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CONTINUOUS-VALUED BINARY DECISION PROCEDURES

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Review of Economic Studies, forthcoming.

SOCIAL SCIENCE WORKING PAPER 214

May 1978

Rev. September 1979

2nd Rev. February 1980

ABSTRACT

Conditions have been given elsewhere which guarantee that binary decision procedures have a simple structure. Here we show that a continuity requirement together with some weak algebraic regularity conditions ensures that a binary procedure is locally simple. Additional topological assumptions are given which require that the binary procedure is a simple game.

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I. INTRODUCTION

A Bergson-Samuelson social welfare function maps each n-tuple of continuous individual preference orderings into a continuous, transitive binary relation over alternative states of an economy. Arrow and his successors dropped the continuity requirement and demanded instead that the social ordering vary in a natural way with individual preferences. These requirements, the independence of irrelevant alternatives and citizen sovereignty axioms, together with the requirement that the social preference relation satisfy transitivity or some weaker rationality condition, imply either that power is concentrated to an extreme extent or that the social choice process is indecisive.

In this paper we eliminate the transitivity assumption employed in the Bergson-Samuelson formulation and instead concentrate on the axiom that the social preference relation be continuous. Specifically, we examine the nature of a function that maps each n-tuple of individual preferences into a continuous binary relation that does not necessarily satisfy any rationality condition. When this function depends in a "natural" way on individual preferences, we show that power, although not necessarily concentrated, is distributed in a dichotomous fashion among individuals; each coalition can either determine the social preference relation on <u>every</u> pair of alternatives or on <u>no</u> pair of alternatives. Furthermore, any alternative x will be socially preferred to an alternative y <u>only</u> if those who prefer x to y constitute one of the powerful coalitions. Thus, such procedures have the form of a simple game.

The primary way in which our procedures depend on individual preferences is to satisfy binaryness. Binaryness, a weak form of Arrow's independence of irrelevant alternatives axiom, requires that the social preference on any two-alternative subset depend only on individual preferences on that subset. Many well-known social choice procedures and game theoretic solution concepts satisfy binariness. Among these processes are those satisfying the weak axiom of revealed preference, the core-selecting and the von Neumann-Morgenstern solution selecting choice functions [3], and Schwartz's GOCHA set [4].

It is known that any binary procedure is characterized by a decisiveness structure which describes the powers of all coalitions [1]. It turns out that the decisiveness structure needed to obtain this characterization is generally large and complex. However, some familiar binary procedures are simply characterized by the family of "winning" subsets. The best known example is, of course, absolute majority rule in which society chooses x over y if and only if a majority strictly prefers x to y. In [1] conditions on binary procedures are given that indicate when a family of winning sets completely describes the behavior of a binary procedure.

In this paper the characterization in terms of winning sets will be obtained when the social preference relation satisfies the following continuity property. Define an asymmetric binary relation P

*The authors would like to thank Peter Hammond and an anonymous referee for their helpful comments. Support from National Science Foundation Grants SOC78-24787 and SOC79-07366 is gratefully acknowledged.

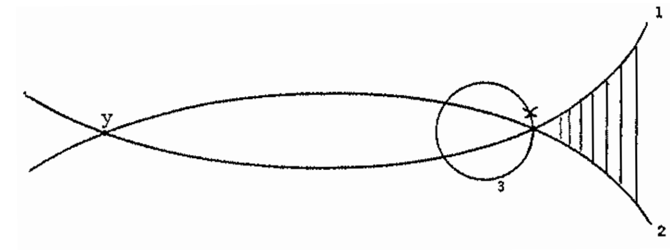
on a topological space X to be <u>continuous</u> if P is open in $X \otimes X$. Equivalently, if R is the reflexive completion of P, P and R are continuous if R is closed in $X \otimes X$. If P is continuous then for each x, $P(x) = \{y \in X | \sim xRy\}$ and $Q(x) = \{y \in X | \sim yRx\}$ are open. The converse is however, not generally true. We now give two examples to provide additional motivation.

