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TRANSITIVE PERMUTATION GROUPS AND EQUIPOTENT VOTING RULES

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I. Introduction

The connection between two-alternative voting rules and permutation groups has been fruitfully explored by Bartoszynski [1]. Given a voting rule F, defined on a set N of voters, its associated permutation group G_F on N may contain structurally revealing information about F. The best known example of this occurs when G_F is the full group of permutations on N, in which case F is said to be anonymous. This anonymity condition is generally accepted in social choice theory as a statement of fairness or equal power among the voters.

In this paper we propose a more general notion of fairness, called equipotency, by means of the condition that G_F be a transitive permutation group. After presenting a streamlined general development of the "committee decomposition" results given in [1], we prove that transitivity of G_F is a sufficient condition for equal distribution of "power" in a voting rule. This strengthens and considerably simplifies a result proved in [1], while suggesting that this transitivity is an important way of axiomatizing fairness among voters. After further strengthening the connection between transitivity and equal power by looking at simple games, we consider the class of equipotent voting rules. We show by example that if the anonymity condition in May's characterization [2] of simple majority rule is replaced by equipotency, then additional voting rules are possible. Finally, with the aid of a new and natural social choice condition, we obtain a new characterization of simple majority rule based on equipotency rather than anonymity. The reader who is primarily interested in these latter results may proceed directly to the penultimate section of the paper after absorbing the definition of a transitive permutation group.

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II. Notation and Group Theoretic Preliminaries

Let $N = \{1,2,\ldots,n\}$ be a set of voters attempting to choose between the two alternatives in the set $X = \{x,y\}$. For convenience we use the set $C = \{-1,0,1\}$ to denote, respectively, "preference for y over x," "indifference," and "preference for x over y." Then a profile of voter preferences is simply an n-tuple $d = (d_1,d_2,\ldots,d_n) \in C^n$ and a voting rule or social choice function is a function $F: C^n \to C$.

For convenience we shall assume, unless otherwise stated, that F is dual, by which we mean that $F(-d) = -F(d)\ \forall\ d \in C^n$. Duality requires that the alternatives x and y be treated symmetrically by the voting rule. Imposing duality is not, for our purposes, a serious loss of generality since without it we could study separately the structure of the "1" (x over y) and "-1" (y over x) vote.

A permutation group G on N is a set of bijections on N which contains the identity permutation, contains the inverse of each of its members, and is closed under composition. Given $g \in G$ and $i \in N$, we let ig denote the image of i under g (composition will be performed from left to right). When convenient we shall also use cycle notation for permutations whereby, for example, (123) denotes the permutation which maps 1 to 2, 2 to 3, and 3 to 1, leaving all other elements fixed. We use (1) to denote the identity permutation. The group of all permutations on a set S will be denoted by $\tau(S)$.

Given $g \in \tau(N)$ and $d = (d_1, d_2, \ldots, d_n) \in C^n$, let $d_g \in C^n$ be defined by $d_g = (d_{1g}, d_{2g}, \ldots, d_{ng})$. Thus d_g permutes the preferences of the voters in accordance with the way g permutes the voters themselves.[1] To each (dual) voting rule $F: C^n \to C$ we associate the permutation group G_F on N as follows:

$$G_F = \{g \in \tau(N) \mid F(d_g) = F(d) \ \forall\, d \in C^n\}.$$

The following example should prove useful in clarifying ideas which have been and will be presented in general form.

<u>Example 0</u>: Let $F: C^9 \to C$ be the voting rule defined verbably as follows. The voter set $N = \{1,2,\ldots,9\}$ is partitioned into three "committees" denoted by $N_1(1) = \{1,2,3\}$, $N_2(1) = \{4,5,6\}$, and $N_3(1) = \{7,8,9\}$ (the parenthetical "1" indicates that higher level "committees" may be forthcoming). Each of these subsets operates under simple majority rule to determine its preference, after which a three member "committee of committees" $N(1) = \{N_1(1), N_2(1), N_3(1)\}$ is formed to reach a final decision, again by simple majority rule. It can be checked that for this rule F, the group G_F can be described by

$$G_F = \{g \in \tau\,(n) \mid \forall\, i = 1,2,3,\, g \text{ maps } N_i(1) \text{ onto some } N_j(1)\,(j = 1,2,3)\}.$$

Thus, (123) (4859) (67) $\in G_F$, but (14) (25) (789) $\notin G_F$.

