rules that derive from the Supreme Court decision in the famous Arizona v. California case (1963). Note that maximization of

aggregate profits of downstream users is generally inconsistent with Pareto optimality, particularly when there is market power, e.g., as in the case of the Imperial Valley Irrigation District, a major force in the winter fruit farm market of the U.S., and the largest user of Colorado River water. The situation gets worse if there are externalities in the agricultural and power markets or if there are other imperfections in these markets.

3. π may be interpreted as the payoff in terms of social welfare associated with operating the reservoir. In this case π is the total expected surplus which equals expected consumer's surplus plus expected producers' surplus. The use of total expected surplus involves the usual difficulties of partial equilibrium welfare economics. Such problems include the need to use compensated demand curves, aggregating areas under demand curves over all consumers, the interactions with other markets and the like. There are further complications posed by the multi-periods nature of the problem: the lack of contingent claims markets to internalize uncertainty with respect to prices of future inputs and outputs means that we have the added problem of dealing with expectations involving diverse subjective probability distributions. Finally, if there are many reservoirs taking only the output of this particular reservoir into consideration in measuring social welfare is inappropriate. In this case, we are only considering the output of a part of the industry rather than the whole. This leads to special problems of measuring consumers' surplus.
4. π may be the utility of the reservoir manager over profits from

the operation of the dam. The concavity of π introduces risk aversion directly in the analysis. No social welfare argument may be made from this interpretation unless we consider this reservoir as part of a competitive market and no imperfections in any input and output markets. In this case, the usual classical welfare arguments applies to the economy and efficiency and unbiasedness are assured.

We pointed out that interpretations of the properties and results in this chapter differ according to how π is being interpreted. However, we shall show that from a technical point of view all that is needed to derive the formal results are concavity and/or linearity, and separability and/or the nonnegatively of the second mixed partials of the objective function.

The Model

The manager of the reservoir is maximizing at every stage[1] p, $1 \leq p \leq n$, a profit function $\pi(y_p, x_p - y_p)$, concave in both its arguments such that $\pi_{12} \geq 0$. The maximization is subject to an upper constraint x^u and a lower constraint x^m on the reservoir storage level.

The optimization is conducted as follows:

1. The manager observes the reservoir level, x_p, at the start of the period.

2. He calculates the optimal release in the period y_p^*,

[1] Following dynamic programming tradition, p is counted in reverse order from the terminal point.

$0 \leq y_p^*(x_p) \leq x_p$, taking into consideration the following factors:

a. The one period objective function[2]
   $[\pi(y_p, x_p - y_p) - c(\bar{x})]$ where $c(\bar{x})$ is the strictly convex annualized cost of construction;

b. The costs of violating the upper and lower constraints $c_1(z)$ and $c_2(z)$. Each cost is related, respectively, to the cost of disposing or importing water to compensate for excesses or deficiencies in water storage;

c. The probability distribution of the inflow $\phi_e(\cdot)$;

d. The evaporation rate k.

3. He implements the optimizing decision y_p^* by releasing water from the reservoir.

4. Toward the end of the period p, the manager has enough information to observe the inflow e_p. Then he makes the following decisions:

   If, as a result of his decision, the water level in the reservoir falls below x^m, he imports water at the cost of $c_2(z)$ to make up for the deficiency (z). He then starts period $(p+1)$ with a water stock equal to x^m. $c_2(z)$ is assumed to be a convex and increasing function of z with $c_2(0) = 0$ and $c_2'(0)$ is finite and positive. This assumption holds whether a constant or increasing net price for water is assumed. The case of rising net price of water is being

[2] When the profit function is separable, it will be expressed as: $g(y) + h(x)$.

considered to account for the increasing difficulty of importing larger amounts of water from further locations. See Figure 1. If, however, the optimizing decision results in water stock exceeding x^u, the manager disposes of the excess water (z) at the cost of $c_1(z)$. He then starts the next period (p + 1) with a water stock equal to x^u. $c_1(z)$ is assumed to be a convex function of z with $c_1(0) = 0$ and $c_1'(0)$ finite. This is consistent with a situation where the export price of water net of transportation cost is constant or decreasing because of the increasing difficulty of marketing larger quantities of water. The net export price may eventually be negative. See Figure 2. The sequence of events and decisions are illustrated in Figure 3.

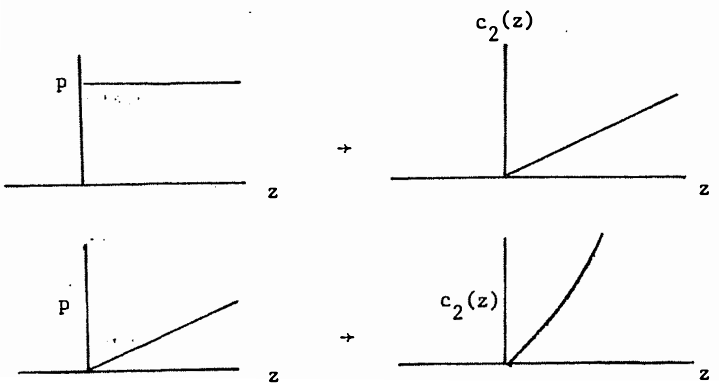

Figure 1

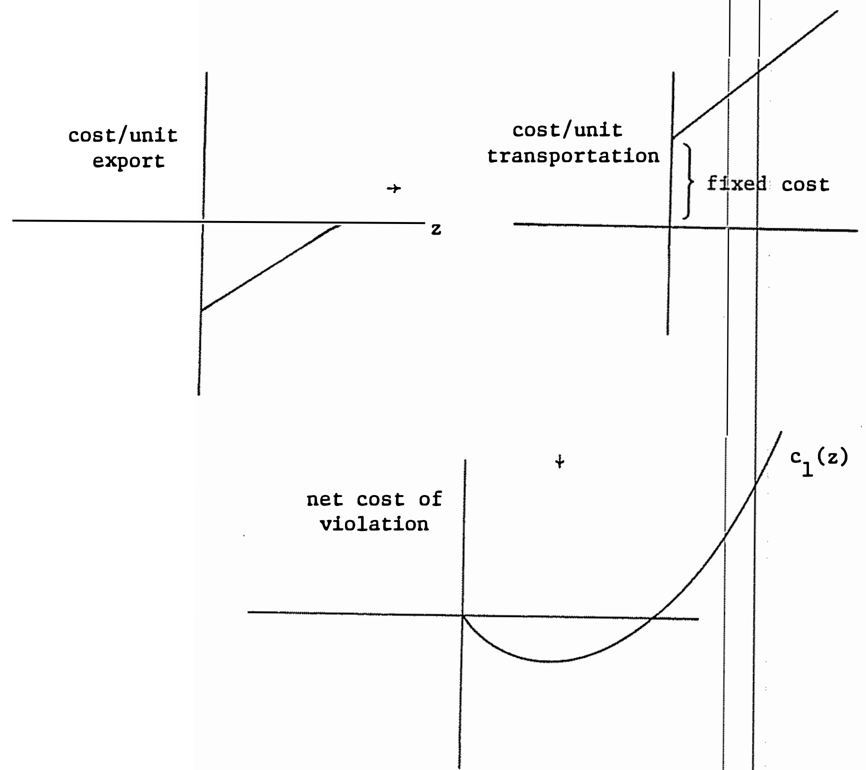

Figure 2

The Continuity Equation

This is the mass balance equation for water in any period p, $1 \leq p \leq n$, and is given by:

$$x_{p-1} = r(x_p - y_p) + e_p + i_p - m_p \qquad (17)$$

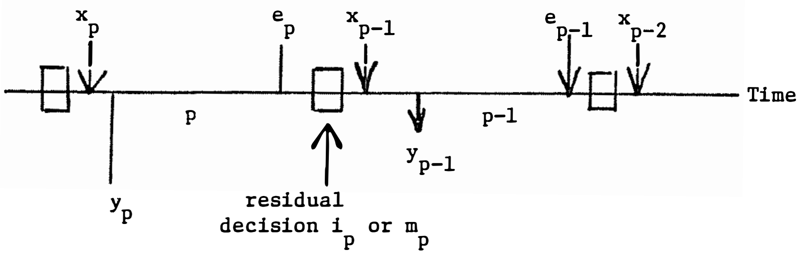

Figure 3

where: $r = 1 - k$ and k is the constant evaporation rate, i_p is the amount of imported water, m_p is the amount of exported water and x_p is the level of the reservoir at the end of stage $(p+1)$ and after implementing the importing and exporting decisions. Or, equivalently x_p is the water stock at the start of period p.

$$\text{If (a) } i_p > 0 \text{ then } m_p = 0, \quad x_{p-1} = x^m$$
$$\text{and} \quad i_p = x^m - rx_p - e_p + ry_p \quad . \qquad (18)$$

$$\text{If (b) } m_p > 0 \text{ then } i_p = 0, \quad x_{p-1} = x^u$$
$$\text{and} \quad m_p = rx_p + e_p - ry_p - x^u \quad . \qquad (19)$$

$$\text{If (c) } i_p = m_p = 0 \text{ then } x_{p-1} = rx_p + e_p - ry_p \quad . \qquad (20)$$

A concave salvage value function of the terminal stock of water v(x) will be added to account for the concern of the planners for future generations. v(x) will also prevent the use of water to the point where its marginal profitability is zero. Moreover, it will be assumed that the manager does not import or export water unless he must. In the static case, this implies that the marginal salvage value at x^u must not be less than the marginal benefit from exporting water $v'(x^u) \geq -c_1'(0)$. It also means that the marginal salvage value of water at x^m must not be greater than the marginal cost of importing water $v'(x^m) \leq c_2'(0)$. Clearly, if these conditions do not hold, exporting and importing water becomes profitable and should be included in the optimizing framework of the problem. Whether to import or export water, in this model, is merely a residual decision taken at the end of each period.