<u>Example 1: (Absolute Majority Rule)</u>

For each pair of alternatives $x, y \in X$, society has xPy (x strictly over y) if and only if a majority strictly prefers x to y. Suppose individual preference orderings are continuous, and that xPy. Then there is a neighborhood U of (x,y) in $X \otimes X$ such that a majority prefers u to v for any $(u,v) \in U$. Hence uPv for any $(u,v) \in U$, which shows that P is open in $X \otimes X$. Thus absolute majority rule is a continuous binary relation when individual preferences are continuous.

<u>Example 2: (Simple Majority Rule)</u>

Under this voting rule, xPy if and only if more individuals prefer x to y than prefer y to x. Let the space of alternatives X be the plane R^2 endowed with its Euclidean topology. The set P(y) need not be open even if each $P_i(y)$ is, as can be seen from the following three person example.



Let individuals 1, 2 and 3 have indifference curves as shown in the figure and note that, under simple majority rule, $x \in P(y)$. Now take a sequence of alternatives $\{w^n\}$ in the shaded region converging to x and note that $w^n \in X - P(y)$ for each n. If P(y) were open the limit of $\{w^n\}$ would have to be contained in $X - P(y)$, a contradiction. For a similar example and restrictions on preferences which ensure that simple majority rule is continuous see Kelly [2].

Example 2 indicates that common procedures can aggregate continuous preferences into discontinuous binary relations. By comparison with example 1 we see that one source of discontinuity is that the choice procedure decides "too often." It turns out that this property of deciding "too often" is what makes the associated decisiveness structures (or constitutions) complicated. It hardly needs to be mentioned in this context that absolute majority rule is a simple game whereas "simple" majority rule is not.

II. NOTATION AND DEFINITIONS

Let X denote a set of alternatives and T a topology for X, such that (X,T) is Hausdorff and has the property that all nonempty open sets contain infinitely many elements. Let $N = \{1,2,...n\}$ be a finite set of individuals with each $i \in N$ possessing a preference relation R_i that is contained in a set Ω_i of admissible reflexive binary relations on X. For each $R_i \in \Omega_i$ we denote the asymmetric and symmetric parts by P_i and I_i respectively.

Let $D = \{(x,y) \in X \otimes X | x = y\}$ be the diagonal of $X \otimes X$. We assume that each Ω_i satisfies the following two domain conditions:

D1. Given any $(x,y) \in X \otimes X - D$, there exists an open neighborhood U of (x,y) and binary relations $\{R^1, R^2, R^3\} \subseteq \Omega_i$ satisfying, for every $(u,v) \in U$,

$$uP^1v, \; uI^2v, \text{ and } vP^3u.$$

D2. Given any $(x,y) \in X \otimes X - D$, and sequences $\{x_n\} \subseteq X - \{x\}$ and $\{y_n\} \subseteq X - \{y\}$ such that $x_n \to x$ and $y_n \to y$. Then there exists $R \in \Omega_i$, a subsequence $\{u_n\}$ of $\{x_n\}$, and a subsequence $\{v_n\}$ of $\{y_n\}$ for which

$$xIy \text{ and } u_nPv_n \text{ for all } n.$$

Property D1 has two important aspects: (1) it allows indifference sets to have interiors, and (2) it allows any ordering of any pair of

alternatives. Thus, if $X \subseteq R^m$ then, domains consisting of only strictly convex preferences, individualistic (selfish) preferences, and monotone ("more is better") preferences fail to satisfy D1. Property D2 is, to our knowledge, new to the literature. We shall show in a subsequent section that D1 and D2 are satisfied if Ω_i contains preference relations satisfying a weak convexity condition.

A <u>binary decision rule</u> is a mapping F of $\Omega = \Omega_1 \otimes \Omega_2 \otimes \ldots \otimes \Omega_n$ into asymmetric binary relations that satisfies

<u>Binaryness</u>: $\forall \pi, \pi' \in \Omega$ and $(x,y) \in X \otimes X$, $[xP_iy \Leftrightarrow xP'_iy$ and $yP_ix \Leftrightarrow yP'_ix \; \forall \; i \in N] \Rightarrow [xF(\pi)y \Leftrightarrow xF(\pi')y$ and $yF(\pi)x \Leftrightarrow yF(\pi')x]$.