Having made the connection between voting rules and permutation groups, we now present some general permutation group results that will be needed for later application. We refer the reader to [3] for fuller exposition.

Let G be a permutation group on N. The following definition is central to the results that follow.

<u>Definition</u>: G is <u>transitive</u> if $\forall\, i, j \in N$, $\exists\, g \in G$ such that $ig = j$.[2]

By way of illustration, it can be checked that G_F for Example 0 is transitive; but that if one the committees, say $N_3(1)$, operated under <u>absolute</u> majority rule, the resultant group G_F would no longer be transitive (there would, for instance, be no $g \in G_F$ with $1g = 7$).

We now describe how a permutation group G can be used to decompose the underlying set N into independent "blocks" in a natural algebraic fashion. If $G = G_F$ for some voting rule F then, as developed in [1], these blocks can be thought of as "committees" whose decisions will be aggregated to eventually determine a final outcome. It may be helpful to keep this interprelation (and Example 0) in mind as we preceed, but we carry out our current development for an arbitrary permutation group G on N.

For any i, j $\in$ N, let $g_{ij} \in \tau(N)$ denote the permutation that interchanges i and j and leaves everything else fixed. It is easy to check that the relation $\sim$ on N defined by $i \sim j \iff g_{ij} \in G$ is an equivalence relation (for the transitivity of ~ , note that $g_{ik} = g_{ij}g_{jk}g_{ij}$). Thus N is partitioned under ~ into disjoint equivalence sets $\{N_i(1)\}_{i=1}^{n_1}$.

It is easy to check that for any $g \in G$ and any $N_i(1)$ with $1 \leq i \leq n_1$, the image of $N_i(1)$ under g is $N_k(1)$ for some $1 \leq k \leq n_1$ (every "block" either maps onto itself or onto some other block). In the language of permutation group theory, the $\{N_i(1)\}_{i=1}^{n_1}$ are called <u>sets of imprimitivity</u> for G. To formalize this idea of permutation of blocks, let $N(1) = \{1,2,...,n_1\}$ and define, for each $g \in G$, a permutation $\overline{g} \in \tau(N(1))$ according to $i\overline{g} = k \Longleftrightarrow (N_i(1))g = N_k(1)$. We then obtain a permutation group on N(1) defined by

$$\overline{G}(1) = \{\overline{g} \in \tau(N(1)) \mid g \in G\}.^3$$

In the case where $G = G_F$, the group $\overline{G}(1)$ can be thought of as the set of permutations among the committees remaining invariant under F.

We can now apply the same construction to $\overline{G}(1)$ that was applied to G, first partitioning N(1) into sets $\{N_i(2)\}_{i=1}^{n_2}$ by means of the equivalence relation on N(1) induced by transpositions $g_{ij} \in \overline{G}(1)$ and then obtaining a permutation group $\overline{G}(2)$ on $N(2) = \{1,2,...,n_2\}$. We continue the process until we reach a stage r where $n_r = n_{r+1}$ (the equivalence sets are all singletons), at which point we stop since no additional decomposition can occur. The group $\overline{G} = \overline{G}(r)$ is then a permutation group on $N(r) = \{1,2,...,n_r\}$ and we call it the <u>imprimitive residue</u> of G. In Example 0, it can be checked that

$N(1) = \{1,2,3\}$, $\overline{G}_F(1) = \tau(N(1))$, $N(2) = \{1\}$, and $\overline{G}_F(2) = \overline{G}_F$ is the trivial permutation group on one element. The following lemma will be useful in the next section.

<u>Lemma 1</u>. A permutation group G is transitive $\Longleftrightarrow$ its imprimitive residue $\overline{G}$ is transitive.