The lines of this analysis will follow the traditional methods employed by dynamic programming formulations [12]. First, the existence and uniqueness of the solution for p = 1 and p = 2 and the concavity of the expected net discounted revenue functions will be established. This will pave the way for an inductive proof for the existence and uniqueness of the solution to the n-period problem. Next, it will be shown that for an infinite period problem the sequence of the expected net discounted revenue function converges under the assumed regularity conditions. This establishes the existence and uniqueness of the solution for the infinite period problem. Finally, maximization of the n-period expected discounted net revenue function will define the optimal size of the reservoir.

The One Period Problem

Let $f_1(x_1)$ be the expected revenue from the release of an optimal quantity of water including revenue from the hydroelectric operation of the reservoir. Let

$$f_0(x_0) = v(x_0) \tag{21}$$

where v(x) is the concave salvage value function indicating the worth of the terminal stock of water to the future generations.

Define

$$f_1(x_1,\bar{x}) = \max_{\substack{0 \le y_1 \le x_1 \\ x^m \le x_1 \le x^u}} G_1(y_1,\ x_1 - y_1;\bar{x}) \tag{22}$$

where

$$G_1(y_1,x_1 - y_1,\bar{x}) = [\pi(y_1,x_1-y_1) + \beta\int_0^{x^m - rx_1 + ry_1} v(x^m)\phi_e de$$

$$+ \beta\int_{x^m - rx_1 + ry_1}^{x^u - rx_1 + ry_1} v(rx_1 + e - ry_1)\phi_e de \quad + \beta\int_{x^u - rx_1 + ry_1}^{\infty} v(x^u)\phi\, de$$

$$- \beta\int_{x^u - rx_1 + ry_1}^{\infty} c_1(rx_1 + e - ry_1 - x^u)\phi_e de$$

$$- \beta\int_0^{x^m - rx_1 + ry_1} c_2(x^m - rx_1 - e + ry_1)\,\phi_e de] - c(\bar{x}). \tag{23}$$

We have the following proposition:

Proposition 1

If a) $\pi(y,x-y)$ is concave in the first argument and strictly concave in the second and $\pi_{12} \ge 0$;

b) $c_1(z)$, $c_2(z)$ are convex and $c_1(0) = c_2(0) = 0$;

c) $v'(x^u) \ge c_1'(0)$;

d) $v'(x^m) \le c_2'(0)$;

where the primes denote the derivatives of the functions with respect to the arguments then :

1) there exists a unique interior maximum $y_1^*(x_1)$ for $G_1(y_1,x_1 - y_1)$;

2) $0 \le \dfrac{dy_1^*}{dx_1} \le 1$

Moreover if

e) $x^u = g(\bar{x})$, $x^m = h(\bar{x})$, and $0 \le h' \le g' \le r$

then 3) $-1 < \dfrac{dy_1^*}{d\bar{x}} \le 0$

Proof of 1):

From (23) we have:

$$\frac{dG_1}{dy_1} = \pi_1 - \pi_2 - \beta r\int_0^{x^m - rx_1 + ry_1} c_2'(x^m - rx_1 + ry_1 - e)\phi_e de$$

$$- \beta r\int_{x^m - rx_1 + ry_1}^{x^u - rx_1 + ry_1} v'(rx_1 + e - ry_1)\phi_e de + \beta r\int_{x^u - rx_1 + ry_1}^{\infty} c_1'(rx_1 + e - ry_1 - x^u)\phi_e de \tag{24}$$

The primed functions denote their derivatives and all functions are parameterized by $\bar{x}$. The optimal release policy[3] $y_1^*(x_1)$ is defined by

$$\frac{dG_1}{dy_1} = 0 \qquad (25)$$

Second Order Conditions:

To show that $y_1^*(x_1)$ is a regular maximum, observe that[4]

$$\frac{d^2G_1}{dy_1^2} = \pi_{11} - 2\pi_{12} + \pi_{22} - \beta r^2[v'(x^u) + c_1'(0)]\,\phi_e(x^u - rx_1 + ry_1)$$

$$-\beta r^2[c_2'(0) - v'(x^m)]\phi_e(x^m - rx_1 + ry_1) - \beta r^2 \int_{x^u - rx_1 + ry_1}^{\infty} c_1''(rx_1 + e - ry_1 - x^u)\phi_e de$$

$$-\beta r^2 \int_0^{x^m - rx_1 + ry_1} c_2''(x^m - rx_1 + ry_1 - e)\phi_e de + \beta r^2 \int_{x^m - rx_1 + ry_1}^{x^u - rx_1 + ry_1} v''(x^m - rx_1 + ry_1 - e)\phi_e de. \qquad (26)$$

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[3](24) illustrates the effect of incorporating the stock of water in the profit function. Consider Case 1: $\pi = g(y) + h(x-y)$, g and h are concave; Case 2: $\pi = g(y)$. Then the expression of (24) in Case 1 is less than that of Case 2 by $h' > 0$. Since the g functions are identical in the two cases then

$$\left.\frac{dG_1}{dy_1}\right|_{\text{Case 1}} < \left.\frac{dG_1}{dy_1}\right|_{\text{Case 2}} \quad \text{everywhere.}$$

This implies that $y_1^*\big|_{\text{Case 1}} < y^*\big|_{\text{Case 2}}$. However, when the profit function is separable: $\pi = g(y) + h(x)$, the optimal release policy will not be effected when the g functions are identical.

[4]Consider the two cases in footnote 2. We have

$$\left.\frac{d^2G_1}{dy_1^2}\right|_{\text{Case 1}} < \left.\frac{d^2G_1}{dy_1^2}\right|_{\text{Case 2}}$$

However, when the profit function is separable: $g(y) + h(x)$, the second partial will be the same provided all the g functions are identical.

We have $c_1'', c_2'' \geq 0$ by convexity; $v'' \leq 0$, $\pi_{11} \leq 0$, $\pi_{22} < 0$ and $\pi_{12} \geq 0$ by assumption. Also if, as we have reasonably argued before,

$$v'(x^u) \geq -c_1'(0) \qquad (27)$$

and

$$v'(x^m) \leq c_2'(0) \qquad (28)$$

then

$$\frac{d^2G_1}{dy_1^2} < 0. \qquad (29)$$

Therefore, $y_1^*(x_1)$ is a regular maximum.

Assumption (27) implies that the marginal salvage value of the stock of water at x^u, at the terminal time, must not be less than the net marginal benefit from exporting water. This must be the case if the interest of the future generation (represented by the terminal stock) is to be safeguarded against profitable water export. Assumption (28) states that the marginal salvage value of the stock of water at x^m, at the terminal time, must not be greater than the net marginal cost of importing water. This relationship is reasonable if the planner is not pushed to import water beyond $x = x^m$. The important assumption in both (27) and (28) is that the manager does not import or export water unless he must. This is because (27) and (28) also imply that $c_1'(0) \leq v'(x_1) \leq c_2'(0)$, $\forall x_1$, $x^m \leq x_1 \leq x^u$, which means that it is not profitable to engage in importing or exporting water in the permissible region of x_1.

Notice that it does not matter whether $c_1'(0)$ is positive or negative provided that (27) holds.