When there is no danger of ambiguity, $F(\pi)$ shall be denoted as P and the reflexive completion of $F(\pi)$ as R. F is <u>continuous-valued at (x,y)</u> if for any $\pi \in \Omega$, xPy implies the existence of a neighborhood V of (x,y) such that uPv for all $(u,v) \in V$. F is <u>continuous-valued</u> if F is continuous-valued at each $(x,y) \in X \otimes X$, i.e., if F maps Ω into continuous binary relations.

It was shown in [1] that any mapping from n-tuples of weak orders into asymmetric binary relations is characterized by the binary constitution defined as follows. Let $\mathcal{B} = \{(A,B) | A \subseteq N, B \subseteq N, A \cap B = \emptyset\}$. Then a binary constitution is a mapping $C : X \otimes X - D \to 2^{\mathcal{B}}$.

Any binary decision rule F is characterized by a binary constitution in the sense that there is one and only one C such that

(1) $xF(\pi)y \Longleftrightarrow (\{i|xP_iy\}, \{i|yP_ix\}) \in C(x,y)$.

We next give two definitions which will help us characterize continuous binary decision rules by very simple constitutions.

Definition: F is neutral on $K \subseteq X \otimes X$ if $\forall \pi, \pi' \in \Omega$ and $\forall (x,y)$, $(z,w) \in K$, $[xP_iy \Leftrightarrow zP_i'w$ and $yP_ix \Leftrightarrow wP_i'z \;\; \forall i \in N] \Rightarrow [xF(\pi)y \Leftrightarrow zF(\pi')w$ and $yF(\pi)x \Leftrightarrow wF(\pi')z]$.

Definition: F is decisive on $K \subseteq X \otimes X$ if $\forall \pi, \pi' \in \Omega$ and $\forall (x,y) \in K$, $[xP_iy \Leftrightarrow xP_i'y \;\; \forall i \in N$ and $xF(\pi)y] \Rightarrow xF(\pi')y$.

A binary decision rule is called simple on a subset $K \subseteq X \otimes X$ if there is a collection of subsets $\mathcal{W}$ of N such that for each $(x,y) \in K$, $xF(\pi)y \Leftrightarrow \{i \in N|xP_iy\} \in \mathcal{W}$. It is easy to show that a binary decision rule is neutral and decisive on K if and only if it is simple on K.

We shall show, by considering localizations of the above concepts, that certain continuous-valued binary decision rules exhibit a local form of simplicity. We need the following definitions.

Definition: F is decisive at $(x,y) \in X \otimes X - D$ if $\forall \pi, \pi' \in \Omega$, $[xP_iy \Leftrightarrow xP_i'y \; \forall i \in N$ and $xF(\pi)y] \Rightarrow xF(\pi')y$.

Definition: F is semi-neutral near $(x,y) \in X \otimes X - D$ if there is a neighborhood U of (x,y) such that $\forall (u,v) \in U$, $[xP_iy \Leftrightarrow uP_i'v$ and $yP_ix \Leftrightarrow vP_i'u \; \forall i \in N] \Rightarrow [xF(\pi)y \Leftrightarrow uF(\pi')v]$.

Definition: F is simple near $(x,y) \in X \otimes X - D$ if there is a collection $W(x,y)$ of coalitions and a neighborhood U of (x,y) such that $\forall (u,v) \in U$, $uF(\pi)v \Leftrightarrow \{i \in N|uP_iv\} \in W(x,y)$. A coalition $A \in W(x,y)$ is winning for x against y.

Definition: F is locally simple if it is simple near every (x,y) $\in X \otimes X - D$.

The following lemma restates the above neutrality and decisiveness definitions in the language of binary constitutions and connects these concepts to local simplicity.