<u>Proof</u>. By the nature of our definition of G, it suffices to show that G is transitive $\Longleftrightarrow \overline{G}(1)$ is transitive. Assuming G is transitive, consider any two equivalence sets $N_i(1)$ and $N_j(1) \subseteq N$, and choose $i_1 \in N_i(1)$ and $j_1 \in N_j(1)$. By transitivity of G, there exists $g \in G$ such that $i_1 g = j_1$. It follows directly that for $\overline{g} \in \overline{G}(1)$, $i\overline{g} = j$, so $\overline{G}(1)$ is transitive. Conversely, if $\overline{G}(1)$ is transitive, consider any $i,j \in N$. Let $N_{i_1}(1)$ and $N_{j_1}(1)$ be the equivalence sets of N containing, respectively, i and j. Then i_1 and j_1 are in N(1) and by transitivity of $\overline{G}(1)$ on N(1), there exists $h \in \overline{G}(1)$ with $i_1 h = j_1$. Choose any $g \in G$ with $\overline{g} = h$ and we must have $ig = k \in N_{j_1}(1)$. It follows that $g' = gg_{kj} \in G$ and $ig' = j$, so G is transitive.

Q.E.D.

III. <u>Transitivity and Equipotency</u>

If individual voters cannot be indifferent between x and y, then we obtain the "dichotomous" voting rules considered in [1]. Letting $B = \{1,-1\}$, we can view such rules as functions $F:B^n \to C$. In this case, the power index of Shapley and Shubik [4] may be used to measure the relative power p_i of each voter $i \in N$. For this index to be meaningful we require (in addition to duality) two more social choice conditions on F. G   c,d ε C^n, we write $c \leq d$ if $c_i \leq d_i$ $\forall$ $i = 1,2,\ldots,n$. Also if $k = 0,1,2,\ldots,n$, we let 1^k denote the n-tuple in B^n whose first k components are 1's with the rest being -1's.

<u>Definition</u>: F is <u>monotonic</u> if $x \leq y \Rightarrow F(x) \leq F(y)$.

<u>Definition</u>: F is <u>Pareto</u> if $F(1^n) = 1$ and $F(-1^n) = -1$.

Given $F:B^n \to C$ monotonic, Pareto, and dual, define for each $g \in G_F$ the integer $k = [g]$ as follows:

$$[g] = k \iff F(1^k{}_g) = 1 \text{ and } F(1^{k-1}{}_g) \neq 1.$$

The well-definedness of [g] follows from monotonicity, Pareto and the definition of $_F$. We can now define the power of voter i by

$$p_i = \frac{1}{n!}\,|\{g \in G_F \mid [g]g = i\}|.$$

The "story" behind p_i is the now familiar one first described in [4]. Thus p_i is the percentage of time voter i will cast the "pivotal" vote assuming that all permutations of voting orders are equally likely and that all voters preceding i vote as i does.

<u>Theorem 1</u>. If there exists $\sigma \in G_F$ such that $i\sigma = j$, then $p_i = p_j$.

<u>Proof</u>. Given any $g \in G$ and a specific σ with $i\sigma = j$, we first show that $[g\sigma] = [g]$. Indeed, $F(1^k{}_{g\sigma}) = 1 \iff F(1^k{}_g) = 1$ by definition of $\sigma \in G_F$ and we therefore have $[g\sigma] = [g]$. Let $G_i = \{g \in G_F \mid [g]g = i\}$. Then $g \in G_i \iff [g]g = i \iff [g\sigma]g = i \iff [g\sigma]g\sigma = i\sigma = j \iff g\sigma \in G_j$, so σ is a bijection between G_i and G_j. Thus $p_i = |G_i| = |G_j| = p_j$.

Q.E.D.

By the definition of transitivity, we immediately obtain:

<u>Theorem 2</u>. Given $F:B^n \to C$ monotonic, Pareto, and dual. Then G_F transitive $\Rightarrow p_i = p_j$ $\forall$ $i,j \in N$.

Theorem 2 provides considerable motivation for our soon to be formulated definition of an equipotent voting role. Before providing added motivation, we note that Theorem 2 along with Lemma 1 considerably simplifies and strengthens a result obtained in [1]. There it was shown that for F as in Theorem 2, sufficient conditions for $p_i = p_j$ $\forall$ $i,j \in N$ are:

i) The imprimitive residue of $\overline{G}_F$ of G_F is either degenerate ($n_r = 1$) or cyclic of order n_r.

ii) The various equivalence sets $N_i(s)$ satisfy $|N_i(s)| = |N_j(s)|$ $\forall$ $1 \leq s \leq r$ and $1 \leq i, j \leq n_s$.