To show that $y_1^*(x_1)$ is an interior maximum, it is sufficient to show:

$$\frac{dG_1}{dy_1}(0,x_1) > 0 \tag{30}$$

and

$$\frac{dG_1}{dy_1}(x_1,0) < 0\,. \tag{31}$$

Or from (24):

$$\frac{dG_1}{dy_1}(0,x_1) = \pi_1(0,x_1) - \pi_2(0,x_1) - \beta r \int_0^{x^m - rx_1} c_2'(x^m - rx_1 - e)\phi_e de$$

$$-\beta r \int_{x^m - rx_1}^{x^u - rx_1} v'(rx_1 + e_1)\phi_e de + \beta r \int_{x^u - rx_1}^{\infty} c_1'(rx_1 + e - x^u)\phi_e de > 0 \tag{32}$$

$$\forall \quad x_1,\ x^m \leq x_1 \leq x^u$$

This is trivially satisfied if $\pi(y, x-y)$ is a neoclassical function, which implies that $\lim_{y_1 \to 0} \pi_1(y_1, x_1 - y_1) \to \infty$ and $\lim_{y_1 \to x_1} \pi_2(y_1, x_1 - y_1) \to \infty$. Generally, however, the assumption that (30) and (31) are satisfied is reasonable in terms of an intuitive economic argument. This is demonstrated by rearranging the terms of (32) as follows:

$$\pi_1(0,x_1) - \beta r \int_0^{x^m - rx_1} c_2'(x^m - rx_1 - e)\phi_e de \geq \pi_2(0,x_1)$$

$$- \beta r \int_{x^u - rx_1}^{\infty} c_1'(rx_1 + e \quad x^u)\phi_e de + \beta r \int_{x^m - rx_1}^{x^u - rx_1} v'(vx_1 + e)\phi_e de. \tag{33}$$

The economic interpretation of (33) is that the net marginal profitability of releasing water exceeds that of storing it at any stock of water between x^m and x^u, providing water release is zero. Even at $x_1 = x^m$, this must be true if large scale damage to the downstream users is to be avoided. To see that this interpretation is correct, we have to remember that:

$$c_2'(z) \begin{cases} > 0 & \forall\ z \geq 0 \\ = 0 & \forall\ z < 0 \end{cases} \tag{34}$$

also

$$c_1'(z) \begin{cases} > 0 & \forall\ z \geq 0 \\ = 0 & \forall\ z < 0 \end{cases} \tag{35}$$

and

$$v'(z) \begin{cases} > 0 & \forall\ x^m \leq z \leq x^u \\ = 0 & \text{otherwise.} \end{cases} \tag{36}$$

This means that (33) can be rewritten and the limits of integration changed as follows:

$$\pi_1(0,x_1) - \beta r \underset{e}{E}\{c_2'(z)\} \geq \pi_2(0,x_1) - \beta r \underset{e}{E}\{c_1'(z)\} + \beta r \underset{e}{E}\{v'(z)\}. \tag{37}$$

Equation (37) is essentially what the previous economic interpretation asserts. One might notice the peculiar range of the salvage value function, but this range facilitates a smooth induction argument. It can be clarified by reinterpreting the salvage value function as follows:

$$V(z) = \begin{cases} v(x^m) - c_2(z) & \forall\ z,\ z < x^m \\ v(z) & \forall\ z,\ x^m \leq z \leq x^u \\ v(x^u) - c_1(z) & \forall\ z,\ z > x^u \end{cases} \tag{38}$$

where $v(z)$ is defined as before. Thus, (37) can be rewritten as follows:

$$\pi_1(0,x_1) \geq \pi_2(0,x_1) + \underset{e}{E}\{V(z)\}. \tag{39}$$

which is a formalization of the preceding argument.

On the other hand, (31) implies that

$$\pi_2(x_1,0) - \beta r\int_{x^u}^{\infty} c_1'(e - x^u) + \beta r\int_{x^m}^{x^u} v'(e)\phi_e de > \pi_1(x_1,0) - \beta r\int_0^{x^m} c_2'(x^m - e)\phi_e de. \tag{40}$$

This is true if π is a neoclassical objective function. Also, using the previous argument, this is equivalent to either:

$$\pi_2(x_1,0) - \beta r \underset{e}{E}\{c_1'(z)\} + \beta r \underset{e}{E}\{v'(z)\} > \pi_1(x_1,0) - \beta r \underset{e}{E}\{c_2'(z)\}$$

or

$$\pi_2(x_1,0) + \beta r \underset{e}{E}\{V(z)\} > \pi_1(x_1,0) \quad . \tag{41}$$

(41) states that the marginal profitability of storing the last unit of water exceeds that of releasing it, assuming all x_1 is released.

Proof of 2):

In this section, the effect on the optimal release policy of a parametric change in the starting stock of water x_1 or in the physical capacity of the reservoir $\bar{x}$ will be investigated. Differentiating the first order conditions (eq. 24) with respect to x_1 gives:

$$(\pi_{11} - \pi_{21})\frac{dy_1^*}{dx_1} = \beta r^2\left(-1 + \frac{dy_1^*}{dx_1}\right)\Big[\int_0^{x^m - rx_1 + ry_1} c_2''(x^m - rx_1 + ry_1 - e)\phi_e de$$
$$+ c_2'(0)\,\phi(x^m - rx_1 + ry_1) - \int_{x^m - rx_1 + ry_1}^{x^u - rx_1 + ry_1} v''(rx_1 + e - ry_1)\phi_e de$$
$$+ v'(x^u)\phi(x^u - rx_1 + ry_1) - v'(x^m)\phi(x^m - rx_1 + ry_1)$$
$$+ \int_{x^u - rx_1 + ry_1}^{\infty} c_1''(rx_1 + e - ry_1 - x^u)\phi_e de + c_1'(0)\phi(x^u - rx_1 + ry_1)$$
$$+ \frac{\pi_{12} - \pi_{22}}{\beta r^2}\Big] \quad . \tag{42}$$

or, equivalently[5]

$$\frac{dy_1^*}{dx_1} = \frac{\frac{d^2G_1}{dy_1^2} - (\pi_{11} - \pi_{21})}{\frac{d^2G_1}{dy_1^2}} \quad . \qquad (43)$$

From (31) we have $\frac{d^2G_1}{dy_1^2} < \pi_{11} - \pi_{21} \leq 0.$ (44)

Therefore, (43) implies that $0 < \frac{dy_1^*}{dx_1} \leq 1.$ (45)

Proof of 3):

Let $x^u = g(\bar{x})$ and $x^m = h(\bar{x})$, such that $0 \leq h' \leq g' \leq r$ (46)

Differentiating (23) with respect to $\bar{x}$ and using (26), we have[6]

$$\frac{dy_1^*}{d\bar{x}} \cdot \frac{d^2G_1}{dy_1^2} = \beta rh'\int c_2''\phi_e de + \beta rh'[c_2'(0) - v(h(\bar{x}))]\phi + \beta rg'\int c_1''\phi_e de + \beta rg'[c_1'(0) + v'(g(\bar{x}))]\phi \qquad (47)$$

---

[5]Consider the two cases of footnote 3:

$$\left.\frac{dy_1^*}{dx_1}\right|_{\text{Case 2}} = 1 - \frac{g''}{\left.\frac{d^2G_1}{dy_1^2}\right|_{\text{Case 2}}}$$

$$\left.\frac{dy_1^*}{dx_1}\right|_{\text{Case 1}} = 1 - \frac{g''(1 - \frac{dy}{d(x-y)})}{\left.\frac{d^2G_1}{dy_1^2}\right|_{\text{Case 1}}} = \left.\frac{dy_1^*}{dx_1}\right|_{\text{Case 2}} + g''\frac{\frac{dy}{d(x-y)}}{\left.\frac{d^2G_1}{dy_1^2}\right|_{\text{Case 1}}}$$

From footnote 3: $\left.\frac{d^2G_1}{dy_1^2}\right|_{\text{Case 1}} < \left.\frac{d^2G_1}{dy_1^2}\right|_{\text{Case 2}}$ therefore $\left.\frac{dy_1^*}{dx_1}\right|_{\text{Case 2}} > \left.\frac{dy_1^*}{dx_1}\right|_{\text{Case 1}}$,

only if $\frac{dy}{d(x-y)}$ is unambiguously negative otherwise it is ambiguous.

[6]To keep the expressions simple, we shall drop the arguments of the functions and the integral limits in such expressions whenever it is unambiguous to do so.

From (27), (28), and the convexity of c_1 and c_2, the right-hand side of (47) is positive, which implies that $\frac{dy_1^*}{d\bar{x}}$ is negative. Moreover, for each term in the right-hand side of (47), there is a corresponding term in the expression of $\frac{d^2G_1}{dy_1^2}$ with opposite sign and weight equal to either $\frac{r}{h'}$ or $\frac{r}{g'}$, which, by assumption, are greater than 1. Thus, comparing the expression on the right-hand side of (47) with the expression of $\frac{d^2G_1}{dy_1^2}$ we conclude that

$$-1 < \frac{dy_1^*}{d\bar{x}} \leq 0. \qquad (48)$$

(End of Proof of Proposition 1)

This result has been obtained by placing some restrictions on the derivatives of h and g; these are $0 \leq h' \leq r$, and $0 \leq g' \leq r$. These assumptions will be justified on the following basis:

a) the non-negativity restriction on $g'(\bar{x})$ is reasonable. This is because increasing the physical capacity of the reservoir, for the same inflow and hydrology of the river basin, offers the opportunity to increase $x^u = g(\bar{x})$ and hence, the hydroelectric power potential of the reservoir. This increase in x^u must not be greater than 1 in order to avoid decreasing the designed free board capacity $(\bar{x} - x^u)$ of the reservoir. To illustrate further, consider the case where $g' = \alpha$ (α is a constant), and the inflow in the period before last brings the total storage to $\bar{x}$. The storage after evaporation in this case is $r\bar{x}$. Hence, if $g' = \alpha \geq r$ or $\alpha\bar{x} \geq r\bar{x}$ which means that $x^u \geq r\bar{x}$, then there is no need to export water under all conditions where $x_1 \leq \bar{x}$. That is, the natural process of evaporation under these conditions provides an automatic excess water disposal. Such a situation is imaginary and will not be considered any further. Thus, it seems reasonable to accept the assumption that g' is bounded in the range $0 \leq g' < r$.