Lemma 1. Let C be the binary constitution corresponding to F. Then

i) F is decisive at $(x,y) \Longleftrightarrow [(A,B) \in C(x,y) \Rightarrow (A,B') \in C(x,y)$ $\forall (A,B') \in \mathcal{B}]$.

ii) F is semi-neutral near $(x,y) \Longleftrightarrow \exists$ a neighborhood V of (x,y) such that $\forall (u,v) \in V$, $C(x,y) = C(u,v)$.

iii) F is locally simple $\Longleftrightarrow \forall (x,y) \in X \otimes X - D$, F is decisive at (x,y) and semi-neutral near (x,y).

Proof. i) and ii) are direct translations between the language of binary decision rules and the language of binary constitutions.

iii): Let $\mathcal{D}(x,y) = \{A \subseteq N \mid (A,B) \in C(x,y) \ \forall\ B \subseteq N - A\}$. Then $\forall (x,y) \in X \otimes X - D$, F locally simple $\Rightarrow$ F simple near $(x,y) \Rightarrow W(x,y) = \mathcal{D}(u,v)\ \forall\ (u,v)$ in a neighborhood V of $(x,y) \Rightarrow$ F is decisive at (x,y) and F is semi-neutral near (x,y). Conversely if F is decisive at and semi-neutral near some $(x,y) \in X \otimes X - D$, then i) and ii) show that F is simple near (x,y), with $W(x,y) = \mathcal{D}(x,y)$. Hence F is locally simple if F is decisive at and semi-neutral near every $x \in X \otimes X - D$.

Q.E.D.

Throughout the next section we adopt a convenient shorthand method of representing a preference configuration π. If, as will be usual, we want to have $xP_iy\ \forall\ i \in A$, $yP_ix\ \forall\ i \in B$ and $xI_iy\ \forall\ i \in N - A \cup B$, we shall write

A	B	N - (A ∪ B)
x	y	xy
y	x	

If in addition we want to have a sequence of points $\{u^n\}$ converging to x in such a way that for n large enough, xP_iu^n and $u^nP_iy\ \forall i \in A$, and $xI_iu^n\ \forall i \in N - A$, we shall write

A	B	N - (A ∪ B)
x	y	xy[u^n]
[u^n]	x[u^n]	
y		

Finally if a subset of individuals, A, is required to be indifferent among a subset Y of alternatives we shall write

A
(Y)

III. MAIN RESULTS

In this section we show that each member of a class of continuous-valued binary decision rules is locally simple and, under further topological restrictions, globally simple. We use several lemmas.

Lemma 2: If F is continuous-valued at (x,y) and $(A,B) \in C(x,y)$, $B \subseteq B'$, and $A \cap B' = \emptyset$, then $\exists$ a neighborhood V of $(x,y) \ni (A,B') \in C(u,v)$ $\forall\ (u,v) \in V$ for which $u \neq x$ and $v \neq y$.

Proof: Suppose not. Then $\exists\ (x_n,y_n) \to (x,y) \ni (A,B') \notin C(x_n,y_n)\ \forall n$, $x_n \neq x\ \ \forall n$, and $y_n \neq y\ \ \forall n$. Condition D1 applied to all members of A, B, and $N - (A \cup B')$, and condition D2 applied to all members of $B' - B$, imply that $\exists\ \pi \in \Omega$ and subsequences $\{u_n\}$ of $\{x_n\}$ and $\{v_n\}$ of $\{y_n\}$ such that

A	B	B' - B	N - (A ∪ B')
x and [u_n]	y and [v_n]	xy and [v_n]	xy[u_n][v_n]
y [v_n]	x [u_n]	[u_n]	

Then $v_nRu_n\ \ \forall n \Rightarrow yRx$, contrary to $(A,B) \in C(x,y) \Rightarrow xPy$.

Q.E.D.

Lemma 2 establishes the somewhat surprising result that if a binary decision rule is continuous-valued at a point (x,y) then it is "almost" decisive in most of a neighborhood of that point. The "almost" decisiveness of lemma 2 is a strong restriction on binary decision rules and many familiar choice processes, such as the simple majority rule described in example 2, fail to satisfy it.