Since condition i) alone implies transitivity (using Lemma 1) it is clear that Theorem 2 greatly simplifies the result of [1]; and Theorem 2 is genuinely stronger if we can exhibit a transitive G_F whose imprimitive residue $\overline{G}_F$ is neither degenerate nor cyclic. This is provided by:

<u>Example 1</u>. Let $N = \{1,2,3,4,5,6\}$ and let the voting rule $F:B^6 \to C$ be simple majority rule <u>except</u> that in cases of a 3 to 3 "tie,"

a): $d_1 = d_2$ or $d_3 = d_4$ or $d_5 = d_6 \Rightarrow F(d) = 0$

b): a) inapplicable and $|\{i \mid d_i = 1\} \cap \{1,3,5\}|$ odd $\Rightarrow F(d) = 1$

c): a) inapplicable and $|\{i \mid d_i = 1\} \cap \{1,3,5\}|$ even $\Rightarrow F(d) = -1$.

It can be checked that F (which is not as frightening as it looks) is monotone, Pareto, dual, and has (using cycle notation)

$G_F = \langle(1),(12)(34),(13)(24),(14)(23),(12)(56),(15)(26),(16)(25),$
$(34)(56),(35)(46),(36)(45)\rangle$, where $\langle\cdot\rangle$ denotes "the group generated by." Since G_F is clearly transtive and $G_F = \overline{G}_F$, Example 1 achieves its purpose.

In order to strengthen further the implication from transitivity to equipotency, we make the following observations. While use of the Shapley-Shubik index is natural in the context of permutation groups, there are various other well-developed measures of power (eg., [5] and [6]) defined on simple n-person games and hence applicable to $F:B^n \to C$.[4] Without giving details, we note that for a simple game $v:2^N \to \{0,1\}$, the group G_v of permutations on N which preserve winning coalitions can be defined. Any $\sigma \in G_v$ with $i\sigma = j$ will provide a bijection between the winning coalitions containing i and those containing j. For any "reasonable" power index (including all those referred to above) it must then be the case that all players have equal power whenever G_v is transitive.

In [1] an example is given of a voting rule $F:B^n \to B$ (and hence of a simple n-person game) in which players have equal power with respect to a variety of power indices, but whose associated permutation group is not transitive. Since this example is not dual (its associated game v turns out to be improper), a converse to theorem 2 is still possible. A weaker result which seems likely is that $p_i = p_j \; \forall \, i,j \in N$ and condition (ii) above will imply transitivity of G_F. The situation is further complicated by the fact that, for the more general case of

$F:C^n \to C$, there is to date no satisfactory definition for measuring power. If we had such a power index (or a variety of them), it might well be the case that equal powers would imply transitivity or at least that no counterexample (common to all indices) would be evident. Isbell [7] has also considered simple games whose associated permutation groups are transitive and he suggestively calls such games "fair." In any event, we have completed our motivational case for defining a voting rule to be equipotent if its associated permutation group is transitive; and we formally do so in the next section. Even if transitivity should turn out to be more restrictive than equality of individual powers, the results obtained should be of independent interest.

IV. Equipotency and Simple Majority Rule

The important paper by May [2] characterizes a voting rule $F:C^n \to C$ as simple majority rule if and only if F is anonymous, dual, and strongly monotonic. Anonymity means that $G_F = \tau(N)$ (so any two voters can interchange their votes without affecting the outcome), duality has already been defined, and F is strongly monotonic if and only if F is monotonic and $\forall c,d \in C^n$,

$$F(c) = 0 \text{ and } d > c \Rightarrow F(d) = 1$$
$$F(c) = 0 \text{ and } c > d \Rightarrow F(d) = -1.$$[5]

A major motivation for our paper was the question of what happens in May's theorem if anonymity is weakened to the requirement that G_F is a transitive permutation group. Let $F:C^n \to C$ be a not necessarily dual voting rule throughout this section.

<u>Definition</u>: $F:C^n \to C$ is <u>equipotent</u> if G_F is transitive.

<u>Example 2</u>. Let $F:C^3 \to C$ be defined as simple majority rule except that

$$F(1,-1,0) = F(0,1,-1) = F(-1,0,1) = 1$$

and

$$F(-1,1,0) = F(0,-1,1) = F(1,0,-1) = -1.$$

Then ${}_F = \{(1), (123), (132)\}$, a transitive group on $N = \{1,2,3\}$, so F is equipotent.