b) The non-negativity of $h'(x)$ is more straightforward. This is because the minimum pool requirement $x^m = h(\bar{x})$ is dictated by the minimum hydrostatic head required for the operation of a particular turbine on one hand and the salinity control on the other. Neither of these requirements is affected negatively by the increase in the physical capacity of the reservoir. x^m can be expected to stay constant or increase slightly to account for the increase in salinity brought about by a larger stock of water. Moreover, increasing $\bar{x}$ is expected to weaken the overall constraints on the system. Hence, the control volume $x^c = x^u - x^m$ is expected to increase. Therefore, $g' \geq h'$. However, by the previous discussion in (a), $g' \leq r$, which implies that $0 \leq h' \leq g' \leq r$. In the previous sections, it has been argued that the assumptions responsible for our seemingly counterintuitive

results, $-1 < \frac{dy_1^*}{d\bar{x}} \leq 0$, are reasonable. The meaning of the result itself follows. Given the same inflow and river basin hydrology and starting with the same stock of water x, the increase in the physical capacity of the reservoir $\bar{x}$ has resulted in:

1) weakening the upper constraints x^u,

2) strengthening the lower constraints x^m.

This situation leads to a reduction in risk of having excess water and an increase in risk of having to import water, which can only lead to a reduction in the optimal release policy $y^*(x_1)$.

Lemma 1

If $y_1^*(x_1)$ exists and is unique and $\pi_{11} = \pi_{12}$ or both identically vanish, then the optimal release rule is linear of the form $y_1^*(x_1) = x_1 - a_1$, where a_1 is a constant dictated by the hydrology of the stream, the size of the reservoir, and the specific form of the profit function.

Proof:

From (43), if $\pi_{11} = \pi_{12} \equiv 0$, then $\frac{dy_1^*}{dx_1} = 1$ and

$$y_1^*(x_1) = x_1 - a \qquad (49)$$

a is a constant dictated by the hydrology of the river basin, the size of the reservoir and the specific form of π.

Thus, the celebrated linear decision rule, used so often in chance constraint models, emerges as the optimizing decision rule when a specific form of the objective function π is used in this model.

Considering the interpretation given to π earlier:

1. In the first case, where π is the expected net social revenue from operating the reservoir, π_{12} may be zero if either the second mixed partials of expected social revenue and expected social cost functions are identically equal, as evaluated by the central planners, or that both second mixed partials vanishes. The latter case may be argued on the bases that there is no reason for marginal expected cost to be affected by a change in the water head left in the reservoir after the release and used for power generation. Moreover, $\pi_{11} \equiv 0$ if either the second partials with respect to the releases of the expected social revenue and expected social cost functions are identically equal or if both partials vanishes. The latter is consistent with a situation where both functions are characterized by fixed proportion and there is a perfectly competitive market for agricultural products.
2. In the second case, where π reflect the aggregate expected profit of downstream users who own the water in the reservoir, π_{12} may be zero if the marginal profitability in agriculture is unaffected by a change in the stock of water which remains after the release. Moreover $\pi_{11} \equiv 0$, if the production function of the downstream farmers is characterized by fixed proportions and that farmers sell their product in perfectly competitive market.
3. In the third case, where π is the total surplus, since we are talking about the areas under compensated demand curves conditions such as

$\pi_{12} \equiv 0$ if the utility is separable and $\pi_{11} \equiv 0$ if the marginal utility from the payoff which arises from release is linear. This case arises if risk neutrality with respect to uncertainty in agricultural prevails.

Proposition 2

a) If assumptions (a) - (d) in proposition 1 hold, then the expected return $f_1(x_1;\bar{x})$ has the following characteristics:

1) $\frac{df_1}{dx_1} = \pi_1\{y_1^*, x_1 - y_1^*\}$;

2) $f_1(x_1;\bar{x})$ is strictly concave in x_1.

b) If assumption (e) in proposition 1 holds, and

c) if $x^u = g(\bar{x})$, g is concave, and

d) if $x^m = h(\bar{x})$ = constant, then

3) $f_1(x_1,\bar{x})$ is strictly concave in $\bar{x}$.

Proof of 1):

The existence and uniqueness of the solution to the two period problem depends on the nature of the expected net return function in the last period f_1. Therefore, in the following, the concavity of f_1 with respect to x_1 and, under some assumptions, with respect to $\bar{x}$, shall be shown. From (25) we have

$$f_1(x_1;\bar{x}) = \pi(y_1^*, x_1 - y_1^*) + \beta \int_0^{x^m - rx_1 + ry_1^*} v(x^m)\phi_e de$$

$$+ \beta \int_{x^m - rx_1 + ry_1^*}^{x^u - rx_1 + ry_1^*} v(rx_1 + e - ry_1^*)\phi_e de$$

$$+ \beta \int_{x^u - rx_1 + ry_1^*}^{\infty} v(x^u)\phi_e de - \beta \int_{x^u - rx_1 + ry_1^*}^{\infty} c_1(rx_1 + e - ry_1^* - x^u)\phi_e de$$

$$- \beta \int_0^{x^m - rx_1 + ry_1^*} c_2(x^m - rx_1 - e + ry_1^*)\phi_e de - c(\bar{x}) \quad . \qquad (50)$$

Therefore,

$$\frac{df_1}{dx_1} = \pi_1 \frac{dy_1^*}{dx_1} + (-1 + \frac{dy_1^*}{dx_1}) [-\beta r \int_{x^m - rx_1 + ry_1^*}^{x^u - rx_1 + ry_1^*} v'(rx_1 + e - ry_1^*)\phi_e de$$

$$+ \beta r \int_{x^u - rx_1 + ry_1^*}^{\infty} c_1'(rx_1 + e - ry_1^* - x^u)\phi_e de$$

$$- \beta r \int_0^{x^m - rx_1 + ry_1^*} c_2'(x^m - rx_1 - e + ry_1^*)\phi_e de - \pi_2] \quad . \qquad (51)$$

However, from (24), the bracketed term in (51) equals $(-\pi_1)$, then

$$\frac{df_1}{dx_1} = \pi_1(y^*_1, x_1 - y^*_1) \quad . \qquad (52)$$

Proof of 2).

Differentiating (52) with respect to x_1, we have

$$\frac{d^2 f_1}{dx_1^2} = (\pi_{11} - \pi_{12}) \frac{dy_1^*}{dx_1} + \pi_{12}. \qquad (53)$$

Also, from (43)

$$\frac{dy_1^*}{dx_1} = \frac{\frac{d^2 G_1}{dy_1^2} - (\pi_{11} - \pi_{21})}{\frac{d^2 G_1}{dy_1^2}} \quad .$$

Substituting for $\frac{dy_1^*}{dx_1}$ in eq. (53), we have

$$\frac{d^2 G_1}{dy_1^2} \cdot \frac{d^2 f_1}{dx_1^2} = [\pi_{11} \frac{d^2 G_1}{dy_1^2} - (\pi_{11} - \pi_{21})^2]. \qquad (54)$$

However, (23) shows that

$$\frac{d^2 G_1}{dy_1^2} < \pi_{11} - \pi_{12} < 0 \ , \qquad (55)$$

or equivalently,

$$\left| \frac{d^2 G_1}{dy_1^2} \right| > \pi_{11} - \pi_{12} \quad .$$

Also since $\pi_{12} \geq 0$, then

$$\pi_{11} > \pi_{11} - \pi_{12} \quad .$$

Therefore

$$\pi_{11} \frac{d^2 G_1}{dy_1^2} > (\pi_{11} - \pi_{21})^2 \quad .$$

And, hence,

$$\frac{d^2 f_1}{dx_1^2} < 0 \qquad (56)$$

That is, f_1 is strictly concave in x_1.