The remaining developments in this section have the purpose of strengthening the "almost" decisiveness at a point obtained in lemma 2 to simplicity near the point. The following weak version of monotonicity is both natural and useful in this endeavor.

Definition: F is semimonotonic at $(x,y) \in X \otimes X - D$ if $(A,B) \in C(x,y) \Longrightarrow (A,B') \in C(x,y)$ for all $B' \subseteq B$. F is semimonotonic if it is semimonotonic at all $(x,y) \in X \otimes X - D$.

Theorem 1: Suppose that F is semimonotonic at $(x,y) \in X \otimes X - D$, that F is continuous-valued, and that $(A,B) \in C(x,y)$. Then for any $B' \subseteq N - A$, $\exists$ a neighborhood V of (x,y) such that

i) $(A,B') \in C(u,v)\ \forall (u,v) \in V$ with $u \neq x$ and $v \neq y$, and

ii) $(B',A) \notin C(v,u)\ \forall (u,v) \in V$.

Proof: By lemma 2 and the definition of semimonotonic at (x,y), there exists V satisfying i). To prove ii), let $(u,v) \in V$ and choose $(u^n,v^n) \in V$ with $u^n \to u$ and $v^n \to v$, and $u_n \neq u$ and $v_n \neq v\ \forall n$. Condition D1 allows us to pick $\pi \in \Omega$ such that

A	B'	N - (A ∪ B')
$u[u^n]$	$v[v^n]$	$uv[u_n][v_n]$
$v[v^n]$	$u[u^n]$	

Then by i) and continuity at (u,v),

$$u^nPv^n \Rightarrow u^nRv^n \Rightarrow uRv \Rightarrow \sim vPu \Rightarrow (B',A) \notin C(v,u).$$

Q.E.D.

Part (i) of the theorem says that the first coordinates of the pairs in C(x,y) make up a set of winning coalitions for pairs (u,v) of alternatives in V for which $u \neq x$ and $v \neq y$. In addition, (ii) says these coalitions are blocking at all (u,v) in V. To obtain the desired simplicity near (x,y) we must show that these coalitions are winning at any $(u,v) \in V$. This requires one more condition on F.

Definition: F is strong without individual indifference (SWII) at $(x,y) \in X \otimes X - D$ if $\forall A \subseteq N\ (A^c,A) \notin C(y,x) \Longrightarrow (A,A^c) \in C(x,y)$. F is SWII if it is SWII at each $(x,y) \in X \otimes X - D$.

This condition requires that a binary decision rule must state a preference between a pair of alternatives whenever no individual is indifferent between the pair. Clearly this eliminates such procedures as absolute majority rule with n even. However, as real decision rules are expected to always choose between alternatives, SWII may not be requiring too much.

Lemma 3: Suppose F is continuous-valued and SWII. Then $(A,B) \in C(x,y) \Rightarrow \exists$ a neighborhood V of $(x,y) \ni (A,A^c) \in C(u,v)\ \forall (u,v) \in V$. In particular, $(A,B) \in C(x,y) \Rightarrow (A,A^c) \in C(x,y)$.

Proof: Let $(u,v) \in V$ where V is the neighborhood of theorem 1. By theorem 1 (ii) $(A^c,A) \notin C(v,u)$. Then SWII implies $(A,A^c) \in C(u,v)$.

Q.E.D.

Theorem 2: If F is semimonotonic at (x,y), and F is continuous-valued and SWII, then F is decisive at (x,y) and semi-neutral near (x,y).