It is easily checked that the voting rule of Example 2 is both dual and strongly monotonic, so weakening anonymity to equipotency allows a class of voting rules that properly includes simple majority rule. We now propose two social choice conditions which, in conjuction with equipotency, will once again characterize simple majority rule.

<u>Definition</u>: $F:C^n \to C$ is <u>stable</u> if $\forall$ $c,d \in C^n$ with $c + d \in C^n$,

$$F(c) = F(d) \Rightarrow F(c + d) = F(c) = F(d).$$

<u>Definition</u>: $F:C^n \to C$ is <u>unanimous</u> if $F(1^n) = 1$, $F(-1^n) = -1$ and $F(0) = 0$.

Stability requires that agreement of F on two profiles implies agreement on their algebraic sum, provided this sum remains in C^n. In particular, agreement on two "disjoint" profiles (their non-indifferent members do not overlap) forces agreement on their sum. Also "cancellation" of an oppositely voting individual in the two profiles is also permissible ($-1 + 1 = 0 \in C$), but addition of the same non-indifferent preferences of an individual is not ($1 + 1 = 2 \notin C$). Unanimity says that if voters are unanimous in their preference (including "preference" for indifference), then the social decision must support this preference.

Clearly unanimity is implied by Pareto and duality, but we are no longer assuming duality.

<u>Theorem 3</u>. $F:C^n \to C$ is equipotent, stable, and unanimous $\Leftrightarrow$ F is simple majority rule.

<u>Proof</u>. It is easily checked that simple majority rule implies the stated conditions, so we concentrate on the left to right implication. For $i,j \in N$ with $i \neq j$, let $d^{i,j}$ denote the $d \in C^n$ with $d_i = 1$, $d_j = -1$, and all other coordinates 0. We first show that $F(d^{i,j}) = 0$. <u>Suppose that $F(d^{i,j}) = 1$</u>, and choose $\sigma \in G_F$ with $i\sigma = j$ using equipotency of F. Let m be the order of σ (m is the smallest positive integer for which $\sigma^m = (1)$). Since F is invariant under the action of $\sigma^{-1} \in G_F$, we have

$$1 = F(d^{i,j}) = F(d^{i,i\sigma}) = F(d^{i\sigma,i\sigma^2}) = \ldots = F(d^{i\sigma^{m-1},i}).$$

By stability of F we then have

$$F(d^{i,i\sigma^2}) = 1 \qquad (\text{since } d^{i,i\sigma^2} = d^{i,i\sigma} + d^{i\sigma,i\sigma^2})$$

$$F(d^{i,i\sigma^3}) = 1 \qquad (\text{since } d^{i,i\sigma^3} = d^{i,i\sigma^2} + d^{i\sigma^2,i\sigma^3})$$

$$\vdots$$

$$F(d^{i,i\sigma^{m-1}}) = 1 \quad (\text{by a straightforward induction argument}).$$

This give us $F(d^{i\sigma^{m-1},i}) = F(d^{i,i\sigma^{m-1}}) = 1$ and hence by stability, $F(0) = F(d^{i\sigma^{m-1},i} + d^{i,i\sigma^{m-1}}) = 1$, which violates unanimity. Hence $F(d^{i,j}) \neq 1$ and a completely analogous argument using -1 gives

$F(d^{i,j}) \neq -1$. Thus $F(d^{i,j}) = 0 \ \forall \ i,j \in N$ with $i \neq j$. We now introduce the notation

$$1(d) = \{i \in N \mid d_i = 1\} \text{ and } -1(d) = \{i \in N \mid d_i = -1\}.$$