Proof of 3):

Also from (24), substituting for $x^u = g(\bar{x})$ and $x^m = h(\bar{x})$ and differentiating with respect to $\bar{x}$, we have

$$\frac{df_1}{d\bar{x}} = -c'(\bar{x}) - \beta h'(\bar{x}) \int_0^{h(\bar{x})-rx_1+ry_1^*} c_2'(h(\bar{x}) - rx_1 - e + ry_1^*)\phi_e de$$

$$+ \beta g'(\bar{x}) \int_{g(\bar{x})-rx_1+ry_1^*}^{\infty} c_1'(rx_1 + e - ry_1^* - g(x))\phi_e de$$

$$+ \beta h'(\bar{x}) \int_0^{h(\bar{x}) - rx_1 + ry_1^*} v'(h(\bar{x})\phi_e de + \beta g'(\bar{x}) \int_{g(\bar{x}) - rx_1 + ry_1^*}^{\infty} v'(g(\bar{x})\phi_e de. \qquad (57)$$

In particular if $h' = 0$, then

$$\frac{df_1}{d\bar{x}} = -c'(\bar{x}) + \beta g'(\bar{x}) \int_{g(\bar{x}) - rx_1 + ry*_1}^{\infty} c'\phi_e de + \beta g'(\bar{x}) \int_{g(\bar{x}) - rx_1 + ry_1^*}^{\infty} v'(g(\bar{x})\phi_e de. \quad (58)$$

In general, however,

$$\frac{d^2 f_1}{d\bar{x}^2} = c'' - \beta h'' \int c_2'\phi_e de - \beta h'(h' + r \frac{dy_1^*}{d\bar{x}}) \int c_2''\phi_e de$$

$$+ \beta h'(h' + r \frac{dy_1^*}{d\bar{x}})(v'(h) - c_2'(0))\phi(h - rx_1 + ry_1^*)$$

$$+ \beta g'' \int c_1'\phi_e de$$

$$- \beta g'(g' + r \frac{dy_1^*}{d\bar{x}}) \int c_1''\phi_e de - \beta g'(g' + r \frac{dy_1^*}{d\bar{x}})(v'(g)$$

$$+ c_1'(0)\phi(g - rx_1 + ry_1^*)$$

$$+ \beta h'' \int v'\phi_e de + [h']^2 \int v''(\phi_e de$$

$$+ \beta g'' \int v'\phi_e de + [g']^2 \int v''\phi_e de. \qquad (59)$$

It can be shown that $g' + r \frac{dy_1^*}{d\bar{x}} > 0$ while $h' + r \frac{dy_1^*}{dx}$ is ambiguous, which makes the sign of $\frac{d^2 f_1}{d\bar{x}^2}$ indeterminate. However, it is obvious that $\frac{d^2 f_1}{d\bar{x}^2} < 0$, under assumption that x^m is a constant. Thus, under plausible assumptions, f_1 is shown to be strictly concave in $\bar{x}$ (as well as x_1).[7]

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[7] $\frac{d^2 f_1}{dx^2}$ is also negative if $h' = g' = $ constant.

Proposition 3

Under the assumptions of proposition 2, there exists a unique optimal size $\bar{x}_1^*$ for the reservoir which maximizes $f_1(x_1,\bar{x})$.

Proof:

If it is assumed, as in the first model, that $\exists\, y_0$ such that $\pi_1(y_0) = 0$ and that $\exists\, x_0 \ni v'(x_0) = 0$, then $\bar{x}$ is bounded by 0 and $y_0 + x_0$. This implies that f_1 is defined on a compact set $0 \leq \bar{x} \leq y_0 + x_0$. If the assumptions of proposition 2 hold, then f_1 is a strictly concave function in $\bar{x}$ defined on a compact set. Therefore, it must have a unique maximum $\bar{x}_1^*$.

This ends the analysis of the one-period problem. It appears that the inclusion of the water stock in the profit function, although it affected the optimal policy and size of the reservoir, did not make substantial difference to the technical conditions needed to get the usual inventory dynamic programming results. Inspecting (33) and (40), the conditions which insure interior maximum, enhance this observation. Certainly, for a neoclassical profit function, the finite terms $\pi_2(0,x_1)$ and $\pi_2(x_1,0)$ do not make either of the inequalities (33) and (40) more stringent or relaxed. For any other concave function, the inclusion of the water stock makes (33) more stringent while relaxing (40). Thus, the concavity of π with respect to the water stock and that $\pi_{12} \geq 0$ are all the additional requirements needed to get the usual inventory dynamic programming results.

Summary of the One-Period Problem

It has been shown that a unique solution $y_1^*(x_1)$ for the functional equation (1) exists and is unique if

1) $v'(x^u) \geq -c_1'(0)$

2) $v'(x^m) \leq c_2'(0)$.

Moreover, it has been shown that

a) $0 < \frac{dy_1^*}{dx_1} \leq 1$

b) f_1 is strictly concave in x_1.

Furthermore, it has been shown that if

3) $x^u = g(\bar{x})$,

4) $x^m = h(\bar{x})$ $\quad 0 \leq h' \leq g' \leq r$

then

c) $-1 < \frac{dy_1^*}{d\bar{x}} \leq 0$.

In particular, if

5) $h' = 0$ and g is concave

then

d) f_1 is strictly concave in $\bar{x}$; and

e) $f_1(\bar{x})$ has a unique maximum, $\bar{x}_1^*$, provided $\bar{x}$ is bounded above.

The Two Period Horizon

In this case, the continuity equation is[8]

$$x_1 = r(x_2 - y_2) + e + i_2 - m_2 \quad . \qquad (60)$$

Define

$$f_2(x_2) = \max_{0 \le y_2 \le x_2} G_2(y_2, x_2 - y_2)$$

where

$$G_2(y_2, x_2-y_2) = \max_{0 \le y_2 \le x_2} [\pi(y_2, x_2 - y_2) + \beta \int_0^{x^m - rx_2 + ry_2} f_1(x^m)\phi_e de \qquad (61)$$

$$+ \beta \int_{x^u - rx_2 + ry_2}^{\infty} f_1(x^u)\phi_e de$$

$$+ \beta \int_{x^m - rx_2 + ry_2}^{x^u - rx_2 + ry_2} f_1(rx_2 + e - ry_2)\phi_e de - \beta \int_{x^u - rx_2 + ry_2}^{\infty} c_1(rx_2 + e - ry_2 - x^u)\phi_e de$$

$$- \beta \int_0^{x^m - rx_2 + ry_2} c_2(x^m - rx_2 - e + ry_2)\phi_e de - c(\bar{x}),$$

[8] Assume that at the start of every period the manager knows the actual inflow. However, he only knows the probability distribution of the inflow for future periods. Then a redefinition of terms and a relabeling of periods leaves the analysis intact. For example, in the two period case, x_2 is the starting stock of water, after observing e_2 and correcting for deficiencies or surplus in the previous period. Therefore, $x_1 = r(x_2 - y_2) + e_1 + i_2 - m_2$. Relabelling e_i by e_{i+1} gives $x_1 = r(x_2 - y_2) + e_2 + i_2 - m_2$, which is the original continuity equation. However, it must be noted that since e_T is now known with certainty, the decision in the last period is deterministic, not stochastic.

Proposition 4

If assumptions (a - d) in proposition 1 are accepted, the following results hold:

1) There exists a unique interior maximum $y_2^*(x_2)$,

2) $y_2^*(x) \le y_1^*(x)$,

3) $0 < \frac{dy_2^*}{dx_2} \le 1$,

4) Further, if $\pi_{11} = \pi_{12} \equiv 0$ then the optimal release rule takes the form

$$y_2^*(x_2) = x_2 - a_2, \quad \forall x_2, \; x^m \le x_2 \le x^u .$$

a_2 is a constant dictated by the hydrology of the stream, the size of the reservoir and the specific form of the profit function π.

Moreover, if assumption (3) in proposition 1 holds then,

5) $-1 < \frac{dy_2^*}{d\bar{x}} \le 0$.

Proof of 1):

When (61) is compared with (22), the two expressions for the optimal return function in the one period and the two period case are identical except that f_1 replaces v_1 wherever v_1 occurs in expression (22). Moreover, since both v_1 and f_1 are concave, it can be verified that under identical assumptions, all the qualitative results of the one period problem also hold in the two period case. In particular:

$$\frac{dG_2}{dy_2} = \pi_1(y_2, x_2 - y_2) - \pi_2(y_2, x_1 - y_2) \tag{62}$$

$$- \beta r \int_0^{x^m - rx_2 + ry_2} c_2'(x^m - rx_2 + ry_2 - e)\, \phi_e de$$

$$+ \beta r \int_{x^u - rx_2 + ry_2}^{\infty} c_1'(rx_2 + e - ry_2 - x^u)\phi_e de$$

$$- \beta r \int_{x^m - rx_2 + ry_2}^{x^u - rx_2 + ry_2} f_1'(rx_2 + e - ry_2)\phi_e de \quad .$$

The primes denote the derivatives of the functions with respect to their arguments. Thus, $y_2^*(x_2)$ is defined by $\frac{dG_2}{dy_2} = 0$. Similarly,

$$\frac{d^2G_2}{dy_2^2} = \pi_{11} - 2\pi_{12} + \pi_{23} - \beta r^2[f_1'(x^u + c_1'(0)]\phi_e(x^u - rx_1 + ry_1)$$

$$- \beta r^2[c_2'(0) - f_1'(x^m)]\phi_e(x^m - rx_1 + ry_1)$$

$$- \beta r^2 \int_{x^u - rx_1 + ry_1}^{\infty} c_1''(rx_1 + e - ry_1 - x^u)\phi_e de$$

$$- \beta r^2 \int_0^{x^m - rx_1 + ry_1} c_2''(x^m - rx_1 + ry_1 - e)\, \phi_e de$$

$$+ \beta r^2 \int_{x^m - rx_1 + ry_1}^{x^u - rx_1 + ry_1} f_1''(x^m - rx_1 + ry_1 - e)\, \phi_e de \quad . \tag{63}$$