Proof: To get decisiveness at (x,y), consider $(A,B) \in C(x,y)$. By lemma 3 we have $(A,A^c) \in C(x,y)$. Semimonotonicity then implies that $(A,B') \in C(x,y)$ whenever $A \cap B' = \phi$. By lemma 1 (i) F is then decisive at (x,y). For semi-neutrality near (x,y), first note that $\exists$ a neighborhood V of (x,y) such that $(A,B) \in C(x,y) \Rightarrow (A,A^c) \in C(u,v)\ \forall\ (u,v) \in V$ (lemma 3) and $C(x,y) \subseteq C(u,v)\ \forall\ (u,v) \in V$ (lemma 2). By lemma 1 (i) we need only to show that there exists a neighborhood V' of (x,y) such that $C(u,v) \subseteq C(x,y)\ \forall\ (u,v) \in V'$. Suppose not. Then

$$\exists\ (u^n,v^n) \in V \ni u^n \to x,\ v^n \to y \text{ and } C(u^n,v^n) \not\subseteq C(x,y)\ \forall n.$$

Since N is finite $\exists$ $(A,B) \notin C(x,y)$ and a subsequence (x^n,y^n) of (u^n,v^n) such that $(A,B) \in C(x^n,y^n)$. By semimonotonicity we also have $(A,A^c) \notin C(x,y)$. By our choice of V, $(A,A^c) \in C(x^n,y^n)$. Now D1 allows us to pick $\pi \in \Omega$ such that

$\underline{A}$	$\underline{A^c}$
$x[x^n]$	$y[y^n]$
$y[y^n]$	$x[x^n]$

Then $(A,A^c) \in C(x^n,y^n) \Rightarrow x^nPy^n \Rightarrow x^nRy^n \Rightarrow xRy \Rightarrow (A^c,A) \notin C(y,x) \Rightarrow (A,A^c) \in C(x,y)$. This contradicts $(A,A^c) \notin C(x,y)$. Hence $C(u,v) \subseteq C(x,y)$ $\forall\ (u,v)$ in a neighborhood V' of (x,y). Intersecting V and V' and invoking lemma 1 (ii), we have F semi-neutral near (x,y).

Q.E.D.

Lemma 1 (iii) and theorem 2 together will imply that a continuous-valued, SWII, and semimonotonic F is locally simple. To provide a condition for F to be simple, we use the following lemma. (The lemma was suggested by Peter Hammond.)

Lemma 4: Let K be a connected component of $X \otimes X - D$. If F is semi-neutral near each $(x,y) \in K$, then F is neutral on K.

Proof: For any $(x,y) \in K$, define

$$E(x,y) = \{(w,z) \in K \mid C(x,y) = C(w,z)\}.$$

Since F is semi-neutral near each $(w,z) \in E(x,y)$, E(x,y) is open in K by lemma 1 (ii). Because $A = \{E(x,y) \mid (x,y) \in K\}$ is a covering of the connected set K by open, disjoint sets, A must contain only one set, K. This proves that F is neutral on K.

Q.E.D.

Lemma 4 and theorem 2 immediately yield the main result.

Theorem 3: Suppose F is continuous-valued, SWII, and semimonotonic. Then F is

(i) locally simple,

(ii) simple on any connected $K \subseteq X \otimes X - D$, and

(iii) simple on $X \otimes X$ if $X \otimes X - D$ is connected.

Proof: Part (i) follows from lemma 1 (iii) and theorem 2. Theorem 2 also implies that F is decisive. Hence F is simple on any subset of $X \otimes X$ on which F is neutral. Therefore (ii) follows from lemma 4. Part (iii) follows from (ii), since F is trivially neutral on $X \otimes X$ if F is neutral on $X \otimes X - D$.

Q.E.D.

The following corollary provides a condition for F to be simple on $X \otimes X$ when X is in $\mathcal{R}^m$.

Corollary: Suppose that $X \subseteq \mathcal{R}^m$ is a connected set of full dimension, and that F is continuous-valued, SWII, and semimonotonic. If $m > 1$, then F is simple on $X \otimes X$.

Proof: F is trivially simple on $X \otimes X$ if it is simple on $X \otimes X - D$. Hence, by theorem 3 (iii), we need only show that $X \otimes X - D$ is connected. Now, $X \otimes X$ is a connected set of dimension 2m. The diagonal D is the intersection of m orthogonal hyperplanes in $\mathcal{R}^{2m}$ with $X \otimes X$. Hence D is of dimension $(2m - 1) - (m - 1) = m$. Now, $X \otimes X$ can be disconnected by the deletion of an affine subset only if the subset is of dimension at least $\dim (X \otimes X) - 1 = 2m - 1$. Since $\dim(D) = m < 2m - 1$ for $m > 1$, $X \otimes X - D$ is connected for $m > 1$.