We have so far shown that $|1(d)| = |-1(d)| = 1 \Rightarrow F(d) = 0$. A straightforward inductive application of stability then establishes $|1(d)| = |-1(d)| \Rightarrow F(d) = 0$. Now consider $d \in C^n$ with $|1(d)| = 1$ and $|-1(d)| = 0$. Since F is equipotent, F must take the same value on all such d. Repeated use of stability and an application of unanimity ($F(1^n) = 1$) then requires that $F(d) = 1$. Similar arguments establish generally that $1(d) \neq \emptyset$, $-1(d) = \emptyset \Rightarrow F(d) = 1$ and $1(d) = \emptyset$, $-1(d) \neq \emptyset \Rightarrow F(d) = -1$ (thus F has the strong Pareto property). Finally, consider $d \in C^n$ with $|1(d)| > |-1(d)|$. Suppose that $F(d) = 0$ and choose $c \in C^n$ with $1(c) = -1(d)$, $-1(c) \subseteq 1(d)$, and $|1(c)| = |-1(c)|$. Then $F(d) = 0$, $F(c) = 0$ (by earlier results), and $c + d \in C^n$ with $1(c+d) \neq \emptyset$ and $-1(c+d) = \emptyset$. Stability requires $F(c+d) = 0$, contradicting the strong Pareto result obtained above. Thus we cannot have $F(d) = 0$. Suppose that $F(d) = -1$ and choose $c \in C^n$ with $1(c) = \emptyset$, $-1(c) \subseteq 1(d)$, and $|-1(c)| = |1(d)| - |-1(d)|$. Then $c + d \in C^n$ and $|1(c+d)| = |-1(c+d)|$, so $F(c + d) = 0$. But stability and $F(c) = F(d) = -1$ provide the desired contradiction, so we cannot have $F(d) = -1$. It follows that $|1(d)| > |-1(d)| \Rightarrow F(d) = 1$ and by analogous arguments, $|1(d)| < |-1(d)| \Rightarrow F(d) = -1$. We have shown overall that

$$f(d) = \begin{cases} 0 & \text{if } |1(d)| = |-1(d)| \\ 1 & \text{if } |1(d)| > |-1(d)| \\ -1 & \text{if } |1(d)| < |-1(d)| \end{cases}$$

so F is indeed simple majority rule.

Q.E.D.

In the spirit of [2] we now show that equipotency, stability, and unanimity are independent by exhibiting voting rules which satisfy all combinations of two of these three conditions. Example 2 given previously can be seen to be equipotent and unanimous, but stability fails since $F(1,-1,0) = F(0,1,-1) = 1$, but $F(1,0,-1) = -1$. For another such example where the stability fails at 0 rather than at 1, consider

Example 3: Let $F^{(n)}$ denote simple majority rule for n voters and let $F:C^6 \to C$ be the representative system defined by

$$F(d) = F^{(3)}(F^{(2)}(c_1,c_2),F^{(2)}(c_3,c_4),F^{(2)}(c_5,c_6)).$$

It can be checked that F is equipotent and unanimous, but $F(1,0,-1,0,0,0) = F(0,1,0,0,-1,0) = 0$ and $F(1,1,-1,0,-1,0) = -1$, so F is not stable.

A simple example which is stable and unanimous, but not equipotent is provided by

Example 4: Let $F:C^2 \to C$ be simple majority rule except that $F(1,-1) = 1$ and $F(-1,1) = -1$. Then $G_F = \{(1)\}$ and fails to be transitive.

Finally, we need not work hard to obtain an equipotent, stable, but nonunanimous voting rule.

Example 5: Let $F:C \to C$ be defined by

(a) $F(1) = F(0) = 1$, $F(-1) = -1$.

(b) $F(d) = 0 \ \forall \ d \in C$.

Each of 5(a) and 5(b) is trivially equipotent and stable, but fails to be unanimous. Less trivial and more realistic examples could also be supplied to establish the independence of equipotency, stability, and unanimity.

V. Conclusions

We have taken cues from two important papers in social choice theory in what we have done. Bartoszynski's development [1], which we have streamlined somewhat, led to results connecting transitivity and equipotency. Consideration of May's Theorem [2] then evolved into an alternative characterization of simple majority rule in which transitivity (equipotency) plays an important role. The results obtained indicate that the condition of permutation group transitivity provides a natural and significant statement of individual equity in social choice theory (and in game theory).

Another important aspect of the results in [1] is the decomposition into a hierarchical committee structure which they provide. These results have been viewed as an alternative approach to the theory of representative systems as developed in Fishburn [8, chapters 3 and 4]. Since the committees obtained in a group theoretic decomposition are disjoint, this approach will succeed only for representative systems whose various committees (at all hierarchical levels) are disjoint. Thus Example 2 is a representative system according to the following structure (recall that $F^{(k)}$ is simple majority rule with k voters):

$$F(d_1,d_2,d_3) = F^{(3)}(F^{(2)}(d_1,F^{(2)}(d_1,d_2)),F^{(2)}(d_2,F^{(2)}(d_2,d_3)), F^{(2)}(d_3,F^{(2)}(d_3,d_1))).$$

The group theoretic decomposition for F has each voter being his own committee and clearly does not reflect the essential nature of this representative system.