Moreover, at x^u, the benefit from releasing the last unit of $y_1^*(x_1)$ must exceed the marginal benefit from exporting water. If this is not the case, then it becomes profitable to export water rather than release it to downstream users. Hence,

$$f_1'(x^u) = \pi_1(y_1^*(x^u), x^u - y_1^*(x^u)) \geq -c_1'(0). \tag{64}$$

Similarly at x^m, once $y_1^*(x^m)$ is released, the marginal benefit from releasing an extra unit of water must be less than the marginal cost of violating the lower constraint (the price of water import). If this is not the case, it becomes profitable to import water and release it to downstream users. Therefore,

$$f_1'(x^m) = \pi_1\{y_1^*(x^m),\ x_1 - y_1^*(x^m)\} \leq c_1'(0). \tag{65}$$

These conditions motivate the same economic behavior as that in (32) and (33) ; it is not profitable to engage in importing or exporting water in the permissable range of x_1, $x^m \leq x_1 \leq x^u$. However, from (63) , these conditions imply

$$\frac{d^2G_2}{dy_2^2} < 0. \tag{66}$$

Thus, $y_2^*(x_1)$ is a regular maximum. Moreover, it can be shown that the relations $\frac{dG_2}{dy_2}(0,x_1) > 0$ and $\frac{dG_2}{dy_2}(x_2,0) < 0$ hold, and are based on the same economic arguments presented in the one period case. Hence, $y_2^*(x_2)$ is an interior maximum.

Proof of 2):

To prove that $y_2^*(x) \leq y_1^*(x)$, notice that if $v(x) = 0$, then from (24) and (62), $\frac{dG_2}{dy} < \frac{dG_1}{dy}$ everywhere and hence $y_2^*(x) < y_1^*(x)$. See Figure 4.

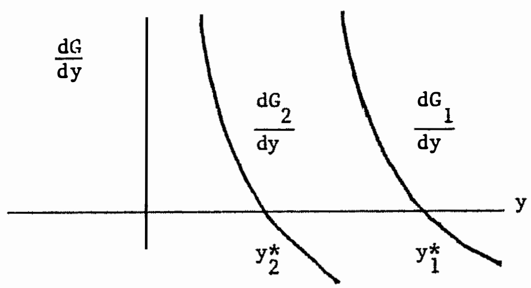

Figure 4

In general, however, if $\frac{dG_1}{dy}$ is evaluated at y_2^*, it can be proven that $\left.\frac{dG_1}{dy}\right|_{y_2 = y_2^*} > 0$. Since $\frac{d^2G_i}{dy^2} < 0$, $i = 1,2$ everywhere and $\frac{dG_i}{dy} = 0$ has only one solution, the following inequality must hold:

$$y_1^*(x) > y_2^*(x). \qquad (67)$$

See Figure 4. This result has already been implied by the previous analysis, where it has been shown that

$$0 < \frac{dy_1^*}{dx_1} \leq 1 \quad \forall x,\ x^m \leq x_1 \leq x^u.$$

Hence, it is economical to release some of the unit increase in initial storage rather than retaining the entire storage increase. Therefore, the marginal expected return from releasing some of the unit increase

in initial storage and storing the rest must exceed the marginal expected increase in salvage value due to the storage of the whole unit increase,

$$f_1'(x_1) > v'(x_1), \quad \forall x_1, \quad x^m \leq x_1 \leq x^u \quad .$$

Hence,

$$\int_{x^m-rx+ry}^{x^u-rx+ry} f_1'(rx + e - ry)\phi_e de > \int_{x^m-rx+ry}^{x^u-rx+ry} v'(rx + e - ry)\phi_e de. \qquad (68)$$

(68) holds because the arguments of both f_1' and v' lie in the interval x^m to x^u for the specific range of the random variable e defined by the limits of integration.

Comparing the first order conditions in the one period and the two period cases, the previous argument implies that $y_2^*(x) \leq y_1^*(x)$.

Proof of 3):

From (62), it is found that

$$\frac{d^2G_2}{dy_2^2}\frac{dy_2^*}{dx_2} = \frac{d^2G_2}{dy_2^2} - (\pi_{11} - \pi_{21}) \quad . \qquad (69)$$

It follows that

$$0 < \frac{dy_2^*}{dx_1} \leq 1. \qquad (70)$$

Proof of 4):

Notice that if $\pi_{11} = \pi_{21} \equiv 0$, then

$$\frac{dy_2^*}{dx_2} = 1 \qquad (71)$$

and

$$y_2^* = x_2 - a_2, \quad \forall x_2, \quad x^m \le x_2 \le x^u \quad . \qquad (72)$$

a_2 is dictated by the hydrology of the river basin, the size of the reservoir and the specific form of the profit function.

Proof of 5):

If $x^u = g(\bar{x})$, $x^m = h(\bar{x})$, $0 \le h' \le g' \le r$, then

$$-1 < \frac{dy_2^*}{d\bar{x}} \le 0. \qquad (73)$$

Proposition 5

Under the assumptions (a - e) of proposition 1:

1) $\dfrac{df_2}{dx_2} = \pi_2\{y_2^*(x_2), x_2 - y_2(x_2)\}$

2) f_2 is strictly concave in x_2.

3) If, in addition $x^u = g(\bar{x})$, $x^m = h(\bar{x})$, g is concave and h is a constant, then f_2 is strictly concave in $\bar{x}$.

4) $f_2(x) > f_1(x)$.

Proof of 1):

Differentiating (61), we have

$$\frac{df_2}{dx_2} = \pi_1 \frac{dy_2^*}{dx_2} + \left(1 - \frac{dy_2^*}{dx_2}\right)\pi_1 = \pi_1(y_2^*, x_2 - y_2^*) \quad . \qquad (74)$$

Since $\dfrac{df_1}{dx_1} = \pi_1(y_1^*(x_1), x_1 - y_1^*(x_1))$ and $y_1^*(x) > y_2^*(x)$, then by the concavity of π, $\dfrac{df_{2(x)}}{dx} > \dfrac{df_{1(x)}}{dx}$. (75)

Proof of 2):

From (74) and (69),

$$\frac{d^2f_2}{dx_2} = \frac{1}{\dfrac{d^2G_2}{dy_2^*}}\left[\pi_{11}\frac{d^2G_2}{dy_2^2} - (\pi_{11} - \pi_{21})^2\right] \qquad (76)$$

This can be shown to be negative, in a manner similar to that employed in the one period case. Thus, f_2 is concave in x_2.

Proof of 3):

Also, if $x^u = g(\bar{x})$, and $x^m = h(\bar{x})$, then

$$\begin{aligned}\frac{df_2}{d\bar{x}} = {} & \beta h'(\bar{x})\int_0^{h(\bar{x})-rx_2+ry_2^*} f_1'(h(\bar{x}))\phi_e de + \beta g'(\bar{x})\int_{g(\bar{x})-rx_2+ry_2^*}^{\infty} f_1'(g(\bar{x}))\phi_e de \\ & + \beta g'(\bar{x})\int_{g(\bar{x})-rx_2+ry_2^*}^{\infty} c_1'(rx_2 + e - ry_2^* - g(\bar{x}))\phi_e de \\ & - \beta h'(\bar{x})\int_0^{h(\bar{x})-rx_2+ry_2^*} c_2'(h(\bar{x}) - rx_2 - e + ry_2^*)\phi_e de - c'(\bar{x}).\end{aligned} \qquad (77)$$

Therefore, it follows that

$$\begin{aligned}\frac{d^2f_2}{d\bar{x}^2} = {} & \beta h'\left(h' + r\frac{dy_2^*}{d\bar{x}}\right)[f_1'(h(\bar{x})) - c_2'(0)]\phi(h(\bar{x}) - rx_1 + ry_1^*) \\ & - \beta g'\left(g' + r\frac{dy_2^*}{d\bar{x}}\right)[f_1'(g(\bar{x})) + c_1'(0)]\phi(g(\bar{x}) - rx_1 + ry_1^*) \\ & - \beta h''\int c_2'\phi_e de - \beta h'\left(h' + r\frac{dy_1^*}{d\bar{x}}\right)\int c_2''\phi_e de\end{aligned}$$

$$+ \beta g'' \int c_1' \phi_e de - \beta g'(g' + r \frac{dy_1^*}{d\bar{x}}) \int c_1'' \phi_e de$$

$$+ \beta h'' \int f_1' \phi_e de + \beta [h']^2 \int f_1'' \phi_e de$$

$$+ \beta g'' \int f_1' \phi_e de + \beta [g']^2 \int f_1'' \phi_e de. \qquad (78)$$

In particular, if $h' = 0$ and g is concave, and since $f_1'(g(\bar{x})) + c_1'(0) \geq 0$ and $f_1'(h(\bar{x})) - c_2'(0) \leq 0$, we conclude that

$$\frac{d^2 f_2}{d\bar{x}^2} < 0 \qquad (79)$$

i.e. f_2 is strictly concave in $\bar{x}$.