Q.E.D.

The following example illustrates how F may not be simple on $X \otimes X$ if $X \subseteq \mathcal{R}^m$ and $m = 1$. Let $X = [0, 1]$, $N = \{1,2,3\}$, and Ω be the family of continuous weak orders on X. Define

$$W_1 = \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\},$$

$$W_2 = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}, \text{ and}$$

$$W(x,y) = \begin{cases} W_1 & \text{if } 0 \le x < y \le 1 \\ W_2 & \text{if } 0 \le y < x \le 1. \end{cases}$$

Lastly, define F by $xF(\pi)y \Leftrightarrow \{i \in N | xP_i y\} \in W(x,y)$. This F is not simple on $X \otimes X$, since $W(x,y)$ is not invariant on $X \otimes X - D$. However, F is continuous-valued, SWII, semimonotonic, locally simple, and simple on the two connected components $\{(x,y) | 0 \le x < y \le 1\}$ and $\{(x,y)\ 0 \le y < x \le 1\}$.

We conclude this section with a partial converse to theorem 3.

Theorem 4: If F is locally simple and individual preferences are continuous, then F is semimonotonic and continuous-valued. (Note that the domain Ω need not satisfy D1 and D2 for this theorem to hold.)

Proof: F is semimonotonic because it is decisive. Hence we need to show only that F (π) is continuous for any $\pi \in \Omega$. Let $(x,y) \in X \otimes X - D$, and let $W(x,y)$ be the collection of winning coalitions on a neighborhood V of (x,y). Let π be a profile of continuous preferences and suppose xPy. Then $A = \{i \in N | xP_i y\} \in W(x,y)$. For each $i \in A$ note that P_i is an open set for which $(x,y) \in P_i \subseteq X \otimes X - D$. Then uPv for any $(u,v) \in \bigcap_{i \in A} (P_i \cap V)$. Hence P is open in $X \otimes X - D$.

Q.E.D.

IV. ECONOMIC PREFERENCES

In this section we show that when X is a convex subset of R^m, interpretable as a set of public commodity bundles, then standard assumptions on individual preferences imply D1 and D2. Specifically, let $\overline{\Omega}$ be the set of binary relations on X that are reflexive, complete, transitive, continuous, and satisfy the convexity property

C. $\quad xRy \Rightarrow (tx + (1-t)y)Ry \quad \forall x \neq y,\ 0 < t < 1.$

Notice that C allows the interior of indifference sets to be nonempty. The preferences of $\overline{\Omega}$ are representable by continuous utility functions that are quasiconcave, i.e., continuous functions $u : R^m \to R$ for which

$$\{x \in X | u(x) \geq \alpha\} \text{ is convex } \forall\ \alpha \in R.$$

We must prove the following result.

<u>Lemma 5</u>: $\overline{\Omega}$ satisfies D1 and D2.

<u>Proof</u>: For any fixed $p \in R^m$, define the utility function $u(w;p)$ to be

$$u(w;p) = -(w-p) \cdot (w-p).$$

The preferences represented by $u(w;p)$ are clearly in $\overline{\Omega}$. Let $(x,y) \in X \otimes X - D$. Let $u^1(w) = u(w;x)$, $u^2(w) = 0$, and $u^3(w) = u(w;y)$. The preferences R^1, R^2, and R^3 that correspond to u^1, u^2, and u^3 satisfy the requirements of D1.