The stability condition presented here is, to our knowledge, new. It does have aspects in common with the consistency property given by Young [9] and Smith [10] in their Borda count characterizations. Nonetheless, our characterization of simple majority rule appears to be quite different from either May's characterization or what falls out from the Borda count characterizations applied with only two alternatives.

It is hoped that replacing the commonly used anonymity condition with our weaker transitivity condition may lead to interesting characterizations of other well known voting rules as well as a clean characterization of "fair" representative systems (see [8]).

We conclude by posing a "backwards" question which might have interesting applications in social choice theory. Given a permutation group G on N, what can we say about the collection of voting rules $F:C^n \to C$ such that $G_F = G$ and are there group theoretic properties which will force the presence of various well-established social choice conditions?

FOOTNOTES

1. More precisely, if $ig = j$, then voter i in the profile d_g takes the preference of voter $ig = j$ in the profile d. Thus, strictly speaking, d_g permutes d as $g^{-1} \in G_F$ permutes N.

2. Transitivity in the context of permutation groups appears to have no direct connection with transitivity of a binary relation. Since the terminology is standard, we stay with it with the belief that no serious confusion will result.

3. If we let β denote the homomorphism from G into $\overline{G}(1)$ defined by $g\beta = \overline{g}$, we can let $G(1) \subseteq G$ be the kernel of β. It follows directly that $G(1) \cong \tau(N_1(1)) \otimes \tau(N_2(1)) \otimes \cdots \otimes \tau(N_{n_1}(1)$ and that $G/G(1) \cong \overline{G}(1)$. Bartoszynski [1] claims that $G \cong G(1) \otimes \overline{G}(1)$, which is not technically correct, through G is isomorphic to a wreath product of $G(1)$ and $\overline{G}(1)$ when G is a transitive permutation group.

4. Specifically, a voting rule $F:B^n \to C$ can be associated with the simple game v defined as follows. For $S \subseteq N$, let $x_S = \{(x_i,\ldots,x_n) \in B^n \mid x_i = 1 \Longleftrightarrow i \in S\}$.

Then let

$$v(S) = \begin{cases} 1 \text{ if } F(x_S) = 1 \\ 0 \text{ otherwise} \end{cases}$$

Note that v as defined treats indifference under F as a "no" vote. Also, v will be proper $(v(S) = 1 \Longrightarrow v(N \backslash S) = 0)$ if F is dual; v will be nontrivial $(v(N) = 1)$ if F is Pareto; and v will be monotone $(S \subseteq T \Longrightarrow v(S) \leq v(T))$ if F is monotone.

5. By $c > d$ we mean $c_i \geq d_i$ for all i and $c_j > d_j$ for some j.

REFERENCES

[1] Bartoszynski, R., "Power Structure in Dichotomous Voting," Econometrica 40 (1972):1003-1019.

[2] May, K. O., "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision," Econometrica 20 (1952): 680-684.

[3] Hall, Jr., M., The Theory of Groups 2nd ed. (Bronx, New York: Chelsea Publications, 1967).

[4] Shapley, L.S., and M. Shubik, "A Method of Evaluating the Distribution of Power in a Committee System," The American Political Science Review 48 (1954):787-792.

[5] Banzhaf, J.F., "Weighted Voting Doesn't Work: A Mathematical Analysis," Rutgers Law Review 19 (1965):317-343.

[6] Deegan, J., and E. Packel, "A New Index of Power for Simple n-Person Games," (mimeo, The University of Rochester, 1977).

[7] Isbell, J. R., "Homogeneous Games. II," Proceedings of the American Math. Society 11 (1960):159-161.

[8] Fishburn, P.C., The Theory of Social Choice (Princeton, New Jersey: Princeton University Press, 1973).

[9] Young, H. P., "An Axiomatization of Borda's Rule," Journal of Economic Theory 9 (1974):43-52.

[10] Smith, J. H., "Aggregation of Preferences and Variable Electorate," Econometrica 41 (1973):1027-1041.