Proof of 4):

f_1 and f_2 can be rewritten as follows:

$$f_1 = \operatorname*{Max}_{y_1} [\pi + E\{V\}]$$

$$f_2 = \operatorname*{Max}_{y_2} [\pi + E\{f_1\}]$$

where V is defined as in (38). An equivalent expression for f_2 is

$$f_2 = \operatorname*{Max}_{y_2} [\pi + \operatorname*{Max}_{y_1} \{\pi\} + \operatorname*{Max}_{y_1} E\{V\}]$$

$$= \operatorname*{Max}_{y_2} [f_1 + \operatorname*{Max}_{y_1} \{\pi\}]$$

hence

$$f_2 > f_1 \quad . \qquad (80)$$

Proposition 6

Under the assumption of proposition 3,

1) $\exists$ a unique optimal size $\bar{x}_2^*$ for the reservoir which maximizes the total expected return $f_2(\bar{x})$;

2) $\bar{x}_2^* > \bar{x}_1^*$.

Proof of 1):

In this case, $\bar{x}_2$ is bounded below by 0 and above by $2y_0 + x_0$. Thus, $f_2(\bar{x})$ is defined on a compact set. However, $f_2(\bar{x})$, under the assumptions of proposition 4, is strictly concave. Thus, $f_2(\bar{x})$ must posess a unique maximum $\bar{x}_2^*$ on its convex and compact domain.

Proof of 2):

x_2^* and $\bar{x}_1^*$ are defined by $\frac{df_2}{d\bar{x}} = 0$ and $\frac{df_1}{d\bar{x}} = 0$ respectively.

or, equivalently by

$$-c'(\bar{x}) + \beta g'(\bar{x}) \int_{g(\bar{x})-rx_2+ry_2^*}^{\infty} c_1'(rx_2 + e - ry_2^* - g(\bar{x})\phi_e de$$

$$+ \beta g'(\bar{x}) \int_{g(\bar{x})-rx_2+ry_2^*}^{\infty} f_1'(g(\bar{x}))\phi_e de = 0 \quad , \qquad (81)$$

and

$$-c'(\bar{x}) + \beta g'(\bar{x}) \int_{g(\bar{x})-rx_1+ry_1^*}^{\infty} c_1'(rx_1 + e - ry_1^* - g(\bar{x}))\phi_e de$$

$$+ \beta g'(\bar{x}) \int_{g(\bar{x})-rx_1+ry_1^*}^{\infty} v'(g(\bar{x}))\phi_e de = 0. \qquad (82)$$

Since $y_2^*(x) < y_1^*(x)$, therefore

$$c_1'(rx + e - ry_2^* - g(\bar{x})) > c_1'(rx + e - ry_1^* - g(\bar{x}))$$

and

$$g(\bar{x}) - rx + ry_2^* < g(\bar{x}) - rx + ry_1^*.$$

As a result

$$\int_{g(\bar{x})-rx+ry_2^*}^{\infty} c_1'(rx + e - ry_2^* - g(\bar{x}))\phi_e de \quad \int_{g(\bar{x})-rx+ry_1^*}^{\infty} c_1'(rx + e - ry_1^* - g(\bar{x}))\phi_e de.$$

Also, since

$$f_1'(g(\bar{x})) > v'(g(\bar{x}))$$

then

$$\int_{g(\bar{x})-rx+ry_2^*}^{\infty} f_1'(g(\bar{x}))\phi_e de > \int_{g(\bar{x})-rx+ry_1^*}^{\infty} v'(g(\bar{x}))\phi_e de.$$

From (81) and (82), these results imply that

$$\frac{df_2(\bar{x}^*)}{d\bar{x}} > \frac{df_1(\bar{x}^*)}{d\bar{x}} \quad \forall\ \bar{x}_1^*; \quad 0 \leq \bar{x}^* \leq 2y_0 + x_0 \tag{83}$$

and hence

$$\bar{x}_2^* > \bar{x}_1^*. \tag{84}$$

<u>The n Period Problem</u>

For an arbitrary n, the continuity equation is given by

$$x_{n-1} = r(x_n - y_n) + e_n + i_n - m_n.$$

Define

$$f_n(x_n) = \text{Max } G_n(y_n, x_n - y_n)$$

where

$$G_n(y_n, x_n - y_n) = \pi(y_n, x_n - y_n) + \beta \int_0^{x^m - rx_n + ry_n} f_{n-1}(x^m)\phi_e de$$

$$+ \beta \int_{x^u - rx_n + ry_n}^{\infty} f_{n-1}(x^u)\phi_e de + \beta \int_{x^m - rx_n + ry_n}^{x^u - rx_n + ry_n} f_{n-1}(rx_n + e - ry_n)\phi_e de$$

$$- \beta \int_{x^u - rx_n + ry_n}^{\infty} c_1(rx_n + e - ry_n - x^u)\phi_e de$$

$$- \beta \int_0^{x^m - rx_n + ry_n} c_2(x^m - rx_n - e + ry_n)\phi_e de - c(\bar{x}). \tag{85}$$

Then $y_n^*(x_n)$ is defined by $\frac{dG_2}{dy_2} = 0$ or equivalently by

$$\pi_1(y_n, x_n - y_n) - \pi_2(y_n, x_n - y_n) = \beta r \int_{x^u - rx_n + ry_n}^{\infty} c_1'(rx_n + e - ry_n$$

$$- x^u)\phi_e de - \beta r \int_0^{x^m - rx_n + ry_n} c_2'(x^m - rx_n + ry_n - e)\phi_e de$$

$$- \beta r \int_{x^m - rx_n + ry_n}^{x^u - rx_n + ry_n} f_{n-1}'(rx_n + e - ry_n)\phi_e de. \tag{86}$$

Using a straightforward induction argument [31], the following propositions can be proven:

Proposition 7

If assumptions (a - d) in proposition 1 hold, then

1) $\exists$ a unique interior maximum $y^*_n(x_n)$

2) $0 < \frac{dy^*_n}{dx_n} \leq 1$

3) $y^*_n(x) \leq y^*_{n-1}(x)$

4) If $\pi_{11} = \pi_{12} \equiv 0$, then the optimal release rule is of the form

$$y^*_n(x_n) = x_n - a_n$$

where a_n is a constant dictated by the hydrology of the river basin, the size of the reservoir, and the specific form of the profit function.

Proposition 8

Under the assumptions of proposition 1 (a - e), if g is concave and h is a constant, then

1) f_n is strictly concave in x_n and $\bar{x}$

2) $\frac{df_n}{dx_n} = \pi_2(y^*_n(x_n), x_n - y^*_n(x_n))$

3) $f'_n > f'_{n-1}$.

Proposition 9

Under the assumptions of proposition 3:

1) $\exists$ a unique optimal size $\bar{x}^*_n$ for the reservoir which maximizes the total expected return $f_n(\bar{x})$

2) $\bar{x}^*_n > \bar{x}^*_{n-1}$.

The Infinite Stage Process

In this section the following functonal equation will be discussed.

$$f(x) = \underset{0\leq y\leq x}{\text{Max}} \Big[\pi(y,x-y) + \beta\int_0^{x^m-rx+ry} f(x^m)\phi_e de + \beta\int_{x^u-rx+ry}^{\infty} f(x^u)\phi_e de$$

$$- \beta\int_{x^u-rx+ry}^{\infty} c_1(rx + e - ry - x^u)\phi_e de + \beta\int_{x^m-rx+ry}^{x^u-rx+ry} f(rx + e - ry)\phi_e de$$

$$- \beta\int_0^{x^m-rx+ry} c_2(x^m - rx - e - ry)\phi_e de - c(\bar{x})\Big]. \qquad (87)$$

Proposition 10

There is a unique solution to (85) which is bounded for x in any finite real interval. This solution, f(x), is continuous and concave.

The proof of this proposition is well known and follows closely the development given in Bellman [31]. Define the sequence $\{f_n(x)\}$ as follows:

$$f_{n+1}(x) = \underset{0\leq y\leq x}{\text{Max}}\ G(y,x - y, f_n). \qquad n = 0,1,2,...$$

where $f_0(x) = v(x)$ and $f_0(x)$ is continuous over $x \geq 0$. Then it can be shown that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for $x \geq 0$ and is the solution of $f(x) = \underset{0\leq y\leq x}{\text{Max}}\ G(y,x,f)$. Moreover, the convergence of $f_n(x)$ is uniform.