To show D2, let $\{(x_n,y_n)\}$ be a sequence converging to (x,y) such that $x_n \neq x\ \forall n$ and $y_n \neq y\ \forall n$. Define the line

$$L = \{w \in X | \exists\ t \in R \ni w = tx + (1-t)y\}.$$

Then three cases must be considered:

I. $\{n | x_n \in X-L\}$ is infinite,

II. $\{n | y_n \in X-L\}$ is infinite, and

III. $\{n | (x_n,y_n) \in L \otimes L\}$ is infinite.

<u>Case I</u>: In this case there exists $p \in R^m-\{0\}$ such that $(y-x) \cdot p = 0$ and

$$\{n | (x_n - x) \cdot p > 0\} \text{ is infinite.}$$

Without loss of generality assume $(x_n - x) \cdot p > 0\ \forall n$. Define the linear function

$$u(w;x,p) = (w - x) \cdot p.$$

The preference relation R represented by $u(w;x,p)$ is in $\overline{\Omega}$. By the choice of p, xIy. For any n let

$$n'(n) = \min \{m | u(y_m;x,p) < u(x_n;x,p)\}.$$

Such an n' exists because u is continuous, $y_m \to y$, and $u(y;x,p) = 0 < u(x_n;x,p)$. Letting $v_n = y_{n'(n)}$, we now have $x_n P v_n\ \forall n$. Hence R satisfies the requirements of D2.

<u>Case II</u>: This case can be dealt with in the same way as was case I.

<u>Case III</u>: We must break up this case into three subcases:

IIIa. $\{n | \exists t > 1 \ni x_n = tx + (1-t)y\}$ is infinite

IIIb. $\{n | \exists t < 0 \ni y_n = tx + (1-t)y\}$ is infinite

IIIc. $\{n | \exists\ t_x$ and $t_y \ni 0 < t_x, t_y < 1$ and $\ni x_n = t_x x + (1-t_x)y$ and $y_n = t_y x + (1-t_y)y\}$ is infinite.

IIIa. We can assume that $\forall n \ \exists\ t_n > 1 \ni x_n = t_n x + (1-t_n)y$. Let

$$u(w) = \max(0,(x-y)\cdot(w-x)).$$

Then the preferences R represented by u(w) are in $\overline{\Omega}$, and xIy. Because $u(x_n) = (t_n-1)\|x-y\|^2 > 0 = u(y)$, u is continuous, and $y_n \to y$, we have $x_n P y_n$ $\forall n$ sufficiently large. Hence R satisfies D2.

IIIb. We can assume that $\forall n \ \exists\ t_n < 0 \ni y_n = t_n x + (1-t_n)y$. Let

$$u(w) = \min(0,(x-y)\cdot(w-y)).$$

Then the preferences R represented by u(w) are in $\overline{\Omega}$, and xIy. Because $u(y_n) = t_n\|x-y\|^2 < 0 = u(x)$, u is continuous, and $y_n \to y$, we have $x_n P y_n$ $\forall n$ sufficiently large. Hence R satisfies D2.

IIIc. We can assume that $\forall n \ \exists\ 0 < t_{x_n}, t_{y_n} < 1$ such that

$$x_n = t_{x_n} x + (1-t_{x_n})y, \text{ and}$$
$$y_n = t_{y_n} x + (1-t_{y_n})y.$$

Let $p = 1/2(x+y)$ and consider u(w;p) as defined above. The preferences R represented by u(w;p) are in $\overline{\Omega}$, and xIy. For any n let

$$n'(n) = \min\{m \mid u(y_m;p) < u(x_n;p)\}.$$

The number n'(n) exists because $u(x_n;p) > u(x;p) = u(y;p)$, u is continuous, and $y_m \to y$. Letting $v_n = y_{n'(n)}$, we have $x_n p v_n$ $\forall n$. Hence R satisfies D2.

Q.E.D.

We now immediately have

Theorem 5: Suppose that $X \subseteq R^m$ is convex, that F is SWII, semimonotonic, and U-continuous, and that $\Omega = \overline{\Omega}$. Then F is locally simple, and F is simple if $m > 1$.

It seems to us that our results suggest that continuous-valued decision rules have a significantly "simpler" structure than has been generally suspected. Clearly the continuity axiom excludes some real decision procedures, as does SWII. However, our results may allow the application of the analytical machinery available for simple games to a wide class of decision procedures.

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