Therefore, since each function in the sequence is continuous and concave, f(x) is continuous. To show the similarity of (87) to the problem discussed by Bellman [31], the following theorem is stated:

Bellman's Theorem

The functional equation

$$f(x) = \underset{y \geq x}{\text{Min}} \left[k(y - x) + z \left[\int_y^\infty p(s-y)\phi(s)ds + f(0) \int_y^\infty \phi(s)ds + \int_0^y f(y - s)\phi(s)ds\right]\right]$$

has a unique solution which is bounded for x contained in any finite interval. The solution f(x) is continuous. Assumptions:

K(y-x) and P(x-y) are convex.

Proposition 11

In the case of an infinite planning horizon and under the assumption that $y_n^*(x)$ exists and is unique for any arbitrary n:

1) there exists a unique optimal policy $y^*(x)$ where

$$y_n^*(x) \to y^*(x), \qquad x^m \leq x \leq x^u$$

2) $0 < \frac{dy^*}{dx} \leq 1$

3) $-1 \leq \frac{dy^*}{d\bar{x}} < 0.$

Proof:

Since for any arbitrary n, we have $x^m \leq x_n \leq x^u$; it follows that

$y_n^*(x_n)$ has an upper bound equal to x_n and a lower bound equal to 0. It has also been shown that the sequence $\{y_n^*\}$ is a non-decreasing sequence such that

$$y_1^*(x) \geq y_2^*(x) \geq y_3^*(x) \geq \ldots.$$

Since each y_i^* is bounded below, $y_n^*(x)$ converges to $y^*(x)$ [26], where $y^*(x)$ is the solution of

$$\pi_1(y, x - y) - \pi_2(y, x - y) - \beta r \int_0^{x^m - rx + ry} c_2'(x^m - rx + ry - e)\phi_e de$$
$$+ \beta r \int_{x^u - rx + ry}^{\infty} c_1'(rx + e - ry - x^u)\phi_e de - \beta r \int_{x^m - rx + ry}^{x^u - rx + ry} f'(rx + e - ry)\phi_e de = 0. \tag{88}$$

The proof of the comparative statics results in the infinite stage process is similar to the proof previously outlined for the two period case.

Proposition 12

There exists a unique optimal size $\bar{x}^*$ for the reservoir which maximizes $f(x; \bar{x})$.

Proof:

Since the assumptions of the model make each member of the sequence $\{f_n(\bar{x})\}$ concave, $f(\bar{x})$ is also concave. The next step is to prove that $\bar{x}$ is bounded. Assume as before that there exists y_0 such that $\pi_1(y_0) = 0$ and $\bar{x}_0$ such that $\pi_2(\bar{x}_0) = 0$. The discounted gross revenue realised must be less than the gross revenue when the reservoir is always operating at y_0, because of the cost of importing and exporting water. Thus,

$$\text{realized gross revenue} \leq \frac{\pi(y_0,x_0)}{(1-r)} \qquad \forall\ \bar{x}\ .$$

Define $\bar{\bar{x}}$ by $\frac{c(\bar{\bar{x}})}{(1-r)} = \frac{\pi(y_0,x_0)}{(1-r)}$;

then

$$\text{realized gross revenue} \leq \frac{c(\bar{\bar{x}})}{(1-r)} \ . \qquad \forall\ \bar{x}\ .$$

That is if $\bar{x} > \bar{\bar{x}}$, then the realized net revenue must be negative and hence $\bar{\bar{x}}$ bounds $\bar{x}$.

The Long Term Distribution and the Case of the Linear Decision Rule

The process we are dealing with is represented by the continuity equation

$$x_{p-1} = rx_p - ry^* + e_p + i_p - m_p .$$

This is a discrete time, continuous state Markov process. Therefore the usual "ergodic theorem" could not be employed to find the long-run distribution of the water stock.

In this section it is shown that the long-run distribution exists and can be derived for a special class of objective functions. This class of functions corresponds to the case when π_{11} and $\pi_{12} \equiv 0$.

Proposition 13

If the assumptions of proposition 12 hold, and if $\pi_{11} = \pi_{12} \equiv 0$, then there exists a long run distribution for the water stock in the reservoir given by

$$P(x = x^m) = \Phi(x^m + ra)$$

$$P(x = x^u) = 1 - \Phi(x^u + ra)$$

and $\quad x \sim \phi(x + ra)$ for x, $x^m < x < x^u$

where a is a constant.

Proof:

We have seen that

$$i_p > 0 \iff m_p = 0 \text{ or } rx_p - ry^*(x_p) + e_p < x^m$$

$$\iff e_p < x^m - rx_p + ry^*(x_p).^{9} \qquad (89)$$

Moreover, we have seen that separability and linearity of π implies a linear decision rule of the form

$$y^*(x_p) = x_p - a \quad ^{10} \qquad (90)$$

Then, from (89) and substituting for $y^*(x_p)$ from (90), we have

$$i_p > 0 \iff e_p < x^m - ra. \qquad (91)$$

Therefore, it follows that

$$P(i_p > 0) = \Phi(x^m - ra) \ . \qquad (92)$$

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[9] This is Iff statement, because importing and exporting actions are not optimizing decisions, but rather a penalty imposed by the stochastic nature of the inflow to correct for deficiencies or surpluses after the decisions are taken.

[10] Notice that a is the same from period to period only in the long run for the infinite planning horizon case. However, in the finite case $y_p^*(x_p) = x_p - a_p$.

In a similar fashion, it is possible to show that

$$P(m_p > 0) = 1 - \Phi(x^u - ra) \tag{93}$$

and

$$P(i_p = 0, m_p = 0) = \Phi(x^u - ra) - \Phi(x^m - ra). \tag{94}$$

However, we know that

$$P(i > 0) = P(x_{p-1} = x^m),\ P(m_p > 0) = P(x_{p-1} = x^u) \tag{95}$$

and

$$P(i_p = 0, m_p = 0) = P(x^m < x_{p-1} < x^u)$$

Therefore,

$$P(x_{p-1} = x^m) = \Phi(x^m - ra) \tag{96}$$

$$P(x_{p-1} = x^u) = 1 - \Phi(x^u - ra) \tag{97}$$

and x_{p-1} is distributed as $\phi_e(x_{p-1} - ra) \quad x^m \leq x_p \leq x^u$. (98)

(96–98) show that the distribution of x_{p-1} has two mass points at x^u and x^m and is continuously distributed with $\phi_e(x_{p-1} - ra)$ in the range of (x^m, x^u). That is, the distribution of x is given by

$$\left.\begin{array}{ll} \Phi(x^m+ra) & \text{at } x = x^m \\ \phi(x+ra) & x^m < x < x^u \\ 1 - \Phi(x^u+ra) & x = x^u \end{array}\right\} \tag{99}$$

and

$$E(x) = x^m\Phi(x^m - ra) + x^u\{1 - \Phi(x^u - ra)\} + \int_{x^m}^{x^u} x\,(x - ra)dx.$$

The expression above could only be evaluated if a specific form for the profit function is postulated. It is also necessary to simulate the dynamic program for a large number of periods p until

$$(a_p - a_{p-1}) \to 0 .$$

Using the simulated value of a and postulating a specific form for the inflow distribution (e.g. log-normal or χ^2) after calibrating with

actual data, the solution is found by: (1) select the optimal policy, given a particular physical size of the reservoir $\bar{x}$ (i.e. $y^* = x - a(\bar{x})$); (2) obtain the optimal size of the reservoir $\bar{x}^*$. The selection of $\bar{x}^*$ defines a exactly; therefore, the distribution of x is determined and so is E(x).

<u>Conclusion and Summary</u>

It has been demonstrated that chance constrained programming can be incorporated within the usual dynamic programming formulation by transforming the chance constraints into a penalty function that is added to the criterion function to be maximized. Moreover, it has been found that allowing for importing and exporting of water from the reservoir provides an economic rational for the penalty function and provides acceptable economic interpretation to the technical requirements for the solution of the maximization problem. Allowing for evaporation losses, the manager of the reservoir maximizes a criterion function which reflects benefits from water releases to agriculture and from the water stock in the reservoir for power generation. Within the chance constrained dynamic programming, the manager solves for the dual problem of optimal operating policy and optimal size of the reservoir. The procedure of maximization is similar to that of two-step programming in that water import and export is considered a residual decision to correct for the violation of the constraints. Specifically, it does not pay to engage in importing or exporting water unless violation of the constraints occur as a result of implementing the optimal policy. These conditions, together with concavity of the criterion function and convexity of the penalty function, are found sufficient to get all the usual dynamic programming results, such as the existence, uniqueness, monotonicity, and convergence of the optimal policy. It has also been demonstrated that the usual dynamic programming results extend to the optimal size of the reservoir under these and some other plausible conditions. Assuming the criterion function to be separable and linear in water releases, the optimal operation policy is found to be linear. Moreover, under this condition, it has been demonstrated that the long-run distribution of the water stock in the reservoir exists and is derived. Finally, another model is presented in the Appendix which incorporates the chance constrained problem into a planning model by finding a deterministic equivalent to the chance constraints. It has been demonstrated, that for an infinite sized reservoir, the optimal operating policy exists and is unique. Moreover, a formula for the long-run distribution of the water stock is derived and some bounds on the expected value are developed.

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